

NONLINEAR, NONHOMOGENEOUS PARAMETRIC NEUMANN PROBLEMS

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ABSTRACT. We consider a parametric nonlinear Neumann problem driven by a nonlinear nonhomogeneous differential operator and with a Caratheodory reaction $f(t, x)$ which is p -superlinear in x without satisfying the usual in such cases Ambrosetti-Rabinowitz condition. We prove a bifurcation type result describing the dependence of the positive solutions on the parameter $\lambda > 0$, we show the existence of a smallest positive solution \bar{u}_λ and investigate the properties of the map $\lambda \rightarrow \bar{u}_\lambda$. Finally we also show the existence of nodal solutions.

1. Introduction

In this paper we study the following nonlinear parametric Neumann problem

$$(P_\lambda) \quad \begin{cases} -\operatorname{div} a(Du(z)) + \lambda |u(z)|^{p-2} u(z) = f(z, u(z)) & \text{in } \Omega, \\ \frac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega, \lambda > 0, 1 < p < \infty. \end{cases}$$

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Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with a C^2 -boundary $\partial\Omega$. The map $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is continuous, strictly monotone and satisfies certain other regularity conditions which are listed in hypotheses **H(a)** (see Section 2). These hypotheses are general enough to incorporate in our framework many differential operators of interest, such as the p -Laplacian. Also $\lambda > 0$ is a parameter and $f(t, z)$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$ $z \rightarrow f(z, x)$ is measurable and for almost all $z \in \Omega$, $x \rightarrow f(z, x)$ is continuous) which exhibits a $(p - 1)$ -superlinear growth in the x -variable, but without satisfying the usual in such case Ambrosetti-Rabinowitz condition (AR-condition for short).

Our work here was motivated by a recent paper of Motreanu-Motreanu-Papageorgiou [18], who produced constant sign and nodal solutions. Our results here complement and improve those of [18]. More precisely, the authors in [18] produced positive solutions for problem (P_λ) but did not give the precise dependence of the set of positive solutions on the parameter $\lambda > 0$. Here, we prove a bifurcation-type theorem for large values of λ , which gives a complete picture of the set of positive solutions as the parameter varies. Moreover, in [18] nodal (that is, sign-changing) solutions were produced only for the particular case of equations driven by the p -Laplacian. In contrast here, we generate nodal solutions for the general case. We stress that the p -Laplacian differential operator is homogeneous, while the differential operator in (P_λ) is not. Hence, the methods and techniques used in [18] fail in the present setting, and so a new approach is needed. Finally, we mention that a bifurcation near infinity for a different class of p -Laplacian Dirichlet problems was recently produced by Gasinski-Papageorgiou [12].

In the next section, we review the main mathematical tools which will be used in this paper. We also present the hypotheses on the map $y \rightarrow a(y)$ and state some useful consequences of them.

2. Mathematical background

Let $(X, \|\cdot\|)$ be a Banach space and X^* be its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X) and \xrightarrow{w} will designate weak convergence.

Let $\varphi \in C^1(X)$. We say that $x^* \in X$ is a critical point of φ if $\varphi'(x^*) = 0$. If $x^* \in X$ is a critical point of φ then $c = \varphi(x^*)$ is a critical value of φ . We say that φ satisfies the "Palais-Smale condition" (the PS-condition for short), if the following holds:

"every sequence $\{u_n\}_{n \geq 1} \subseteq X$ such that $\{\varphi(u_n)\}_{n \geq 1}$ is bounded in \mathbb{R} and $\varphi'(u_n) \rightarrow 0$ in X^* as $n \rightarrow \infty$ admits a strongly convergent subsequence."

This compactness-type condition on the functional φ leads to a deformation theorem from which one can derive the minimax theory of the critical values of φ . One of the main results in that theory is the so called "mountain pass theorem", which we recall here:

THEOREM 2.1. *If $\varphi \in C^1(X)$ satisfies the PS-condition, $u_0, u_1 \in X$ with $\|u_1 - u_0\| > \rho > 0$ and*

$$\begin{aligned} \max\{\varphi(u_0), \varphi(u_1)\} &< \inf\{\varphi(u) : \|u - u_0\| = \rho\} =: m_\rho, \\ \text{and } c &:= \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)) \text{ where} \\ \Gamma &= \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = u_1\}, \end{aligned}$$

then $c \geq m_\rho$ and c is a critical value of φ .

The main spaces that we will use in the analysis of problem (P_λ) are the Sobolev space $W^{1,p}(\Omega)$ and the Banach space $C^1(\overline{\Omega})$. The latter is an ordered Banach space with positive cone

$$\mathcal{C}_+ = \{u \in C^1(\overline{\Omega}) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\}.$$

This cone has a nonempty interior given by

$$\text{int } \mathcal{C}_+ = \{u \in \mathcal{C}_+ : u(z) > 0 \text{ for all } z \in \overline{\Omega}\}.$$

By $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}(\Omega)$, that is

$$\|u\| = \left[\|u\|_p^p + \|Du\|_p^p \right]^{\frac{1}{p}} \text{ for all } u \in W^{1,p}(\Omega),$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$ (or $L^p(\Omega, \mathbb{R}^N)$).

Also, by $\|\cdot\|$ we denote the \mathbb{R}^N -norm. However, no confusion is possible, since it will be clear from the context which norm is used. The inner product in \mathbb{R}^N will be denoted by $(\cdot, \cdot)_{\mathbb{R}^N}$.

Let $\theta \in C^1(0, \infty)$ and assume that there exist constants $\widehat{C}, C_0, C_1, C_2 > 0$ such that

$$(2.1) \quad \widehat{C} \leq \frac{t\theta'(t)}{\theta(t)} \leq C_0 \text{ and } C_1 t^{p-1} \leq \theta(t) \leq C_2 (1 + t^{p-1}) \text{ for all } t > 0,$$

with $1 < p < \infty$. The hypotheses on the map $a : \mathbb{R}^N \rightarrow \mathbb{R}^N$ are the following:

H(a) : $a(y) = a_0(\|y\|)y$ for all $y \in \mathbb{R}^N$ with $a_0(t) > 0$ for all $t > 0$ and

(i) $a_0 \in C^1(0, \infty)$, $t \rightarrow ta_0(t)$ is strictly increasing in $(0, \infty)$, $ta_0(t) \rightarrow 0^+$ as $t \rightarrow 0^+$ and

$$\lim_{t \rightarrow 0^+} \frac{ta_0'(t)}{a_0(t)} > -1;$$

(ii)

$$\|\nabla a(y)\| \leq C_3 \frac{\theta(\|y\|)}{\|y\|} \text{ for some } C_3 > 0 \text{ and all } y \in \mathbb{R}^N \setminus \{0\};$$

(iii)

$$\frac{\theta(\|y\|)}{\|y\|} \|\xi\|^2 \leq (\nabla a(y) \xi, \xi)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N \setminus \{0\}, \text{ all } \xi \in \mathbb{R}^N;$$

(iv) if $G_0(t) = \int_0^t s a_0(s) ds$, then there exists $\xi_0 > 0$ such that

$$pG_0(t) - t^2 a_0(t) \geq -\xi_0 \text{ for all } t > 0;$$

(v) there exist $\tau, q \in (1, p)$ such that

$$t \rightarrow G_0\left(t^{\frac{1}{\tau}}\right) \text{ is convex and } \lim_{t \rightarrow 0^+} \frac{G_0(t)}{t^q} = 0.$$

Remarks: The hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) were motivated by the regularity results of Lieberman [15] (p. 320) and the nonlinear maximum principle of Pucci-Serrin [25] (pp. 111, 120). Hypotheses $\mathbf{H}(\mathbf{a})$ (iv), (v) are particular for our problem here, but they are quite general and are satisfied by many differential operators of interest (see the example 3 below). Hypotheses $\mathbf{H}(\mathbf{a})$ imply that the primitive $G_0(\cdot)$ is strictly convex and strictly increasing.

We set

$$G(y) = G_0(\|y\|) \text{ for all } y \in \mathbb{R}^N.$$

Evidently $G(\cdot)$ is convex and differentiable on \mathbb{R}^N . We have

$$\nabla G(y) = G'_0(\|y\|) \frac{y}{\|y\|} = a_0(\|y\|) y \text{ for all } y \in \mathbb{R}^N \setminus \{0\}, \nabla G(0) = 0.$$

Since $G(\cdot)$ is convex and $G(0) = 0$, we have

$$(2.2) \quad G(y) \leq (a(y), y)_{\mathbb{R}^N} \text{ for all } y \in \mathbb{R}^N.$$

The next lemma summarizes the main properties of $G(\cdot)$ and is an easy consequence of hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii).

LEMMA 2.2. *If hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) hold, then:*

(a) *the map $y \rightarrow a(y)$ is continuous and strictly monotone, hence maximal monotone too;*

(b)

$$\|a(y)\| \leq C_4 \left(1 + \|y\|^{p-1}\right) \text{ for some } C_4 > 0 \text{ and all } y \in \mathbb{R}^N;$$

(c)

$$(a(y), y)_{\mathbb{R}^N} \geq \frac{C_1}{p-1} \|y\|^p \text{ for all } y \in \mathbb{R}^N.$$

This lemma together with (2.1) and (2.2) leads to the following growth estimates for $G(\cdot)$:

COROLLARY 2.3. *If hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) hold, then*

$$\frac{C_1}{p(p-1)} \|y\|^p \leq G(y) \leq C_5(1 + \|y\|^p) \text{ for some } C_5 > 0, \text{ all } y \in \mathbb{R}^N.$$

Examples: The following maps $a(y)$ satisfy hypotheses $\mathbf{H}(\mathbf{a})$:

(a) $a(y) = \|y\|^{p-2} y$ with $1 < p < \infty$.

