Fischer decomposition in generalized fractional Clifford Analysis*

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Abstract

In this paper we present the basic tools of a fractional function theory in higher dimensions by means of a fractional correspondence to the Weyl relations via Gelfond-Leontiev operators of generalized differentiation. A Fischer decomposition is established. Furthermore, we give an algorithm for the construction of monogenic homogeneous polynomials of arbitrary degree.

This paper is dedicated to Professor Daniel Alpay, on the occasion of his 60th birthday.

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1 Introduction

What is nowadays called (classic) Clifford analysis, or hypercomplex function theories consists in the establishment of a function theory for null-solutions of the Dirac operator. Among other things it represents a higher-dimensional generalization of complex analysis, where these null-solutions, also called monogenic functions, replace the notion of a holomorphic function. While such functions can very well describe problems of a particle with internal $SU(2)$-symmetries, higher-order symmetries are beyond the reach of this theory. Although many modifications were suggested over the years (such as Yang-Mills theory), they could not address the principal problem, id est, the need of a n-fold factorization of the d'Alembert operator. One way to approach this problem is using generalized fractional calculus.

In the last decades the interest in generalized fractional calculus increased substantially. This fact is in part due to some of its interesting special cases and its applications to different topics of analysis. The basic operators of this calculus, the so-called generalized (multiple) operators of fractional integration are defined by means of convolutional type single integrals involving the Meijer functions $G^{m,0}_{m,m}$ as kernels. The classical Riemann-Liouville fractional integrals are a particular case of these general fractional integrals. In a more general case, the H-functions of Fox appear as kernel functions of those convolutional type single integrals. These same generalized fractional integral operators can be also considered as compositions of an arbitrary number of commuting Erdélyi-Kober fractional integrals written as repeated integrals without any use of special functions. Since such compositions arise quite often in several problems and branches of different areas of applied mathematics, this relationship proves to be the key to applications of the fractional calculus. From the long list of applications we have: differential and integral equations, operational calculi, convolutional calculi, integral transforms, univalent functions and special functions, among many more.

Nevertheless, in the used setting one often disregards the geometry of the domain. To overcome this difficulty, the authors proposed in [12, 19] to treat fractional calculi by means of Clifford analysis. Clifford analysis is a generalization of classical complex analysis in the plane to the case of an arbitrary dimension $d \in \mathbb{Z}$ (in the case

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of negative dimensions, one is dealing with the so-called super Clifford analysis). At the heart of the theory lies the Dirac operator D on $\mathbb{R}^d$, a conformally invariant first-order differential operator which plays the same role in classical Clifford analysis as the Cauchy-Riemann operator ∂_z does in complex analysis. Over the last decades F. Sommen and his collaborators developed a method for establishing a higher-dimensions function theory based on the so-called Weyl relations [3, 5, 6]. In more restrictive settings, this is called Howe dual pair technique (see [17]). Its focal point is the construction of an operator algebra (classically $\mathfrak{osp}(1|2)$) and to establish a Fischer decomposition. But this approach runs immediately into problems in the setting of Dirac operators with fractional derivatives. The main reason is that in general they do not allow a construction of a Howe dual pair. Hereby, the principal problem is not the invariance under a fractional spin group, but the construction of a Super-Lie-algebra $\mathfrak{osp}(1|2)$ for general fractional Dirac operators. This requires the application of a new approach and new methods, in particular the establishment of fractional Sommen-Weyl relations.

The traditional Fischer decomposition in harmonic analysis yields an orthogonal decomposition of the space of homogeneous polynomials of fixed degree of homogeneity in terms of spaces of harmonic homogeneous polynomials. In classical continuous Clifford analysis one obtains a refinement yielding an orthogonal decomposition with respect to the so-called Fischer inner product of homogeneous polynomials in terms of spaces of monogenic polynomials, i.e., null solutions of the Dirac operator (see [3]). Generalizations of the Fischer decomposition in other frameworks can be found, for example, in [1,5,6,9,12,16,17,19] and are mainly based on the establishment of raising and lowering operators acting on a ground state.

The aim of this paper is to present the building blocks of a function theory for fractional Dirac operators based on Gelfond-Leontiev operators of generalized differentiation. These operators were studied in the 80s and 90s and allow particular realizations in form of the classical Caputo and Riemann-Liouville fractional derivatives. In this paper we will show that Dirac operators based on these fractional derivatives still allow for a Fischer decomposition, and while not providing an $\mathfrak{osp}(1|2)$ algebra realization, they still permit the explicit construction of the basis of fractional homogeneous monogenic polynomials acting on a ground state. The proposed algorithm will construct these fractional homogeneous monogenic polynomials as linear combinations of fractional homogeneous polynomials. To show the application of this algorithm, MATLAB programs for the three-dimensional case are provided.

The structure of the paper reads as follows: in the Preliminaries we recall some basic facts about Clifford analysis and generalized fractional calculus. In Section 3, we introduce the corresponding Sommen-Weyl relations for the generalized fractional setting, the fractional correspondence to the Fischer decomposition, and its extension to a fractional Almansi decomposition. Based on the calculation of the dimension of the space of fractional homogeneous monogenic polynomials we conclude the paper with the construction of an algorithm for the explicit expression of the basis for the space of fractional homogeneous monogenic polynomials.

2 Preliminaries

2.1 Analysis in higher dimensions

It is well known that the treatment of the two-dimensional vector spaces $\mathbb{R}^2$ in terms of complex numbers has the advantage of providing a multiplication operator. Appropriate higher-dimensional associative analogues of the complex numbers are the real Clifford algebras. For details about Clifford algebras and basic concepts of the associated function theory we refer the interested reader to [2, 3, 11].

Let $\{e_1, \ldots, e_d\}$ be the standard basis of the Euclidean vector space $\mathbb{R}^d$. The associated Clifford algebra $\mathbb{R}_{0,d}$ is the free algebra generated by $\mathbb{R}^d$ modulo $x^2 = -||x||^2$. The defining relation induces the multiplication rules

$$e_i e_j + e_j e_i = -2\delta_{i,j}, \quad i, j = 1, \ldots d,$$

where $\delta_{i,j}$ denoted the Kronecker symbol. In particular, as we have $e_i^2 = -1$, the standard basis vectors operate as imaginary units.

