Conditioned invariance and detectability subspaces in the behavioral approach ⋆

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Abstract: In this paper we extend the state space geometric notions of conditioned invariance and detectability subspaces to the behavioral framework. This is achieved based on the existing notions of behavioral tracking observer and of behavioral asymptotic observer introduced by Valcher, Willems, Trentelman and Trumph, combined with a notion of behavioral invariance introduced here. Moreover, we provide characterizations for the newly defined behavioral properties.

Keywords: Behavior, invariance, observer, conditioned-invariance, detectability

1. INTRODUCTION

The theory of behavioral observers developed in (Valcher and Willems, 1999) and (Trumpf et al., 2011) is an important contribute to incorporate classical state space concepts into the behavioral framework. In this paper we try to pursue this effort by introducing and characterizing the geometric notions of conditioned invariance and detectability subspaces ((Basile and Marro, 1969) and (Trentelman et al., 2001)) in the behavioral approach.

A first attempt to define behavioral conditioned invariance was made in (Pereira and Rocha, 2013); however it turned out that this first definition was not strong enough since the simple existence of an observer guaranteed conditioned invariance. This weakness was due to the proposed definition of behavioral invariance that served as basis for the notion of conditioned invariance.

Here we introduce a new definition of behavioral invariance. Roughly speaking, we shall say that a sub-behavior $\mathcal{V}$ of a behavior $\mathcal{B}$ is $\mathcal{B}$-invariant if $\mathcal{B}$ is autonomous modulo $\mathcal{V}$. Combining this with the behavioral definition of tracking observer ((Valcher and Willems, 1999), (Trumpf et al., 2011)) as well as with the classical notion of conditioned invariant subspace, we define a conditioned invariant behavior as an invariant behavior with respect to the error dynamics of a suitable (behavioral) observer, and provide a complete characterization of this latter (nontrivial) property.

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Moreover, we also introduce and characterize (behavioral) detectability subspaces based on the behavioral definition of asymptotic observer (Trumpf et al., 2011) and on the classical notion of detectability subspace. Such subspaces are in fact behaviors up to which the error dynamics of a suitable observer is stable. In the geometric approach to state space systems, detectability subspaces play an important role in state estimation in the presence of disturbances. The extension of the classical results to the behavioral framework is currently under our investigation.

The paper is organized as follows: we start by giving the relevant preliminaries on behaviors in Section II and, in Section III, the behavioral notion of invariance is introduced. Section IV gives an overview of behavioral observers and Sections V and VI are dedicated to conditioned invariance and detectability subspaces, respectively; finally, Section VII contains our concluding remarks.

2. PRELIMINARIES

As is well known, the central object in the behavioral theory is the system behavior, which is defined as the set of all admissible systems signals. In this paper we consider behaviors $\mathcal{B}$ that are linear subspaces of the universe $\mathcal{U} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, for some $\mathtt{w} \in \mathbb{N}$, consisting of the solutions of systems of linear, homogeneous differential equations with constant coefficients in $\mathtt{w}$ variables.

This means that there exists a positive integer g and a $g \times \mathtt{w}$ matrix $R(s)$ with entries in the ring $\mathbb{R}[s]$ of polynomials in s, i.e., $R(s) \in \mathbb{R}^{g\times \mathtt{w}}[s]$, such that

$$\mathcal{B} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) : R\left(\tfrac{d}{dt}\right) w = 0 \right\}.$$

In other words, $\mathcal{B}$ is the kernel of the operator $R\left(\frac{d}{dt}\right)$ defined over $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, and we write $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$. Whenever the context is clear we omit the indeterminate s and the operator $\frac{d}{dt}$.