This map corresponds to the p -Laplace differential operator defined by

$$\Delta_p u = \operatorname{div} \left(\|Du\|^{p-2} Du \right) \text{ for all } u \in W^{1,p}(\Omega).$$

(b) $a(y) = \|y\|^{p-2} y + \|y\|^{q-2} y$ with $1 < q < p < \infty$.

This map corresponds to the (p, q) -Laplacian defined by

$$\Delta_p u + \Delta_q u \text{ for all } u \in W^{1,p}(\Omega).$$

Such operators arise in many physical applications (see Cherfils-Ilyasov [6] and the references therein). Recently there have been existence and multiplicity results for equations driven by such operators. We mention the works of Aizicovici-Papageorgiou-Staicu [4], Cingolani-Degiovanni [7], Mugnai-Papageorgiou [22], Papa-georgiou-Radulescu [23], Sun [26].

(c) $a(y) = \left(1 + \|y\|^2\right)^{\frac{p-2}{2}} y$ with $1 < p < \infty$.

This map correspond to the generalized p -mean curvature differential operator defined by

$$\operatorname{div} \left(\left(1 + \|Du\|^2\right)^{\frac{p-2}{2}} Du \right) \text{ for all } u \in W^{1,p}(\Omega).$$

(d) $a(y) = \|y\|^{p-2} y + \frac{\|y\|^{q-2} y}{1 + \|y\|^q}$ with $1 < q \leq p$.

$$(e) a(y) = \begin{cases} \|y\|^{p-1} y & \text{if } \|y\| < 1 \\ 2\|y\|^{p-2} y - \|y\|^{p-3} y & \text{if } 1 < \|y\| \end{cases} \text{ with } 1 < p < \infty.$$

Let $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ be the nonlinear map defined by

$$(2.3) \quad \langle A(u), y \rangle = \int_{\Omega} (a(Du), Dy)_{\mathbb{R}^N} dz \text{ for all } u, y \in W^{1,p}(\Omega).$$

From Papageorgiou-Rocha-Staicu [24] we have:

PROPOSITION 2.4. *If hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) hold, then the map $A : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ defined by (2.3) is demicontinuous, strictly monotone (hence maximal monotone too) and of type $(S)_+$ (that is, if $u_n \xrightarrow{w} u$ in $W^{1,p}(\Omega)$ and*

$$\limsup_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle \leq 0,$$

then $u_n \rightarrow u$ in $W^{1,p}(\Omega)$ as $n \rightarrow \infty$).

Let $f_0 : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a Carathéodory function such that

$$|f_0(z, x)| \leq a_0(z) \left(1 + |x|^{r-1}\right) \text{ for a. a. } z \in \Omega, \text{ all } x \in \mathbb{R}$$

with $a_0 \in L^\infty(\Omega)_+$ and $1 < r < p^*$, where

$$p^* := \begin{cases} \frac{Np}{N-p} & \text{if } p < N \\ +\infty & \text{if } p \geq N. \end{cases}$$

We set $F_0(z, x) = \int_0^x f_0(z, s) ds$ and consider the C^1 -functional $\varphi_0 : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \int_{\Omega} G(Du(z)) dz - \int_{\Omega} F_0(z, u(z)) dz \text{ for all } u \in W^{1,p}(\Omega).$$

From Motreanu-Papageorgiou [21], we have:

PROPOSITION 2.5. *If hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) hold, $\varphi_0 : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ is as defined above and $u_0 \in W^{1,p}(\Omega)$ is a local $C^1(\overline{\Omega})$ -minimizer of φ_0 , that is, there exists $\rho_0 > 0$ such that*

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in C^1(\overline{\Omega}) \text{ with } \|h\|_{C^1(\overline{\Omega})} \leq \rho_0,$$

then $u_0 \in C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and u_0 is also a $W^{1,p}(\Omega)$ -minimizer of φ_0 , that is, there exists $\rho_1 > 0$ such that

$$\varphi_0(u_0) \leq \varphi_0(u_0 + h) \text{ for all } h \in W^{1,p}(\Omega) \text{ with } \|h\| \leq \rho_1.$$

Let X be a Banach space, $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets

$$\varphi^c = \{u \in X : \varphi(u) \leq c\},$$

$$K_\varphi = \{u \in X : \varphi'(u) = 0\},$$

$$K_\varphi^c = \{u \in K_\varphi : \varphi(u) = c\}.$$

For every topological pair (Y_1, Y_2) with $Y_2 \subset Y_1 \subset X$ and every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th} -relative singular homology group with integer coefficients.

Given an isolated $u \in K_\varphi^c$, the *critical groups of φ at u* are defined by

$$C_k(\varphi, u) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}), \text{ for all integers } k \geq 0,$$

where U is a neighborhood of u such that $K_\varphi \cap \varphi^c \cap U = \{u\}$.

The excision property of the singular homology implies that the above definition is independent of the particular choice of the neighborhood U .

Finally we outline some additional notations used in this paper. By $|\cdot|_N$ we denote the Lebesgue measure on \mathbb{R}^N .

Given $x \in \mathbb{R}$, we define

$$x^\pm = \max\{\pm x, 0\}.$$

For $u \in W^{1,p}(\Omega)$ we set $u^\pm(\cdot) = u(\cdot)^\pm$. We know that

$$u^\pm \in W^{1,p}(\Omega), \quad |u| = u^+ + u^-, \quad u = u^+ - u^-.$$

Given a measurable function $h : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ (for example, a Carathéodory function), we define

$$N_h(u)(\cdot) = h(\cdot, u(\cdot)) \text{ for all } u \in W^{1,p}(\Omega)$$

(the Nemitskii operator corresponding to h). Evidently $z \rightarrow N_h(u)(z) = h(z, u(z))$ is measurable.

3. Positive solutions

In this section, we prove a bifurcation-type theorem describing the set of positive solutions of (P_λ) as $\lambda > 0$ varies. We impose the following conditions on the reaction $f(z, x)$:

(H₁): $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that $f(z, 0) = 0$ a.e. in Ω , $f(z, x) > 0$ for all $x > 0$ and

(i) there exists $a \in L^\infty(\Omega)_+$ such that

$$f(z, x) \leq a(z)(1 + x^{r-1}) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ with } p < r < p^*;$$

(ii) if $F(z, x) = \int_0^x f(z, s) ds$ then

$$\lim_{x \rightarrow +\infty} \frac{F(z, x)}{x^p} = +\infty \text{ uniformly for a.a. } z \in \Omega;$$

(iii) there exist $\mu \in \left(\max\left\{(r-p)\frac{N}{p}, 1\right\}, p^*\right)$ and $\beta_0 > 0$ such that

$$\beta_0 \leq \liminf_{x \rightarrow +\infty} \frac{f(z, x)x - pF(z, x)}{x^\mu} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exist $\widehat{\delta}_0$ and $\widehat{C}_0 > 0$ such that

$$f(z, x) \geq \widehat{C}_0 x^{q-1} \text{ for a.a. } z \in \Omega, \text{ all } x \in [0, \widehat{\delta}_0],$$

with $q \in (1, p)$ as in hypothesis **H(a)**(v).

Remarks: Since in this section we are looking for positive solutions and all the above hypotheses concern the positive half-axis $\mathbb{R}_+ = [0, \infty)$, we may assume, without any loss of generality, that $f(z, x) = 0$ for a.a. $z \in \Omega$, all $x \leq 0$.

Hypotheses **(H₁)**(ii), (iii) imply that

$$\lim_{x \rightarrow \infty} \frac{f(z, x)}{x^{p-1}} = \infty \text{ uniformly for a.a. } z \in \Omega,$$

that is, for a.a. $z \in \Omega$ $f(z, \cdot)$ is $(p-1)$ -superlinear. Usually superlinear problems are treated using the so-called Ambrosetti-Rabinowitz condition (AR-condition, for short). We recall that the AR-condition (the unilateral version) says that there exist $\eta > p$ and $M > 0$ such that

$$(3.1) \quad 0 < \eta F(z, x) \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } x \geq M \text{ and } \operatorname{ess\,inf} F(\cdot, M) > 0.$$

From (3.1) through integration, we obtain the weaker condition

$$(3.2) \quad C_6 x^\eta \leq F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq M \text{ with } C_6 > 0.$$

By (3.2) and since $\eta > p$, we infer that the much weaker condition $(\mathbf{H}_1)(ii)$ holds.

Hypotheses $(\mathbf{H}_1)(ii)$, (iii) together are weaker than the AR condition and allow us to include in our framework superlinear functions with "slower" growth near $+\infty$ (see the examples bellow).

Suppose that the AR-condition holds. We may assume that $\eta > \max\left\{(r-p)\frac{N}{p}, 1\right\}$. We have

$$\begin{aligned} \frac{f(z, x)x - pF(z, x)}{x^\eta} &= \frac{f(z, x)x - \eta F(z, x)}{x^\eta} + (\eta - p) \frac{F(z, x)}{x^\eta} \\ &\geq (\eta - p) C_6 \text{ for a.a. } z \in \Omega, \text{ all } x \geq M. \end{aligned}$$

So, hypothesis $(\mathbf{H}_1)(iii)$ holds.

Hypothesis $(\mathbf{H}_1)(iv)$ implies that the reaction $f(z, \cdot)$ exhibits a concave term near zero. Therefore our hypotheses (\mathbf{H}_1) incorporate the case of equations with competing nonlinearities ("concave-convex problems").

We mention that similar or different extensions of the AR-superlinearity condition can be found in Aizicovici-Papageorgiou-Staicu [3], Costa-Magalhães [8], Li-Yang [16], and Mugnai-Papageorgiou [22].

Examples: The following functions satisfy hypotheses (\mathbf{H}_1) . For the sake of simplicity we drop the z -dependence:

$$\begin{aligned} f_1(x) &= x^{q-1} + x^{r-1} \text{ for all } x \geq 0 \text{ with } 1 < q < p < r < p^*, \\ f_2(x) &= \begin{cases} x^{q-1} & \text{if } x \in [0, 1] \\ x^{p-1}(\ln x + 1) & \text{if } 1 < x \end{cases} \text{ with } 1 < q < p. \end{aligned}$$

Note that f_2 does not satisfy the AR-condition.

We introduce the following two sets:

$$\mathcal{L} = \{\lambda > 0 : (P_\lambda) \text{ admits a positive solution}\}$$

and, for $\lambda \in \mathcal{L}$,

$$\mathcal{S}(\lambda) = \text{the set of positive solutions of } (P_\lambda).$$

We start with a useful observation concerning the solution set $\mathcal{S}(\lambda)$.

PROPOSITION 3.1. *If hypotheses $\mathbf{H}(\mathbf{a})$ (i), (ii), (iii) and (\mathbf{H}_1) hold, then*

$$\mathcal{S}(\lambda) \subseteq \text{int}C_+.$$

PROOF. Let $\lambda \in \mathcal{L}$ and $u \in \mathcal{S}(\lambda)$. We have

$$(3.3) \quad -\text{div } a(Du(z)) + \lambda u(z)^{p-1} = f(z, u(z)) \text{ a.e. in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega$$

(see Motreanu-Papageorgiou [20]). From Hu-Papageorgiou [14] and Winkert [27], we have that $u \in L^\infty(\Omega)$. So, we can apply the regularity result of Lieberman [15] (p. 320) and infer that $u \in C_+ \setminus \{0\}$.

Since $f \geq 0$ (see hypotheses (\mathbf{H}_1)), from (3.3) we have

$$(3.4) \quad \text{div } a(Du(z)) \leq \lambda u(z)^{p-1} \text{ for a.a. } z \in \Omega.$$

Let

$$\chi(t) = ta_0(t) \text{ for all } t > 0.$$

Then from the one-dimensional version of hypothesis $\mathbf{H}(\mathbf{a})$ (iii) we have

$$t\chi'(t) = t^2 a_0'(t) + ta_0(t) \geq C_1 t^{p-1},$$

hence

$$(3.5) \quad \int_0^t s\chi'(s) ds = t\chi(t) - \int_0^t \chi(s) ds = t^2 a_0(t) - G_0(t) \geq \frac{C_1}{p} t^p \text{ for all } t \geq 0.$$

Let

$$H(t) = t^2 a_0(t) - G_0(t) \text{ and } H_0(t) = \frac{C_1}{p} t^p \text{ for all } t \geq 0.$$

Let $s \in (0, 1)$ and consider the sets

$$D_s = \{t \in (0, 1) : H(t) \geq s\} \text{ and } D_s^0 = \{t \in (0, 1) : H_0(t) \geq s\}.$$

From (3.5) we see that $D_s^0 \subseteq D_s$, hence we have successively: $\inf D_s^0 \leq \inf D_s$, $H^{-1}(s) \leq H_0^{-1}(s)$, and

$$\int_0^\delta \frac{1}{H^{-1}\left(\frac{\lambda}{p} s^p\right)} ds \geq \int_0^\delta \frac{1}{H_0^{-1}\left(\frac{\lambda}{p} s^p\right)} ds = C_7 \int_0^\delta \frac{ds}{s} = +\infty \text{ for some } C_7 > 0.$$

Because of (3.4) we can apply the strong maximum principle of Pucci-Serrin [25] (p. 111) and deduce that $u(z) > 0$ for all $z \in \Omega$. Then invoking the boundary point theorem of Pucci-Serrin [25] (p. 120), we conclude that $u \in \text{int } C_+$. Therefore $\mathcal{S}(\lambda) \subseteq \text{int } C_+$.

Next we show that \mathcal{L} is nonempty and prove a structural property of \mathcal{L} , namely that \mathcal{L} is a half-line.

PROPOSITION 3.2. *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_1) hold, then*

$$\mathcal{L} \neq \emptyset, \text{ and } \lambda \in \mathcal{L} \text{ implies that } [\lambda, +\infty) \subseteq \mathcal{L}.$$

PROOF. We consider the following auxiliary Neumann

$$(3.6) \quad -\operatorname{div} a(Du(z)) + u(z)^{p-1} = 1 \text{ in } \Omega, \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, u > 0.$$

Let $K_p : L^p(\Omega) \rightarrow L^{p'}(\Omega)$ ($\frac{1}{p} + \frac{1}{p'} = 1$) be the nonlinear map defined by

$$K_p(u)(\cdot) = |u(\cdot)|^{p-2} u(\cdot) \text{ for all } u \in L^p(\Omega).$$

Clearly K_p is continuous and strictly monotone and so is $K_p|_{W^{1,p}(\Omega)}$ which implies that $K_p|_{W^{1,p}(\Omega)}$ is maximal monotone. Let $V : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)^*$ be defined by

$$V(u) = A(u) + K_p(u) \text{ for all } u \in W^{1,p}(\Omega).$$

Using Proposition 2.4, from Gasinski-Papageorgiou [11] (p.320), we conclude that $V(\cdot)$ is maximal monotone. Also, we have

$$\begin{aligned} \langle V(u), u \rangle &= \langle A(u), u \rangle + \|u\|_p^p \geq \frac{C_1}{p-1} \|Du\|_p^p + \|u\|_p^p \text{ (see Lemma 2.2)} \\ &\geq C_8 \|u\|_p^p \text{ for some } C_8 > 0 \end{aligned}$$

hence $V(\cdot)$ is coercive. Then from in [11] (p.320), we have that $V(\cdot)$ is surjective.

So, we can find $\bar{u} \in W^{1,p}(\Omega)$, $\bar{u} \neq 0$ such that $V(\bar{u}) = 0$, hence

$$(3.7) \quad A(\bar{u}) + |\bar{u}|^{p-2} \bar{u} = 1.$$

On (3.7) we act with $-\bar{u}^-$ and obtain

$$\frac{C_1}{p-1} \|D\bar{u}^-\|_p^p + \|\bar{u}^-\|_p^p \leq 0 \text{ (see Lemma 2.2),}$$

hence

$$\bar{u} \geq 0, \bar{u} \neq 0.$$

Then (3.7) becomes

$$A(\bar{u}) + \bar{u}^{p-1} = 1,$$

hence \bar{u} is a positive solution of the auxiliary problem (3.6).

As in the proof of Proposition 3.1, using the nonlinear regularity theory (see [14], [27] and [15]) and the nonlinear maximum principle (see [25]), we have $\bar{u} \in \operatorname{int} C_+$. So, we can find $C_9 > 0$ such that

$$\bar{u}(z) \geq C_9 \text{ for all } z \in \bar{\Omega}.$$

Let

$$\lambda_0 = \frac{1 + \|N_f(\bar{u})\|_\infty}{C_9^{p-1}}$$

(see hypothesis (\mathbf{H}_1) (i)). Then

$$(3.8) \quad A(\bar{u}) + \lambda_0 \bar{u}^{p-1} \geq N_f(\bar{u}) \text{ in } W^{1,p}(\Omega)^*.$$

Using $\bar{u} \in \text{int } C_+$, we introduce the following truncation of the reaction $f(z, \cdot)$:

$$(3.9) \quad f_0(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ f(z, x) & \text{if } 0 \leq x \leq \bar{u}(z) \\ f(z, \bar{u}(z)) & \text{if } \bar{u}(z) < x. \end{cases}$$

This is a Carathéodory function. Let

$$F_0(z, x) = \int_0^x f_0(z, s) ds$$

and consider the C^1 -functional $\varphi_0 : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\varphi_0(u) = \int_{\Omega} G(Du(z)) dz + \frac{\lambda_0}{p} \|u\|_p^p - \int_{\Omega} F_0(z, u(z)) dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (3.9) it is clear that φ_0 is coercive. Also, using the Sobolev embedding theorem, we see that φ_0 is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_0 \in W^{1,p}(\Omega)$ such that

$$(3.10) \quad \varphi_0(u_0) = \inf \{ \varphi_0(u) : u \in W^{1,p}(\Omega) \}.$$

By virtue of $\mathbf{H}(\mathbf{a})(v)$ and $(\mathbf{H}_1)(iv)$, given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) \in (0, \widehat{\delta}_0]$ such that

$$G_0(t) \leq \frac{\varepsilon}{q} t^q \text{ for all } t \in [0, \delta],$$

hence

$$(3.11) \quad G(y) \leq \frac{\varepsilon}{q} \|y\|^q \text{ for all } y \in \mathbb{R}^N \text{ with } \|y\| \leq \delta.$$

Given $u \in \text{int } C_+$, we can find $t \in (0, 1]$ small such that

$$(3.12) \quad tu \leq \bar{u}, \quad tu(z) \in (0, \delta] \text{ and } t \|Du(z)\| \in [0, \delta] \text{ for all } z \in \bar{\Omega}$$

(recall $u, \bar{u} \in \text{int } C_+$ and use Lemma 3.3 of Filippakis-Kristaly-Papageorgiou [10]). Then we have

$$(3.13) \quad \begin{aligned} \varphi_0(tu) &= \int_{\Omega} G(tDu(z)) dz + \frac{\lambda_0 t^p}{p} \|u\|_p^p - \int_{\Omega} F_0(z, tu(z)) dz \\ &\leq \frac{\lambda_0 t^p}{p} \|u\|_p^p - \frac{t^q}{q} \left[\widehat{C}_0 \|u\|_q^q - \varepsilon \|Du\|_q^q \right] \end{aligned}$$