A vector space basis for $\mathbb{R}_{0,d}$ is given by the set

$$\{e_\emptyset = 1, e_A = e_{l_1} e_{l_2} \ldots e_{l_r} : A = \{l_1, l_2, \ldots, l_r\} \subseteq M = \{1, \ldots, d\},\ 1 \leq l_1 < \ldots < l_r \leq d\}.$$

Each $a \in \mathbb{R}_{0,d}$ can be written in the form $a = \sum_A a_A e_A$, with $a_A \in \mathbb{R}$. Now, we introduce the complexified Clifford algebra $\mathbb{C}_d$ as the tensor product

$$\mathbb{C} \otimes \mathbb{R}_{0,d} = \left\{ w = \sum_A w_A e_A,\ w_A \in \mathbb{C},\ A \subseteq M \right\},$$

where the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e., $ie_j = e_j i$ for all $j = 1, \ldots, d$.

The conjugation in the Clifford algebra $\mathbb{C}_d$ is defined as the automorphism

$$w \mapsto \overline{w} = \sum_A \overline{w}_A \, \overline{e}_A,$$

where $\overline{w}_A$ denotes the usual complex conjugation and $\overline{e}_A = \overline{e}_{l_r} \, \overline{e}_{l_{r-1}} \ldots \overline{e}_{l_1}$, where $\overline{e}_\emptyset = 1$ and $\overline{e}_j = -e_j$ for $j = 1, \ldots, d$. For a vector $w = \sum_{j=1}^d w_j e_j$ we have $w\overline{w} = \|w\|^2 := \sum_{j=1}^d \|w_j\|^2$. Hence, each non-zero vector $w = \sum_{j=1}^d w_j e_j$ has an unique multiplicative inverse given by $w^{-1} = \frac{\overline{w}}{\|w\|^2}$.

An $\mathbb{C}_d$-valued function f over a non-empty domain $\Omega \subset \mathbb{R}^d$ is written as $f = \sum_A f_A e_A$, with components $f_A : \Omega \to \mathbb{C}$. Properties such as continuity are understood component-wisely. For example, $f = \sum_A f_A e_A$ is continuous if and only if all components f_A are continuous. Next, we recall the Euclidean Dirac operator $D = \sum_{j=1}^d e_j \, \partial_{x_j}$, which factorizes the d-dimensional Euclidean Laplacian, i.e., $D^2 = -\Delta = -\sum_{j=1}^d \partial_{x_j}^2$. A $\mathbb{C}_d$-valued function f is said to be *left-monogenic* if it satisfies $Df = 0$ on Ω (resp. *right-monogenic* if it satisfies $fD = 0$ on Ω).

From now on, we define $x^\alpha = \sum_{i=1}^d x_i^\alpha e_i \in \mathbb{C}^d$, where each component x_i^α is a fractional power which should be understood as

$$x_i^\alpha = \begin{cases} \exp(\alpha \ln |x_i^\alpha|); & x_i^\alpha > 0 \\ 0; & x_i^\alpha = 0 \\ \exp(\alpha \ln |x_i^\alpha| + \mathbf{i}\alpha\pi); & x_i^\alpha < 0 \end{cases}, \tag{1}$$

with $0 < \alpha < 1$, and $\mathbf{i}$ represents the imaginary unit. The authors remark that in the present manuscript they restrict themselves to the case of $\alpha \in]0, 1[$. Indeed, for values of α outside this range one can always reduce it to the previous case via $\alpha = [\alpha] + \tilde{\alpha}$, where $[\alpha]$ denotes its integer part and $\tilde{\alpha} \in]0, 1[$.

2.2 Generalized fractional derivatives

In this section we recall some basic facts about generalized fractional calculus (for more details we refer [15]). We start with the following definition of generalized differentiation and integration operators.

Definition 2.1 *[10, 15] Let the function*

$$\varphi(\lambda) = \sum_{k=0}^{\infty} \varphi_k \, \lambda^k$$

be an entire function with order $\rho > 0$ and $\sigma \neq 0$ such that $\lim_{k\to\infty} k^{\frac{1}{\rho}} \sqrt[k]{|\varphi_k|} = (\sigma e \rho)^{\frac{1}{\rho}}$. Then, the operation

$$f(z) = \sum_{k=0}^{\infty} a_k \, z^k \quad \xrightarrow{\ D_\varphi\ } \quad D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \frac{\varphi_{k-1}}{\varphi_k} \, z^{k-1} \tag{2}$$

is said to be the Gelfond-Leontiev (G-L) operator of generalized differentiation with respect to the function φ and the corresponding G-L operator, or integration, is

$$I_\varphi f(z) = \sum_{k=0}^{\infty} a_k \frac{\varphi_{k+1}}{\varphi_k} \, z^{k+1}. \tag{3}$$

From the theory of entire functions, the conditions required for φ should be given in terms of $\limsup_{k\to\infty}$. However, we assume that the limit exits and, therefore, $\lim_{k\to\infty} \sqrt[k]{\left|\frac{\varphi_{k-1}}{\varphi_k}\right|} = 1$. By the Cauchy-Hadamard formula, both series, (2) and (3), have the same radius of convergence $R > 0$.

Example 2.2 *Consider a function* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *analytic in the disk* $\Delta_R = \{z : |z| < R\}$. *Then, particular cases of series (2) and (3) are respectively the Riemann-Liouville (R-L) fractional derivative and integral of order* $\delta > 0$ *of* f. *They have the form*

$$D^\delta f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+1)}{\Gamma(k-\delta+1)} z^{k-\delta}, \qquad R^\delta f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} z^{k+\delta},$$

and the usual differentiation $D = \frac{d}{dx}$, $\delta = 1$ *is of the same form with a multiplier* $\frac{\Gamma(k+1)}{\Gamma(k)}$. *Moreover, due to the interchange formula between the Riemann-Liouville fractional derivative and the Caputo fractional derivative (see Expression (2.4.1) in [13]), it is possible also to obtain a particular case of series (2) and (3) in a form of a Caputo fractional derivative and integral of order* $\delta > 0$ *of* f.

