A family of graded epistemic logics

Mario R. F. Benevides[2,3]

PESC/COPPE - Inst. de Matemática — Universidade Federal do Rio de Janeiro

Alexandre Madeira[1,4]

QuantaLab and HASLab INESC TEC, U. Minho & CIDMA, U. Aveiro, Portugal

Manuel A. Martins[1,5]

CIDMA, Dep. Mathematics, U. Aveiro, Portugal

Abstract

Multi-Agent Epistemic Logic has been investigated in Computer Science [5] to represent and reason about agents or groups of agents knowledge and beliefs. Some extensions aimed to reasoning about knowledge and probabilities [4] and also with a fuzzy semantics have been proposed [6,13].
This paper introduces a parametric method to build graded epistemic logics inspired in the systematic method to build Multi-valued Dynamic Logics introduced in [11,12]. The parameter in both methods is the same: an action lattice [9]. This algebraic structure supports a generic space of agent knowledge operators, as choice, composition and closure (as a Kleene algebra), but also a proper truth space for possible non bivalent interpretation of the assertions (as a residuated lattice).

Keywords: Epistemic Logic, Action Lattice, Modal Logics

1 Introduction

The analysis and the applications of concepts such as agent's knowledge, everybody's knowledge and common knowledge became a stimulating research field, particularly in the last decades, when *epistemic logics* emerged. Although, the work of Hintikka [8] can be considered the founder of modern modal epistemic logic, most

[1] This work is supported by ERDF European Regional Development Fund, through the COMPETE Programme, and by National Funds through FCT - Portuguese Foundation for Science and Technology - within projects POCI-01-0145-FEDER-016692 and UID/MAT/04106/2013. A. Madeira is also supported by the FCT individual grant SFRH/BPD/103004/2014

[2] This work is supported by the Brazilian research agencies CNPq, FAPERJ and CAPES.

[3] Email: mario@cos.ufrj.br

[4] Email: amadeira@inesctec.pt

[5] Email: martins@ua.pt

of these logics are heavily influenced by the work of Halpern et al [5] on modal logics of knowledge in a multi-agent systems framework. Modal logics of knowledge describe how an agent reasons about his own knowledge and about the knowledge of other agents. We say that an agent knows a fact φ if φ is true in every state that the agent considers possible. "The intuition is that if an agent does not have complete knowledge about the world, he will consider a number of possible worlds. These are his candidates for the way the world actually is" [5].

Much of the agreement and cooperation in a group of agents is reached considering the interaction among the agents and the increasing group knowledge acquisition. A fact φ is mutual knowledge in a group of agents, if each agent knows φ. This group knowledge is also known as everybody's knowledge. Suppose, for instance, that each participant in a conference knows that the lecturer will arrive late. The fact that the lecturer will arrive late is mutual knowledge among the participants, but each participant may think that he is the only one who knows about that. However, suppose that one of the participants makes an announcement for the audience: "The lecturer told me that he will arrive late". From this moment onwards, each participant knows that each participant knows that the lecturer will arrive late, and each participant knows that each participant knows that each participant knows that the lecturer will arrive late, and so on. The participant's statement turned the fact that was mutually known into a common knowledge fact.

There are many situations where we have uncertainty in our knowledge and beliefs. It is not unusual to believe in some fact with some grade of possibility. For instance, *Anne believes that her father has a strong preference for Bob, which means that she believes that he will give a sweet to Bob rather than to Clara. In a scale from 0 to 5, her belief is 4.* This kind of belief is not true or false. In this work we deal with graded knowledge, but atomic propositions are true or false.

In [5] Multi-Agent Epistemic Logics has been investigated, to represent and reason about agents or groups of agents knowledge and beliefs. There are many proposals to extend these logics with uncertainty. Some extensions aimed to reasoning about knowledge and probabilities [4]. In general, this is accomplished extending the language with weighted formulas and adding probabilities to the semantics. There are other attempts that provide a fuzzy or many valued semantics [6,13]. This work goes in the later direction.

The work of Fitting [6] proposes a many valued modal logic where the truth values are taken from a lattice.It is presented two semantics, one where the atomic propositions are many valued and a second one where the accessibility relation also is many valued. Also, in [3], it is presented a many-valued modal logic over a finite residuated lattice. In [13] it is introduced an epistemic logic based on the work of Fitting. It differs from ours because they work with a particular lattice. Another related work that uses a complete, distributive lattices as semantics for epistemic and doxastic logics is presented in [7]. More recently, some interesting works have appeared to deal with many valued dynamic epistemic logic [16,10].

In [11,12] it is proposed a method to build Multi-valued Dynamic Logics. Inspired on this method, we introduce a method to build graded Multi-Agent Epistemic logics. Both methods are based on Action Lattices [9]. Using action lattices, we are able to support a generic space of agent knowledge operators, as choice,

composition and closure (as a Kleene algebra), but also a proper truth space for possible non bivalent interpretation of the assertions (as a residuated lattice). We use matricial algebra to be able to introduce knowledge representations as weighted graphs, which enables us to capture a wide class of weighted scenarios, from the classic bivalent perspective of knowledge, to other structured, discrete and continuous, domains. It should be notice that, in this work, we only deal with the epistemic notions of knowledge and their duals.

This paper is organized as follows. Section 2 presents all the background needed about multi-agent epistemic logic. Section 3, introduces our method for building graded Multi-Agent Epistemic logics. It also provides some concepts on Kleene algebras and action lattices. Section 4 illustrates the use of our method with two examples. Section 5 discusses some conditions where classical axioms of Multi-Agent Epistemic Logic are valid and points out some future works.

2 Multi-Agent Epistemic Logic

Multi-agent epistemic logic has been investigated in Computer Science [5] to represent and reason about agents or groups of agents knowledge and beliefs.