(see (3.11), (3.12) and hypothesis $\mathbf{H}(\mathbf{a})(iv)$). We choose

$$\varepsilon \in \left(0, \frac{\widehat{C}_0 \|u\|_q^q}{\|Du\|_q^q} \right).$$

Then from (3.13) it follows

$$(3.14) \quad \varphi_0(tu) \leq \frac{\lambda_0 t^p}{p} \|u\|_p^p - C_{10} t^q \text{ for some } C_{10} = C_{10}(u) > 0.$$

Since $q < p$ (see hypothesis $\mathbf{H}(\mathbf{a})(iv)$), choosing $t \in (0, 1)$ even smaller if necessary, from (3.14) we see that

$$\varphi_0(tu) < 0.$$

which implies

$$\varphi_0(u_0) < 0 = \varphi_0(0) \quad (\text{see (3.10)})$$

hence

$$u_0 \neq 0.$$

From (3.10) we have $\varphi'_0(u_0) = 0$, hence

$$(3.15) \quad A(u_0) + \lambda_0 |u_0|^{p-2} u_0 = N_{f_0}(u_0).$$

On (3.15) we act with $-u_0^- \in W^{1,p}(\Omega)$ and obtain

$$\frac{C_1}{p-1} \|Du_0^-\|_p^p + \lambda_0 \|u_0^-\|_p^p \leq 0 \quad (\text{see Lemma 2.2 and (3.9)}),$$

hence

$$u_0 \geq 0, \quad u_0 \neq 0.$$

Also, on (3.15) we act with $(u_0 - \bar{u})^+ \in W^{1,p}(\Omega)$. We obtain

$$\begin{aligned} & \left\langle A(u_0), (u_0 - \bar{u})^+ \right\rangle + \lambda_0 \int_{\Omega} u_0^{p-1} (u_0 - \bar{u})^+ dz \\ &= \int_{\Omega} f_0(z, u_0) (u_0 - \bar{u})^+ dz \\ &= \int_{\Omega} f(z, \bar{u}) (u_0 - \bar{u})^+ dz \quad (\text{see (3.9)}) \\ &\leq \left\langle A(\bar{u}), (u_0 - \bar{u})^+ \right\rangle + \lambda_0 \int_{\Omega} \bar{u}^{p-1} (u_0 - \bar{u})^+ dz \quad (\text{see (3.8)}), \end{aligned}$$

hence

$$\begin{aligned} & \int_{\{u_0 > \bar{u}\}} (a(Du_0) - a(D\bar{u}), Du_0 - D\bar{u})_{\mathbb{R}^N} \\ &+ \lambda_0 \int_{\{u_0 > \bar{u}\}} (u_0^{p-1} - \bar{u}^{p-1}) (u_0 - \bar{u}) dz \\ &\leq 0, \end{aligned}$$

therefore

$$|\{u_0 > \bar{u}\}|_N = 0,$$

and we conclude that

$$u_0 \leq \bar{u}.$$

So, we have proved that

$$u_0 \in [0, \bar{u}] := \{u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq \bar{u}(z) \text{ for a.a. } z \in \Omega\}, \quad u_0 \neq 0.$$

Then equation (3.15) becomes

$$A(u_0) + \lambda_0 u_0^{p-1} = N_f(u_0) \quad (\text{see (3.9)})$$

therefore

$$u_0 \in \mathcal{S}(\lambda_0) \subseteq \text{int } C_+$$

(see Proposition 3.1) and so

$$\lambda_0 \in \mathcal{L}.$$

Now let $\lambda \in \mathcal{L}$ and $\eta > \lambda$. Then there exists $u_\lambda \in \mathcal{S}(\lambda) \subseteq \text{int } C_+$ (see Proposition 3.1). We have

$$(3.16) \quad A(u_\lambda) + \eta u_\lambda^{p-1} \geq A(u_\lambda) + \lambda u_\lambda^{p-1} = N_f(u_\lambda) \text{ in } W^{1,p}(\Omega)^*.$$

We truncate $f(z, \cdot)$ at $u_\lambda(z)$ and reasoning as above with \bar{u} replaced by u_λ and using (3.16) instead of (3.8), via the direct method, we produce

$$u_\eta \in [0, u_\lambda] \cap \mathcal{S}(\eta) \subseteq [0, u_\lambda] \cap \text{int } C_+.$$

Therefore $\eta \in \mathcal{L}$ and so we conclude that $[\lambda, +\infty) \subseteq \mathcal{L}$.

A useful by-product of the above proof is the following corollary:

COROLLARY 3.3. *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_1) hold, $\eta > \lambda \in \mathcal{L}$ and $u_\lambda \in \mathcal{S}(\lambda) \subseteq \text{int } C_+$, then we can find $u_\eta \in \mathcal{S}(\eta) \subseteq \text{int } C_+$ such that $u_\eta \leq u_\lambda$.*

In fact, we can improve the conclusion of this corollary provided we strengthen a little the hypotheses on the reaction $f(z, x)$. The new hypotheses on the reaction $f(z, x)$ are the following:

- (H₂):** $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$
 $f(z, 0) = 0$, $f(z, x) > 0$ for all $x > 0$, hypotheses **(H₂)** (i) – (iv) are the same as **(H₁)** (i) – (iv) and
 (v) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $z \in \Omega$ the function $x \rightarrow f(z, x) + \xi_\rho x^{p-1}$ is nondecreasing on $[0, \rho]$.

Remarks: Note that if for a.a. $z \in \Omega$, $f(z, \cdot) \in C^1(0, \infty)$ and $f_x(z, \cdot)$ is $L^\infty(\Omega)$ –bounded on compact subsets of $(0, \infty)$, the hypothesis **(H₂)** (v) is automatically satisfied. So, the two examples given after hypotheses **(H₁)** satisfy **(H₂)** (v).

PROPOSITION 3.4. *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_2) hold, $\eta > \lambda \in \mathcal{L}$ and $u_\lambda \in \mathcal{S}(\lambda) \subseteq \text{int } C_+$, then we can find $u_\eta \in \mathcal{S}(\eta) \subseteq \text{int } C_+$ such that*

$$u_\lambda - u_\eta \in \text{int } C_+.$$

PROOF. From Corollary 3.3, we know that there exists $u_\eta \in \mathcal{S}(\eta) \subseteq \text{int } C_+$ such that

$$(3.17) \quad u_\eta \leq u_\lambda.$$

Let $\delta > 0$ and set $u_\eta^\delta = u_\eta + \delta \in \text{int } C_+$. Let $\rho = \|u_\lambda\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $(\mathbf{H}_2)(v)$. We have

$$\begin{aligned}
& -\text{div } a(Du_\eta^\delta) + (\lambda + \xi_\rho)(u_\eta^\delta)^{p-1} \\
& \leq -\text{div } a(Du_\eta) + \eta u_\eta^{p-1} - (\eta - \lambda)u_\eta^{p-1} + \xi_\rho u_\eta^{p-1} + \sigma(\delta) \text{ with } \sigma(\delta) \rightarrow 0^+ \text{ as } \delta \rightarrow 0^+ \\
& \leq -\text{div } a(Du_\eta) + (\eta + \xi_\rho)u_\eta^{p-1} - (\eta - \lambda)m_\eta^{p-1} + \sigma(\delta) \text{ with } m_\eta = \min_{\bar{\Omega}} u_\eta > 0 \\
& \leq -\text{div } a(Du_\eta) + (\eta + \xi_\rho)u_\eta^{p-1} \text{ for } \delta > 0 \text{ small} \\
& = f(z, u_\eta) + \xi_\rho u_\eta^{p-1} \text{ (since } u_\eta \in \mathcal{S}(\eta)) \\
& \leq f(z, u_\lambda) + \xi_\rho u_\lambda^{p-1} \text{ (see (3.17) and hypothesis } (\mathbf{H}_2)(v)) \\
& = -\text{div } a(Du_\lambda) + \xi_\rho u_\lambda^{p-1} \text{ (since } u_\lambda \in \mathcal{S}(\lambda)),
\end{aligned}$$

hence

$$u_\eta^\delta \leq u_\lambda \text{ for all } \delta > 0 \text{ small (see Damascelli [9], p.495)}$$

therefore

$$u_\lambda - u_\eta \in \text{int } C_+.$$

Let

$$\lambda_* = \inf \mathcal{L}.$$

In what follows, for every $\lambda > 0$, $\varphi_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ is the energy functional defined by

$$\varphi_\lambda(u) = \int_{\Omega} G(Du(z)) dz + \frac{\lambda}{p} \|u\|_p^p - \int_{\Omega} F(z, u(z)) dz \text{ for all } u \in W^{1,p}(\Omega).$$

Evidently $\varphi_\lambda \in C^1(W^{1,p}(\Omega))$.