Example 2.3 *Let* $\varphi(\lambda)$ *be a Mittag-Leffler function of the form*

$$\varphi(\lambda) = E_{\frac{1}{\rho},\mu}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}, \qquad \rho > 0, \quad \mu \in \mathbb{C},$$

with $\mathrm{Re}(\mu) > 0$. *Then* $\varphi_k(\lambda) = \frac{1}{\Gamma\left(\mu+\frac{k}{\rho}\right)}$ *and operators (2), (3) turn into the so-called Dzrbashjan-Gelfond-Leontiev (D-G-L) operators of differentiation and integration:*

$$D_{\rho,\mu} f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k-1}{\rho}\right)} z^{k-1}, \qquad I_{\rho,\mu} f(z) = \sum_{k=0}^{\infty} a_k \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k+1}{\rho}\right)} z^{k+1}, \tag{4}$$

studied in [7, 8, 15].

In [15,18] the author studied the connections between the D-G-L operators (4) and the so-called Erdélyi-Kober (E-K) fractional integrals and derivatives. In [15], the author presented transmutation operators relating the Riemann-Liouville (R-L) fractional integrals $R^{\frac{1}{\rho}}$ with the D-G-L generalized integrations $L_{\rho,1}$, and $L_{\rho,\mu}$, which where given in terms of E-K operators.

The above statements lead us to consider the fractional Dirac operator $D^\alpha = \sum_{j=1}^{d} e_i D_i^\alpha$, where D_i^α represents the G-L generalized derivative (2) with respect to the coordinate x_i^α. Analogous to the Euclidean case a $\mathbb{C}_d$-valued function u is called *fractional left-monogenic* if it satisfies $D^\alpha u = 0$ on Ω (resp. *fractional right-monogenic* if it satisfies $uD^\alpha = 0$ on Ω). As can be seen from the above exposition the most common fractional derivatives arise as special cases in our studies. We start with the discussion of one of the most important tools in Clifford analysis, the *Fischer decomposition.*

3 Sommen-Weyl relations and Fractional Fischer decomposition

The aim of this section is to provide the basic tools for a function theory for the fractional Dirac operator defined via generalized Gelfond-Leontiev differentiation operators.

3.1 Fractional Sommen-Weyl relations

The standard approach to the establishment of a function theory in higher dimensions is the construction of the analogues to the Euler and Gamma operators and the establishment of the corresponding Sommen-Weyl relations.

In order to achieve our goal, we want to study the commutator and the anti-commutator between x^α (see (1)) and $D^\alpha = \sum_{i=1}^{d} D_i^\alpha e_i$, where D_i^α is the G-L differentiation operator (see Definition 2.1) with respect to x_i^α. To this effect we start with the following fractional relations:

$$\left[D_i^\alpha, x_j^\alpha \right] (x_r^\alpha)^l = \left(D_i^\alpha x_j^\alpha - x_j^\alpha D_i^\alpha \right) (x_r^\alpha)^l = \begin{cases} 0, & \text{if } i \neq j, \\ \varphi(l,0)\, (x_r^\alpha)^l, & \text{if } i = j \ \wedge \ i \neq r, \\ \varphi_D(l+1, l-1)\, (x_r^\alpha)^l, & \text{if } i = j = r, \end{cases} \tag{5}$$

with $l \in \mathbb{N}$, $i = 1, \ldots, d$, $0 < \alpha < 1$, $\varphi(l,k) = \frac{\varphi_l}{\varphi_k}$, and $\varphi_D(l+1, l-1) = \varphi(l+1, l) - \varphi(l, l-1)$.

Example 3.1 *For the case of Mittag-Leffler functions (see Example 2.3) we have* $\varphi(a,b) = \frac{\Gamma(a\alpha+1)}{\Gamma(b\alpha+1)}$.

From (5) we get the following Sommen-Weyl relations for x, D^α and $(\underline{x}^\alpha)^{\underline{l}} = \prod_{i=1}^d (x_i^\alpha)^{l_i}$, with $\underline{l} = (l_1, \ldots, l_d)$, and $l = |\underline{l}| = l_1 + \ldots + l_d$:

$$\{D^\alpha, x\} \ (\underline{x}^\alpha)^{\underline{l}} = -\left(\sum_{r=1}^{d} \varphi_D(l_r+1, l_r-1) + 2\mathbb{E}^\alpha \right) (\underline{x}^\alpha)^{\underline{l}}$$
$$[D^\alpha, x] \ (\underline{x}^\alpha)^{\underline{l}} = -\left(\sum_{r=1}^{d} \varphi_D(l_r+1, l_r-1) + 2\Gamma^\alpha \right) (\underline{x}^\alpha)^{\underline{l}}, \tag{6}$$

where $\mathbb{E}^\alpha$, Γ^α are, respectively, the fractional Euler and Gamma operators

$$\mathbb{E}^\alpha = \sum_{r=1}^{d} x_r^\alpha \, D_r^\alpha, \qquad \Gamma^\alpha = -\sum_{r<s} e_r \, e_s \, (x_r^\alpha \, D_s^\alpha - D_r^\alpha \, x_s^\alpha). \tag{7}$$

This can be easily checked by

$$\begin{aligned} \{D^\alpha, x\} \ (\underline{x}^\alpha)^{\underline{l}} &= -\sum_{r=1}^{d} (D_r^\alpha \, x_r^\alpha - x_r^\alpha \, D_r^\alpha)\, (\underline{x}^\alpha)^{\underline{l}} - 2\sum_{r=1}^{d} x_r^\alpha \, D_r^\alpha \ (\underline{x}^\alpha)^{\underline{l}} \\ &= -\left(\sum_{r=1}^{d} \varphi_D(l_r+1, l_r-1) + 2\mathbb{E}^\alpha \right) (\underline{x}^\alpha)^{\underline{l}}, \end{aligned}$$

and

$$\begin{aligned} [D^\alpha, x] \ (\underline{x}^\alpha)^{\underline{l}} &= -2\sum_{r<s} e_r \, e_s \ (D_r^\alpha x_s^\alpha - x_r^\alpha D_s^\alpha) \ (\underline{x}^\alpha)^{\underline{l}} - \sum_{r=1}^{d} (D_r^\alpha x_r^\alpha - x_r^\alpha D_r^\alpha) \ (\underline{x}^\alpha)^{\underline{l}} \\ &= -\left(\sum_{r=1}^{d} \varphi_D(l_r+1, l_r-1) + 2\Gamma^\alpha \right) (\underline{x}^\alpha)^{\underline{l}}. \end{aligned}$$