2.0.1 Language and Semantics

Definition 2.1 *The epistemic language consists of a set Φ of countably many proposition symbols, a finite set $\mathcal{A}$ of agents, the boolean connectives $\neg$ and $\wedge$, a modality K_a for each agent a. The formulas are defined as follows:*

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi_1 \wedge \varphi_2 \mid K_a\varphi \mid C_G\varphi$$

where $p \in \Phi$, $a \in \mathcal{A}$ and $G \subseteq \mathcal{A}$.

The standard connectives can be presented as abbreviations, namely $\bot \equiv \neg\top$, $\varphi \vee \phi \equiv \neg(\neg\varphi \wedge \neg\phi)$, $\varphi \to \phi \equiv \neg(\varphi \wedge \neg\phi)$ and $E_G\varphi \equiv \bigwedge_{a\in G} K_a\varphi$.

The intuitive meaning of the modal formulas are:

- $K_a\varphi$ - agent a knows φ;
- $E_G\varphi$ - every agent $a \in G$ knows φ;
- $C_G\varphi$ - it is common knowledge among all members of group G that it is the case that φ.

We also introduce, by definition, the dual operators $B\varphi \equiv \neg K\neg\varphi$ and $M_G\varphi \equiv \neg E_G\neg\varphi$.

Definition 2.2 *A multi-agent epistemic* frame *is a tuple $\mathcal{F} = (W, R_a)$ where*

- *W is a non-empty set of states;*
- *R_a is a binary relation over W, for each agent $a \in \mathcal{A}$;*

  We also define the following relations

- $R_G = \bigcup_{a\in G} R_a$
- $R_G^* = (R_G)^*$, *where $(R_G)^*$ is the reflexive, transitive closure of R_G.*

Definition 2.3 *A* multi-agent model *is a pair* $\mathcal{M} = (\mathcal{F}, \mathbf{V})$*, where* $\mathcal{F}$ *is a frame and* $\mathbf{V}$ *is a valuation function* $\mathbf{V} : \Phi \to 2^W$*.*

In most applications of multi-agent epistemic logic the relations R_a are equivalence relations. In this case, models are called *epistemic models* and, in these structures, if G is not the empty group of agents, R_G^* coincides with R_G^+, for R_G^+ being the transitive closure of R_G.

Definition 2.4 *Given a multi-agent model* $\mathcal{M} = \langle S, R_a, V \rangle$*. The notion of satisfaction* $\mathcal{M}, s \models \varphi$ *is defined as follows*

- $\mathcal{M}, s \models p$ *iff* $s \in V(p)$
- $\mathcal{M}, s \models \neg\phi$ *iff* $\mathcal{M}, s \not\models \phi$
- $\mathcal{M}, s \models \phi \wedge \psi$ *iff* $\mathcal{M}, s \models \phi$ *and* $\mathcal{M}, s \models \psi$
- $\mathcal{M}, s \models K_a\phi$ *iff for all* $s' \in S : sR_as' \Rightarrow \mathcal{M}, s' \models \phi$
- $\mathcal{M}, s \models C_G\phi$ *iff for all* $s' \in S : sR_G^*s' \Rightarrow \mathcal{M}, s' \models \phi$

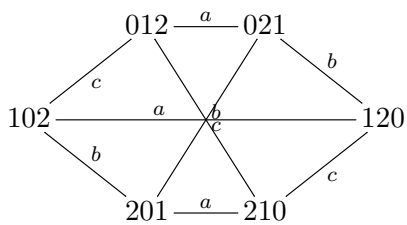
It is easy to see that $\mathcal{M}, s \models E_G\phi$ *iff for all* $s' \in S : sR_Gs' \Rightarrow \mathcal{M}, s' \models \phi$*.*

Example 1 (An adaptation from [17]) *Suppose a father has three envelopes, each containing:* **0**, **1** *and* **2** *dollars inside respectively.The father has three children:* **a***nne,* **b***ob and* **c***lara. Each child receives one envelope and do not know content of the envelopes of the other children.*

We use proposition symbols $0_x, 1_x, 2_x$ *for* $x \in \{a, b, c\}$ *meaning "child* x *has envelope* **0**, **1** *or* **2***. We name each state by the envelope that each child has in that state, for instance* 012 *is the state where child* **a** *has* **0***, child* **b** *has* **1** *and child* **c** *has* **2***. A state name underlined means current state. The following epistemic model represents the epistemic state of each agent*[6] *.*

Hexa $= \langle S, R_a, R_b, R_c, V \rangle$*:*

- $S = \{012, 021, 102, 120, 201, 210\}$
- $R_a =$ $\{(012, 012), (012, 021), (021, 021), \dots\}$, ...
- $V(0_a) = \{012, 021\}$, $V(1_a) =$ $\{102, 120\}$, ...



It is not difficult to see that $012 \models B_b0_a$ *and* $012 \models B_aK_c2_c$ *hold, but* $021 \models E_{ac}2_b$ *does not hold.*

3 Parametric construction of Graded Epistemic Logics

We introduce, in this paper, a parametric method to build graded epistemic logics inspired in the systematic method to build multi-valued dynamic logics introduced in [11,12]. Both methods are based in the same parameter: an action lattice [9].