PROPOSITION 3.5. *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_1) hold, then $\lambda_* > 0$*

PROOF. We argue by contradiction. So, suppose that $\lambda_* = 0$ and let $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq (0, \infty) \subseteq \mathcal{L}$ be such that $\lambda_n \downarrow 0$ as $n \rightarrow \infty$. We can find $u_n \in \mathcal{S}(\lambda_n)$ for $n \geq 1$, such that $\{u_n\}_{n \in \mathbb{N}}$ is nondecreasing and

$$(3.18) \quad \varphi_{\lambda_n}(u_n) < 0 \text{ for all } n \geq 1$$

(see the last part of the proof of Proposition 3.2). From (3.18) we have

$$(3.19) \quad -\int_{\Omega} pF(z, u_n) dz \leq -\int_{\Omega} pG(Du_n) dz - \lambda_n \|u_n\|_p^p \text{ for all } n \geq 1.$$

Since $u_n \in \mathcal{S}(\lambda_n)$ for $n \geq 1$, we have

$$A(u_n) + \lambda_n u_n^{p-1} = N_f(u_n)$$

hence

$$(3.20) \quad \int_{\Omega} f(z, u_n) u_n dz = \int_{\Omega} (a(Du_n), Du_n)_{\mathbb{R}^N} + \lambda_n \|u_n\|_p^p \text{ for all } n \geq 1.$$

Adding (3.19) and (3.20), we obtain

$$\int_{\Omega} [f(z, u_n) u_n - pF(z, u_n)] dz \leq \int_{\Omega} [(a(Du_n), Du_n)_{\mathbb{R}^N} - pG(Du_n)] dz,$$

hence

$$\int_{\Omega} [f(z, u_n) u_n - pF(z, u_n)] dz \leq \xi_0 \text{ for all } n \geq 1 \text{ (see } (\mathbf{H}(\mathbf{a})) \text{ (iv))}.$$

From hypotheses (\mathbf{H}_1) (i), (iii), we see that we can find $\beta_1 \in (0, \beta_0)$ and $C_{11} > 0$ such that

$$(3.21) \quad \beta_1 x^\mu - C_{11} \leq f(z, x) x - pF(z, x) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

Using (3.21) and (3.20), we have

$$\|u_n\|_{\mu}^{\mu} \leq C_{12} \text{ for some } C_{12} > 0, \text{ all } n \geq 1,$$

hence

$$(3.22) \quad \{u_n\}_{n \in \mathbb{N}} \subseteq L^{\mu}(\Omega) \text{ is bounded.}$$

First suppose that $p < N$. It is clear from hypothesis (\mathbf{H}_1) (iii) that without any loss of generality we may assume that $\mu < r < p^*$. Let $t \in (0, 1)$ such that

$$(3.23) \quad \frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{p^*}.$$

Invoking the interpolation inequality (see for example, Gasinski-Papageorgiou [11], p. 905), we have

$$\|u_n\|_r \leq \|u_n\|_{\mu}^{1-t} \|u_n\|_{p^*}^t \text{ for all } n \geq 1.$$

Then using (3.22) and the Sobolev embedding theorem we have

$$(3.24) \quad \|u_n\|_r^r \leq C_{13} \|u_n\|^{tr} \text{ for some } C_{13} > 0, \text{ all } n \geq 1.$$

Hypothesis (\mathbf{H}_1) (i) implies that

$$(3.25) \quad f(z, x) \leq C_{14} (1 + x^r) \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0, \text{ some } C_{14} > 0.$$

From (3.20) and (3.25), we have

$$\int_{\Omega} (a(Du_n), Du_n)_{\mathbb{R}^N} + \lambda_n \|u_n\|_p^p \leq C_{15} (1 + \|u_n\|_r^r) \text{ for some } C_{15} > 0, \text{ all } n \geq 1,$$

hence

$$(3.26) \quad \frac{C_1}{p-1} \|Du_n\|_p^p \leq C_{16} (1 + \|u_n\|^{tr}) \text{ for some } C_{16} > 0, \text{ all } n \geq 1,$$

(see Lemma 2.2 and (3.24)). Recall that $u \rightarrow \|u\|_{\mu} + \|Du\|_p$ is an equivalent norm on the Sobolev space $W^{1,p}(\Omega)$ (see for example, Gasinski-Papageorgiou [11] (p.227)). So, from (3.22) and (3.26) we have

$$(3.27) \quad \|u_n\|_p^p \leq C_{17} (1 + \|u_n\|^{tr}) \text{ for some } C_{17} > 0, \text{ all } n \geq 1.$$

From (3.23) and the hypothesis on μ (see (\mathbf{H}_1) (iii)) it follows that $tr < p$. Therefore from (3.27), we infer that

$$(3.28) \quad \{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

If $p \geq N$, then $p^* = +\infty$ and $W^{1,p}(\Omega) \hookrightarrow L^\theta(\Omega)$ for all $\theta \in [1, +\infty)$. Then the previous argument works if we replace p^* by $\eta > r > \mu$ and we choose $t \in (0, 1)$ such that

$$\frac{1}{r} = \frac{1-t}{\mu} + \frac{t}{\eta},$$

that is,

$$tr = \frac{\eta(r-\mu)}{\eta-\mu}.$$

Note that $\frac{\eta(r-\mu)}{\eta-\mu} \rightarrow r - \mu$ as $\eta \rightarrow +\infty = p^*$. But by hypothesis (\mathbf{H}_1) (iii), $r - \mu < p$. Therefore for $\eta > r$ large, we have $tr < p$ and so again (3.28) holds.

By virtue of (3.28) and by passing to a subsequence if necessary, we may assume that

$$(3.29) \quad u_n \xrightarrow{w} u \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u \text{ in } L^r(\Omega) \text{ as } n \rightarrow \infty.$$

Recall that

$$(3.30) \quad A(u_n) + \lambda_n u_n^{p-1} = N_f(u_n) \text{ for all } n \geq 1.$$

On (3.30) we act with $u_n - u \in W^{1,p}(\Omega)$, pass to the limit as $n \rightarrow \infty$ and use (3.29) to obtain

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u \rangle = 0,$$

hence

$$(3.31) \quad u_n \rightarrow u \text{ in } W^{1,p}(\Omega) \text{ as } n \rightarrow \infty$$

(see Proposition 2.4). So, if in (3.30) we pass to the limit as $n \rightarrow \infty$ and use (3.31) and the fact that $\lambda_n \downarrow 0$, we obtain

$$(3.32) \quad A(u) = N_f(u).$$

By the nonlinear regularity theory (see Lieberman [15]) it follows that

$$u \in C_+.$$

Claim: $u \neq 0$.

From hypotheses (\mathbf{H}_1) (i), (iv), we see that we can find $C_{18} > 0$ such that

$$f(z, x) \geq \widehat{C}_0 x^{q-1} - C_{18} x^{r-1} \text{ for a.a. } z \in \Omega, \text{ all } x \geq 0.$$

Motivated by this unilateral growth estimate, we introduce the following auxiliary Neumann problem

$$(3.33) \quad \begin{aligned} -\operatorname{div} a(Du(z)) + \lambda_1 u(z)^{p-1} &= \widehat{C}_0 u(z)^{q-1} - C_{18} u(z)^{r-1} \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= 0 \text{ on } \partial\Omega, u > 0. \end{aligned}$$

Let $\psi : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ be the energy functional for problem (3.33) defined by

$$\psi(u) = \int_{\Omega} G(Du(z)) dz + \frac{\lambda_1}{p} \|u\|_p^p - \frac{\widehat{C}_0}{q} \|u^+\|_q^q + \frac{C_{18}}{r} \|u^+\|_r^r \quad \text{for all } u \in W^{1,p}(\Omega).$$

Since $q < p < r$, it is clear that ψ is coercive (see Corollary 2.3). Also, by the Sobolev embedding theorem, we see that ψ is sequentially weakly lower semicontinuous. So, we can find $\tilde{u} \in W^{1,p}(\Omega)$ such that

$$(3.34) \quad \psi(\tilde{u}) = \inf \{ \psi(u) : u \in W^{1,p}(\Omega) \}.$$

Because $q < p < r$, as in the proof of Proposition 3.2, we check that

$$\psi(\tilde{u}) < 0 = \psi(u), \quad \text{hence } \tilde{u} \neq 0.$$

From (3.34), we have

$$\psi'(\tilde{u}) = 0$$

hence

$$(3.35) \quad A(\tilde{u}) + \lambda_1 \|\tilde{u}\|^{p-2} \tilde{u} = \widehat{C}_0 (\tilde{u}^+)^{q-1} - C_{18} (\tilde{u}^+)^{r-1}.$$

On (3.35) we act with $-\tilde{u}^- \in W^{1,p}(\Omega)$, and using Lemma 2.2, we obtain

$$\tilde{u} \geq 0, \tilde{u} \neq 0.$$

Then (3.35) becomes

$$A(\tilde{u}) + \lambda_1 \tilde{u}^{p-1} = \widehat{C}_0 \tilde{u}^{q-1} - C_{18} \tilde{u}^{r-1},$$

hence \tilde{u} is a positive solution of (3.33) and

$$\tilde{u} \in \text{int } C_+$$

(by nonlinear regularity [15] and the nonlinear maximum principle [25]). Moreover, as in Aizicovici-Papageorgiou-Staicu [4], we conclude that $\tilde{u} \in \text{int } C_+$ is the unique positive solution of (3.33).