From (7) we derive, via straightforward calculations, the remaining Sommen-Weyl relations

$$[x, \mathbb{E}^\alpha] \ (\underline{x}^\alpha)^{\underline{l}} = -\sum_{i=1}^{d} e_i \, \varphi_D \, (l_i+1, l_i-1) \ x_i^\alpha \ (\underline{x}^\alpha)^{\underline{l}}, \tag{8}$$
$$[D^\alpha, \mathbb{E}^\alpha] \ (\underline{x}^\alpha)^{\underline{l}} = -\sum_{i=1}^{d} e_i \, \varphi(l_i, l_i-1) \, \varphi_D \, (l_i, l_i-2) \ (x_i^\alpha)^{-1} \ (\underline{x}^\alpha)^{\underline{l}}. \tag{9}$$

Now, the standard procedure would be to construct a finite dimensional Lie-superalgebra generated by x and D^α which, in the classical case leads to an algebra isomorphic to $\mathfrak{osp}(1|2)$. To achieve this, we would need to have the following relations:

$$[x, \mathbb{E}^\alpha] \ (\underline{x}^\alpha)^{\underline{l}} = -x \ (\underline{x}^\alpha)^{\underline{l}}, \qquad [D^\alpha, \mathbb{E}^\alpha] \ (\underline{x}^\alpha)^{\underline{l}} = D^\alpha \ (\underline{x}^\alpha)^{\underline{l}}.$$

However, relations (8) and (9) show that this is only possible if

$$\varphi_D \, (l_i+1, l_i-1) = 1, \qquad \varphi(l_i, l_i-1) \, \varphi_D \, (l_i, l_i-2) = 1.$$

This means that one cannot follow that path in the general case. Despite this setback, we can observe that the polynomial space generated by $(\underline{x}^\alpha)^{\underline{l}}$ allows a decomposition into homogeneous spaces Π_l, where $\Pi_0 = \text{span}\{[1]\}$, and $\Pi_l = \text{span}\left\{x^{\underline{\beta}}[1]\right\}$ with $\underline{\beta} = (\beta_1, \ldots, \beta_d)$ and $|\underline{\beta}| = \beta_1 + \ldots + \beta_d$, and where $[1]$ is the ground-state (usually, given by 1).

3.2 Fractional Fischer decomposition

The observation in the end of the previous subsection opens a path for defining a Fischer inner product for fractional homogeneous polynomials. A fractional Fischer inner product of two fractional homogeneous polynomials P and Q is given by:

$$\langle P(x), Q(x) \rangle = \text{Sc}\left[\overline{P\,(D^\alpha)}\, Q(x) \right]\Big|_{x=0}, \tag{10}$$

where $P(D^\alpha)$ is a differential operator obtained by replacing in the polynomial P each variable x_j^α by its corresponding fractional derivative D_j^α. From (10) we immediately get that for any polynomial P_{l-1} of homogeneity $l-1$ and any polynomial Q_l of homogeneity l it holds

$$\langle x\, P_{l-1},\, Q_l\rangle = \langle P_{l-1}, D^\alpha Q_l\rangle\,. \tag{11}$$

This fact allows us to prove the following result:

Theorem 3.2 *For each $l \in \mathbb{N}_0$ we have $\Pi_l = \mathcal{M}_l + x\,\Pi_{l-1}$, where Π_l denotes the space of fractional homogeneous polynomials of degree l and $\mathcal{M}_l$ denotes the space os fractional monogenic homogeneous polynomials of degree l. Moreover, the subspaces $\mathcal{M}_k$ and $x\,\Pi_{l-1}$ are orthogonal with respect to the Fischer inner product (10).*

Proof: Since $\Pi_l = x\,\Pi_{l-1} + (x\,\Pi_{l-1})^\perp$, it suffices to prove that $(x\,\Pi_{l-1})^\perp = \mathcal{M}_l$. Assume that $P_l \in \Pi_l$ is in $(x\,\Pi_{l-1})^\perp$. Then, we have $\langle x\,P_{l-1},\, P_l\rangle = 0$, for all $P_{l-1} \in \Pi_{l-1}$. From (11) we get $\langle P_{l-1},\, D^\alpha P_l\rangle = 0$, for all $P_{l-1} \in \Pi_{l-1}$. Hence, we obtain that $D^\alpha P_l = 0$, that is $P_l \in \mathcal{M}_l$. This means that $(x\,\Pi_{l-1})^\perp \subset \mathcal{M}_l$. Conversely, take $P_l \in \mathcal{M}_l$. Then, for every $P_{l-1} \in \Pi_{l-1}$ we have that:

$$\langle x\,P_{l-1},\, P_l\rangle = \langle P_{l-1},\, D^\alpha P_l\rangle = \langle P_{l-1},\, 0\rangle = 0,$$

from which it follows that $\mathcal{M}_l \subset (x\,\Pi_{l-1})^\perp$. Therefore $\mathcal{M}_l = (x\,\Pi_{l-1})^\perp$.

■

In consequence, we obtain the fractional Fischer decomposition with respect to the fractional Dirac operator D^α.