[6] We omit the reflexive loops in the picture

3.1 Kleene algebras, action lattices and graded knowledges representation

Action lattices support a generic space of agent knowledge operators, as choice, composition and closure (as a Kleene algebra), but also a proper truth space for possible non bivalent interpretation of the assertions (as a residuated lattice). Observe that the original motivations of Kozen to introduce Action Lattices were very different for these ones. Originally, the residues were introduced within *Action Algebra* [15] as a necessary technicality to obtain a finitely-based equational variety to reason about imperative programs. Then, Kozen adjusted this notion into the *Action Lattice* in [9] by introducing and axiomatizing a meet operation, in order to recover the closeness by matricial formation of the Kleene Algebras [2]. We overview, in the following, the action algebra with some relevant examples in the context of our purpose. A lot of other examples and properties can be found in [11]. The structure of Kleene algebra will be used to model the set of agent knowledge operators over a set of agents $\mathcal{A}$. In our setting, the valuations of propositions are crisp, i.e., true or false. This forces the integrability on the action lattices adopted.

$$a + (b + c) = (a + b) + c \quad (1)$$
$$a + b = b + a \quad (2)$$
$$a + a = a \quad (3)$$
$$a + 0 = 0 + a = a \quad (4)$$
$$a; (b; c) = (a; b); c \quad (5)$$
$$a; 1 = 1; a = a \quad (6)$$
$$a; (b + c) = (a; b) + (a; c) \quad (7)$$
$$(a + b); c = (a; c) + (b; c) \quad (8)$$
$$a; 0 = 0; a = 0 \quad (9)$$
$$1 + a + (a^*; a^*) \leq a^* \quad (10)$$
$$a; x \leq x \Rightarrow a^*; x \leq x \quad (11)$$
$$x; a \leq x \Rightarrow x; a^* \leq x \quad (12)$$
$$a; x \leq b \Leftrightarrow x \leq a \to b \quad (13)$$
$$a \to b \leq a \to (b + c) \quad (14)$$
$$(x \to x)^* = x \to x \quad (15)$$
$$a \cdot (b \cdot c) = (a \cdot b) \cdot c \quad (16)$$
$$a \cdot b = b \cdot a \quad (17)$$
$$a \cdot a = a \quad (18)$$
$$a + (a \cdot b) = a \quad (19)$$
$$a \cdot (a + b) = a \quad (20)$$

Fig. 1. Axiomatisation of action lattices (from [9])

Definition 3.1 (Kleene Algebra) *A* Kleene algebra *is an idempotent (and thus partially ordered) semiring endowed with a closure operator $*$, i.e. it consists of a tuple $(A, +, ;, 0, 1, *)$ where A is a set, $+$ and $;$ are binary operations, $*$ is an unary operation and $0, 1$ are constants satisfying the axioms (1)–(12) (the relation $\leq$ is the natural order induced by the operation $+$: $a \leq b$ iff $a + b = b$).*

Note that (4) implies that 0 is the minimum element in any Kleene algebra. Conway shown in [2] that we can endow the class of all matrices over a Kleene algebra with a Kleene structure. We recall this procedure here: given a Kleene algebra $\mathbf{A} = (A, +, ;, 0, 1, *)$ we define a Kleene algebra $\mathbb{M}_n(\mathbf{A}) = (M_n(\mathbf{A}), \mathbf{+}, \mathbf{;}, \mathbf{0}, \mathbf{1}, \mathbf{*})$ as follows:

(i) $M_n(\mathbf{A})$ is the space of $(n \times n)$-matrices over $\mathbf{A}$

(ii) for any $A, B \in M_n(\mathbf{A})$, define $M = A + B$ by $M_{ij} = A_{ij} + B_{ij}$, $i, j \leq n$.

(iii) for any $A, B \in M_n(\mathbf{A})$, define $M = A \mathbf{;} B$ by $M_{ij} = \sum_{k=1}^{n} (A_{ik}; B_{kj})$ for any $i, j \leq n$.

(iv) $\mathbf{1}$ and $\mathbf{0}$ are the $(n \times n)$-matrices defined by $\mathbf{1}_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$ and $\mathbf{0}_{ij} = 0$, for any $i, j \leq n$.

(v) for any $M = [a] \in \mathbb{M}_1(\mathbf{A})$, $M^* = [a^*]$; for any $M = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \in M_n(\mathbf{A})$, $n > 1$, where A and D are square matrices, define

$$M^* = \left[\begin{array}{c|c} F^* & F^* \,;\, B \,;\, D^* \\ \hline D^*; C; F^* & D^* + (D^* \,;\, C \,;\, F^* \,;\, B \,;\, D^*) \end{array}\right]$$

where $F = A + B \,;\, D^* \,;\, C$. Note that this construction is recursively defined from the base case ($n = 2$) where the operations of the base action lattice $\mathbf{A}$ are used.

In the present work we take advantage of this matricial algebra to be able to operate knowledge representations as weighted graphs or, more precisely, weighted labelled transition systems. As we will see, this abstract structure capture a wide class of weighted scenarios, from the classic bivalent perspective of knowledge, to other structured, discrete and continuous, domains.

Moreover, as stated, we are interesting in the definition of Graded Epistemic logics with non necessarily boolean degrees of truth. In this view, in order to be able to interpret other logical connectives, we extend our Kleene Algebra of knowledge with some additional structure - namely, with a residue for the interpretation of the logical implication and an infimum to interpret the logical conjunction. This can be found in the following notion of Action Lattice introduced by D. Kozen in [9]. Note, however, that the seminal motivation for this definition was quite distinct of the stated one. In particular, it aimed to adjust the finitely-based equational variety "Action Algebra" of Pratt [15], to an algebra closed under the matricial constructions. Let us recall this notion:

Definition 3.2 *A* action lattice *is a tuple* $\mathbf{A} = (A, +, ;, 0, 1, *, \rightarrow, \cdot)$*, where* A *is a set,* 0 *and* 1 *are constants,* $*$ *is an unary operation in* A *and* $+, ;, \rightarrow$ *and* $\cdot$ *are binary operations in* A *satisfying the axioms enumerated in Figure 5, where the relation* $\leq$ *is induced by* $+$*:* $a \leq b$ *iff* $a + b = b$*. An* integral action lattice *consists of an action lattice satisfying* $a \leq 1$*.*

Beyond the bivalent $\{0, 1\}$-action lattice we consider the following two action lattice that will be used to illustrate our method in Section 4. More examples and properties of action lattices can be found in [11].

Definition 3.3 (Ł - the Łukasiewicz arithmetic lattice) *The* Łukasiewicz arithmetic lattice *is the structure* $\boldsymbol{Ł} = ([0, 1], \max, \odot, 0, 1, *, \rightarrow, \min)$*, where*

- $x \rightarrow y \;=\; \min(1, 1 - x + y)$,
- $x \odot y = \max(0, y + x - 1)$ *and*
- $x^* = 1$.

Definition 3.4 ($\mathbf{W}_k$ finite Wajsberg hoops) *We consider now an action lattice endowing the finite* Wajsberg hoops *[1] with a suitable star operation. Hence, for a fix natural $k > 0$ and a generator a, we define the structure $\mathbf{W}_k = (W_k, +, ; , 0, 1, *, \rightarrow, \cdot)$, where $W_k = \{a^0, a^1, \cdots, a^k\}$, $1 = a^0$ and $0 = a^k$, and for any $m, n \leq k$,*

- $a^m + a^n = a^{min\{m,n\}}$
- $a^m; a^n = a^{min\{m+n,k\}}$
- $(a^m)^* = a^0$
- $a^m \rightarrow a^n = a^{max\{n-m,0\}}$
- $a^m \cdot a^n = a^{max\{m,n\}}$

3.2 A method to build Graded Epistemic Logic

In this section we introduce a method to build multi-agent epistemic logics parameterized by an action lattice. The "on-demand grading" of the logic is only reflected in its semantics; the syntax is the same as in the standard case. The proposition assignment is crisp and only the agent's relations are graded on the underlying action lattice. This non orthodox feature is naturally expressed on the definition of satisfaction.

Let us fix a complete action lattice $\mathbf{A} = (A, +, ; , 0, 1, *, \rightarrow, \cdot)$. We introduce, in the following, a method to generate an $\mathbf{A}$-graded epistemic logic $\mathcal{GE}(\mathbf{A})$:

- Signatures $(\mathrm{At}, \mathrm{Ag})$ where At is a set of atomic propositions and Ag is a finite set of agents.
- Sentences are the standard sentences of Multi-Agent Epistemic Logic:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid K_a\varphi \mid B_a\varphi \mid E_G\varphi \mid M_G\varphi \mid C_G\varphi$$

  where $p \in \mathrm{At}$, $a \in \mathrm{Ag}$, $G \subseteq \mathrm{Ag}$. Note that, here we are explicitly considering the or connective and the dual operators of the ones introduced in Definition 2.1 . Actually, here these operators are not definable because we do not have, in general, a negation.
- Models are structures (W, R, V) where W is a non empty set of states, with cardinality n; R is an Ag-family of $(n \times n)$-matrices of $\mathbb{M}(\mathbf{A})$ and $V : \mathrm{At} \times W \rightarrow \{0, 1\}$ is a valuation function. We use the notation $R_a(w, w')$ to denote the cell (w, w') of the matrix R_a.
- Satisfaction:
  · $(w \models \bot) = 0$
  · $(w \models p) = V(p, w)$, for any $p \in \mathrm{At}$
  · $(w \models \varphi \wedge \varphi') = (w \models \varphi) \cdot (w \models \varphi')$
  · $(w \models \rho \vee \rho') = (w \models \rho) + (w \models \rho')$
  · $(w \models \varphi \rightarrow \varphi') = (w \models \varphi) \rightarrow (w \models \varphi')$
  · $(w \models K_a\, \varphi) = \bigwedge_{w' \in W} \big(R_a(w, w') \rightarrow (w' \models \varphi)\big)$
  · $(w \models B_a\, \varphi) = \bigvee_{w' \in W} \big(R_a(w, w'); (w' \models \varphi)\big)$
  · $(w \models E_G\, \varphi) = \bigwedge_{w' \in W} \big(R_G(w, w') \rightarrow (w' \models \varphi)\big)$
  · $(w \models M_G\, \varphi) = \bigvee_{w' \in W} \big(R_G(w, w'); (w' \models \varphi)\big)$
  · $(w \models C_G\, \varphi) = \bigwedge_{w' \in W} \big(R^*_G(w, w') \rightarrow (w' \models \varphi)\big)$
  for $R_G = \sum_{a \in G} R_a$

4 Examples

We have already discussed an example of epistemic logic in the background section. Such example can be seen as an instantiation of our method over the $\{0, 1\}$ standard action lattice (see [11]). We present two more examples, namely one that deals with discrete degrees of knowledge and, on the same context, another one that admits knowledge ranging over a continuous scale.

Example 2 *Consider here the Graded Epistemic Logic generated by the Wajsberg hoop* $\mathbf{W}_5$ *over* $\{a^0, a^1, a^2, a^3, a^4, a^5\}$ *(Definition 3.4). Recall that the order in* $\mathbf{W}_5$ *is* $a^5 < a^4 < a^3 < a^2 < a^1 < a^0$*. In order to simplify the example, we denote* a^k *by* $5 - k$*, for* $k = 0, \ldots, 5$*. This logic is useful to reasoning about the following variant of Example 2.*
Suppose now that the children are jealous and they have the following beliefs:

(i) **a***nne believes that the father has a strong preference for* **b***ob, which means that she believes that he will give the envelop with higher value to bob than to* **c***lara. In a scale from 0 to 5, her belief is 4; Conversely, her belief that the envelop bob received has a smaller value is 1.*

(ii) **c***lara also believes that the father has a preference for* **b***ob . In a scale from 0 to 5, her belief is 3; and conversely, her belief that the envelop bob received has a smaller value is 1. But if she has the envelop* **2** *then she believes that the father has no preference between* **a***nne and* **b***ob; in that case her belief is 4.*

(iii) **b***ob does not believe that the father has any preference between* **a***nne and* **c***lara. So his belief is 3 indifferently about any situation.*

The following draws represent the beliefs of **a***nna,* **b***ob and* **c***lara. We draw it separately for clarity sake. Moreover, we omit the reflexive loops in the picture with value 5.*

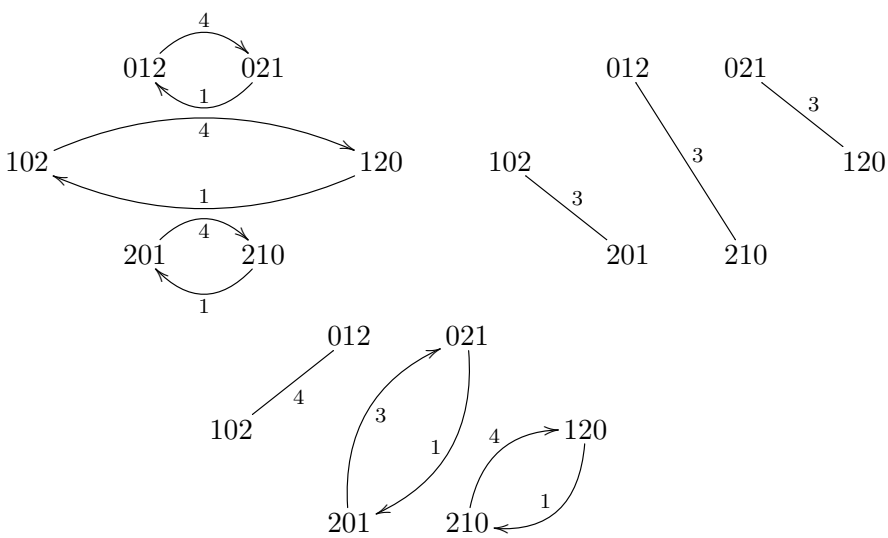


Fig. 2. anna's, **b**ob's and **c**lara's beliefs

We evaluate some formulas in this model. In order to simplify the calculations we use the fact that $a^5 \to x = a^0$*(i.e.,* $0 \to x = 5$*) and* $a^5; x = a^5$ *(i.e.,* $0; x = 0$*).*

$$
\begin{aligned}
012 \models B_b 0_a = \quad & \bigvee \{R_b(012,012); 012 \models 0_a, R_b(012,210); 210 \models 0_a\} = \bigvee\{5;5,3;0\} = 5 \\
012 \models B_a K_c 2_a = \quad & \bigvee \{R_a(012,012); 012 \models K_c 2_a, R_a(012,021); 021 \models K_c 2_a\} \\
= \quad & \bigvee \{5; \bigwedge \{R_c(012,012) \to 012 \models 2_a, R_c(012,102) \to 102 \models 2_a\}, \\
& 4; \bigwedge \{R_c(021,021) \to 021 \models 2_a, R_c(021,201) \to 201 \models 2_a\}\} \\
= \quad & \bigvee \{5; \bigwedge \{5 \to 0, 4 \to 0\}, 4; \bigwedge \{5 \to 0, 1 \to 5\}\} \\
= \quad & \bigvee \{a^0; \bigwedge \{a^0 \to a^5, a^1 \to a^5\}, a^1; \bigwedge \{a^0 \to a^5, a^4 \to a^0\}\} \\
= \quad & \bigvee \{a^0; a^5, a^1; a^5\} = a^5 (= 0)
\end{aligned}
$$

To calculate $M_{ac}2_b$ at 021 we first calculate the matrix of $R_{ac} = R_a + R_c$.

$R_{ac} =$

	012	021	102	120	201	210
012	5	4	4	0	0	0
021	1	5	0	0	1	0
102	4	0	5	4	0	0
120	0	0	1	5	0	1
201	0	0	0	4	5	4
210	0	0	0	4	1	5

Then we have,

$$
\begin{aligned}
021 \models M_{ac} 2_b = \quad & \bigvee \{R_{ac}(021,012); 012 \models 2_b, R_{ac}(021,021); 021 \models 2_b, \\
& R_{ac}(021,201); 201 \models 2_b\} = \bigvee\{1;5,5;5,1;0\} = 5
\end{aligned}
$$

If we consider the group knowledge we have

$$
\begin{aligned}
021 \models E_{ac} 2_b = \quad & \bigwedge \{R_{ac}(021,012) \to 012 \models 2_b, R_{ac}(021,021) \to 021 \models 2_b, \\
& R_{ac}(021,201) \to 201 \models 2_b\} \\
= \quad & \bigwedge\{1 \to 0, 5 \to 5, 1 \to 0\} = \bigwedge\{4,5,4\} = 4
\end{aligned}
$$

Example 3 *Consider now the Graded Epistemic Logic generated by the Łukasiewicz arithmetic lattice* $\boldsymbol{Ł} = ([0,1], \max, \odot, 0, 1, *, \to, \min)$ *(Definition 3.3). This logic is adequate to reasoning about knowledge expressed in the continuous scale* $[0,1]$*. Let us look to the following variant of Example 2.*
Suppose now that the children have the following beliefs:

(i) **a***nne believes that the father has a strong preference for* **b***ob, which means that she believes that he will give the envelop with higher value to bob than to* **c***lara. Her belief is* $\frac{4}{5}$*; moreover her belief that the value is less is* $\frac{1}{5}$

(ii) **c***ath also believes that the father has a preference for* **b***ob. Her belief is* $\frac{3}{5}$*. But if she has the envelop* **2** *then she believes that the father has no preference between* **a***nne and* **b***ob.In such case her belief is* 1*.*

(iii) **b***ob does not believe that the father has any preference between* **a***nne and* **c***lara. So, his beliefs are all* 1*.*

The draws in figure 3 represent the beliefs of **a***nna,* **b***ob and* **c***lara. We draw it separately for clarity sake.*