Note that different operators may give rise to the same behavior. In particular $\ker R = \ker UR$ for any unimodular matrix U. Moreover, $\mathcal{B}_1 = \ker R_1 \subset \mathcal{B}_2 = \ker R_2$ if and only if there exists a polynomial matrix $\bar{R}$ such that $R_2 = \bar{R}R_1$. Since every polynomial matrix R can be brought to the form $\left[\begin{smallmatrix} F \\ 0 \end{smallmatrix}\right]$, where F is a full row rank polynomial matrix, by pre-multiplication by a suitable unimodular matrix U, one has that

$$\ker R = \ker UR = \ker \left[\begin{smallmatrix} F \\ 0 \end{smallmatrix}\right] = \ker F.$$

Hence every behavior can be regarded as the kernel of an operator associated to a full row rank polynomial matrix (Polderman and Willems, 1998).

The notion of autonomy plays a crucial role in this paper. A behavior $\mathcal{B}$ is called autonomous if the future of any trajectory in $\mathcal{B}$ is completely determined by its past. A formal definition of this property is given next.

Definition 1. A behavior $\mathcal{B}$ is *autonomous* if for every $w \in \mathcal{B}$ we have that $w(t) = 0$ for all $t \leq 0$ implies $w = 0$.

The following characterization of autonomy is given in (Polderman and Willems, 1998).

Lemma 2. A behavior $\mathcal{B} = \ker R$ is autonomous if and only if R has full column rank over $\mathbb{R}[s]$.

Another important property is the one of stability, which is also defined and characterized in (Polderman and Willems, 1998) as follows.

Definition 3. A behavior $\mathcal{B}$ is said to be *stable* if for every $w \in \mathcal{B}$ we have that $\lim_{t \to +\infty} w(t) = 0$.

Lemma 4. A behavior $\mathcal{B} = \ker R$ is stable if and only if $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}_0^+ := \{\lambda \in \mathbb{C} : Re(\lambda) \geq 0\}$.

Note that it follows from the previous lemmas that every stable behavior must be autonomous.

3. INVARIANCE

In the classical state-space case, given an autonomous system

$$\tfrac{d}{dt}x = Ax, \tag{1}$$

with $A \in \mathbb{R}^{n \times n}$, and where $x(t) \in \mathcal{X} = \mathbb{R}^n$ is the state vector at time t, the notion of invariance is defined as follows (Trentelman et al., 2001).

Definition 5. A subspace $\mathcal{V}_{\mathcal{X}}$ of $\mathcal{X}$ is said to be *invariant* with respect to system (1), or simply *A-invariant*, if

$$[x \text{ satisfies (1)},\ x(0) \in \mathcal{V}_{\mathcal{X}}] \Rightarrow [x(t) \in \mathcal{V}_{\mathcal{X}},\ \forall t \geq 0].$$

This means that the state subspace $\mathcal{V}_{\mathcal{X}}$ is invariant under the dynamics induced by state space equations. In the behavioral approach we consider the invariance of a given sub-behavior $\mathcal{V}$ under the dynamics associated to a behavior $\mathcal{B}$.

In (Rocha and Wood, 1997) the notion of hermetic sub-behavior was presented in the context of multidimensional (nD) systems. Here, we adapt that definiton to the 1D case and hermetic sub-behaviors are called invariant.

Definition 6. Let $\mathcal{B}$ be a behavior. A behavior $\mathcal{V}$ is said to be $\mathcal{B}$*-invariant* if the following condition holds:

$$\mathcal{V} \subset \mathcal{B} \text{ and } \left[w_{|(-\infty,0]} \in \mathcal{V}_{|(-\infty,0]},\ w \in \mathcal{B}\right] \Rightarrow [w \in \mathcal{V}],$$

where, as usual, $w_{|(-\infty,0]}$ denotes the restriction of w to the interval $(-\infty, 0]$ and $\mathcal{V}_{|(-\infty,0]}$ is the set of restrictions of the trajectories $v \in \mathcal{V}$ to the same interval.