Theorem 3.3 *Let P_l be a fractional homogeneous polynomial of degree l. Then:*

$$P_l = M_l + x\,M_{l-1} + (x)^2\, M_{l-2} + \ldots + (x)^l\, M_0, \tag{12}$$

where each M_j denotes the fractional monogenic polynomial of degree j. More specifically,

$$M_0 \in \Pi_0, \qquad \textit{and} \qquad M_l \in \{u \in \Pi_l :\ D^\alpha u = 0\}\,.$$

The spaces represented in (12) are orthogonal to each other with respect to the Fischer inner product (10). Moreover, the above decomposition can be represented in form of an infinite triangle:

$$\begin{array}{ccccccccc}
\Pi_0 & & \Pi_1 & & \Pi_2 & & \Pi_3 & \\
 & & & & & & & \\
\mathcal{M}_0 & \xleftarrow{\quad D^\alpha \quad} & x\,\mathcal{M}_0 & \xleftarrow{\quad D^\alpha \quad} & |x|^2\,\mathcal{M}_0 & \xleftarrow{\quad D^\alpha \quad} & (x)^3\,\mathcal{M}_0 & \ldots \\
 & & \oplus & & \oplus & & \oplus & \\
 & & \mathcal{M}_1 & \xleftarrow{\quad D^\alpha \quad} & x\,\mathcal{M}_1 & \xleftarrow{\quad D^\alpha \quad} & |x|^2\,\mathcal{M}_1 & \ldots \\
 & & & & \oplus & & \oplus & \\
 & & & & \mathcal{M}_2 & \xleftarrow{\quad D^\alpha \quad} & x\,\mathcal{M}_2 & \ldots \\
 & & & & & & \oplus & \\
 & & & & & & \mathcal{M}_3 & \ldots
\end{array}$$

While in the classic case all the summands in the same row are isomorphic to $\mathrm{Pin}(d)$-modules, and each row is an irreducible module for the Howe dual pair $\mathrm{Pin}(d) \times \mathfrak{osp}(1|2)$ (see [4] for more details), the same cannot be said at the moment for the fractional case. The reason for this fact was already stated in the previous subsection where we showed that for a general φ we do not have a finite dimensional superalgebra generated by x and D^α isomorphic to $\mathfrak{osp}(1|2)$.

Nevertheless, we have that all the summands in the same row are indeed modules. The authors conjecture is that these modules are invariant under a certain "fractional" Spin group which does not coincide with the classical one. However, such a study is beyond the scope of the present paper.

The Dirac operator shifts all spaces in the same row to the left, the multiplication by x shifts them to the right, and both of these actions establish isomorphisms between the modules.

From Theorem 3.3 we can derive the following direct extension to the fractional case of the Almansi decomposition:

Theorem 3.4 *For any fractional polyharmonic polynomial P_l of degree $l \in \mathbb{N}_0$ in a starlike domain Ω in $\mathbb{R}^d$ with respect to* 0*, i.e.,*

$$(D^\alpha)^2 P_l = 0, \qquad in\ \Omega,$$

there exist uniquely fractional harmonic functions $P_0, P_1, \ldots, P_{l-1}$ such that

$$P_l = P_0 + |x|^2 P_1 + \ldots + |x|^{2(l-1)} P_{l-1}, \quad in\, \Omega.$$

3.3 Explicit formulae

The aim of this subsection is to give an explicit algorithm for the construction of the projection $\pi_{\mathcal{M}}(P_l)$ of a given fractional homogeneous polynomial P_l into the space of fractional homogeneous monogenic polynomials $\mathcal{M}_l$. In order to reach our goal, we start by looking at the dimension of the space of fractional homogeneous monogenic polynomials of degree l. From the Fischer decomposition (12) we get:

$$\dim(\mathcal{M}_l) = \dim(\Pi_l) - \dim(\Pi_{l-1}),$$

with the dimension of the space of fractional homogeneous polynomials of degree l given by:

$$\dim(\Pi_l) = \frac{(l+d-1)!}{l!\,(d-1)!}.$$

This leads to the following theorem:

Theorem 3.5 *The space of fractional homogeneous monogenic polynomials of degree l has dimension*

$$\dim(\mathcal{M}_l) = \frac{(l+d-1)! - l(l+d-2)!}{l!\,(d-1)!} = \frac{(l+d-2)!}{l!\,(d-2)!}.$$

In the classical setting (see [3, 12, 19]) one usually considers the following scheme for the monogenic projection

$$r = a_0 P_l + a_1 x^\alpha D^\alpha P_l + a_2 (x^\alpha)^2 (D^\alpha)^2 P_l + \ldots + a_l (x^\alpha)^l (D^\alpha)^l P_l,$$

with $a_j \in \mathbb{R}, j = 0, \ldots, l$, and $a_0 = 1$. Let us take a closer look at the case $d = 3$ and $l = 3$,

$$r = a_0\, P_3 + a_1\, x D^\alpha P_3 + a_2\, (x)^2\, (D^\alpha)^2\, P_3 + a_3\, (x)^3\, (D^\alpha)^3\, P_3,$$

where $x = e_1 x_1^\alpha + e_2 x_2^\alpha + e_3 x_3^\alpha$ and $P_3 = (x_1^\alpha)^2\, (x_2^\alpha)^1\, (x_3^\alpha)^0$. In this case, we wish to obtain the real coefficients a_0, a_1, a_2, a_3 such that $D^\alpha P_3 = 0$. Taking into account that

$$\begin{aligned}
&D^\alpha\, (\underline{x}^\alpha)^3 = e_1\, \varphi(2,1)\, x_1^\alpha\, x_2^\alpha + e_2\, \varphi(1,0)\, (x_1^\alpha)^2\,,\\
&(D^\alpha)^2\, (\underline{x}^\alpha)^3 = -\varphi(2,0)\, x_2^\alpha,\\
&(D^\alpha)^3\, (\underline{x}^\alpha)^3 = -e_2\, \varphi(2,0)\, \varphi(1,0),\\
&D^\alpha x D^\alpha\, (\underline{x}^\alpha)^3 = -e_1\, \varphi(2,1)\, x_1^\alpha\, x_2^\alpha\, (2\varphi(2,1) + 2\varphi(1,0))\\
&\qquad - e_2 \left[\varphi(1,0)\, (x_1^\alpha)^2\, (\varphi(3,2) + 2\varphi(1,0) + \varphi(2,1)) + \varphi(2,0)\, (x_2^\alpha)^2\right] + e_3 \varphi(2,0) x_2^\alpha\, x_3^\alpha,\\
&D^\alpha\, (x)^2\, (D^\alpha)^2\, (\underline{x}^\alpha)^3 = e_1\, \varphi(2,0)\, \varphi(2,1)\, x_1^\alpha x_2^\alpha + e_2\, \varphi(2,0) \left[\varphi(1,0)\, (x_1^\alpha)^2 + \varphi(3,2)\, (x_2^\alpha)^2 + \varphi(1,0)\, (x_3^\alpha)^2\right]\\
&\qquad + e_3\, \varphi(2,0)\, \varphi(2,1)\, x_2^\alpha\, x_3^\alpha,\\
&D^\alpha\, (x)^3\, (D^\alpha)^3\, (\underline{x}^\alpha)^3 = e_2\, \varphi(2,0)\, \varphi(1,0) \left[- (x_1^\alpha)^2\, (\varphi(3,2) + 2\varphi(1,0)) - (x_2^\alpha)^2\, (\varphi(3,2) + 2\varphi(1,0))\right.\\
&\qquad \left. - (x_3^\alpha)^2\, (\varphi(3,2) + 2\varphi(1,0))\right],
\end{aligned}$$

