*Proceedings of the 17th International Conference
on Computational and Mathematical Methods
in Science and Engineering, CMMSE 2017
4–8 July, 2017.*

Matrices related to orthogonal hypercomplex polynomial systems

I. Cação[1], H. R. Malonek[1] and G. Tomaz[1,2]

[1] *CIDMA-Center for Research and Development in Mathematics and Applications, University of Aveiro, Portugal*

[2] *UDI-Research Unit for Inland Development, Polythecnic Institute of Guarda, Portugal*

emails: isabel.cacao@ua.pt, hrmalon@ua.pt, gtomaz@ipg.pt

Abstract

We show how special well known matrices, namely, the creation and shift matrices play an important role on a matrix representation of orthogonal systems of polynomials with a hypercomplex variable and values in a Clifford algebra.

Key words: hypercomplex polynomials, matrix representation, creation matrix, shift matrix

MSC 2000: 30G35, 65F60, 11B83.

1 Introduction

In [7] the author constructs orthogonal bases of polynomials in the space of square integrable functions that are in the kernel of a generalized Cauchy-Riemann operator in the unit ball of $\mathbb{R}^{n+1}$. The construction process relies on building blocks that do not belong, in general, to the kernel of the referred operator. By using results established in [3] for these building blocks, we generalize the algebraic approach developed in [1] and stress the role of the well-known *creation matrix* and *shift matrix* in this representation. We recall that the so-called creation matrix H and the shift matrix J are defined by

$$(H)_{il} = \begin{cases} i, & i = l+1 \\ 0, & i \neq l+1 \end{cases} \qquad \text{and} \qquad (J)_{il} = \begin{cases} 1, & i = l+1 \\ 0, & i \neq l+1, \end{cases}$$

$i, l = 0, \ldots, m$, respectively. Although their simple structure, these matrices appear naturally in a matrix decomposition that represents polynomials in arbitrary dimension and in the framework of non-commutative algebras.

 ISBN: 978-84-617-8694-7

2 Basic concepts

Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^n$ endowed with a non-commutative product according to the multiplication rules $e_k e_l + e_l e_k = -2\delta_{kl},\ k, l = 1, \ldots, n$, where δ_{kl} is the Kronecker symbol. The associative 2^n-dimensional Clifford algebra $\mathcal{C}\ell_{0,n}$ over $\mathbb{R}$ is the set of numbers of the form $a = \sum_A a_A e_A,\ a_A \in \mathbb{R}$, with $A \subseteq \{1, \cdots, n\}$, $e_A = e_{l_1} e_{l_2} \cdots e_{l_r}$, where $1 \leq l_1 < \cdots < l_r \leq n$ and $e_\emptyset =: e_0 =: 1$. The vector space $\mathbb{R}^{n+1}$ is embedded in $\mathcal{C}\ell_{0,n}$ by identifying $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ with the so-called *paravectors* $x = x_0 + \underline{x} \in \mathcal{A}_n := \mathrm{span}_{\mathbb{R}}\{1, e_1, \ldots, e_n\} \subset \mathcal{C}\ell_{0,n}$, where $\underline{x} = x_1 e_1 + \ldots + x_n e_n$ is called a *vector*. The conjugate $\bar{x}$ and the norm $|x|$ of x are given by $\bar{x} = x_0 - \underline{x}$ and $|x| = (x\bar{x})^{1/2} = (\bar{x}x)^{1/2} = \left(\sum_{k=0}^n x_k^2\right)^{1/2}$, respectively. The generalized Cauchy-Riemann operator and its conjugate are given, respectively, by $\overline{\partial} := \frac{1}{2}(\partial_0 + \partial_{\underline{x}})$ and $\partial := \frac{1}{2}(\partial_0 - \partial_{\underline{x}})$, where $\partial_0 := \frac{\partial}{\partial x_0}$ and $\partial_{\underline{x}} := e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$.

We consider $\mathcal{C}\ell_{0,n}-$valued functions defined in an open subset $\Omega \subseteq \mathbb{R}^{n+1} \cong \mathcal{A}_n$, i.e. functions of the form $f(z) = \sum_A f_A(z) e_A$ with $f_A(z)$ real valued. Continuously differentiable functions f that satisfy the equation $\overline{\partial} f = 0$ (resp. $f\overline{\partial} = 0$) are called (left) *monogenic* (resp. right monogenic) and constitute the analogue of the class of holomorphic functions in higher dimensions. For more details, see [2, 5].

Let f be a monogenic function that is hypercomplex-differentiable in some domain $\Omega \subset \mathbb{R}^{n+1}$ in the sense of [6]. Then f is real-differentiable and its (hypercomplex) derivative is given by $f' = \partial f$ in Ω.