Thus, when $\mathcal{V}$ is $\mathcal{B}$-invariant, the trajectories of $\mathcal{B}$ whose past is compatible with $\mathcal{V}$ (i.e., $w_{|(-\infty,0]} \in \mathcal{V}_{|(-\infty,0]}$) must remain in $\mathcal{V}$ in the future. This is expressed in the following characterization of invariance that was given in (Rocha and Wood, 1997, Thm. 7).

Proposition 7. Let $\mathcal{B}$ and $\mathcal{V} \subset \mathcal{B}$ be two behaviors. The following statements are equivalent:

(1) $\mathcal{V}$ is $\mathcal{B}$-invariant.
(2) $\mathcal{B}/\mathcal{V}$ is autonomous.

Remark 8. It follows from this proposition that if $\mathcal{B}$ is itself autonomous then any sub-behavior $\mathcal{V}$ of $\mathcal{B}$ is $\mathcal{B}$-invariant.

In (Oberst, 1990, Thm. 2.56) it was shown that if $\mathcal{V}$ is a sub-behavior of $\mathcal{B}$, then the quotient of the two behaviors $\mathcal{B}/\mathcal{V}$ also admits the structure of a behavior. Hence, $\mathcal{B}$-invariance can also be characterized in terms of the polynomial matrices associated to the operators that define the behaviors. Indeed, if $\mathcal{V} = \ker V$ and $\mathcal{B} = \ker \bar{R}V$, with V full row rank, as mentioned in (Rocha and Wood, 2001, Lemma 2.13), the quotient behavior $\mathcal{B}/\mathcal{V}$ is isomorphic to $\ker \bar{R}$. This leads to the following characterization.

Proposition 9. Given two behaviors $\mathcal{V} = \ker V$ and $\mathcal{B} = \ker \bar{R}V$, with V full row rank, the following statements are equivalent:

(1) $\mathcal{V}$ is $\mathcal{B}$-invariant.
(2) the matrix $\bar{R}$ is full column rank.

4. OBSERVERS

In this section we recall some elementary notions of the theory of observers in the state space approach (Trentelman et al., 2001) as well as in the behavioral approach (Trumpf et al., 2011) and (Valcher and Willems, 1999).

For this purpose it is necessary to consider systems with different variables. Therefore, in the sequel, when denoting a behavior the corresponding system variable will be made explicit by means of a subscript.

Starting with the state space approach, consider a state space system Σ with state space $\mathcal{X}$ described by

$$\begin{cases} \frac{d}{dt}x = Ax \\ y = Cx \end{cases} \tag{2}$$

where x is the state (to be estimated), y is the (measured) output, and A and C are real matrices of suitable dimensions. The corresponding behavior is

$$\mathcal{B}_{(y,x)} = \ker \begin{bmatrix} 0 & \frac{d}{dt}I - A \\ I & -C \end{bmatrix}.$$

Definition 10. A system Ω with state space $\widehat{\mathcal{X}} = \mathcal{X}$, and behavior $\widehat{\mathcal{B}}_{(y,\widehat{x})}$ given by an equation of the form

$$\tfrac{d}{dt}\widehat{x} = P\widehat{x} + Ry, \tag{3}$$

is said to be a *state observer* for Σ if $\widehat{x}$ is to be understood as an estimate of x. Moreover, defining the error of the estimate by $e = \widehat{x} - x$ and the corresponding *state estimation error behavior* $\mathcal{B}_e^{ss} = \{e = \widehat{x} - x : \exists y \text{ s.t. } (y,x) \in \mathcal{B}_{(y,x)}, (y,\widehat{x}) \in \widehat{\mathcal{B}}_{(y,\widehat{x})}\}$, Ω is said to be:

- a *tracking state observer* for Σ if, for any pair of initial values $(x(0), \widehat{x}(0))$ satisfying $e(0) = 0$, one has $e(t) = 0$ for all $t \geq 0$, i.e.,
$$[e(0) = 0,\ e \in \mathcal{B}_e^{ss}] \Rightarrow [e(t) = 0,\ \forall t \geq 0].$$
- an *asymptotic state observer* for Σ if it is a tracking state observer, and $\lim\limits_{t\to+\infty} e(t) = 0$ for every initial condition $e(0)$.

This is a simplified version of the definitions presented in the literature (as, for instance, the one given in (Trentelman et al., 2001), where the dynamics of the state estimate $\widehat{x}$ is not necessarily of first order).

Consider now a linear time-invariant differential system with behavior $\mathcal{B}_{(w_1,w_2)}$, where the system variable $w = (w_1, w_2)$ is partitioned into measured variables w_1 and to-be-estimated variables w_2, with $\mathtt{w}_1$ and $\mathtt{w}_2$ components, respectively.

In the behavioral approach, the notions of observer, tracking observer and asymptotic observer of w_2 from w_1 for $\mathcal{B}_{(w_1,w_2)}$ are defined as follows (Trumpf et al., 2011).

Definition 11. Given a linear time-invariant differential behavior $\mathcal{B}_{(w_1,w_2)}$, let $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ be a behavior such that the universe $\mathcal{U}_{\widehat{w}_2}$ coincides with the universe $\mathcal{U}_{w_2} = \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$ of the variable w_2. $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ is said to be an *observer* of w_2 from w_1 (for $\mathcal{B}_{(w_1,w_2)}$) if $\widehat{w}_2$ is to be understood as an estimate of w_2. Moreover, defining the error of the estimate by $e = \widehat{w}_2 - w_2$ and the corresponding *error behavior*

$$\begin{aligned}\mathcal{B}_e = \{e = \widehat{w}_2 - w_2 : \exists w_1 \text{ s.t.}\\ (w_1,w_2) \in \mathcal{B}_{(w_1,w_2)}, (w_1,\widehat{w}_2) \in \widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}\},\end{aligned}$$

$\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ is said to be:

- a *tracking observer* of w_2 from w_1 if, whenever $(w_1,w_2) \in \mathcal{B}_{(w_1,w_2)}$ and $(w_1,\widehat{w}_2) \in \widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ with $\widehat{w}_2(t) = w_2(t)$ for $t \in (-\infty, 0]$, then $\widehat{w}_2(t) = w_2(t)$, $\forall t \in \mathbb{R}$, in other words
$$[e_{|(-\infty,0]} \equiv 0,\ e \in \mathcal{B}_e] \Rightarrow [e(t) = 0,\ \forall t \in \mathbb{R}],$$
i.e., $\mathcal{B}_e$ is autonomous.
- an *asymptotic observer* of w_2 from w_1 if $\lim\limits_{t\to+\infty} e(t) = 0$, for all $e \in \mathcal{B}_e$, i.e, $\mathcal{B}_e$ is stable.

In case a tracking observer of w_2 from w_1 for $\mathcal{B}_{(w_1,w_2)}$ exists, we shall say that w_2 is *trackable* from w_1 in $\mathcal{B}_{(w_1,w_2)}$. In (Valcher and Willems, 1999), the following test is given for trackability.

Proposition 12. Let $\mathcal{B}_{(w_1,w_2)}$ be described by

$$R_2(\tfrac{d}{dt})w_2 = R_1(\tfrac{d}{dt})w_1$$

with $R_1 \in \mathbb{R}^{g\times \mathtt{w}_1}[s]$ and $R_2 \in \mathbb{R}^{g\times \mathtt{w}_2}[s]$ polynomial matrices, then w_2 is trackable from w_1 for $\mathcal{B}_{(w_1,w_2)}$ if and only if R_2 has full column rank.

A trivial observer is the behavior $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ described by $R_2(\frac{d}{dt})\widehat{w}_2 = R_1(\frac{d}{dt})w_1$. The error behavior corresponding to this observer is $\mathcal{B}_e = \ker R_2$, which coincides with the *hidden behavior* of w_2 defined in (Trumpf et al., 2011),

$$\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \{w_2 \mid (0, w_2) \in \mathcal{B}_{(w_1,w_2)}\}.$$

Hence Proposition 12 means that trackability is equivalent to the autonomy of the hidden behavior, cf (Trumpf et al., 2011, Def. 4.2 and Prop. 4.3).

It turns out that, when applied to state space systems, the behavioral notion of observer does not coincide with the one of state observer (cf Definition 10). This can be illustrated by the following simple example.

Example 13. Let $\mathcal{B}_{(y,x)}$ be described by

$$\begin{cases} \frac{d}{dt}x = x \\ y = x \end{cases}$$

and $\widehat{\mathcal{B}}_{(y,\widehat{x})}$ by

$$\tfrac{d}{dt}\widehat{x} = 0\widehat{x} + 0y$$

This behavior is a tracking observer for x from y for $\mathcal{B}_{(y,x)}$ in the behavioral setting, but it is not a tracking state observer for $\mathcal{B}_{(y,x)}$. Indeed, note that the x trajectories in $B_{(y,x)}$ are of the form $x(t) = e^t x(0)$, whereas the $\widehat{x}$ trajectories in $\widehat{\mathcal{B}}_{(y,\widehat{x})}$ are constant. Therefore

$$\widehat{x}(t) = x(t) \text{ for } t \in (-\infty, 0] \Rightarrow \widehat{x} \equiv x \equiv 0.$$

However, for the trajectories $x(t) = e^t$ and $\widehat{x}(t) \equiv 1$ it happens that

$$[x(0) = \widehat{x}(0)], \text{ but } [x(t) = e^t \neq 1 = \widehat{x}(t) \text{ for } t > 0].$$

5. CONDITIONED INVARIANCE

The aim of this section is to introduce conditioned invariance in the behavioral framework based on the definitions of behavioral invariance (cf Definition 6) and behavioral observer (cf Definition 11), and of the notion of conditioned invariance defined in the context of state space systems ((Basile and Marro, 1969),(Trentelman et al., 2001)). This latter rests on the concept of observer modulo a subspace of the state space that we next define with a slight adaptation.

Definition 14. Consider a state space system Σ described by (2). A system Ω with equation (3) is said to be an *observer* for $x/\mathcal{V}_\mathcal{X}$, where $\mathcal{V}_\mathcal{X}$ is a subspace of $\mathcal{X}$, if, for any pair of initial values $(x_0, \widehat{x}_0)$ satisfying $\widehat{x}_0 - x_0 \in \mathcal{V}_\mathcal{X}$, we have $\widehat{x}(t) - x(t) \in \mathcal{V}_\mathcal{X}$ for all $t \geq 0$.

In terms of the previously defined error behavior $\mathcal{B}_e^{ss}$, the condition in this definition can be restated as

$$[e(0) \in \mathcal{V}_\mathcal{X},\ e \in \mathcal{B}_e^{ss}] \Rightarrow [e(t) \in \mathcal{V}_\mathcal{X},\ \forall t \in \mathbb{R}],$$

which corresponds to the invariance of $\mathcal{V}_\mathcal{X}$ under the error dynamics as defined in the state space context, see Definition 5.

Given a subspace $\mathcal{V}_\mathcal{X}$ of $\mathcal{X}$, the existence of an observer for $x/\mathcal{V}_\mathcal{X}$ is not guaranteed.

Definition 15. A subspace $\mathcal{V}_\mathcal{X}$ of $\mathcal{X}$ is called *conditioned invariant* if there exists an observer for $x/\mathcal{V}_\mathcal{X}$.

The following behavioral definition of observer modulo $\mathcal{V}$ arises naturally from the behavioral definition of observer (Definition 11) and the classical definition of observer for $x/\mathcal{V}_\mathcal{X}$ (Definition 14).

Definition 16. Let $\mathcal{B}_{(w_1,w_2)}$ and $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ be two linear time-invariant differential behaviors for which the universes $\mathcal{U}_{w_2}$ and $\mathcal{U}_{\widehat{w}_2}$ of the variables w_2 and $\widehat{w}_2$, resp., coincide. Define the error behavior $\mathcal{B}_e$ as in Definition 11, and let $\mathcal{V}_e$ be a sub-behavior of $\mathcal{B}_e$. The behavior $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ is said to be an *observer of w_2 modulo $\mathcal{V}_e$ from w_1 for* $\mathcal{B}_{(w_1,w_2)}$ if

$$\begin{bmatrix} e_{|(-\infty,0]} \in (\mathcal{V}_e)_{|(-\infty,0]} \\ e \in \mathcal{B}_e \end{bmatrix} \Rightarrow [e \in \mathcal{V}_e].$$

The previous definition of observer modulo $\mathcal{V}_e$ corresponds to saying that $\mathcal{V}_e$ is $\mathcal{B}_e$-invariant. By Proposition 7 this leads to the next lemma.

Lemma 17. With the previous notation, the behavior $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ is an observer of w_2 modulo $\mathcal{V}_e$ from w_1 for $\mathcal{B}_{(w_1,w_2)}$ if and only if $\mathcal{B}_e/\mathcal{V}_e$ is autonomous.

In (Blumthaler and Oberst, 2009; Blumthaler, 2010) the concepts of T-autonomy and T-observer were introduced as follows. Let T be a multiplicatively closed subset of $\mathcal{D} \setminus \{0\}$, where $\mathcal{D}$ is a polynomial ring. A behavior $\mathcal{B}$ is said to be T-*autonomous* if there exists $t \in T$ such that

$$t\left(\tfrac{d}{dt}\right)(\mathcal{B}) = \{0\}. \tag{4}$$

Further, a T-*observer* is an observer such that the corresponding error behavior is T-autonomous.

Our definitions of behavioral invariance and observer modulo $\mathcal{V}$ have some resemblances with these concepts, but differ therefrom as explained next.

Recall that, according to Proposition 9, a sub-behavior $\mathcal{V} = \ker V\left(\frac{d}{dt}\right)$ of $\mathcal{B} = \ker R\left(\frac{d}{dt}\right)$ is $\mathcal{B}$-invariant if $\mathcal{B}/\mathcal{V}$ is autonomous. By (Oberst, 1990; Wood, 2000) this is equivalent to saying that $V\left(\frac{d}{dt}\right)(\mathcal{B})$ is an autonomous behavior. In turn, by (Wood et al., 1999), this means that there exists a nonzero polynomial $r \in \mathbb{R}[s]$ such that

$$\left(r\left(\tfrac{d}{dt}\right) V\left(\tfrac{d}{dt}\right)\right)(\mathcal{B}) = \{0\}. \tag{5}$$

Whereas (4) implies that $\mathcal{B}$ is an autonomous behavior (with a scalar annihilator $t\left(\frac{d}{dt}\right)$ in the special class T), (5) does not imply the autonomy of $\mathcal{B}$, but rather the autonomy of $V(\mathcal{B})$, without any specification for the corresponding scalar annihilators $r\left(\frac{d}{dt}\right)$.

Analogous to the state space case, in the behavioral context we define the conditioned invariance of a behavior $\mathcal{V}$ as the existence of a behavioral observer modulo $\mathcal{V}$.

Definition 18. Let $\mathcal{B}_{(w_1,w_2)}$ be a linear time-invariant differential behavior with measured variable w_1, and to-be-estimated variable w_2 in a universe $\mathcal{U}_{w_2}$. A behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is said to be *conditioned invariant* if there exists a (behavioral) observer of w_2 modulo $\mathcal{V}$ from w_1 for $\mathcal{B}_{(w_1,w_2)}$.

It follows from the previous definitions that, given $\mathcal{B}_{(w_1,w_2)}$, a behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is conditioned invariant with respect to $\mathcal{B}_{(w_1,w_2)}$, if there exists an observer of w_2 from w_1 for $\mathcal{B}_{(w_1,w_2)}$ such that $\mathcal{V}$ is invariant with respect to the corresponding error behavior $\mathcal{B}_e$.

Example 19. Consider the behavior $\mathcal{B}_{(w_1,w_2)}$ described by $R_2(\frac{d}{dt})w_2 = R_1(\frac{d}{dt})w_1$ with

$$R_2(\tfrac{d}{dt}) = (\tfrac{d}{dt}+1)\begin{bmatrix}\tfrac{d}{dt}+2 & \tfrac{d}{dt}+3\end{bmatrix} \text{ and } R_1 = [1].$$

Since R_2 has not full column rank, then w_2 is not trackable from w_1 in $\mathcal{B}_{(w_1,w_2)}$.

Considering the trivial observer $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ described by $R_2(\frac{d}{dt})\widehat{w}_2 = R_1(\frac{d}{dt})w_1$, the error behavior is $\mathcal{B}_e = \ker R_2$. Let $\mathcal{V}$ be the sub-behavior of $\mathcal{B}_e$ described by

$$\mathcal{V} = \ker\begin{bmatrix}\tfrac{d}{dt}+2 & \tfrac{d}{dt}+3\end{bmatrix}.$$

Since $\mathcal{B}_e/\mathcal{V} \simeq \ker(\frac{d}{dt}+1)$ is autonomous, then $\mathcal{V}$ is $\mathcal{B}_e$-invariant which implies by definition that $\mathcal{V}$ is a conditioned invariant behavior.

In order to characterize conditioned invariance it is important to know which error behaviors can be obtained by designing a suitable observer. This question has been addressed in (Trumpf et al., 2011).

Definition 20. Let $\mathcal{B}_{(w_1,w_2)}$ be a linear time-invariant differential behavior with observed variable w_1, and to-be-estimated variable w_2 in a universe $\mathcal{U}_{w_2}$. A behavior $\mathcal{E} \subset \mathcal{U}_{w_2}$ is said to be an *achievable error behavior* if there exists an observer $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ of w_2 from w_1 with error behavior $\mathcal{B}_e$ such that $\mathcal{E} = \mathcal{B}_e$.

Proposition 21. (Trumpf et al., 2011, Prop. 3.5) Let $\mathcal{B}_{(w_1,w_2)}$ be a linear time-invariant differential behavior with observed variable w_1, and to-be-estimated variable w_2 in a universe $\mathcal{U}_{w_2}$. Then the behavior $\mathcal{E} \subset \mathcal{U}_{w_2}$ is an achievable error behavior if and only if $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) \subset \mathcal{E}$.

Therefore $\mathcal{V}$ is conditioned invariant if and only if it is $\mathcal{E}$-invariant, for some achievable error behavior $\mathcal{E}$. By Proposition 21 this immediately leads to the following result (Pereira and Rocha, 2013).

Proposition 22. Let $\mathcal{B}_{(w_1,w_2)}$ be a behavior with observed variable w_1, and to-be-estimated variable w_2 in a universe $\mathcal{U}_{w_2}$. A behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is conditioned invariant if and only if there exists a behavior $\mathcal{E} \supset \mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ such that $\mathcal{V}$ is $\mathcal{E}$-invariant.

Since $\mathcal{E} = \mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ is trivially an achievable error behavior, a necessary but not sufficient condition for conditioned invariance of a behavior contained in the hidden behavior is a direct consequence of the previous proposition.

Proposition 23. Let $\mathcal{B}_{(w_1,w_2)}$ be behavior with observed variable w_1, and to-be-estimated variable w_2 in a universe $\mathcal{U}_{w_2}$. A behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is conditioned invariant only if $\mathcal{V}$ is $\mathcal{N}_{w_2}$-invariant.

Now, recall that it follows from Proposition 12 that the existence of a tracking observer is equivalent to the autonomy of the hidden behavior. Moreover, by Remark 8, if the hidden behavior $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$ is autonomous then any behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right)$-invariant. This leads to the following result.

Proposition 24. Let $\mathcal{B}_{(w_1,w_2)}$ be a behavior such that a tracking observer for w_2 from w_1 exists. Then a behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is conditioned invariant only if $\mathcal{V} \subset \mathcal{N}_{w_2}$.

A full characterization of condition invariance is given in the next proposition.

Proposition 25. Let $\mathcal{B}_{(w_1,w_2)}$ be a behavior with hidden behavior $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2$. Then a behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ described by $\mathcal{V} = \ker V$ is conditioned invariant if and only if $\exists q \in \mathbb{R}[s]$ such that $\ker R_2 \subset \ker qV$.

Proof. Taking into account the results in the Preliminaries, we may assume without loss of generality that V has full row rank.

"If part:" If the condition holds, by Proposition 21 $\mathcal{E} = \ker qV$ is an achievable error behavior such that $\mathcal{E}/\mathcal{V} \simeq \ker qI$ is autonomous. By Proposition 7 this implies that $\mathcal{V}$ is $\mathcal{E}$-invariant and the result follows by Proposition 22.

"Only if part:" Assume that $\mathcal{V}$ is conditioned invariant. By Propositions 22 and 7, there exists an error behavior $\mathcal{E}$ such that $\mathcal{E} \supset \ker R_2$ and $\mathcal{E} \supset \mathcal{V}$, and, moreover, $\mathcal{E}/\mathcal{V}$ is autonomous. Let $\mathcal{E} = \ker E$, for some polynomial matrix E. Then there exist polynomial matrices $\bar{E}$ and F such that

$$E = \bar{E}R_2 \quad \text{and} \quad E = FV,$$

with F full column rank since $\mathcal{E}/\mathcal{V}$ is autonomous. Let U be a polynomial matrix such that $Q = UF$ is a square nonsingular polynomial matrix. Then $UE = UFV = QV$. Denote the determinant and the adjoint matrix of Q by $\det Q$ and Q^{adj}, respectively. The previous equation can be written as

$$Q^{adj}U\bar{E}R_2 = (\det Q)V.$$

This implies that $LR_2 = qV$, with $L = Q^{adj}U\bar{E}$ and $q = \det Q$ and so $\exists q \in \mathbb{R}[s]$ such that $\ker R_2 \subset \ker qV$, (cf. Preliminaries).

6. DETECTABILITY SUBSPACES

In this section we define detectability subspaces as behaviors up to which the error dynamics of a suitable observer is stable.

Definition 26. Let $\mathcal{B}_{(w_1,w_2)}$ be a behavior and let $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ be an observer of w_2 from w_1 for $\mathcal{B}_{(w_1,w_2)}$. Define the error behavior $\mathcal{B}_e$ as in Definition 11. Given a behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$, the observer $\widehat{\mathcal{B}}_{(w_1,\widehat{w}_2)}$ is called *asymptotic modulo* $\mathcal{V}$, if

$$\mathcal{V} \subset \mathcal{B}_e \quad \text{and} \quad \mathcal{B}_e/\mathcal{V} \text{ is stable.}$$

Definition 27. Let $\mathcal{B}_{(w_1,w_2)}$ be a linear time-invariant differential behavior with measured variable w_1, and to-be-estimated variable w_2 in a universe $\mathcal{U}_{w_2}$. A behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ is said to be a *detectability subspace of* $\mathcal{B}_{(w_1,w_2)}$ if $\mathcal{B}_{(w_1,w_2)}$ admits an asymptotic observer modulo $\mathcal{V}$.

Detectability subspace have a characterization similar to the one given in Proposition 25 for conditioned invariance.

Proposition 28. