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Nonlinear periodic problems superlinear at $+\infty$ and sublinear at $-\infty$

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Abstract: We consider a nonlinear periodic problem driven by a nonlinear, nonhomogeneous differential operator with a reaction which exhibits an asymmetric growth at $+\infty$ and at $-\infty$. It is $(p-1)$-superlinear near $+\infty$ and $(p-1)$-sublinear near $-\infty$. A particular case of our problem is that of periodic equations with the scalar p-Laplacian and an asymmetric nonlinearity.
Using variational methods and Morse theory, we prove the existence of at least three nontrivial solutions.

Keywords: Asymmetric reaction, nonhomogeneous differential opertator, C-condition, critical groups, homotopy equivalent, mountain pass theorem.

MSC2010: 34B15, 34B18, 34C25, 58E05.

Dedicated to the memory of Professor Francesco S. De Blasi

1 Introduction

In this paper we examine the following nonlinear periodic problem

$$\begin{cases} -\left(a\left(\left|u'\left(t\right)\right|\right)u'\left(t\right)\right)' = f\left(t,u\left(t\right)\right) \text{ a.e. on } T := [0,b] \\ u\left(0\right) = u\left(b\right), \, u'\left(0\right) = u'\left(b\right). \end{cases} \tag{1.1}$$

In the above problem, the differential operator is in general nonhomogeneous and incorporates as a special case the scalar $p-$Laplacian. The reaction $f\left(t,x\right)$ is a Carathéodory function (i.e., for all $x \in \mathbb{R}$, $t \to f\left(t,x\right)$ is measurable and for a.a. $t \in T$, $x \to f\left(t,x\right)$ is continuous).

Our aim is to study the existence and multiplicity of solutions when the reaction $f\left(t,x\right)$ exhibits an asymmetric behavior near $+\infty$ and near $-\infty$ and it is $\left(p-1\right)-$superlinear in the positive direction (i.e., as $x \to +\infty$) and $\left(p-1\right)-$sublinear in the negative direction (i.e., as $x \to -\infty$).

Multiplicity results for nonlinear periodic problems driven by the scalar $p-$Laplacian were proved by Aizicovici-Papageorgiou-Staicu [1], [3], [4], Del Pino-Manasevich-Murua [8], Gasinski [11], Gasinski-Papageorgiou [13], and Yang [20].

In all the above mentioned papers, the reaction of the problem exhibits a similar growth near $+\infty$ and $-\infty$. Recently, Aizicovici-Papageorgiou-Staicu [5], [6], studied periodic eigenvalue problems driven by a nonhomogeneous differential operator.

Equations with an asymmetric reaction were studied in the context of semilinear (i.e., $p = 2$) Neumann problems. We mention the works of Dong [9], de Figueiredo-Ruf [7], Perera [17] and Villegas [19]. Of these, only Perera [17] proves a multiplicity result.

Our approach uses variational methods based on critical point theory together with suitable truncation techniques and Morse theory (critical groups).

2 Mathematical Background and Hypotheses

Let $(X, \|.\|)$ be a Banach space and $(X^*, \|.\|_*)$ its topological dual. By $\langle .,. \rangle$ we denote the duality brackets for the pair (X^*, X), and $\xrightarrow{w}$ denotes weak convergence in X.

Let $\varphi \in C^1(X)$. A real number c is said to be a critical value of φ if there exists $x^* \in X$ such that $\varphi'(x^*) = 0$ and $\varphi(x^*) = c$. We say that $\varphi \in C^1(X)$ satisfies the *C-condition*, if the following is true:

> "every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1}$ is bounded in $\mathbb{R}$ and
> $$(1 + \|x_n\|)\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$
> admits a strongly convergent subsequence."

This is in general weaker than the more common Palais-Smale condition. Nevertheless, the $C-$condition suffices to prove a deformation theorem and from it derive the minimax theory of certain critical values of $\varphi \in C^1(X)$. One such minimax theorem, which we recall for future use, is the so called "mountain pass theorem".

Theorem 2.1. *If $\varphi \in C^1(X)$ satisfies the C-condition, $x_0, x_1 \in X$, $\rho > 0$, $\|x_1 - x_0\| > \rho$, $\max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) : \|x - x_0\| = \rho\} = \eta_\rho$, and $c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t))$ where*

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geq \eta_\rho$ and c is a critical value of φ.

Let (Y_1, Y_2) be a topological pair such that $Y_2 \subseteq Y_1 \subseteq X$. For every integer $k \geq 0$, by $H_k(Y_1, Y_2)$ we denote the k^{th}-relative singular homology group with integer coefficients for the pair (Y_1, Y_2). Recall that $H_k(Y_1, Y_2) = 0$ for all integers $k < 0$.

Let $\varphi \in C^1(X)$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\begin{aligned} \varphi^c &= \{x \in X : \varphi(x) \leq c\}, \\ \dot{\varphi}^c &= \{x \in X : \varphi(x) < c\} \\ K_\varphi &= \left\{x \in X : \varphi'(x) = 0\right\}, \\ K_\varphi^c &= \{x \in K_\varphi : \varphi(x) = c\}. \end{aligned}$$

The critical groups of φ at an isolated critical point $x \in X$ with $\varphi(x) = c$ (i.e., $x \in K_\varphi^c$) are defined by

$$C_k(\varphi, x) = H_k(\varphi^c \cap U, (\varphi^c \cap U) \setminus \{x\}) \text{ for all } k \geq 0,$$

where U is a neighborhood of x such that $K_\varphi \cap \varphi^c \cap U = \{x\}$. The excision property of singular homology implies that the above definition of critical groups is independent of the particular choice of the neighborhood U. Suppose that $\varphi \in C^1(X)$ satisfies the $C-$condition and $\inf \varphi(K_\varphi) > -\infty$. Let $c < \inf \varphi(K_\varphi)$. The critical groups of φ at infinity, are defined by

$$C_k(\varphi, \infty) = H_k(X, \varphi^c) \text{ for all } k \geq 0.$$

The second deformation theorem (see for example, Gasinski-Papageorgiou [12], p. 628) implies that the above definition of critical groups of φ at infinity is independent of the choice of the level $c < \inf \varphi(K_\varphi)$.

Suppose that K_φ is finite. We define

$$\begin{aligned} M(t, x) &= \sum_{k \geq 0} rank\ C_k(\varphi, x) t^k \text{ for all } t \in \mathbb{R},\ x \in K_\varphi, \\ P(t, \infty) &= \sum_{k \geq 0} rank\ C_k(\varphi, \infty) t^k \text{ for all } t \in \mathbb{R}. \end{aligned}$$

The Morse relation says that

$$\sum_{x \in K_\varphi} M(t, x) = P(t, \infty) + (1 + t) Q(t) \tag{2.1}$$

where $Q(t) = \sum_{k \geq 0} \beta_k t^k$ is a formal series in $t \in \mathbb{R}$ with nonnegative integer coefficients.

In the analysis of problem (1.1) , we will use the Sobolev space

$$W_{per}^{1,p}(T) = \left\{u \in W^{1,p}(T) : u(0) = u(b)\right\},$$

with $1 < p < \infty$. The space $W_{per}^{1,p}(T)$ is embedded compactly into $C(T)$, and so, the evaluations at $t = 0$ and $t = b$ make sense.

In the sequel, for notational economy we set

$$W := W_{per}^{1,p}(T).$$

In addition to the Sobolev space W we will also use the Banach space

$$\widehat{C^1}(T) = C^1(T) \cap W.$$

This is an ordered Banach space with positive cone

$$\widehat{C}_+ = \left\{u \in \widehat{C^1}(T) : u(t) \geq 0 \text{ for all } t \in T\right\}.$$

This cone has a nonempty interior, given by

$$int\ \widehat{C}_+ = \left\{u \in \widehat{C}_+ : u(t) > 0 \text{ for all } t \in T\right\}.$$

Throughout this paper, the norm of the Banach space W will be denoted by $\|.\|$, i.e.,

$$\|u\| = \left(\|u\|_p^p + \|u'\|_p^p\right)^{\frac{1}{p}} \text{ for all } u \in W,$$

with $\|.\|_p$ being the norm of $L^p(T)$.

Given $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. We have

$$x = x^+ - x^-, \text{ and } |x| = x^+ + x^-.$$

Then for every $u \in W$, we define $u^{\pm}(.) = u(.)^{\pm}$ and we have

$$u = u^+ - u^-, \ |u| = u^+ + u^- \text{ and } u^{\pm} \in W.$$

Also, if $h : T \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then we define

$$N_h(u)(.) = h(., u(.)) \text{ for all } u \in W$$

(the Nemytskii map corresponding to $h(t, x)$).

Finally, by $|.|_1$ we denote the Lebesgue measure on $\mathbb{R}$.

Our hypotheses on the map a in problem (1.1) are the following:

$\mathbf{H}(a)$: $a:(0,\infty)\to(0,\infty)$ is a C^1-function such that:

(i) $x\to a(x)x$ is strictly increasing on $(0,\infty)$, $a(x)x\to 0$ as $x\to 0^+$ and

$$\frac{a'(x)x}{a(x)}\to C>-1 \text{ as } x\to 0^+;$$

(ii) there exists $\widehat{C}>0$ and $1<p<\infty$ such that

$$|a(x)x|\le \widehat{C}\left(1+|x|^{p-1}\right) \text{ for all } x\in\mathbb{R};$$

(iii) there exists $C_0>0$ such that

$$a'(x)x^2\ge C_0x^{p-1} \text{ for all } x>0;$$

(iv) if $G_0(x)=\int\limits_0^x a(s)sds$ for all $x\ge 0$, then

$$pG_0(x)-a(x)x^2\ge 0 \text{ for all } x>0.$$

Evidently $G_0(.)$ is strictly convex and strictly increasing on $(0,\infty)$. We set

$$G(x)=G_0(|x|) \text{ for all } x\in\mathbb{R}.$$

Then $G(.)$ is strictly convex and for $x\ne 0$ we have

$$G'(x)=G_0'(|x|)\frac{x}{|x|}=a(|x|)x.$$

So $G(.)$ is the primitive of the function $x\to a(|x|)x$, $x\in\mathbb{R}$. Since $G_0(.)$ is convex and $G_0(0)=0$, we have

$$G_0(x)\le a(x)x^2 \text{ for all } x>0. \tag{2.2}$$

Using (2.1) and hypotheses $\mathbf{H}(a)(ii)$, (iii), we obtain

$$\frac{C_0}{p}|x|^p\le G(x)\le C_1(1+|x|^p) \text{ for all } x\in\mathbb{R} \text{ and some } C_1>0. \tag{2.3}$$

Examples: The following functions satisfy the above hypotheses:

$$\begin{aligned} a_1(x)x &= |x|^{p-2}x \text{ with } 1<p<\infty, \\ a_2(x)x &= |x|^{p-2}x+|x|^{q-2}x \text{ with } 1<q<p<\infty, \\ a_3(x)x &= \left(1+x^2\right)^{\frac{p-2}{2}}x \text{ with } 1<p<\infty, \\ a_4(x)x &= |x|^{p-2}x+\frac{|x|^{p-2}x}{1+|x|^p} \text{ with } 1<p<\infty. \end{aligned}$$

The corresponding potential (primitive) functions are:

$$\begin{aligned} G_1(x) &= \frac{1}{p}|x|^p, \\ G_2(x) &= \frac{1}{p}|x|^p+\frac{1}{q}|x|^q, \\ G_3(x) &= \frac{1}{p}\left[\left(1+x^2\right)^{\frac{p}{2}}-1\right], \\ G_4(x) &= \frac{1}{p}|x|^p+\ln\left(1+|x|^p\right). \end{aligned}$$

Note that $a_1(x)x$ corresponds to the scalar $p-$Laplacian, $a_2(x)x$ corresponds to the scalar $(p,q)-$Laplacian, and $a_3(x)x$ corresponds to the generalized scalar $p-$mean curvature operator.

Let $A:W\to W^*$ be the nonlinear map defined by

$$\langle A(u),v\rangle=\int_0^b a\left(\left|u'(t)\right|\right)u'(t)v'(t)\,dt \text{ for all } u,\, v\in W. \tag{2.4}$$

The following result can be found in Papageorgiou-Rocha-Staicu [16].

Proposition 2.2. *If hypotheses* $\mathbf{H}(a)$ *hold, then the nonlinear operator* $A:W\to W^*$ *defined by* (2.4) *is bounded (i.e., it maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too), and of type* $(S)_+$, *i.e., if* $u_n\xrightarrow{w}u$ *in* W *and*

$$\limsup_{n\to\infty}\langle A(u_n),u_n-u\rangle\le 0,$$

then $u_n\to u$ *in* $W_0^{1,p}(\Omega)$.

Let $f_0:T\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function such that

$$|f_0(t,x)|\le a(t)\left(1+|x|^{r-1}\right) \text{ for a.a. } t\in T, \text{ all } x\in\mathbb{R},$$

with $a \in L^1(T)_+$ and $1 < r < \infty$. We set

$$F_0(t,x) = \int_0^x f_0(t,s)\,ds$$

and consider the C^1-functional $\psi_0 : W \to \mathbb{R}$ defined by

$$\psi_0(u) = \int_0^b G\left(u'(t)\right) dt - \int_0^b F_0(t,u(t))\,dt \text{ for all } u \in W.$$

The following result can be found in Aizicovici-Papageorgiou-Staicu [6]).

Proposition 2.3. *If hypotheses* $\mathbf{H}(\mathbf{a})$ *hold and* $u_0 \in W$ *is a local* $\widehat{C^1}(T)$*-minimizer of* ψ_0 *(i.e., there exists* $\rho_0 > 0$ *such that* $\psi_0(u_0) \le \psi_0(u_0+h)$ *for all* $h \in \widehat{C^1}(T)$ *with* $\|h\|_{\widehat{C^1}(T)} \le \rho_0$*), then* $u_0 \in \widehat{C^1}(T)$ *and it is a local* W*-minimizer of* ψ_0*, (i.e., there exists* $\rho_1 > 0$ *such that* $\psi_0(u_0) \le \psi_0(u_0+h)$ *for all* $h \in W$ *with* $\|h\| \le \rho_1$*).*

The hypotheses on the reaction $f(t,x)$ are the following:

$\mathbf{H}(f)$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(t,0) = 0$ a.e. on T and

(i) there exist $a \in L^1(T)_+$ and $p < r < \infty$ such that

$$|f(t,x)| \le a(t)\left(1 + |x|^{r-1}\right) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R};$$

(ii) $\lim\limits_{x\to+\infty} \frac{F(t,x)}{x^p} = +\infty$ uniformly for a.a. $t \in T$, and there exist $\tau > r - p$ and $\beta_0 > 0$ such that

$$\beta_0 \le \liminf_{x\to+\infty} \frac{f(t,x) - pF(t,x)}{x^\tau} \text{ uniformly for a.a .} t \in T,$$

where

$$F(t,x) := \int_0^x f(t,s)\,ds;$$

(iii) there exist functions $\widehat{\theta}$, $\theta \in L^{\infty}(T)$ such that:

$$\widehat{\theta}(t) \leq \theta(t) \leq 0 \text{ for a.a. } t \in T, \ \theta \neq 0,$$

$$\widehat{\theta}(t) \leq \liminf_{x \to -\infty} \frac{pF(t,x)}{|x|^p} \leq \limsup_{x \to +\infty} \frac{pF(t,x)}{x^p} \leq \theta(t)$$

$$\text{uniformly for a.a. } t \in T,$$

$$\limsup_{x \to -\infty} [pF(t,x) - f(t,x)x] < +\infty$$

$$\text{uniformly for a.a. } t \in T;$$

(iv) there exist constants $\widetilde{\xi}_0$, $\delta_0 > 0$ such that

$$F(t,x) \leq 0 \text{ for a.a. } t \in T, \text{ all } |x| \leq \delta_0, \ F\left(t, -\widetilde{\xi}_0\right) dt < 0$$

and for every $\rho > 0$, there exists $\widehat{\xi}_\rho > 0$ such that for a.a. $t \in T$,

$$x \to f(t,x) + \widehat{\xi}_\rho |x|^{p-2} x$$

is nondecreasing on $[-\rho, \rho]$.