We will evaluate the same formulas as in previous example:

$$
\begin{aligned}
012 \models B_b 0_a = \quad & \bigvee \{R_b(012,012) \odot 012 \models 0_a, R_b(012,210) \odot 210 \models 0_a\} \\
= \quad & \bigvee\{1 \odot 1, 1 \odot 0\} = 1
\end{aligned}
$$

$$
\begin{aligned}
012 \models B_a K_c 2_a = \quad & \bigvee \{R_a(012,012) \odot 012 \models K_c 2_a, R_a(012,021) \odot 021 \models K_c 2_a\} \\
= \quad & \bigvee \{1 \odot \bigwedge \{R_c(012,012) \to 012 \models 2_a, R_c(012,102) \to 102 \models 2_a\}, \\
& \tfrac{4}{5} \odot \bigwedge \{R_c(021,021) \to 021 \models 2_a, R_c(021,201) \to 201 \models 2_a\}\} \\
= \quad & \bigvee \{1 \odot \bigwedge \{1 \to 0, 1 \to 0\}, \tfrac{4}{5} \odot \bigwedge \{1 \to 0, \tfrac{2}{5} \to 1\}\} \\
= \quad & \bigvee \{1 \odot \bigwedge \{0,0\}, \tfrac{4}{5} \odot \bigwedge \{0,1\}\} \\
= \quad & \bigvee \{1 \odot 0, \tfrac{4}{5} \odot 0\} = \bigvee \{0,0\} = 0
\end{aligned}
$$

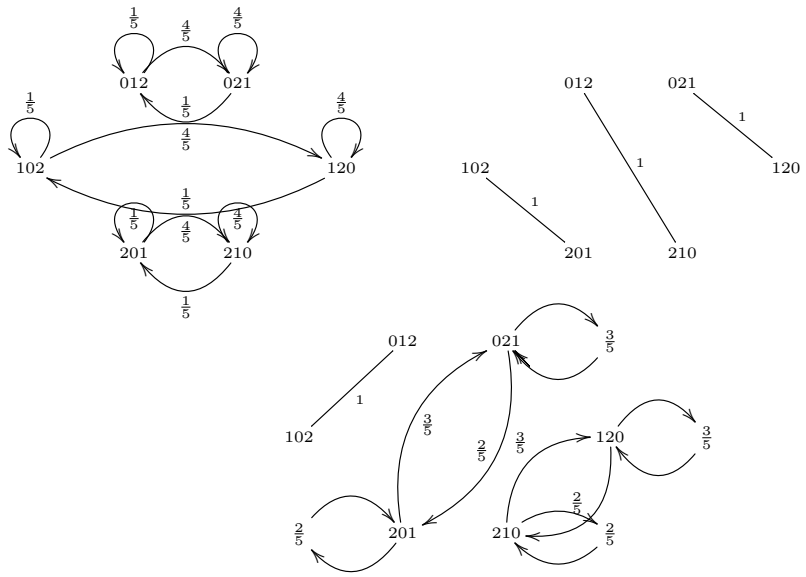


Fig. 3. **a**nna's, **b**ob's and clara's beliefs

To calculate $M_{ac}2_b$ *at* 021 *we first calculate the matrix of* $R_{ac} = R_a + R_c$.

$$R_{ac} = \begin{array}{c|cccccc} & 012 & 021 & 102 & 120 & 201 & 210 \\ \hline 012 & 1 & \frac{4}{5} & 1 & 0 & 0 & 0 \\ 021 & \frac{1}{5} & \frac{4}{5} & 0 & 0 & \frac{2}{5} & 0 \\ 102 & 1 & 0 & 1 & \frac{4}{5} & 0 & 0 \\ 120 & 0 & 0 & \frac{1}{5} & \frac{4}{5} & 0 & \frac{3}{5} \\ 201 & 0 & \frac{3}{5} & 0 & 0 & \frac{2}{5} & \frac{4}{5} \\ 210 & 0 & 0 & 0 & \frac{3}{5} & 1 & \frac{4}{5} \end{array}$$

Then we have,

$$\begin{aligned} 021 \models M_{ac}2_b = \quad & \bigvee \{R_{ac}(021,012) \odot 012 \models 2_b, R_{ac}(021,021) \odot 021 \models 2_b, \\ & R_{ac}(021,201) \odot 201 \models 2_b\} \\ = \quad & \bigvee\{\tfrac{1}{5} \odot 1, \tfrac{4}{5} \odot 1, \tfrac{2}{5} \odot 0\} = \bigvee\{\tfrac{1}{5}, \tfrac{4}{5}, 0\} = \tfrac{4}{5} \end{aligned}$$

If we consider the group knowledge we have

$$\begin{aligned} 021 \models E_{ac}2_b = \quad & \bigwedge \{R_{ac}(021,012) \to 012 \models 2_b, R_{ac}(021,021) \to 021 \models 2_b, \\ & R_{ac}(021,201) \to 201 \models 2_b\} \\ = \quad & \bigwedge\{\tfrac{1}{5} \to 1, \tfrac{4}{5} \to 1, \tfrac{2}{5} \to 0\} = \bigwedge\{1, 1, \tfrac{3}{5}\} = \tfrac{3}{5} \end{aligned}$$

5 How epistemic GE(A) logics are?

The study of each one of these instantiation of the logics generated in the previous section, as logics with 'its own rights', is very challenging. Obviously, there are aspects that have to be studied instantiation-by-instantiation. In this section, however we approach this in a more systematic perspective, trying to respond the question *How epistemic* $\mathcal{GE}(\mathbf{A})$ *logics are?* by studying the validity of the standard axioms of epistemic logic in Fig 5 on the generated logics.

We obtain some generic results for specific classes of generated logics, with respect to specific classes of action lattices and imposing constrains on the achieved models. The latter also happens in the standard epistemic logic, which the completeness is established for a restricted class of models, for instance, the epistemic ones (i.e., models whose accessible relations are equivalence relations) [17].

(i) *All instantiations of propositional tautologies*,

(ii) $K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$,

(iii) $K_a\varphi \to \varphi$,

(iv) $K_a\varphi \to K_aK_a\varphi$ (+ *introspection*),

(v) $\neg K_a\varphi \to K_a\neg K_a\varphi$ (− *introspection*),

(vi) $C_G\varphi \leftrightarrow E_GC_G\varphi$

(vii) $C_G(\varphi \to E_G\varphi) \to (\varphi \to C_G\varphi)$

Fig. 4. Axiomatics of epistemic logic [5,17]

We follow the strategy adopted in [11,12] (in the context of generated graded dynamic logics). The integrability ($a \leq 1$) on action lattices provides a nice proof strategy to work at this generic level: as it is well known, in any integral action lattice, we have

$$(a \to b) = 1 \;\Leftrightarrow\; a \leq b \tag{21}$$

Theorem 5.1 *Let* $\mathbf{A}$ *be an integral* ;*-idempotent,* ;*-commutative action lattice. The property*

(ii) $K_a(\varphi \to \psi) \to (K_a\varphi \to K_a\psi)$

is valid in the logic $\mathcal{GE}(\mathbf{A})$.

Proof. This proof can be extracted from Lemma 9 of [11]. □

In a similar way, but by imposing commutativity on the operation ; we can extract the proof for the axiom (vii):

Theorem 5.2 *Let* $\mathbf{A}$ *be an integral action lattice such that* $; = \cdot$. *Then the property*

(vii) $C_G(\varphi \to E_G\varphi) \to (\varphi \to C_G\varphi)$

is valid in the logic $\mathcal{GE}(\mathbf{A})$.

Proof. This can be directly adapted from Lemma 10 of [11]. □

So, we have to study the remaining axioms, specifically the ones that distinguish epistemic logic from other modal logics - the axioms (iii), (iv), (v) and (vi). In this view, we have to impose further properties on the structure of the models. In particular, we have to generalize the reflexivity and transitivity conditions for our graded setting to guarantee the validity of (iii) and (iv). What the conditions needed for the cases (iii) and (iv) are still in study.

Definition 5.3 Let $\mathbf{A}$ be an action lattice and M be a model in $\mathcal{GE}(\mathbf{A})$. We say that M is *graded-reflexive* if for any $a \in \mathrm{Ag}$, $w \in W$,

$$R_a(w, w) = 1 \tag{22}$$

and that it is *graded-transitive*, whenever any $a \in \mathrm{Ag}$

$$\text{for any } w, w', w'' \in W, R_a(w, w'') \geq R_a(w, w') \;;\; R_a(w', w'') \tag{23}$$

Theorem 5.4 *Let* $\mathbf{A}$ *be an integral action lattice. Then, the axiom*

(iii) $K_a\varphi \to \varphi$,

is valid in graded-reflexive models.

Proof. Since **A** is integral, we have by (21) that it is sufficient to prove that, for any model M, and for any state $w \in W$, $(w \models K_a\varphi) \leq (w \models \varphi)$. In this view, we observe that:

$$
\begin{aligned}
& (w \models K_a\varphi) \\
= \quad & \{ \models \text{defn} \} \\
& \bigwedge_{w' \in W} \big(R_a(w,w') \to (w' \models \varphi)\big) \\
\leq \quad & \{ \text{infimum properties} \} \\
& \big(R_a(w,w) \to (w \models \varphi)\big) \\
= \quad & \{ (22) \} \\
& \big(1 \to (w \models \varphi)\big) \\
= \quad & \{ \text{in any action lattice } 1 \to a = a \text{ (cf. [11])} \} \\
& (w \models \varphi)
\end{aligned}
$$

□

Theorem 5.5 *Let* **A** *be an integral* ;*-commutative action lattice. Then, the axiom*

(iv) $K_a\varphi \to K_aK_a\varphi$ (+ *introspection*),

is valid in graded-transitive models.

Proof. Since **A** is integral, we have by (21) that it is sufficient to prove that, for any model M, and for any state $w \in W$, $(w \models K_a\varphi) \leq (w \models K_aK_a\varphi)$. In this view, we observe that:

$$
\begin{aligned}
& \text{for any } w', w'' \in W, R_a(w,w'') \geq R_a(w,w');R_a(w',w'') \\
\Leftrightarrow \quad & \{ \text{;-commutative} \} \\
& \text{for any } w', w'' \in W,\ R_a(w,w'') \geq R_a(w',w'');R_a(w,w') \\
\Leftrightarrow \quad & \{ a \leq b \Rightarrow b \to c \leq a \to c \text{ (cf. [11])} \} \\
& \text{for any } w', w'' \in W,\ R_a(w,w'') \to (w'' \models \varphi) \leq \\
& (R_a(w',w'');R_a(w,w')) \to (w'' \models \varphi) \\
\Leftrightarrow \quad & \{ \text{infimum properties} \} \\
& \text{for any } w'' \in W, R_a(w,w'') \to (w'' \models \varphi) \leq \\
& \bigwedge_{w' \in W} \big((R_a(w',w'');R_a(w,w')) \to (w'' \models \varphi)\big) \\
\Leftrightarrow \quad & \{ \text{in any action lattice } a \to (b \to c) = (b;a) \to c \text{ (cf. [11])} \} \\
& \text{for any } w'', R_a(w,w'') \to (w'' \models \varphi) \leq
\end{aligned}
$$

$$\bigwedge_{w' \in W} \big(R_a(w,w') \to (R_a(w',w'') \to (w'' \models \varphi))\big)$$

$\Leftrightarrow$ $\{$ inf. monotocity$\}$

$$\bigwedge_{w'' \in W} R_a(w,w'') \to (w'' \models \varphi) \leq$$
$$\bigwedge_{w',w'' \in W} \big(R_a(w,w') \to (R_a(w',w'') \to (w'' \models \varphi))\big)$$