we obtain the equation

$$
\begin{aligned}
0 &= D^\alpha r \\
&= a_0\, D^\alpha\ (\underline{x}^\alpha)^3 + a_1\, D^\alpha x D^\alpha\ (\underline{x}^\alpha)^3 + a_2 D^\alpha\ (x)^2\ (D^\alpha)^2\ (\underline{x}^\alpha)^3 + a_3\, D^\alpha\ (x)^3\ (D^\alpha)^3\ (\underline{x}^\alpha)^3 \\
&= (a_0 + a_1\, D^\alpha x) \left[e_1\, \varphi(2,1) x_1^\alpha\, x_2^\alpha + e_2\, \varphi(1,0)\ (x_1^\alpha)^2\right] + a_2\, D^\alpha\ (x)^2\ [-\varphi(2,0) x_2^\alpha] \\
&\quad + a_3\, D^\alpha\ (x)^3\ [-e_2\, \varphi(2,0)\, \varphi(1,0)] \\
&= e_1\, \varphi(2,1)\, x_1^\alpha\, x_2^\alpha\, [1 - a_1\, (2\varphi(2,1) + 2\varphi(1,0)) + a_2\, \varphi(2,0)] \\
&\quad + e_2\, \varphi(1,0) \Big\{ (x_1^\alpha)^2\, [1 - a_1\, (\varphi(3,2) + 2\varphi(1,0) + \varphi(2,1)) + a_2 \varphi(2,0) - a_3 \varphi(2,0)\, (\varphi(3,2) + 2\varphi(1,0))] \\
&\quad + \varphi(2,1)\, x_2^\alpha\, [-a_1 + a_2\, \varphi(3,2) - a_3\, (\varphi(3,2) + 2\varphi(1,0))] + \varphi(2,0)\ (x_3^\alpha)^2\, [a_2 - a_3\, (\varphi(3,2) + 2\varphi(1,0))] \Big\} \\
&\quad + e_3 \varphi(2,0)\, x_2^\alpha\, x_3^\alpha\, (a_1 + a_2\, \varphi(2,1)),
\end{aligned}
$$

which leads to the following system of linear equations:

$$
\begin{cases}
1 - a_1\ (2\varphi(2,1) + 2\varphi(1,0)) + a_2\, \varphi(2,0) = 0, \\
1 - a_1\ (\varphi(3,2) + 2\varphi(1,0) + \varphi(2,1)) + a_2\, \varphi(2,0) - a_3\, \varphi(2,0)\, (\varphi(3,2) + 2\varphi(1,0)) = 0, \\
-a_1 + a_2\, \varphi(3,2) - a_3\ (\varphi(3,2) + 2\varphi(1,0)) = 0, \\
a_2 - a_3\ (\varphi(3,2) + 2\varphi(1,0)) = 0, \\
a_1 + a_2\, \varphi(2,1) = 0,
\end{cases}
$$

which has no solution if we assume $a_0,\, a_1,\, a_2,\, a_3 \in \mathbb{R}$. This fact is, of course, due to the lack of the $\mathfrak{osp}(1|2)$ property. Using the Fischer decomposition and the explicit knowledge of the dimensions of the spaces (see Theorem 3.5) we can use a more direct approach to determine the fractional homogeneous monogenic polynomial. In fact, any homogeneous polynomial of degree $l = |\underline{l}|$ can be written as

$$
P_l(\underline{x}^\alpha) = \sum_{\underline{l} \in \mathbb{N}_0^d:\ |\underline{l}| = l} (\underline{x}^\alpha)^{\underline{l}}\ a_{\underline{l}}, \qquad a_{\underline{l}} \in \mathbb{R},
$$

with $l = |\underline{l}| = l_1 + \ldots + l_d$ denoting the degree of the polynomial. We now check under which conditions we have $D^\alpha P_l = 0$, i.e.,

$$
\begin{aligned}
0 &= D^\alpha \left(\sum_{\underline{l} \in \mathbb{N}_0^d:\ |\underline{l}| = l} (\underline{x}^\alpha)^{\underline{l}}\ a_{\underline{l}} \right) \\
&= \sum_{\underline{l} \in \mathbb{N}_0^d:\ |\underline{l}| = l} D^\alpha\ (\underline{x}^\alpha)^{\underline{l}}\ a_{\underline{l}} \\
&= \sum_{\underline{l} \in \mathbb{N}_0^d:\ |\underline{l}| = l} \left(\sum_{j=1}^d e_j\, D_j^\alpha \right) \left(\prod_{i=1}^d (x_i^\alpha)^{l_i} \right)\ a_{\underline{l}} \\
&= \sum_{j=1}^d e_j \left(\sum_{\underline{l} \in \mathbb{N}_0^d:\ |\underline{l}| = l} \varphi(l_j, l_j - 1)\ \left(x_j^\alpha\right)^{-1}\ (\underline{x}^\alpha)^{\underline{l}}\ a_{\underline{l}} \right) \\
&= \left[e_1\, \varphi(d, d-1)\, a_{(d,0,\ldots,0)} + e_2\, \varphi(1,0)\, a_{(d-1,1,0,\ldots,0)} + \ldots + e_d\, \varphi(1,0)\, a_{(d-1,0,\ldots,1)} \right]\ (x_1^\alpha)^{d-1} \\
&\quad + \ldots \\
&\quad + \left[e_1\, \varphi(2,1)\, a_{(2,1,\ldots,0)} + e_2\, \varphi(2,1)\, a_{(1,2,1,\ldots,0)} + \ldots + e_d\, \varphi(1,0)\, a_{(1,\ldots,1,0\ldots,0)} \right]\ x_1^\alpha\, x_2^\alpha \cdot \ldots \cdot x_{d-1}^\alpha. \qquad (13)
\end{aligned}
$$

The last equality leads to the following theorem:

Theorem 3.6 *Equation (13) is equivalent to the linear system*

$$
M\, A = 0, \qquad (14)
$$

where $A = [a_{(l_1,\ldots,l_d)}]_{dim(\Pi_l)\times 1}$, $0 = [0]_{dim(\Pi_{l-1})\times 1}$ *are vectors, and* M *is the matrix*

$$
M = \left[M_{(k_1,\ldots,k_d),(l_1,\ldots,l_d)} \right]_{dim(\Pi_{l-1})\times(dim(\Pi_l))},
$$

with entries given by:

$$M_{(k_1,\ldots,k_d),(l_1,\ldots,l_d)} = \begin{cases} e_i\,\varphi(l_i,k_i), & \text{if } k_i = l_i - 1 \wedge k_j = l_j \quad \forall i \neq j, \\ 0, & \text{otherwise,} \end{cases} .$$

Let us now indicate a possible ordering for the rows of system (14). In order to do that, let us consider the ordered set

$$L = \{\underline{L}^i = (l_1^i,\ldots,l_d^i) : \; |\underline{L}^i| = l = l_1^i + \ldots + l_d^i, \; i = 1,2\ldots,dim(\Pi_l)\},$$

where the relation order is given by:

$$\underline{L}^i > \underline{L}^{i+1} \qquad \Leftrightarrow \qquad \left(l_1^i,\ldots,l_d^i\right) > \left(l_1^{i+1},\ldots,l_d^{i+1}\right) \qquad \Leftrightarrow \qquad l_i^i l_2^i \ldots l_d^i > l_1^{i+1} l_2^{i+1} \ldots l_d^{i+1},$$

with

$$l_1^k l_2^k \ldots l_d^k := l_1^k \times 10^{d-1} + l_2^k \times 10^{d-2} + \ldots + l_d^k \times 10^0.$$

Applying this ordering we get the following corollary:

Corollary 3.7 *The matrix M has the structure*

$$M = \begin{pmatrix} M_1 & M_2 \end{pmatrix},$$

where the sub-matrix $M_1 = [m_{ij}^1]_{dim(\Pi_{l-1})\times dim(\Pi_{l-1})}$ is an upper triangular matrix with entries given by:

$$M_1 = \begin{pmatrix} e_1\varphi(l,l-1) & e_2\varphi(1,0) & e_3\varphi(1,0) & e_4\varphi(1,0) & e_5\varphi(1,0) & \cdots & e_d\varphi(1,0) & 0 & 0 & \cdots & 0 \\ 0 & e_1\varphi(l-1,l-2) & 0 & e_2\varphi(2,1) & e_3\varphi(1,0) & \cdots & e_{d-2}\varphi(1,0) & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ 0 & 0 & e_1\varphi(l-1,l-2) & 0 & e_2\varphi(1,0) & \cdots & e_{d-3}\varphi(1,0) & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & e_1\varphi(1,0) \end{pmatrix},$$

and the sub-matrix $M_2 = [m_{ij}^2]_{dim(\Pi_{l-1})\times dim(\Pi_l)}$ has its entries given by:

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ e_2\varphi(l,l-1) & e_3\varphi(1,0) & e_4\varphi(1,0) & \cdots & e_d\varphi(1,0) & 0 & \cdots & 0 \\ 0 & e_2\varphi(l-1,l-2) & e_3\varphi(2,1) & \cdots & e_{d-1}\varphi(1,0) & e_d\varphi(1,0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & e_2\varphi(1,0) & \cdots & e_d\varphi(l,l-1) \end{pmatrix}.$$

For the resolution of system (14) we implement the following algorithm to obtain the coefficients. Since M_1 is upper triangular matrix we can treat $a_{dim(\Pi_{l-1})+1}, \ldots, a_{dim(\Pi_l)}$ as free parameters and obtain the following formula for the coefficients. Let the entry (n,n) of M_1 correspond to the index $[(k_1,k_2,\ldots,k_d),(k_1+1,k_2,\ldots,k_d)]$, then

$$a_n = e_1\,(\varphi(k_1+1,k_1))^{-1}\left[\sum_{j=2}^{d} e_j\,\varphi(k_j+1,k_j)\,a_{(k_1,\ldots,k_j+1,\ldots,k_d)}\right], \tag{15}$$

where

$$a_n \leftrightarrow M_{(n,n)} \leftrightarrow M_{(k_1,l\ldots,k_d),(k_1+1,k_2,\ldots,k_d)}.$$

Example 3.8 *To illustrate the structure of M and A consider the case $d=3$, $l=3$, $\alpha=\frac{1}{2}$, and consider the Mittag-Leffler function, i.e., $\varphi(a,b)=\frac{\Gamma(a\alpha+1)}{\Gamma(b\alpha+1)}$ (see Example 2.3). Taking into account Corollary 3.7, the vector A and the matrixes M_1, M_2 take the form*

$$\begin{aligned} A^T &= \begin{pmatrix} a_{(3,0,0)} & a_{(2,1,0)} & a_{(2,0,1)} & a_{(1,2,0)} & a_{(1,1,1)} & a_{(1,0,2)} & a_{(0,3,0)} & a_{(0,2,1)} & a_{(0,1,2)} & a_{(0,0,3)} \end{pmatrix} \\ &= \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} \end{pmatrix}, \end{aligned}$$

$$M_1 = \begin{pmatrix} e_1\,\varphi(3,2) & e_2\,\varphi(1,0) & e_3\,\varphi(1,0) & 0 & 0 & 0 \\ 0 & e_1\,\varphi(2,1) & 0 & e_2\,\varphi(2,1) & e_3\,\varphi(1,0) & 0 \\ 0 & 0 & e_1\,\varphi(2,1) & 0 & e_2\,\varphi(1,0) & e_3\,\varphi(2,1) \\ 0 & 0 & 0 & e_1\,\varphi(1,0) & 0 & 0 \\ 0 & 0 & 0 & 0 & e_1\,\varphi(1,0) & 0 \\ 0 & 0 & 0 & 0 & 0 & e_1\,\varphi(1,0) \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ e_2\,\varphi(3,2) & e_3\,\varphi(1,0) & 0 & 0 \\ 0 & e_2\,\varphi(2,1) & e_3\,\varphi(2,1) & 0 \\ 0 & 0 & e_2\,\varphi(1,0) & e_3\,\varphi(3,2) \end{pmatrix}.$$

Now we can see that the columns of the matrix M_2 are associated, respectively, to the last four elements of the matrix A. Hence, if we fix a_7, a_8, a_9, a_{10} we can obtain, via formula (15), the remaining elements of the matrix A:

$$\begin{aligned} a_1 &= a_{(3,0,0)} = e_1\,(\varphi(3,2))^{-1}\left[e_2\,\varphi(1,0)\,a_{(2,1,0)} + e_3\,\varphi(1,0)\,a_{(2,0,1)}\right], \\ a_2 &= a_{(2,1,0)} = e_1\,(\varphi(2,1))^{-1}\left[e_2\,\varphi(2,1)\,a_{(1,2,0)} + e_3\,\varphi(1,0)\,a_{(1,1,1)}\right], \\ a_3 &= a_{(2,0,1)} = e_1\,(\varphi(2,1))^{-1}\left[e_2\,\varphi(1,0)\,a_{(1,1,1)} + e_3\,\varphi(2,1)\,a_{(1,0,2)}\right], \\ a_4 &= a_{(1,2,0)} = e_1\,(\varphi(1,0))^{-1}\left[e_2\,\varphi(3,2)\,a_{(0,3,0)} + e_3\,\varphi(1,0)\,a_{(0,2,1)}\right], \\ a_5 &= a_{(1,1,1)} = e_1\,(\varphi(1,0))^{-1}\left[e_2\,\varphi(2,1)\,a_{(0,2,1)} + e_3\,\varphi(2,1)\,a_{(0,1,2)}\right], \\ a_6 &= a_{(1,0,2)} = e_1\,(\varphi(1,0))^{-1}\left[e_2\,\varphi(1,0)\,a_{(0,1,2)} + e_3\,\varphi(3,2)\,a_{(0,0,3)}\right], \end{aligned}$$

and, therefore, we solve system (14). Furthermore, we can use the previous conclusions to obtain the four polynomials which are the basis for the space of fractional homogeneous monogenic polynomials $\mathcal{M}_3$

$$\begin{aligned} V_1^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -e_3\,(x_1^\alpha)^3 - \frac{3}{2}\,(x_1^\alpha)^2\,(x_2^\alpha) + e_3\,\frac{3}{2}\,x_1^\alpha\,(x_2^\alpha)^2 + (x_2^\alpha)^3, \\ V_2^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= e_2\,\frac{2}{3}\,(x_1^\alpha)^3 - (x_1^\alpha)^2\,x_3^\alpha - e_2\,(x_1^\alpha)\,(x_2^\alpha)^2 + e_3\,\frac{4}{\pi}\,x_1^\alpha\,x_2^\alpha\,x_3^\alpha + (x_2^\alpha)^2\,x_3^\alpha, \\ V_3^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -e_3\,\frac{2}{3}\,(x_1^\alpha)^3 - (x_1^\alpha)^2\,x_2^\alpha - e_2\,\frac{4}{\pi}\,x_1^\alpha\,x_2^\alpha\,x_3^\alpha + e_3\,x_1^\alpha\,(x_3^\alpha)^2 + x_2^\alpha\,(x_3^\alpha)^2, \\ V_4^{3,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= e_2\,(x_1^\alpha)^3 - \frac{3}{2}\,(x_1^\alpha)^2\,x_3^\alpha - e_2\,\frac{3}{2}\,x_1^\alpha\,(x_3^\alpha)^2 + (x_3^\alpha)^3. \end{aligned}$$

For convenience of the reader we also give the basic monogenic polynomials for $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively,

$$\begin{aligned} V_1^{1,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -e_3\, x_1^\alpha + x_2^\alpha, \\ V_2^{1,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -e_2\, x_1^\alpha + x_3^\alpha, \end{aligned}$$

$$\begin{aligned} V_1^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -\left(x_1^\alpha\right)^2 + e_3\,\frac{4}{\pi}\, x_1^\alpha\, x_2^\alpha + \left(x_2^\alpha\right)^2, \\ V_2^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -e_2\, x_1^\alpha\, x_2^\alpha + e_3\, x_1^\alpha\, x_3^\alpha + x_2^\alpha\, x_3^\alpha, \\ V_3^{2,\frac{1}{2}}\left(\underline{x}^{\frac{1}{2}}\right) &= -\left(x_1^\alpha\right)^2 - e_2\,\frac{4}{\pi}\, x_1^\alpha\, x_3^\alpha + \left(x_3^\alpha\right)^2. \end{aligned}$$

Remark 3.9 *The above algorithm can be easily implemented. For the convenience of the reader a Matlab program for the D-G-L operators (see Example 2.3) in the three-dimensional case, is available at:*

http://sweet.ua.pt/pceres/Webpage/Main_files/Frac_Code.zip

The main program is coef_frac(d,l,a)*. This program calculates the coefficients of the monogenic homogeneous polynomials that form the basis of the space of fractional homogeneous monogenic polynomials $\mathcal{M}_l$, i.e., it solves the system (14). The input data of this program consists of the dimension of the space (at this moment is fixed to* d=3*), the degree of homogeneity* l*, and the value of α* a*. For the cases presented in the previous example the program should be called, respectively, in the following forms:*

coef_frac(3,3,0.5)   coef_frac(3,1,0.5)   coef_frac(3,2,0.5).

The output of the program is given as cell of 4×4 matrices representing quaternions, where the coefficients for each polynomial are given by each column ordered according to Multi-indices given by the function MultiindexIndexgen*. We opted for the matrix representation of quaternions, but the program can be adapted to any Clifford algebra by using the appropriate matrix representation in terms of a $2^n \times 2^n$ real or complex matrix (the latter for the case of complexified Clifford algebras).*

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