Remarks: Hypotheses $\mathbf{H}(f)(ii)$, (iii) reveal the asymmetric character of the nonlinearity $f(t,.)$. By virtue of hypothesis $\mathbf{H}(f)(ii)$, near $+\infty$, $x \to f(t,x)$ is $(p-1)$ −superlinear. However, note that here we do not use the usual in such cases Ambrosetti-Rabinowitz condition (unilateral version). Instead, we employ a weaker requirement.

Hypothesis $\mathbf{H}(f)(iii)$ implies that for a.a. $t \in T$, near $-\infty$, $x \to f(t,x)$ is $(p-1)$ −sublinear. So, we have, a different growth for $f(t,.)$ in the positive and negative direction, respectively.

3 Three Solutions Theorem

In this section, we establish the existence of three nontrivial solutions for problem (1.1). To this end, let $\varphi : W \to \mathbb{R}$ be the energy functional for problem (1.1) defined by

$$\varphi(u) = \int_0^b G\left(u'(t)\right) dt - \int_0^b F(t, u(t))\, dt, \text{ for all } u \in W.$$

Evidently $\varphi \in C^1(W)$. Also, we consider the following perturbations-truncations of $f(t,.)$:

$$\widehat{f}_+(t,x) = \begin{cases} 0 & \text{if } x \le 0, \\ f(t,x) + x^{p-1} & \text{if } x > 0, \end{cases}$$
$$\widehat{f}_-(t,x) = \begin{cases} f(t,x) + |x|^{p-2}x & \text{if } x < 0 \\ 0 & \text{if } x \ge 0. \end{cases} \tag{3.1}$$

Both are Carathéodory functions. We set

$$\widehat{F}_\pm(t,x) = \int_0^x \widehat{f}_\pm(t,s)\,ds$$

and introduce the C^1-functionals $\widehat{\varphi}_\pm : W \to \mathbb{R}$ by

$$\widehat{\varphi}_\pm(u) = \int_0^b G\left(u'(t)\right)dt + \frac{1}{p}\|u\|_p^p - \int_0^b \widehat{F}_\pm(t,u(t))\,dt, \text{ for all } u \in W.$$

Proposition 3.1. *If hypotheses* $\mathbf{H}(a)$ *and* $\mathbf{H}(f)$ *hold, then the functional* φ *satisfies the* $C-$*condition.*

Proof. Let $\{u_n\}_{n\ge1}$ be a sequence in W such that

$$|\varphi(u_n)| \le M_1 \text{ for some } M_1 > 0, \text{ all } n \ge 1 \tag{3.2}$$

and

$$(1 + \|u_n\|)\,\varphi'(u_n) \to 0 \text{ in } W^* \text{ as } n \to \infty. \tag{3.3}$$

From (3.3), we have

$$\left|\langle A(u_n), h\rangle - \int_0^b f(t,u_n)\,h dt\right| \le \frac{\varepsilon_n \|h\|}{1 + \|u_n\|} \text{ for all } h \in W, \tag{3.4}$$

with $\varepsilon_n \to 0^+$. In (3.4), we choose $h = u_n^+ \in W$. Then

$$-\int_0^b a\left(\left|(u_n^+)'\right|\right)\left((u_n^+)'\right)^2 dt + \int_0^b f\left(t,u_n^+\right)u_n^+ dt \le \varepsilon_n \text{ for all } n \ge 1. \tag{3.5}$$

From (3.2) we have

$$\begin{aligned} &\int_0^b pG\left((u_n^+)'\right)dt + \int_0^b pG\left(-(u_n^-)'\right)dt \\ &-\int_0^b pF\left((u_n^+)'\right)dt - \int_0^b pF\left(t,-(u_n^-)'\right)dt \\ &\le pM_1 \text{ for all } n \ge 1. \end{aligned} \tag{3.6}$$

Adding (3.5) and (3.6), we obtain

$$\begin{aligned} &\int_0^b \left[pG\left((u_n^+)'\right) - a\left(\left|(u_n^+)'\right|\right)\left((u_n^+)'\right)^2\right]dt \\ &+\int_0^b \left[f\left(t,u_n^+\right)u_n^+ - pF\left(t,u_n^+\right)\right]dt \\ &+\int_0^b pG\left(-(u_n^-)'\right)dt - \int_0^b pF\left(t,-u_n^-\right)dt \\ &\le pM_1 \text{ for all } n \ge 1. \end{aligned} \tag{3.7}$$

By virtue of hypothesis $\mathbf{H}(f)(i),(iii)$, given $\varepsilon > 0$, we can find $a_\varepsilon \in L^1(T)$ such that

$$F(t,x) \le \frac{1}{p}(\theta(t)+\varepsilon)|x|^p + a_\varepsilon(t) \text{ for a.a. } t \in T, \text{ all } x \le 0. \tag{3.8}$$

Using (3.8) in (3.7), we obtain

$$\begin{aligned} &\int_0^b \left[pG\left((u_n^+)'\right) - a\left(\left|(u_n^+)'\right|\right)\left((u_n^+)'\right)^2\right]dt \\ &+\int_0^b \left[f\left(t,u_n^+\right)u_n^+ - pF\left(t,u_n^+\right)\right]dt \\ &+C_0\left\|(u_n^-)'\right\|_p^p - \int_0^b \theta(t)\left|u_n^-\right|^p dt - \varepsilon\left\|u_n^-\right\|^p - C_2 \\ &\le pM_1 \text{ for some } C_2>0, \text{ all } n \ge 1, \end{aligned}$$

(see (3.8) and (2.3)), hence

$$\int_0^b \left[pG\left(\left(u_n^+\right)'\right) - a\left(\left|\left(u_n^+\right)'\right|\right)\left(\left(u_n^+\right)'\right)^2\right] dt$$
$$+ \int_0^b \left[f\left(t, u_n^+\right) u_n^+ - pF\left(t, u_n^+\right)\right] dt + \xi_0 \left\|u_n^-\right\|^p$$
$$\leq pM_1 + C_2 =: C_3 \text{ for all } n \geq 1, \text{ some } \xi_0 > 0$$

(see Aizicovici-Papageorgiou-Staicu [6], Lemma 2.1), therefore

$$\int_0^b \left[f\left(t, u_n^+\right) u_n^+ - pF\left(t, u_n^+\right)\right] dt \leq C_3 \text{ for all } n \geq 1 \text{ (see } \mathbf{H}\,(a)\,(iv)\,). \tag{3.9}$$

Hypotheses $\mathbf{H}\,(f)\,(i)\,,(ii)$ imply that we can find $\beta_1 \in (0, \beta_0)$ and $a_1 \in L^1(T)_+$ such that

$$\beta_1 x^\tau - a_1(t) \leq f(t,x)\,x - pF(t,x) \text{ for a.a. } t \in T, \text{all } x \geq 0. \tag{3.10}$$

Returning to (3.9) and using (3.10), we obtain

$$\beta_1 \left\|u_n^+\right\|_\tau^\tau \leq M_2 \text{ for some } M_2 > 0, \text{ all } n \geq 1,$$

hence

$$\left\{u_n^+\right\}_{n\geq 1} \subset L^\tau(T) \text{ is bounded.} \tag{3.11}$$

In (3.4) we choose $h = u_n^+ \in W$ and use hypothesis $\mathbf{H}\,(a)\,(iii)$ to arrive at

$$C_0 \left\|\left(u_n^+\right)'\right\|_p^p - \int_0^b f\left(t, u_n^+\right) u_n^+ dt \leq \varepsilon_n \text{ for all } n \geq 1, \text{(see (2.3))}. \tag{3.12}$$

By virtue of $\mathbf{H}\,(f)\,(i)$ we have

$$f\left(t, u_n^+(t)\right) u_n^+(t) \leq a(t)\left(u_n^+(t) + u_n^+(t)^r\right) \text{ for a.a. } t \in T, \text{ all } n \geq 1. \tag{3.13}$$

Using (3.13) in (3.12) we obtain

$$\left\|\left(u_n^+\right)'\right\|_p^p \leq \varepsilon_n + C_5\left(\left\|u_n^+\right\| + \left\|u_n^+\right\|_r^r\right) \text{ for some } C_5 > 0, \text{ all } n \geq 1. \tag{3.14}$$

From hypothesis $\mathbf{H}(f)(ii)$ it is clear that we can always assume that $\tau \leq r < \infty$. So, we can find $t \in [0,1)$ such that

$$\frac{1}{r} = \frac{1-t}{\tau}. \tag{3.15}$$

Invoking the interpolation inequality (see, for example, Gasinski-Papageorgiou [12], p. 905), we have

$$\left\|u_n^+\right\|_r \leq \left\|u_n^+\right\|_\tau^{1-t} \left\|u_n^+\right\|_\infty^t \text{ for all } n \geq 1,$$

hence

$$\left\|u_n^+\right\|_r^r \leq C_6 \left\|u_n^+\right\|^{tr} \text{ for some } C_6 > 0, \text{all } n \geq 1 \tag{3.16}$$

(see (3.11)). Using (3.16) in (3.14) we have

$$\left\|\left(u_n^+\right)'\right\|_p^p \leq C_7 \left(1 + \left\|u_n^+\right\| + \left\|u_n^+\right\|^{tr}\right) \text{ for some } C_7 > 0, \text{ all } n \geq 1,$$

therefore

$$\left\|u_n^+\right\|^p \leq C_8 \left(1 + \left\|u_n^+\right\| + \left\|u_n^+\right\|^{tr}\right) \text{ for some } C_8 > 0, \text{ all } n \geq 1 \tag{3.17}$$

(see (3.11) and Gasinski-Papageorgiou [12], p. 227)). From (3.15) we have

$$tr = r - \tau < p \text{ (see hypothesis } \mathbf{H}(f)(ii)\text{)}.$$

So, from (3.17) it follows that

$$\left\{u_n^+\right\}_{n\geq 1} \subset W \text{ is bounded.} \tag{3.18}$$

Then from (3.7), (3.18) and hypotheses $\mathbf{H}(a)(iv)$, $\mathbf{H}(f)(i)$, we have

$$\int_0^b pG\left(-\left(u_n^-\right)'\right) dt - \int_0^b F\left(t, -u_n^-\right) dt \leq M_3 \text{ for some } M_3 > 0, \text{ all } n \geq 1.$$

Using (3.18) and Lemma 2.1 of Aizicovici-Papageorgiou-Staicu [6], we have

$$\xi_0 \left\|u_n^-\right\|^p \leq M_4 \text{ for some } M_4, \ \xi_0 > 0, \text{ all } n \geq 1,$$

hence

$$\left\{u_n^-\right\}_{n\geq 1} \subset W \text{ is bounded.} \tag{3.19}$$

From (3.18) and (3.19) it follows that $\{u_n\}_{n\geq 1} \subset W$ is bounded and so, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W \text{ and } u_n \to u \text{ in } C(T). \tag{3.20}$$

In (3.4) we choose $h = u_n - u \in W$, pass to the limit as $n \to \infty$ and use (3.20). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u \rangle = 0,$$

therefore

$$u_n \to u \text{ in } W$$

(see Proposition 2.2). This proves that the functional φ satisfies the $C-$condition. $\square$

Proposition 3.2. *If hypotheses* $\mathbf{H}(a)$ *and* $\mathbf{H}(f)$ *hold, then the functional* $\widehat{\varphi}_+$ *satisfies the* $C-$*condition.*

Proof. Let $\{u_n\}_{n\geq 1}$ be a sequence in W such that

$$|\widehat{\varphi}_+(u_n)| \leq M_5 \text{ for some } M_5 > 0, \text{ all } n \geq 1. \tag{3.21}$$

and

$$(1 + \|u_n\|)\,\widehat{\varphi}_+'(u_n) \to 0 \text{ in } W^* \text{ as } n \to \infty. \tag{3.22}$$

From (3.22) we have

$$\begin{aligned}&\left|\langle A(u_n), h\rangle + \int_0^b |u_n|^{p-2} u_n h dt - \int_0^b \widehat{f}_+(t, u_n) h dt\right| \\ &\leq \frac{\varepsilon_n \|h\|}{1+\|u_n\|} \text{ for all } h \in W, \text{with } \varepsilon_n \to 0^+.\end{aligned} \tag{3.23}$$

In (3.23), we choose $h = -u_n^- \in W$. Then from hypothesis $\mathbf{H}(a)\,(iii)$ and (3.1), we have

$$C_0 \left\|\left(u_n^-\right)'\right\|_p^p + \left\|u_n^-\right\|_p^p \leq \varepsilon_n \text{ for all } n \geq 1$$

hence

$$u_n^- \to 0 \text{ in } W \text{ as } n \to \infty. \tag{3.24}$$

From (3.21) and (3.24) we have

$$\int_0^b pG\left(\left(u_n^+\right)'\right) dt + \left\|u_n^+\right\|_p^p - \int_0^b p\widehat{F}\left(t, u_n^+\right) dt \leq M_6 \tag{3.25}$$

for some $M_6 > 0$, all $n \geq 1$.

Also, if in (3.23), we choose $h = u_n^+ \in W$, then

$$-\int_0^b a\left(\left|(u_n^+)'\right|\right)\left((u_n^+)'\right)^2 dt - \|u_n^+\|_p^p + \int_0^b \widehat{f}_+\left(t, u_n^+\right) dt \le \varepsilon_n \text{ for all } n \ge 1. \quad (3.26)$$

We add (3.25) and (3.26). Using hypothesis $\mathbf{H}(a)\,(iv)$ and (3.1), we obtain

$$\int_0^b \left[f\left(t, u_n^+\right) u_n^+ - pF\left(t, u_n^+\right)\right] dt \le M_7 \text{ for some } M_7 > 0, \text{ all } n \ge 1. \quad (3.27)$$

Using (3.10), we infer that $\{u_n^+\}_{n\ge1} \subset L^\tau(T)$ is bounded. By virtue of hypothesis $\mathbf{H}(f)\,(i)$, we have

$$|f(t,x)\,x| \le a(t)\left[|x| + |x|^r\right] \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}. \quad (3.28)$$

In (3.23), we choose $h = u_n^+ \in W$ and obtain

$$C_0\left\|(u_n^+)'\right\|_p^p + \|u_n^+\|_p^p \le \varepsilon_n + \int_0^b f\left(t, u_n^+\right) u_n^+ dt$$
$$\le C_9\left(1 + \|u_n^+\|_r^r\right) \text{ for some } C_9 > 0, \text{ all } n \ge 1$$

(see (3.26) and (2.3)). Using (3.15), the interpolation inequality and the boundedness of $\{u_n^+\}_{n\ge1} \subset L^\tau(T)$, as in the proof of Proposition 3.1 (see (3.16) and (3.17)), we obtain

$$\|u_n^+\|^p \le C_{10}\left(1 + \|u_n^+\|^{tr}\right) \text{ for some } C_{10} > 0, \text{ all } n \ge 1,$$

hence

$$\{u_n^+\}_{n\ge1} \subset W \text{ is bounded} \quad (3.29)$$

(since $tr = r - \tau < p$, see $\mathbf{H}(f)\,(ii)$). From (3.24) and (3.29) it follows that $\{u_n\}_{n\ge1} \subset W$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W \text{ and } u_n \to u \text{ in } C(T). \quad (3.30)$$

In (3.23) we choose $h = u_n - u \in W$, pass to the limit as $n \to \infty$ and use (3.30). Then

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0,$$

therefore

$$u_n \to u \text{ in } W$$

(see Proposition 2.2). This proves that the functional $\widehat{\varphi}_+$ satisfies the $C-$condition. □

Proposition 3.3. *If hypotheses* $\mathbf{H}(a)$ *and* $\mathbf{H}(f)$ *hold, then the functional* $\widehat{\varphi}_-$ *is coercive.*

Proof. By virtue of hypotheses $\mathbf{H}(f)(i)$ and (iii), given $\varepsilon > 0$ we can find $a_2 \in L^1(T)$ such that

$$F(t,x) \leq \frac{1}{p}(\theta(t)+\varepsilon)|x|^p + a_2(t) \text{ for a.a. } t \in T, \text{ all } x \leq 0. \tag{3.31}$$

Then for all $u \in W$, we have

$$\begin{aligned}\widehat{\varphi}_-(u) &= \int_0^b G\left(u'(t)\right) dt + \frac{1}{p}\|u\|_p^p - \int_0^b \widehat{F}_-(t,u(t))\,dt \\ &\geq \frac{C_0}{p}\left\|u'\right\|_p^p - \frac{1}{p}\int_0^b \theta(t)|u(t)|^p\,dt - \frac{\varepsilon}{p}\|u\|^p - \|a_2\|_1 \\ &\qquad \text{(see (2.2) and (3.31))} \\ &\geq \frac{\xi_0-\varepsilon}{p}\|u\|^p - \|a_2\|_1 \text{ with } \xi_0 > 0 \\ &\qquad \text{(see [6]), Lemma 2.1).}\end{aligned}$$

Choosing $\varepsilon \in (0,\xi_0)$ in the last inequality, we conclude that $\widehat{\varphi}_-$ is coercive. □

Now we are ready to produce two constant sign solutions.

Proposition 3.4. *If hypotheses* $\mathbf{H}(a)$ *and* $\mathbf{H}(f)$ *hold, then problem* (1.1) *has at least two constant sign solutions* $u_0 \in int\,\widehat{C}_+$ *and* $v_0 \in -int\,\widehat{C}_+$ *. Moreover,* v_0 *is a local minimizer of* φ.

Proof. First we show that $u=0$ is a local minimizer of the functional $\widehat{\varphi}_+$. So, let $u \in \widehat{C^1}(T)$ with $\|u\|_{\widehat{C^1}(T)} \leq \delta_0$, where $\delta_0 > 0$ is as postulated by hypothesis $\mathbf{H}(f)(iv)$. Then

$$\begin{aligned}\widehat{\varphi}_+(u) &= \int_0^b G\left(u'(t)\right) dt + \frac{1}{p}\|u\|_p^p - \int_0^b \widehat{F}_+(t,u(t))\,dt \\ &\geq \frac{C_0}{p}\left\|u'\right\|_p^p \text{ (see (2.2) and } \mathbf{H}(f)(iv)),\end{aligned}$$

hence $u = 0$ is a local $\widehat{C^1}(T)-$minimizer of the functional $\widehat{\varphi}_+$, therefore $u = 0$ is a local $W-$minimizer of the functional $\widehat{\varphi}_+$ (see Proposition 2.3). This implies that we can find $\rho \in (0,1)$ small, such that

$$\widehat{\varphi}_+(0) = 0 < \inf\{\widehat{\varphi}_+(u) : \|u\| = \rho\} =: \widehat{\eta}_\rho^+ \tag{3.32}$$

(see Aizicovici-Papageorgiou-Staicu [2], p. 57). For $\xi \in (0,\infty)$ we have

$$\widehat{\varphi}_+(\xi) = -\int_0^b F_+(t,\xi)\,dt \text{ (see (3.1))},$$

hence

$$\widehat{\varphi}_+(\xi) \to -\infty \text{ as } \xi \to +\infty \text{ (see } \mathbf{H}(f)(ii)). \tag{3.33}$$

From Proposition 3.2 we know that $\widehat{\varphi}_+$ satisfies the $C-$condition. This fact together with (3.32) and (3.33) permit the use of Theorem 2.1 (the mountain pass theorem). So, we can find $u_0 \in W$ such that

$$\widehat{\varphi}_+(0) = 0 < \widehat{\eta}_\rho^+ \le \widehat{\varphi}_+(u_0) \tag{3.34}$$

and

$$\widehat{\varphi}_+'(u_0) = 0. \tag{3.35}$$

From (3.34) we see that $u_0 \neq 0$. From (3.35) we have

$$A(u_0) + |u_0|^{p-2}u_0 = N_{\widehat{f}_+}(u_0). \tag{3.36}$$

On (3.36) we act with $-u_0^- \in W$, and use (3.1) and Proposition 3.2 to obtain

$$C_0 \left\|\left(u_0^-\right)'\right\|_p^p + \left\|u_0^-\right\|_p^p \le 0,$$

therefore

$$u_0 \ge 0, \; u_0 \neq 0.$$

Hence, (3.36) becomes

$$A(u_0) = N_f(u_0) \text{ (see (3.1))},$$

and we get

$$\begin{cases} -\left(a\left(|u_0'(t)|\right)u_0'(t)\right)' = f(t,u_0(t)) \text{ a.e. on } T, \\ u_0(0) = u_0(b), u_0'(0) = u_0'(b). \end{cases} \tag{3.37}$$

Then $u_0 \in \widehat{C^1}(T)$. Let $\rho = \|u_0\|_\infty$ and $\xi_\rho > 0$ be as postulated by hypothesis $\mathbf{H}(f)(iv)$. From (3.37) we have

$$-\left(a\left(\left|u_0'(t)\right|\right)u_0'(t)\right)' + \xi_\rho u_0(t)^{p-1} = f(t, u_0(t)) + \xi_\rho u_0(t)^{p-1} \geq 0 \text{ a.e. on } T,$$

hence

$$\left(a\left(\left|u_0'(t)\right|\right)u_0'(t)\right)' \leq \xi_\rho u_0(t)^{p-1} \text{ a.e. on } T. \tag{3.38}$$

From (3.38) and the strong maximum principle of Pucci-Serrin ([18], p.111) it follows that $u_0(t) > 0$ for all $t \in (0, b)$.

Then the boundary point theorem of Pucci-Serrin ([18], p.120) implies that $u_0 \in int\ C_+$. From Proposition 3.3 we know that $\widehat{\varphi}_-$ is coercive. Also, using the Sobolev embedding theorem, we see that $\widehat{\varphi}_-$ is sequentially lower semicontinuous.

So, by the Weierstrass theorem we can find $v_0 \in W$ such that

$$\widehat{\varphi}_-(v_0) = \inf\{\widehat{\varphi}_-(u) : u \in W\}. \tag{3.39}$$

Let $v = -\widetilde{\xi}_0 \in -int\ \widehat{C}_+$ be as in hypothesis $\mathbf{H}(f)(iv)$. We have

$$\widehat{\varphi}_-\left(-\widetilde{\xi}_0\right) = -\int_0^b F\left(t, -\widetilde{\xi}_0\right) dt < 0$$

(see (3.1) and hypothesis $\mathbf{H}(f)(iv)$), hence

$$\widehat{\varphi}_-(v_0) < 0 = \widehat{\varphi}_-(0)$$

(see (3.39)), therefore

$$v_0 \neq 0.$$

From (3.39) we have

$$\widehat{\varphi}_-'(v_0) = 0.$$

which implies

$$A(v_0) + |v_0|^{p-2} v_0 = N_{\widehat{f}_-}(v_0). \tag{3.40}$$

Acting on (3.40) with $v_0^+ \in W$ and using (3.1) and Proposition 3.2, we obtain

$$C_0 \left\|\left(v_0^+\right)'\right\|_p^p + \left\|v_0^+\right\|_p^p \leq 0,$$

hence

$$v_0 \leq 0,\ v_0 \neq 0.$$

Then (3.40) becomes

$$A(v_0) = N_f(v_0),$$

and we get

$$\begin{cases} -\left(a\left(\left|v_0'(t)\right|\right)v_0'(t)\right)' = f\left(t, v_0(t)\right) \text{ a.e. on } T, \\ v_0(0) = v_0(b), v_0'(0) = v_0'(b). \end{cases}$$

Hence $v_0 \in \widehat{C^1}(T)$, and as before, using the results of Pucci-Serrin ([18], pp. 111, 120), we obtain $v_0 \in -int\ \widehat{C}_+$. Note that

$$\varphi \mid_{-\widehat{C}_+} = \widehat{\varphi}_- \mid_{-\widehat{C}_+}.$$

Therefore $v_0 \in -int\ \widehat{C}_+$ is a local $\widehat{C}(T)-$minimizer of φ, and from Proposition 2.3 it follows that $v_0 \in -int\ \widehat{C}_+$ is a local $W-$minimizer of φ. □

Next, using Morse theory, we will produce a third nontrivial solution for problem (1.1). To this end, we start by computing the critical groups of φ at infinity.

Proposition 3.5. *If hypotheses* $\mathbf{H}(a)$ *and* $\mathbf{H}(f)$ *hold, then* $C_k(\varphi, \infty) = 0$ *for all* $k \geq 0$.

Proof. Let $\psi := \varphi \mid_{\widehat{C^1}(T)}$. The regularity properties of solutions of (1.1) imply that

$$K_\psi = K_\varphi = K.$$

Since $\widehat{C^1}(T) \hookrightarrow W$ densely, from Palais [15], we have

$$H_k\left(W, \dot{\varphi}^\alpha\right) = H_k\left(\widehat{C^1}(T), \dot{\varphi}^\alpha\right) \text{ for all } \alpha \in \mathbb{R}, \text{ all } k \geq 0. \tag{3.41}$$

Let $\alpha < \inf \varphi(K) = \inf \psi(K)$. We have

$$C_k(\varphi, \infty) = H_k(W, \varphi^\alpha) = H_k\left(W, \dot{\varphi}^\alpha\right) \text{ for all } k \geq 0, \tag{3.42}$$

$$C_k(\psi, \infty) = H_k\left(\widehat{C^1}(T), \psi^\alpha\right) = H_k\left(\widehat{C^1}(T), \dot{\psi}^\alpha\right) \text{ for all } k \geq 0, \tag{3.43}$$

(see Granas-Dugundji [14], p.407). Then from (3.41), (3.42), (3.43), it follows that in order to prove the proposition, it suffices to show that

$$H_k\left(\widehat{C^1}(T), \psi^\alpha\right) = 0 \text{ for all } k \geq 0,\ \alpha < 0 \text{ with } |\alpha| \text{ big}.$$

To this end, we introduce the following sets

$$\partial B_1^C = \left\{ u \in \widehat{C^1}(T) : \|u\|_{\widehat{C^1}(T)} = 1 \right\},$$

$$\partial B_{1,+}^C = \{ u \in \partial B_1^c : u(t) > 0 \text{ for some } t \in (0,b) \}.$$

Let $h_+ : [0,1] \times \partial B_{1,+}^C \to \partial B_{1,+}^C$ be the homotopy defined by

$$h_+(t,u) = \frac{(1-t)u + t\widehat{u}_0}{\|(1-t)u + t\widehat{u}_0\|_{\widehat{C^1}(T)}} \text{ for all } (t,u) \in [0,1] \times \partial B_{1,+}^C,$$

where $\widehat{u}_0 \in int\ \widehat{C}_+$ with $\|\widehat{u}_0\|_{\widehat{C^1}(T)} = 1$. We have

$$h_+(0,u) = u,\ h_+(1,u) = \widehat{u}_0,$$

hence $\partial B_{1,+}^C$ is contractible in itself. For $u \in \partial B_{1,+}^C$ and $\lambda > 0$ we have

$$\begin{aligned}
\varphi(\lambda u) &= \int_0^b G(\lambda u'(t))\,dt - \int_0^b F(t, \lambda u(t))\,dt \\
&\leq C_{11}\left(1 + \lambda^p \|u'\|_p^p\right) - \int_0^b F(t,\lambda u(t))\,dt \text{ for some } C_{11} > 0 \text{ (see (2.3))} \qquad (3.44) \\
&= C_{11}\left(1 + \lambda^p \|u'\|_p^p\right) - \int_0^b F(t,\lambda u^+(t))\,dt - \int_0^b F(t,-\lambda u^-(t))\,dt.
\end{aligned}$$

By virtue of $\mathbf{H}(f)(i),(ii)$, given any $\xi > 0$, we can find $a_3 \in L^1(T)$ such that

$$F(t,x) \geq \xi x^p - a_3(t) \text{ for a.a. } t \in T, \text{ all } x \geq 0. \qquad (3.45)$$

On the other hand, hypotheses $\mathbf{H}(f)(i),(iii)$ imply that there exist $C_{12} > 0$ and $a_4 \in L^1(T)$ such that

$$F(t,x) \geq -C_{12}|x|^p - a_4(t) \text{ for a.a. } t \in T, \text{ all } x \leq 0. \qquad (3.46)$$

Returning to (3.44) and using (3.45) and (3.46), we have

$$\begin{aligned}
\varphi(\lambda u) &\leq \lambda^p C_{11}\|u'\|_p^p - \lambda^p \xi \|u^+\|_p^p + \lambda^p C_{12}\|u^-\|_p^p + C_{13} \\
&\qquad \text{for some } C_{13} > 0 \qquad (3.47) \\
&\leq \lambda^p\left[C_{11}\|u'\|_p^p + C_{12}\|u^-\|_p^p - \xi\|u^+\|_p^p\right] + C_{13}.
\end{aligned}$$

Since $\xi > 0$ is arbitrary, we choose $\xi > 0$ big such that

$$C_{11} \left\|u'\right\|_p^p + C_{12} \left\|u^-\right\|_p^p < \xi \left\|u^+\right\|_p^p .$$

Then from (3.47) it follows that

$$\varphi(\lambda u) \to -\infty \text{ as } \lambda \to +\infty. \tag{3.48}$$

Hypotheses $\mathbf{H}(f)\,(ii)$, (iii) imply that there exist $\beta_2 \in (0, \beta_0)$, $M_8 > 0$ and $\xi^* > 0$ such that

$$pF(t,x) - f(t,x)\,x \leq -\beta_2 x^\tau \text{ for a.a. } t \in T, \text{ all } x \geq M_8, \tag{3.49}$$

$$pF(t,x) - f(t,x)\,x \leq \xi^* \text{ for a.a. } t \in T, \text{ all } x \leq 0. \tag{3.50}$$

By (3.49) and (3.50) and hypothesis $\mathbf{H}(f)\,(i)$, for every $u \in W$, we have

$$\begin{aligned}
&\int_0^b \left[pF(t,u) - f(t,u)\,u\right] dt = \int_{\{u\leq 0\}} \left[pF(t,u) - f(t,u)\,u\right] dt \\
&+ \int_{\{u\geq M_8\}} \left[pF(t,u) - f(t,u)\,u\right] dt + \int_{\{0<u<M_8\}} \left[pF(t,u) - f(t,u)\,u\right] dt \\
&\leq C_{14} - \beta_2 \int_{\{u\geq M_8\}} u^\tau dt.
\end{aligned} \tag{3.51}$$

Let $i : \widehat{C^1}(T) \to W$ be the embedding map. Hence $i \in \mathcal{L}\left(\widehat{C^1}(T), W\right)$. We see that

$$\psi = \varphi \circ i.$$

From the chain rule, we have

$$\psi' = i^* \varphi'(u) \text{ for all } u \in W. \tag{3.52}$$

Let $\langle .,.\rangle_C$ denote the duality brackets for the pair $\left(\widehat{C^1}(T)^*, \widehat{C^1}(T)\right)$. We have

$$\begin{aligned}
\tfrac{d}{d\lambda}\psi(\lambda u) &= \langle \psi'(\lambda u), u\rangle_C \\
&= \langle i^*\varphi'(\lambda u), u\rangle_C \ \ (\text{see } (3.52)) \\
&= \langle \varphi'(\lambda u), u\rangle_C \\
&= \tfrac{1}{\lambda}\left[\int_0^b a(|\lambda u'|)(\lambda u')^2 dt - \int_0^b f(t,\lambda u)\lambda u dt\right] \\
&\le \tfrac{1}{\lambda}\left[\int_0^b pG(\lambda u') dt - \int_0^b pF(t,\lambda u) dt + C_{14}.\right] \\
&\qquad (\text{see } \mathbf{H}(a)(iv) \text{ and } (3.51)) \\
&= \tfrac{1}{\lambda}\left[p\varphi(\lambda u) + C_{14}\right]
\end{aligned} \tag{3.53}$$

From (3.48) and (3.53), we see that for $\lambda > 0$ big (such that $\varphi(\lambda u) < -\frac{C_{14}}{p}$), we have

$$\frac{d}{d\lambda}\psi(\lambda u) < 0. \tag{3.54}$$

From Proposition 3.3, we have

$$\inf_{-\widehat{C}_+} \psi = \inf_{-\widehat{C}_+} \varphi > -C_{15} \text{ for some } C_{15} > 0.$$

Let

$$\alpha < \min\left\{-C_{15}, -\frac{C_{14}}{p}, \inf_{\overline{B}_1^C} \psi\right\}$$

where

$$\overline{B}_1^C = \left\{u \in \widehat{C^1}(T) : \|u\|_{\widehat{C^1}(T)} \le 1\right\}.$$