$\Leftrightarrow$ $\{$ in any complete action lattice, $x \to (\bigwedge_{i \in I} y_i) = \bigwedge_{i \in I}(x \to y_i)$ (cf. [11])$\}$

$$\bigwedge_{w'' \in W} R_a(w,w'') \to (w'' \models \varphi) \leq$$
$$\bigwedge_{w' \in W} \big(R_a(w,w') \to \bigwedge_{w'' \in W} (R_a(w',w'') \to (w'' \models \varphi))\big)$$

$\Leftrightarrow$ $\{ \models$ defn twice$\}$

$$(w \models K_a\varphi) \leq (w \models K_a K_a \varphi)$$

□

6 Conclusions and future work

This paper starts with a research program on the parametric generation of graded epistemic logics. The approach is based on the application of the method introduced in Section 3, and should be explored as an effective source of logics to reason on agent knowledge scenarios with distinct degrees of Knowledge/Belief. The generality of the method was illustrated with three graded epistemic logics (note that the standard multi-agent epistemic logic corresponds to the instantiation of the action lattice **2**), but a lot of other examples can be considered - from a $\{false, unknown, true\}$-three valued epistemic logic, achieved by instantiating the action lattice **3** to a more 'esoteric' graded epistemic logic to deal with knowledge/belief scenarios involving resource aware constraints (built on the Floyd Warshall algebra - see [11]). Beyond of their philosophical interest, the study of each one of these instantiations as a logic with 'its own rights' is very challenging. Indeed, as discussed in Section 5, it is possible to characterize specific classes of graded epistemic logics (parametric on specific subclasses of action lattices and by imposing further condition on the models) that preserves the essence of the bivalent epistemic logic.

There is, however, a lot of work to do in this line of research. To establish sufficient conditions for validating the negative introspection axiom (and of (vi)) is still work in progress for us. It seems that, beyond of a generalization of the Euclidean property on models, some new conditions should be imposed in the action lattices, particularly with respect to their negation (note that, in its generic form, there is no negation involution in general). The parametric generation of calculus and the study of complexity of generated epistemic logic w.r.t. to specific classes of action lattices are also in our agenda. Another interesting line of research is to investigate the concepts of simulation and bisimulation for our knowledge representations on the lines proposed in [18,14] for generic fuzzy labelled transition systems.

Finally, it would be interesting to investigate whether our approach allows for the representation of epistemic actions. Public announcements or private communications. More interesting is to look for epistemic actions that make sense only in this (or similar) setting. For example, one can think of situations in which the agent has a belief of some grade n, and then some new information 'downgrades' or 'upgrades' this belief (some form of belief revision, but now in a 'graded' fashion).

References

[1] W. J. Blok and I. M. A. Ferreirim. On the structure of hoops. *algebra universalis*, 43(2-3):233–257, 2000.

[2] J. H. Conway. *Regular Algebra and Finite Machines.* Printed in GB by William Clowes & Sons Ltd, 1971.

[3] L. Godo F. Bou, F. Esteva and R.O. Rodriguez. Many-valued modal logic over a nite residuated lattice. *Journal of Logic and Computation*, 21(5):739–790, 2011.

[4] R. Fagin and J. Halpern. Reasoning about knowledge and probability. *Journal of the ACM*, 41(2):340–367, 1994.

[5] R. Fagin, J. Halpern, Y. Moses, and M. Vardi. *Reasoning about Knowledge.* MIT Press, USA, 1995.

[6] Melvin Fitting. Many-valued modal logics. *Fundam. Inform.*, 15(3-4):235–254, 1991.

[7] Michell Guzmán, Stefan Haar, Salim Perchy, Camilo Rueda, and Frank D. Valencia. Belief, knowledge, lies and other utterances in an algebra for space and extrusion. *J. Log. Algebr. Meth. Program.*, 86(1):107–133, 2017.

[8] J. Hintikka. *Knowledge and Belief.* Cornell University Press, Ithaca, N.Y, 1962.

[9] Dexter Kozen. On action algebras. *Logic and Information Flow*, pages 78–88, 1994.

[10] Alexander Kurz and Alessandra Palmigiano. Epistemic updates on algebras. *Logical Methods in Computer Science*, 9(4), 2013.

[11] Alexandre Madeira, Renato Neves, and Manuel A. Martins. An exercise on the generation of many-valued dynamic logics. *J. Log. Algebr. Meth. Program.*, 85(5):1011–1037, 2016.

[12] Alexandre Madeira, Renato Neves, Manuel A. Martins, and Luís Soares Barbosa. A dynamic logic for every season. In Christiano Braga and Narciso Martí-Oliet, editors, *Formal Methods: Foundations and Applications - 17th Brazilian Symposium, SBMF 2014, Maceió, AL, Brazil, September 29-October 1, 2014. Proceedings*, volume 8941 of *Lecture Notes in Computer Science*, pages 130–145. Springer, 2014.

[13] Yoshihiro Maruyama. Reasoning about fuzzy belief and common belief: With emphasis on incomparable beliefs. In *IJCAI 2011, Proceedings of the 22nd International Joint Conference on Artificial Intelligence, Barcelona, Catalonia, Spain, July 16-22, 2011*, pages 1008–1013, 2011.

[14] Haiyu Pan, Yongming Li, and Yongzhi Cao. Lattice-valued simulations for quantitative transition systems. *Int. J. Approx. Reasoning*, 56:28–42, 2015.

[15] Vaughan R. Pratt. Action logic and pure induction. In *JELIA*, volume 478 of *Lecture Notes in Computer Science*, pages 97–120. Springer, 1990.

[16] Umberto Rivieccio. Bilattice public announcement logic. *Advances in Modal Logic*, 10:459–477, 2014.

[17] H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic.* Synthese Library Series, volume 337. Springer, The Netherland, 2008.

[18] Hengyang Wu and Yuxin Deng. Logical characterizations of simulation and bisimulation for fuzzy transition systems. *Fuzzy Sets and Systems*, 301:19–36, 2016.