3 Matrix representation of orthogonal Clifford algebra-valued polynomials

Branching techniques combined with Gelfand-Tsetlin bases approach yield to the monogenic polynomials

$$f_{k,\mu} = X_{n+1,k_n}^{(k-k_n)} X_{n,k_{n-1}}^{(k_n-k_{n-1})} \cdots X_{3,k_2}^{(k_3-k_2)} \zeta^{k_2},$$

where $\zeta := x_1 - x_2\, e_1 e_2$ and μ is an arbitrary sequence of integers $(k_{n+1}, k_n, \ldots, k_3, k_2)$ such that $k = k_{n+1} \geq k_n \geq \ldots \geq k_3 \geq k_2 \geq 0$. These polynomials form an orthogonal basis with respect to a suitable Clifford algebra valued inner product of the space of monogenic polynomials of degree k (see [7]). The building blocks $X_{n+1,j}^{(k-j)},\ j = 0, \ldots, k$, are, in general, non-monogenic polynomials and can be expressed as

$$X_{n+1,j}^{(k-j)}(x) = \sum_{s=0}^{k-j} \binom{k}{j+s} d_{j,s}(n)\, x_0^{k-j-s}\, \underline{x}^s, \quad x \in \mathcal{A}_n \tag{1}$$

 ISBN: 978-84-617-8694-7

where $d_{j,s}(n)$ are suitable real constants (cf. [3]). Given an arbitrary monogenic polynomial $P_j(\underline{x})$, in $\mathbb{R}^n$, of degree j, the system

$$\left\{ \widetilde{X}^{(k)}_{n+1,j}(x) := X^{(k-j)}_{n+1,j}(x)\, P_j(\underline{x}), j = 0, \ldots, k,\ x \in \mathcal{A}_n \right\}_{k\in\mathbb{N}} \tag{2}$$

is formed by monogenic polynomials where again (1) appear as building blocks.

The matrix representation of the building blocks relies on the so-called *shifted generalized Pascal matrix* $S_r(t) = e^{(H+rJ)t}$, $t \in \mathbb{R}, r \in \mathbb{N}_0$, that combines in a single expression both H and J matrices.

For this representation we restrict ourselves to vectors of polynomials up to a certain degree $m \in \mathbb{N}_0$. Therefore, for each $j = 0, \ldots, k$, we consider the vectors $\xi(\underline{x}) = [1\ \underline{x}\ \cdots\ \underline{x}^m]^T$, $\boldsymbol{X}_j(x) = [X^{(0)}_{n+1,j}(x)\ X^{(1)}_{n+1,j}(x)\ \cdots\ X^{(m)}_{n+1,j}(x)]^T$ and the diagonal matrix $\mathcal{D}_j = \mathrm{diag}[d_{j,0}(n)\ d_{j,1}(n) \cdots d_{j,m}(n)]$.

Theorem 3.1. *For each fixed j, the vector $\boldsymbol{X}_j(x)$ can be decomposed in the form*

$$\boldsymbol{X}_j(x) = S_j(x_0)\, \mathcal{D}_j\, \xi(\underline{x}). \tag{3}$$

Proof. For each fixed j and considering $l = k - j$, it holds

$$\partial_0 X^{(0)}_{n+1,j}(x) = 0 \quad \text{and} \quad \partial_0 X^{(l)}_{n+1,j}(x) = (l+j) X^{(l-1)}_{n+1,j}(x), \ \ l > 0.$$

This property leads to the vector differential equation

$$\partial_0 \boldsymbol{X}_j(x) = (H + jJ)\boldsymbol{X}_j(x)$$

whose general solution is

$$\boldsymbol{X}_j(x) = e^{(H+jJ)x_0} \boldsymbol{X}_j(0, \underline{x}).$$

The result follows immediately noting that $X^{(l)}_{n+1,j}(0, \underline{x}) = d_{j,l}(n)\underline{x}^l$. □

The proposed decomposition (3) can be generalized for the whole orthogonal system (2) with the help of block matrices. The result highlights the role of the shift matrix J and its powers. Moreover, the generalized Pascal matrix $P(x_0) = S_0(x_0)$ comes into play as well (see [4]), connecting special simple matrices with real entries with hypercomplex entities.

Acknowledgements

This work was supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e Tecnologia"), within project PEst-OE/MAT/UI4106/2013.

 ISBN: 978-84-617-8694-7

References

[1] L. ACETO, H. R. MALONEK AND G. TOMAZ, *A unified matrix approach to the representation of Appell polynomials*, Integral Transforms Spec. Funct. **26** (2015) 426-441.

[2] F. BRACKX, R. DELANGHE AND F. SOMMEN, *Clifford Analysis*, Pitman, Boston-London-Melbourne, 1982.

[3] I. CAÇÃO, M. I. FALCÃO AND H. R. MALONEK, *Three term recurrence relations for systems of Clifford algebra-valued orthogonal polynomials*, Adv. Appl. Clifford Algebras **27** (2017) 71-85.

[4] I. CAÇÃO, H. R. MALONEK AND G. TOMAZ, *Shifted generalized Pascal matrices in the context of Clifford algebra-valued polynomial sequences*, accepted for publication.

[5] K. GÜRLEBECK, K. HABETHA AND W. SPRÖßIG, *Holomorphic Functions in the Plane and n-Dimensional Space*, (Translated from the 2006 German original), Birkhäuser Verlag, Basel, 2008.

[6] K. GÜRLEBECK AND H. R. MALONEK, *A hypercomplex derivative of monogenic functions in* $\mathbb{R}^{m+1}$ *and its applications*, Complex Variables **39** (1999) 199-228.

[7] R. LÁVIČKA, *Complete Orthogonal Appell Systems for Spherical Monogenics*, Complex Anal. Oper. Theory **6** (2012) 477-489.

  ISBN: 978-84-617-8694-7