From (3.54) we see that we can find a unique $\gamma(u) \ge 1$ such that

$$\begin{aligned}
\psi(\lambda u) &> \alpha \text{ if } \lambda < \gamma(u), \\
\psi(\lambda u) &= \alpha \text{ if } \lambda = \gamma(u), \\
\psi(\lambda u) &< \alpha \text{ if } \lambda > \gamma(u)
\end{aligned}$$

and

$$\psi^\alpha = \left\{\lambda u : u \in \partial B_{1,+}^C,\ \lambda \ge \gamma(u)\right\}. \tag{3.55}$$

By virtue of the implicit function theorem, we have $\gamma \in C\left(\partial B^C_{1,+}, [1,\infty)\right)$. Let

$$V_+ = \left\{\lambda u : u \in \partial B^C_{1,+},\ \lambda \geq 1\right\}.$$

It is easily seen that $\partial B^C_{1,+}$ is a retract of V_+ and V_+ is deformable onto $\partial B^C_{1,+}$ in $\widehat{C^1}(T)$. Then invoking Dugundji [10] (Theorem 6.5, p.325), we infer that $\partial B^C_{1,+}$ is a deformation retract of V_+. Therefore

$$V_+ \text{ and } \partial B^C_{1,+} \text{ are homotopy equivalent.} \tag{3.56}$$

We introduce the homotopy $\widehat{h}_+ : [0,1] \times V_+ \to V_+$ by

$$\widehat{h}_+(t,\lambda u) = \begin{cases} (1-t)\lambda u + t\gamma(u)u & \text{if} \quad \lambda \in [1,\gamma(u)], \\ \lambda u & \text{if} \quad \lambda \geq \gamma(u). \end{cases}$$

Then (cf. (3.55))

$$\begin{aligned} &\widehat{h}_+(0,.) = Id,\ \widehat{h}_+(1,\lambda u) \in \psi^\alpha \text{ for all } \lambda u \in V_+, \\ &\widehat{h}_+(t,.)\,|_{\psi^\alpha} = Id\,|_{\psi^\alpha}. \end{aligned}$$

This means that ψ^α is a strong deformation retract of V_+. Therefore

$$V_+ \text{ and } \psi^\alpha \text{ are homotopy equivalent.} \tag{3.57}$$

From (3.56) and (3.57) it follows that

$$\psi^\alpha \text{ and } \partial B^C_{1,+} \text{ are homotopy equivalent,}$$

hence

$$H_k\left(\widehat{C^1}(T), \psi^\alpha\right) = H_k\left(\widehat{C^1}(T), \partial B^C_{1,+}\right) \text{ for all } k \geq 0 \tag{3.58}$$

(see Granas-Dugundji [14], p. 387). Recall that $\partial B^C_{1,+}$ is contractible in itself. So,

$$H_k\left(\widehat{C^1}(T), \partial B^C_{1,+}\right) = 0 \text{ for all } k \geq 0 \tag{3.59}$$

(see Granas-Dugundji [14], p. 389). From (3.58) and (3.59) it follows that

$$H_k\left(\widehat{C^1}(T), \psi^\alpha\right) = 0 \text{ for all } k \geq 0,$$

therefore

$$C_k(\varphi,\infty) = 0 \text{ for all } k \geq 0$$

(see the first part of the proof). □

Also, we compute the critical groups of $\widehat{\varphi}_+$ at infinity.

Proposition 3.6. *If hypotheses* $\mathbf{H}(a)$ *and* $\mathbf{H}(f)$ *hold, then* $C_k(\widehat{\varphi}_+,\infty)=0$ *for all* $k\geq 0$.

Proof. By virtue of hypotheses $\mathbf{H}(f)(i),(ii)$, given any $\xi>0$, we can find $a_5\in L^1(T)$ such that

$$F(t,x)\geq \xi x^p - a_5(t) \text{ for a.a. } t\in T, \text{ all } x\geq 0.$$

We introduce the set

$$E_+=\left\{u\in W:\|u\|=1,u^+\neq 0\right\}.$$

For $u\in E_+$ and $\lambda>0$, we have

$$\begin{aligned}\widehat{\varphi}_+(\lambda u)&=\int_0^b G(\lambda u'(t))\,dt+\tfrac{\lambda^p}{p}\|u\|_p^p-\int_0^b \widehat{F}_+(t,u)\,dt\\&\leq C_{16}\left(1+\lambda^p\|u'\|_p^p\right)+\tfrac{\lambda^p}{p}\|u\|_p^p-\xi\lambda^p\|u^+\|_p^p+C_{17}\\&\qquad\qquad\text{for some } C_{16},C_{16}>0\\&=\lambda^p\left[\|u'\|_p^p+\tfrac{1}{p}\|u\|_p^p-\xi\|u^+\|_p^p\right]+C_{18},\\&\qquad\qquad\text{with } C_{18}=C_{16}+C_{17}>0.\end{aligned}\tag{3.60}$$

Since $\xi>0$ is arbitrary, we choose $\xi>0$ big such that

$$\left\|u'\right\|_p^p+\frac{1}{p}\|u\|_p^p<\xi\left\|u^+\right\|_p^p.$$

So, from (3.60) it is clear that

$$\widehat{\varphi}_+(\lambda u)\to-\infty \text{ as } \lambda\to+\infty.\tag{3.61}$$

Similarly, as in the proof of Proposition 3.5 (see (3.51)), for all $u\in W$ we have

$$\begin{aligned}&\int_0^b\left[p\widehat{F}_+(t,u)-\widehat{f}_+(t,u)u\right]dt=\int_{\{u>0\}}[pF(t,u)-f(t,u)u]\,dt\\&=\int_{\{0<u<M_8\}}[pF(t,u)-f(t,u)u]\,dt\\&+\int_{\{u\geq M_8\}}[pF(t,u)-f(t,u)u]\,dt, \text{ for some } M_8>0 \text{ big (see (3.49))}\\&\leq C_{19}-\beta_2\int_{\{u\geq M_8\}}u^\tau\,dt, \text{ for some } C_{19}>0\end{aligned}\tag{3.62}$$

Using (3.62), we have

$$\begin{aligned}
&\tfrac{d}{d\lambda}\widehat{\varphi}_+(\lambda u) = \langle \widehat{\varphi}'_+(\lambda u), u\rangle \\
&= \int_0^b a(|\lambda u'|)\lambda (u')^2 dt + \tfrac{\lambda^{p-1}}{p}\|u\|_p^p - \int_0^b \widehat{f}_+(t,\lambda u) u dt \\
&= \tfrac{1}{\lambda}\left[\int_0^b a(|\lambda u'|)(\lambda u')^2 dt + \tfrac{\lambda^p}{p}\|u\|_p^p - \int_0^b \widehat{f}_+(t,\lambda u)\lambda u dt\right] \\
&\le \tfrac{1}{\lambda}\left[\int_0^b pG(\lambda u') dt + \lambda^p\|u\|_p^p - \int_0^b p\widehat{F}_+(t,\lambda u)\lambda u dt + C_{19}\right] \\
&\qquad\qquad\qquad (\text{see } \mathbf{H}(a)(iv) \text{ and } (3.62)) \\
&= \tfrac{1}{\lambda}\left[p\widehat{\varphi}_+(\lambda u) + C_{19}\right].
\end{aligned}$$

So, for $\lambda > 0$ big (such that $\widehat{\varphi}_+(\lambda u) < -\frac{C_{19}}{p}$), we have

$$\frac{d}{d\lambda}\widehat{\varphi}_+(\lambda u) < 0.$$

Let $d < -\frac{C_{19}}{p}$. We can find a unique $\gamma_+(u) > 0$, $\gamma_+ \in C(E_+)$ (by the implicit function theorem) such that

$$\widehat{\varphi}_+(\gamma_+(u)u) = d \text{ for all } u \in E_+. \tag{3.63}$$

Let $D_+ := \{u \in W : u^+ \neq 0\}$ and define

$$\widehat{\gamma}_+(u) = \frac{1}{\|u\|}\gamma_+\left(\frac{u}{\|u\|}\right) \text{ for all } u \in D_+.$$

We have $\widehat{\gamma}_+ \in C(D_+)$, and from (3.63) it follows

$$\widehat{\varphi}_+(\widehat{\gamma}_+(u)u) = d \text{ for all } u \in D_+. \tag{3.64}$$

Moreover, if $\widehat{\varphi}_+(u) = d$, then $\widehat{\gamma}_+(u) = 1$. We set

$$\widetilde{\gamma}_+(u) = \begin{cases} 1 & \text{if } \widehat{\varphi}_+(u) \le d \\ \widehat{\gamma}_+(u) & \text{if } \widehat{\varphi}_+(u) > d. \end{cases} \tag{3.65}$$

for all $u \in D_+$. We see that $\widetilde{\gamma}_+ \in C(D_+)$.