Let $u_1 \in \mathcal{S}(\lambda_1) \subseteq \text{int } C_+$ and consider the Carathéodory function

$$(3.36) \quad k(z, x) = \begin{cases} 0 & \text{if } x < 0 \\ \widehat{C}_0 x^{q-1} - C_{18} x^{r-1}, & \text{if } 0 \leq x \leq u_1(z) \\ \widehat{C}_0 u_1(z)^{q-1} - C_{18} u_1(z)^{r-1} & \text{if } u_1(z) < x. \end{cases}$$

We set

$$K(z, x) = \int_0^x k(z, s) ds$$

and consider the C^1 -functional $\widehat{\gamma} : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\gamma}(u) = \int_{\Omega} G(Du(z)) dz + \frac{\lambda_1}{p} \|u\|_p^p - \int_{\Omega} K(z, u(z)) dz \quad \text{for all } u \in W^{1,p}(\Omega).$$

From (3.36) it is clear that $\widehat{\gamma}(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\widetilde{u}_0 \in W^{1,p}(\Omega)$ such that

$$(3.37) \quad \widehat{\gamma}(\widetilde{u}_0) = \inf \{ \widehat{\gamma}(u) : u \in W^{1,p}(\Omega) \}.$$

As before (see the proof of Proposition 3.2), since $1 < q < p < r$, we have

$$\widehat{\gamma}(\widetilde{u}_0) < 0 = \widehat{\gamma}(0), \text{ hence } \widetilde{u}_0 \neq 0.$$

From (3.37), we have

$$\widehat{\gamma}'(\widetilde{u}_0) = 0$$

hence

$$(3.38) \quad A(\widetilde{u}_0) + \lambda_1 \|\widetilde{u}_0\|^{p-2} \widetilde{u}_0 = N_k(\widetilde{u}_0)$$

On (3.38) we first act with $-\widetilde{u}_0^- \in W^{1,p}(\Omega)$ and then with $(\widetilde{u}_0 - u_1)^+ \in W^{1,p}(\Omega)$ and obtain

$$\widetilde{u}_0 \in [0, u_1] := \{u \in W^{1,p}(\Omega) : 0 \leq u(z) \leq u_1(z) \text{ for a.a. } z \in \Omega\}$$

(see the proof of Proposition 3.2). Using (3.36) and (3.37) we obtain

$$A(\widetilde{u}_0) + \lambda_1 \widetilde{u}_0^{p-1} = \widehat{C}_0 \widetilde{u}_0^{q-1} - C_{18} \widetilde{u}_0^{r-1},$$

hence \widetilde{u}_0 is a positive solution of (3.33), and by the uniqueness of the positive solution of (3.33), it follows that

$$\widetilde{u}_0 = \widetilde{u} \in \text{int } C_+.$$

So, we can say that

$$\widetilde{u} \leq u_1 \leq u_n \text{ for all } n \geq 1$$

(recall that $\{u_n\}_{n \in \mathbb{N}}$ is nondecreasing), hence $\widetilde{u} \leq u$ (see (3.31)), therefore $u \neq 0$. This proves the Claim.

On (3.32) we act with $1 \in \text{int } C_+$. We obtain

$$0 = \int_{\Omega} f(z, u) dz.$$

But our hypotheses on f and the Claim, imply $\int_{\Omega} f(z, u) dz > 0$, a contradiction. This means that $\lambda_* > 0$.

If we use the stronger hypotheses (\mathbf{H}_2) , we can show that for $\lambda \in (\lambda_*, \infty)$, problem (P_λ) admits at least two positive solutions.

PROPOSITION 3.6. *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_2) hold and $\lambda \in (\lambda_*, \infty)$, then (P_λ) admits at least two positive solutions*

$$u_\lambda, \widehat{u}_\lambda \in \text{int } C_+, \quad u_\lambda \neq \widehat{u}_\lambda.$$

PROOF. Let $\eta_1, \eta_2 \in \mathcal{L}$ and assume that $\lambda_* < \eta_1 < \lambda < \eta_2$. From Proposition 3.4, we know that we can find $u_{\eta_1} \in \mathcal{S}(\eta_1) \subseteq \text{int } C_+$ and $u_{\eta_2} \in \mathcal{S}(\eta_2) \subseteq \text{int } C_+$ such that $u_{\eta_1} - u_{\eta_2} \in \text{int } C_+$.

We introduce the following Carathéodory function

$$(3.39) \quad w(z, x) = \begin{cases} f(z, u_{\eta_2}(z)) & \text{if } x < u_{\eta_2}(z) \\ f(z, x) & \text{if } u_{\eta_2}(z) \leq x \leq u_{\eta_1}(z) \\ f(z, u_{\eta_1}(z)) & \text{if } u_{\eta_1}(z) < x. \end{cases}$$

We set

$$W(z, x) = \int_0^x w(z, s) ds$$

and consider the C^1 -functional $\xi_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\xi_\lambda(u) = \int_\Omega G(Du(z)) dz + \frac{\lambda}{p} \|u\|_p^p - \int_\Omega W(z, u(z)) dz \text{ for all } u \in W^{1,p}(\Omega).$$

From (3.39) we see that $\xi_\lambda(\cdot)$ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $u_\lambda \in W^{1,p}(\Omega)$ such that

$$\xi_\lambda(u_\lambda) = \inf \{ \xi_\lambda(u) : u \in W^{1,p}(\Omega) \},$$

hence

$$\xi'_\lambda(u_\lambda) = 0$$

therefore

$$(3.40) \quad A(u_\lambda) + \lambda \|u_\lambda\|^{p-2} u_\lambda = N_w(u_\lambda).$$

On (3.40) we act with $(u_\lambda - u_{\eta_1})^+ \in W^{1,p}(\Omega)$ and with $(u_{\eta_2} - u_\lambda)^+ \in W^{1,p}(\Omega)$, and obtain

$$u_\lambda \in [u_{\eta_2}, u_{\eta_1}] := \{ u \in W^{1,p}(\Omega) : u_{\eta_2}(z) \leq u(z) \leq u_{\eta_1}(z) \text{ for a.a. } z \in \Omega \}.$$

In fact, reasoning as in the proof of Proposition 3.4, we show that

$$u_\lambda - u_{\eta_2} \in \text{int } C_+ \text{ and } u_{\eta_1} - u_\lambda \in \text{int } C_+,$$

hence

$$(3.41) \quad u_\lambda(z) \in \text{int}_{C^1(\bar{\Omega})} [u_{\eta_2}, u_{\eta_1}].$$

Then from (3.39) we see that $u_\lambda \in \mathcal{S}(\lambda) \subseteq \text{int } C_+$. So, we have produced one positive solution for (P_λ) . To produce a second positive solution, we introduce the Carathéodory function $\widehat{k}(\cdot, \cdot)$ defined by

$$(3.42) \quad \widehat{k}(z, x) = \begin{cases} f(z, u_{\eta_2}(z)) & \text{if } x < u_{\eta_2}(z) \\ f(z, x) & \text{if } u_{\eta_2}(z) \leq x. \end{cases}$$

Let

$$\widehat{K}(z, x) = \int_0^x k(z, s) ds$$

and consider the C^1 -functional $\sigma_\lambda : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\widehat{\sigma}_\lambda(u) = \int_\Omega G(Du(z)) dz + \frac{\lambda}{p} \|u\|_p^p - \int_\Omega \widehat{K}(z, u(z)) dz \text{ for all } u \in W^{1,p}(\Omega).$$

As before (see the proof of Proposition 3.2) we can check that

$$(3.43) \quad K_{\widehat{\sigma}_\lambda} \subseteq [u_{\eta_2}] := \{u \in W^{1,p}(\Omega) : u_{\eta_2}(z) \leq u(z) \text{ for a.a. } z \in \Omega\}.$$

From (3.39) and (3.42) we see

$$(3.44) \quad \xi_\lambda |_{[u_{\eta_2}, u_{\eta_1}]} = \widehat{\sigma}_\lambda |_{[u_{\eta_2}, u_{\eta_1}]}.$$

By (3.41) and (3.44) and since u_λ is a minimizer of ξ_λ , it follows that u_λ is a $C^1(\overline{\Omega})$ -minimizer of $\widehat{\sigma}_\lambda$. Invoking Proposition 2.5, we infer that u_λ is a $W^{1,p}(\Omega)$ -minimizer of $\widehat{\sigma}_\lambda$.

We may assume that $K_{\widehat{\sigma}_\lambda}$ is finite or otherwise we have an infinity of positive solutions for problem (P_λ) (see (3.43) and (3.42)). Then, from Aizicovici-Papageorgiou-Staicu [1] (see the proof of Proposition 29), we can find $\rho \in (0, 1)$ small such that

$$(3.45) \quad \widehat{\sigma}_\lambda(u_\lambda) < \inf \{\widehat{\sigma}_\lambda(u) : \|u - u_\lambda\| = \rho\} =: \widehat{m}_\lambda.$$

Hypothesis (\mathbf{H}_2) (ii) implies

$$(3.46) \quad \widehat{\sigma}_\lambda(\xi) \rightarrow -\infty \text{ as } \xi \rightarrow +\infty, \xi \in \mathbb{R}.$$

In addition, minor changes in the first part of the proof of Proposition 3.5, reveal that

$$(3.47) \quad \widehat{\sigma}_\lambda \text{ satisfies the C-condition}$$

(see also Aizicovici-Papageorgiou-Staicu [3]). Then (3.45), (3.46), (3.47) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $\widehat{u}_\lambda \in W^{1,p}(\Omega)$ such that

$$(3.48) \quad \widehat{u}_\lambda \in K_{\widehat{\sigma}_\lambda} \subseteq [u_{\eta_2}] \text{ (see (3.43)) and } \widehat{\sigma}_\lambda(u_\lambda) < \widehat{m}_\lambda \leq \widehat{\sigma}_\lambda(\widehat{u}_\lambda).$$

From (3.48) we see that

$$\widehat{u}_\lambda \in \mathcal{S}(\lambda) \subseteq \text{int } C_+ \text{ (see (3.42)) and } u_\lambda \neq \widehat{u}_\lambda.$$

Next we examine what happens in the critical case $\lambda = \lambda_*$.

PROPOSITION 3.7. *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_1) hold, then $\lambda_* \in \mathcal{L}$ and so, $\mathcal{L} = [\lambda_*, +\infty)$.*

PROOF. Let $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathcal{L}$ and assume $\lambda_n \downarrow \lambda_*$. We can find $u_n \in \mathcal{S}(\lambda_n) \subseteq \text{int } C_+$ such that

$$\varphi_{\lambda_n}(u_n) < 0 \text{ for all } n \geq 1.$$

Then, from the proof of Proposition 3.5, we know that

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$(3.49) \quad u_n \xrightarrow{w} u_* \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow u_* \text{ in } L^r(\Omega) \text{ as } n \rightarrow \infty.$$

We have

$$(3.50) \quad A(u_n) + \lambda_n u_n^{p-1} = N_f(u_n) \text{ for all } n \geq 1.$$

Acting on (3.50) with $u_n - u_* \in W^{1,p}(\Omega)$, passing to the limit as $n \rightarrow \infty$ and using (3.49) we obtain

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - u_* \rangle = 0,$$

hence

$$(3.51) \quad u_n \rightarrow u_* \text{ in } W^{1,p}(\Omega) \text{ as } n \rightarrow \infty$$

(see Proposition 2.4). Also, from the proof of Proposition 3.5), we know that

$$\tilde{u}_0 \leq u_n \text{ for all } n \geq 1$$

(here $\tilde{u}_0 \in \text{int } C_+$ denotes the unique positive solution of the auxiliary problem (3.33)). Then from (3.51) we have

$$\tilde{u}_0 \leq u_*, \text{ hence } u_* \neq 0.$$

If in (3.50) we pass to the limit as $n \rightarrow \infty$ and use (3.51), then

$$A(u_*) + \lambda_* u_*^{p-1} = N_f(u_*),$$

hence $u_* \in \mathcal{S}(\lambda_*) \subseteq \text{int } C_+$ and so $\lambda_* \in \mathcal{L}$, hence $\mathcal{L} = [\lambda_*, +\infty)$.

In fact, we can show that for every $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$ problem (P_λ) admits a smallest positive solution $\bar{u}_\lambda \in \mathcal{S}(\lambda) \subseteq \text{int } C_+$. We will need this fact in the next section where we produce nodal solutions.

PROPOSITION 3.8. *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_1) (resp. (\mathbf{H}_2)) hold, and $\lambda \in \mathcal{L} = [\lambda_*, +\infty)$, then problem (P_λ) admits a smallest positive solution $\bar{u}_\lambda \in \text{int } C_+$ and the map $\lambda \rightarrow \bar{u}_\lambda$ is nonincreasing (resp. decreasing) and right continuous from \mathcal{L} into $C^1(\bar{\Omega})$.*

PROOF. As in Aizicovici-Papageorgiou-Staicu [2]), exploiting the monotonicity of A (see Proposition 2.4), we see that for every $\lambda \in \mathcal{L}$, $\mathcal{S}(\lambda)$ is downward directed, that is, if $u_1, u_2 \in \mathcal{S}(\lambda)$, there exists $u \in \mathcal{S}(\lambda)$ such that $u \leq u_1, u \leq u_2$.

Since we are looking for the smallest positive solution, and since $\mathcal{S}(\lambda)$ is downward directed, without any loss of generality, we may assume that there exists $C_{19} > 0$ such that

$$(3.52) \quad \|u\|_\infty \leq C_{19} \text{ for all } u \in W^{1,p}(\Omega).$$

From Hu-Papageorgiou [13] (p. 178), we know that we can find $\{u_n\}_{n \in \mathbb{N}} \subseteq \mathcal{S}(\lambda)$ such that

$$\inf \mathcal{S}(\lambda) = \inf_{n \geq 1} u_n.$$

We have

$$(3.53) \quad A(u_n) + \lambda u_n^{p-1} = N_f(u_n) \text{ for all } n \geq 1.$$

Because of (3.52) we have

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

So, we may assume that

$$(3.54) \quad u_n \xrightarrow{w} \bar{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ and } u_n \rightarrow \bar{u}_\lambda \text{ in } L^r(\Omega) \text{ as } n \rightarrow \infty.$$

Acting on (3.53) with $u_n - \bar{u}_\lambda \in W^{1,p}(\Omega)$, passing to the limit as $n \rightarrow \infty$ and using (3.54) we obtain

$$\lim_{n \rightarrow \infty} \langle A(u_n), u_n - \bar{u}_\lambda \rangle = 0,$$

hence

$$(3.55) \quad u_n \rightarrow \bar{u}_\lambda \text{ in } W^{1,p}(\Omega) \text{ as } n \rightarrow \infty$$

So, if in (3.53) we pass to the limit as $n \rightarrow \infty$ and use (3.55), then

$$(3.56) \quad A(\bar{u}_\lambda) + \lambda (\bar{u}_\lambda)^{p-1} = N_f(\bar{u}_\lambda).$$

Recall that

$$\tilde{u} \leq u_n \text{ for all } n \geq 1,$$

where $\tilde{u} \in \text{int } C_+$ is the unique positive solution of problem (3.33) with $\lambda_1 = \lambda$. Then, because of (3.55), we have

$$\tilde{u}_0 \leq \bar{u}_\lambda,$$

hence

$$\bar{u}_\lambda \in \mathcal{S}(\lambda) \subseteq \text{int } C_+ \text{ and } \bar{u}_\lambda = \inf \mathcal{S}(\lambda).$$

Next, let $\eta > \lambda$ and let $\bar{u}_\lambda \in \mathcal{S}(\lambda) \subseteq \text{int } C_+$ be the minimal positive solution of (P_λ) . If hypotheses (\mathbf{H}_1) (resp. (\mathbf{H}_2)) hold, then from Corollary 3.3 (resp. Proposition 3.4) we know that we can find $u_\eta \in \mathcal{S}(\eta)$ such that

$$\bar{u}_\lambda \geq u_\eta \text{ (resp. } \bar{u}_\lambda - u_\eta \in \text{int } C_+ \text{)}$$

hence

$$\bar{u}_\lambda \geq \bar{u}_\eta \text{ (resp. } \bar{u}_\lambda - \bar{u}_\eta \in \text{int } C_+ \text{)}.$$

This proves the desired monotonicity of the map $\lambda \rightarrow \bar{u}_\lambda$.

Finally, let $\{\lambda_n\}_{n \in \mathbb{N}} \subseteq \mathcal{L}$ such that $\lambda_n \downarrow \lambda$. We have

$$(3.57) \quad A(\bar{u}_{\lambda_n}) + \lambda_n \bar{u}_{\lambda_n}^{p-1} = N_f(\bar{u}_{\lambda_n}) \text{ for all } n \geq 1.$$

From the proof of Proposition 3.5, we know that

$$(3.58) \quad \{u_{\lambda_n}\}_{n \in \mathbb{N}} \subseteq W^{1,p}(\Omega) \text{ is bounded.}$$

Using (3.57), (3.58) and Proposition 2.4, as before, we can show that for at least a subsequence, we have

$$(3.59) \quad \bar{u}_{\lambda_n} \rightarrow \bar{u} \text{ in } W^{1,p}(\Omega) \text{ and } \bar{u} \in \mathcal{S}(\lambda) \subseteq \text{int } C_+.$$

We claim that $\bar{u} = \bar{u}_\lambda \in \text{int } C_+$. From (3.58), Hu-Papageorgiou [14] (see Proposition 5) and the regularity result of Lieberman [15] (p.320), we know that we can find $\alpha \in (0, 1)$ and $C_{20} > 0$ such that

$$u_n \in C^{1,\alpha}(\bar{\Omega}) \text{ and } \|u_n\|_{C^{1,\alpha}(\bar{\Omega})} \leq C_{20} \text{ for all } n \geq 1.$$

Exploiting the compact embedding of $C^{1,\alpha}(\bar{\Omega})$ into $C^1(\bar{\Omega})$ and using (3.59), we infer that

$$(3.60) \quad \bar{u}_{\lambda_n} \rightarrow \bar{u} \text{ in } C^1(\bar{\Omega}).$$

If $\bar{u}_\lambda \neq \bar{u}$, then we can find $z_0 \in \Omega$ such that $\bar{u}_\lambda(z_0) \neq \bar{u}(z_0)$, hence

$$(3.61) \quad \bar{u}_\lambda(z_0) < \bar{u}_{\lambda_n} \text{ for all } n \text{ large enough (see (3.60)).}$$

But from the previous part of the proof, we have

$$\bar{u}_{\lambda_n} \leq \bar{u}_\lambda \text{ for all } n \geq 1$$

which contradicts (3.61). So, indeed $\bar{u} = \bar{u}_\lambda$ and we have proved the continuity of $\lambda \rightarrow \bar{u}_\lambda$ from \mathcal{L} into $C^1(\bar{\Omega})$.

Summarizing the situation for problem (P_λ) , we can state the following bifurcation-type result.

THEOREM 3.9. (a) *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_1) hold, then there exists $\lambda_* > 0$ such that*

(i) *for every $\lambda \in (0, \lambda_*)$, problem (P_λ) has no positive solutions;*

- (ii) for all $\lambda \geq \lambda_*$, problem (P_λ) has at least one positive solution. Moreover, for every $\lambda \geq \lambda_*$, problem (P_λ) has a smallest positive solution $\bar{u}_\lambda \in \text{int } C_+$ and the map $\lambda \rightarrow \bar{u}_\lambda$ from \mathcal{L} into $C^1(\bar{\Omega})$ is nonincreasing and right continuous.
- (b) If hypotheses **H(a)** and **(H₂)** hold, then there exists $\lambda_* > 0$ such that:
- (i) for every $\lambda \in (0, \lambda_*)$, problem (P_λ) has no positive solutions;
 - (ii) for $\lambda = \lambda_*$, problem (P_λ) has at least one positive solution $u_* \in \text{int } C_+$;
 - (iii) for every $\lambda > \lambda_*$, problem (P_λ) has at least two positive solutions $u_\lambda, \hat{u}_\lambda \in \text{int } C_+$, $u_\lambda \neq \hat{u}_\lambda$. Moreover, for every $\lambda \geq \lambda_*$, problem (P_λ) has a smallest positive solution $\bar{u}_\lambda \in \text{int } C_+$ and the map $\lambda \rightarrow \bar{u}_\lambda$ from \mathcal{L} into $C^1(\bar{\Omega})$ is decreasing and right continuous.

4. Nodal solutions

In this section, by imposing bilateral conditions on the reaction $f(z, \cdot)$ we produce nodal solutions.

So, the new hypotheses on the reaction $f(z, x)$ are the following:

(H₃): $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that for a.a. $z \in \Omega$
 $f(z, 0) = 0$, $f(z, x)x > 0$ for all $x \neq 0$ and

(i) there exists $a \in L^\infty(\Omega)_+$ such that

$$|f(z, x)| \leq a(z) \left(1 + |x|^{r-1}\right) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}, \text{ with } p < r < p^*;$$

(ii) if $F(z, x) = \int_0^x f(z, s) ds$ then

$$\lim_{x \rightarrow \pm\infty} \frac{F(z, x)}{|x|^p} = +\infty \text{ uniformly for a.a. } z \in \Omega$$

(iii) there exist $\mu \in \left(\max\left\{(r-p)\frac{N}{p}, 1\right\}, p^*\right)$ and $\beta_0 > 0$ such that

$$\beta_0 \leq \liminf_{x \rightarrow \pm\infty} \frac{f(z, x)x - pF(z, x)}{|x|^\mu} \text{ uniformly for a.a. } z \in \Omega;$$

(iv) there exists $\hat{\delta}_0$ such that

$$0 < qF(z, x) \leq f(z, x)x \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \hat{\delta}_0,$$

and

$$\text{ess inf}_\Omega F(\cdot, \pm\hat{\delta}_0) > 0$$

with $q \in (1, p)$ as in hypothesis **H(a)** (v).

Remarks: Hypothesis **(H₃)** (iv) is a dual AR-condition near zero. It implies the weak condition

$$(4.1) \quad \hat{C}_0 |x|^q \leq F(z, x) \text{ for a.a. } z \in \Omega, \text{ all } |x| \leq \hat{\delta}_0, \text{ and some } \hat{C}_0 > 0$$

(see [19]). So, now we have a stronger condition near zero (see hypothesis **(H₃)** (iv)). Since the conditions on $f(z, \cdot)$ are now bilateral, reasoning as in

Section 3, we can find $\widehat{\lambda}_* > 0$ such that for all $\lambda \geq \widehat{\lambda}_*$ problem (P_λ) has a biggest negative solution $\bar{v}_\lambda \in -\text{int } C_+$ (in this case the set of negative solutions of (P_λ) is upward directed, that is, if v_1, v_2 are negative solutions of (P_λ) , then there exists a negative solution v of (P_λ) such that $v_1 \leq v, v_2 \leq v$).

In what follows, we set

$$\widetilde{\lambda}_* = \max \{ \lambda_*, \widehat{\lambda}_* \}$$

(see Theorem 3.9). We have the following

THEOREM 4.1. *If hypotheses $\mathbf{H}(\mathbf{a})$ and (\mathbf{H}_3) hold and $\lambda \geq \widetilde{\lambda}_*$, then problem (P_λ) admits a nodal solution $y_\lambda \in C^1(\overline{\Omega})$.*

PROOF. Let $\bar{u}_\lambda \in \text{int } C_+$ and $\bar{v}_\lambda \in -\text{int } C_+$ be the two extremal constant sign solutions of (P_λ) . We introduce the following Carathéodory function

$$(4.2) \quad e(z, x) = \begin{cases} f(z, \bar{v}_\lambda(z)) & \text{if } x < \bar{v}_\lambda(z) \\ f(z, x) & \text{if } \bar{v}_\lambda(z) \leq x \leq \bar{u}_\lambda(z) \\ f(z, \bar{u}_\lambda(z)) & \text{if } \bar{u}_\lambda(z) < x. \end{cases}$$

Let

$$e_\pm(z, x) = e(z, \pm x^\pm)$$

(the positive and negative truncations of $e(z, \cdot)$). We set

$$E(z, x) = \int_0^x e(z, s) ds, \quad E_\pm(z, x) = \int_0^x e_\pm(z, s) ds,$$

and introduce the C^1 -functionals $\psi_\lambda, \psi_\lambda^\pm : W^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$\begin{aligned} \psi_\lambda(u) &= \int_\Omega G(Du(z)) dz + \frac{\lambda}{p} \|u\|_p^p - \int_\Omega E(z, u(z)) dz \text{ for all } u \in W^{1,p}(\Omega), \\ \psi_\lambda^\pm(u) &= \int_\Omega G(Du(z)) dz + \frac{\lambda}{p} \|u\|_p^p - \int_\Omega E_\pm(z, u(z)) dz \text{ for all } u \in W^{1,p}(\Omega). \end{aligned}$$

As before (see the proof of Proposition 3.2), we can show that

$$K_{\psi_\lambda} \subseteq [\bar{v}_\lambda, \bar{u}_\lambda], \quad K_{\psi_\lambda^+} \subseteq [0, \bar{u}_\lambda], \quad K_{\psi_\lambda^-} \subseteq [\bar{v}_\lambda, 0].$$

The extremality of \bar{u}_λ and \bar{v}_λ implies

$$(4.3) \quad K_{\psi_\lambda} \subseteq [\bar{v}_\lambda, \bar{u}_\lambda], \quad K_{\psi_\lambda^+} = \{0, \bar{u}_\lambda\}, \quad K_{\psi_\lambda^-} = \{0, \bar{v}_\lambda\}.$$

Claim: $\bar{u}_\lambda \in \text{int } C_+$ and $\bar{v}_\lambda \in -\text{int } C_+$ are both local minimizers of ψ_λ .

It is clear from (4.2) that ψ_λ^+ is coercive. Also, it is sequentially weakly lower semicontinuous. So, we can find $\bar{u} \in W^{1,p}(\Omega)$ such that

$$(4.4) \quad \psi_\lambda^+(\bar{u}) = \inf \{ \psi_\lambda^+(u) : u \in W^{1,p}(\Omega) \}.$$

By (4.1) and since $q < p$, we see that

$$\psi_\lambda^+(\bar{u}) < 0 = \psi_\lambda^+(0), \text{ hence } \bar{u} \neq 0.$$

From (4.3) and (4.4), it follows that $\bar{u} = \bar{u}_\lambda \in \text{int } C_+$. Note that

$$\psi_\lambda|_{C_+} = \psi_\lambda^+|_{C_+},$$

hence $\bar{u}_\lambda \in \text{int } C_+$ is a local $C^1(\bar{\Omega})$ -minimizer of ψ_λ , therefore $\bar{u}_\lambda \in \text{int } C_+$ is a local $W^{1,p}(\Omega)$ -minimizer of ψ_λ (see Proposition 2.5).

Similarly for $\bar{v}_\lambda \in -\text{int } C_+$, using this time the functional ψ_λ^- . This proves the Claim.

Without any loss of generality, we may assume that $\psi_\lambda(\bar{v}_\lambda) \leq \psi_\lambda(\bar{u}_\lambda)$ (the reasoning is similar if the opposite inequality holds).

We assume that K_{ψ_λ} is finite (otherwise we already have infinitely many distinct nodal solutions, see (4.2) and (4.3)). By virtue of the Claim, we can find $\rho \in (0, 1)$ small such that

$$(4.5) \quad \psi_\lambda(\bar{v}_\lambda) \leq \psi_\lambda(\bar{u}_\lambda) < \inf \{ \psi_\lambda(u) : \|u - \bar{u}_\lambda\| = \rho \} =: \bar{m}_\lambda, \quad \|\bar{v}_\lambda - \bar{u}_\lambda\| > \rho$$

(see Aizicovici-Papageorgiou-Staicu [1]), proof of Proposition 29). Recall that ψ_λ is coercive. Therefore

$$(4.6) \quad \psi_\lambda \text{ satisfies the } C \text{ - condition.}$$

Then (4.5) and (4.6) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $y_\lambda \in W^{1,p}(\Omega)$ such that

$$(4.7) \quad y_\lambda \in K_{\psi_\lambda} \subseteq [\bar{v}_\lambda, \bar{u}_\lambda] \text{ (see (4.3)) and } \bar{m}_\lambda \leq \psi_\lambda(y_\lambda).$$

From (4.5) and (4.7) we see that

$$y_\lambda \in [\bar{v}_\lambda, \bar{u}_\lambda], \quad y_\lambda \notin \{\bar{v}_\lambda, \bar{u}_\lambda\}.$$

Hence, if we show that $y_\lambda \neq 0$, then y_λ is a nodal solution of (P_λ) (recall that \bar{u}_λ and \bar{v}_λ are the extremal constant sign solutions of (P_λ)), and the nonlinear regularity of Lieberman [15] will imply that $y_\lambda \in C^1(\bar{\Omega})$.

Since y_λ is a critical point of mountain pass type for ψ_λ , we have

$$(4.8) \quad C_1(\psi_\lambda, y_\lambda) \neq 0$$

(see Motreanu-Motreanu-Papageorgiou [19]). On the other hand, hypothesis $(\mathbf{H}_3)(iv)$ and the work of Marano-Papageorgiou [17], imply

$$(4.9) \quad C_k(\psi_\lambda, 0) = 0 \text{ for all } k \geq 0.$$

From (4.8) and (4.9) it follows that $y_\lambda \neq 0$ and so $y_\lambda \in C^1(\bar{\Omega})$ is a nodal solution of (P_λ) .

Remarks: Nodal solutions for superlinear Neumann problems driven by the p -Laplacian were obtained by Aizicovici-Papageorgiou-Staicu in [2] (using the AR-condition) and in [3] (without the AR-condition). Theorem 4.1 improves substantially Theorem 3.5 in [18].

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