We introduce the homotopy $\widetilde{h}_+ : [0,1] \times D_+ \to D_+$ by

$$\widetilde{h}_+(t,u) = (1-t)\,u + t\widetilde{\gamma}_+(u)\,u \text{ for all } (t,u) \in [0,1] \times D_+.$$

We have

$$\widetilde{h}_+(0,u) = u,\ \widetilde{h}_+(1,u) \in \widehat{\varphi}_+^d \text{ and } \widetilde{h}_+(t,.)\,|_{\widehat{\varphi}_+^d} = Id\,|_{\widehat{\varphi}_+^d}$$
$$\text{for all } (t,u) \in [0,1] \times D_+.$$

This shows that $\widehat{\varphi}_+^d$ is a strong deformation retract of D_+. Hence

$$D_+ \text{ and } \widehat{\varphi}_+^d \text{ are homotopy equivalent,}$$

therefore

$$H_k(W, D_+) = H_k\left(W, \widehat{\varphi}_+^d\right) \text{ for all } k \geq 0. \tag{3.66}$$

Without any loss of generality we assume that $K_{\widehat{\varphi}_+}$ is finite. (Otherwise we already have a sequence of distinct positive solutions of (1.1).) We choose

$$d < \min\left\{\inf \widehat{\varphi}_+\left(K_{\widehat{\varphi}_+}\right), -\frac{C_{19}}{p}\right\}.$$

Then we have

$$H_k\left(W, \widehat{\varphi}_+^d\right) = C_k(\widehat{\varphi}_+, \infty) \text{ for all } k \geq 0. \tag{3.67}$$

Consider the homotopy $h^* : [0,1] \times D_+ \to D_+$ defined by

$$h^*(t,u) = \frac{(1-t)\,u + t\widehat{u}_0}{\|(1-t)\,u + t\widehat{u}_0\|}$$

where $\widehat{u}_0 \in int\ \widehat{C}_+$. Then $h^*(1,u) = \frac{\widehat{u}_0}{\|\widehat{u}_0\|}$, which proves that D_+ is contractible in itself. Therefore

$$H_k(W, D_+) = 0 \text{ for all } k \geq 0 \tag{3.68}$$

(see Granas-Dugundji [14], p.389). From (3.66), (3.67) and (3.68) we conclude that

$$C_k(\widehat{\varphi}_+, \infty) = 0 \text{ for all } k \geq 0.$$

□

Now we are ready to produce a third nontrivial solution and have the complete multiplicity theorem (three solution theorem) for problem (1.1).

Theorem 3.7. *If hypotheses* $\mathbf{H}(a)$ *and* $\mathbf{H}(f)$ *hold, then problem* (1.1) *has at least three nontrivial solutions* $u_0 \in int\ \widehat{C}_+$, $v_0 \in -int\ \widehat{C}_+$ *and* $y_0 \in \widehat{C^1}(T)$.

Proof. From Proposition 3.4 we already have two constant sign solutions $u_0 \in int\ \widehat{C}_+$ and $v_0 \in -int\ \widehat{C}_+$. We may assume that $K_{\widehat{\varphi}_+} = \{0, u_0\}$, or otherwise we already have a third solution of (1.1) which in fact is positive.

Claim: $C_k(\widehat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z}$ for all $k \geq 0$. Here and in what follows $\delta_{k,j}$ $(k, j \in \mathbb{Z}_+)$ denotes the Kronecker delta. Let

$$d < 0 = \widehat{\varphi}_+(0) < \eta < \widehat{\eta}_\rho^+ \leq \widehat{\varphi}_+(u_0)$$

(see (3.32) and (3.34)). We consider the following triple of sets

$$\widehat{\varphi}_+^d \subseteq \widehat{\varphi}_+^\eta \subseteq W.$$

For this triple, we consider the corresponding long exact sequence of homology groups

$$... \to H_k\left(W, \widehat{\varphi}_+^d\right) \xrightarrow{i_*} H_k\left(W, \widehat{\varphi}_+^\eta\right) \xrightarrow{\partial_*} H_{k-1}\left(\widehat{\varphi}_+^\eta, \widehat{\varphi}_+^d\right) \to ..., \tag{3.69}$$

where i_* is the group homomorphism induced by the inclusion $\left(W, \widehat{\varphi}_+^d\right) \xrightarrow{i} \left(W, \widehat{\varphi}_+^\eta\right)$ and ∂_* is the boundary homomorphism. We have

$$H_k\left(W, \widehat{\varphi}_+^d\right) = C_k(\widehat{\varphi}_+, \infty) \text{ for all } k \geq 0 \text{ (see Proposition 3.6)}, \tag{3.70}$$

$$H_k\left(W, \widehat{\varphi}_+^\eta\right) = C_k(\widehat{\varphi}_+, u_0) \text{ for all } k \geq 0 \text{ (recall that } K_{\widehat{\varphi}_+} = \{0, u_0\}\text{)}, \tag{3.71}$$

$$H_{k-1}\left(\widehat{\varphi}_+^\eta, \widehat{\varphi}_+^d\right) = C_{k-1}(\widehat{\varphi}_+, 0) = \delta_{k-1,0}\mathbb{Z} = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \tag{3.72}$$

(see the proof of Proposition 3.4)).

From (3.70), (3.71), (3.72) and the exactness of (3.69), we see that in (3.69) only the tail of a long sequence (i.e., $k = 1$), is nontrivial. So, we focus on the tail and use the rank theorem to obtain

$$\begin{aligned} &rank\ C_1(\widehat{\varphi}_+, u_0) = rank\ H_1\left(W, \widehat{\varphi}_+^\eta\right) \\ &= rank\ \ker \partial_* + rank\ \operatorname{Im} \partial_* \\ &= rank\ \operatorname{Im} i_* + rank\ \operatorname{Im} \partial_* \text{ (by the exactness of (3.69))} \\ &\leq 0 + 1 \text{ (see (3.70), (3.72))}. \end{aligned} \tag{3.73}$$

On the other hand, from the proof of Proposition 3.4, we know that $u_0 \in int\ \widehat{C}_+$ is a critical point of $\widehat{\varphi}_+$ of mountain pass type. Hence

$$rank\ C_1(\widehat{\varphi}_+, u_0) \geq 1. \tag{3.74}$$

From (3.73) and (3.74), it follows that

$$rank\ C_1(\widehat{\varphi}_+, u_0) = 1,$$

hence

$$C_k(\widehat{\varphi}_+, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0, \tag{3.75}$$

as claimed.

Since $\varphi\mid_{\widehat{C}_+} = \widehat{\varphi}_+\mid_{\widehat{C}_+}$ and $u_0 \in int\ \widehat{C}_+$, we have

$$C_k\left(\varphi\mid_{\widehat{C^1}(T)}, u_0\right) = C_k\left(\widehat{\varphi}_+\mid_{\widehat{C^1}(T)}, u_0\right) \text{ for all } k \geq 0,$$

hence

$$C_k(\varphi, u_0) = C_k(\widehat{\varphi}_+, u_0) \text{ for all } k \geq 0 \text{ (see Palais [15])},$$

therefore

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \geq 0 \text{ (see (3.75))}. \tag{3.76}$$

Recall that v_0 and 0 are local minimizers of φ (see Proposition 3.4 and its proof). Therefore, we have

$$C_k(\varphi, v_0) = C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \geq 0. \tag{3.77}$$

Finally, from Proposition 3.5, we have

$$C_k(\varphi, \infty) = 0 \text{ for all } k \geq 0. \tag{3.78}$$

Suppose that $K_\varphi = \{0, v_0, u_0\}$. From (3.76), (3.77), (3.78) and the Morse relation with $t = -1$ (see (2.1)), we infer that

$$2(-1)^0 + (-1)^1 = 0,$$

which is a contradiction.

So, there exists $y_0 \in K_\varphi$, $y_0 \notin \{0, v_0, u_0\}$.

Therefore y_0 is the third nontrivial solution of (1.1) and $y_0 \in \widehat{C^1}(T)$. □

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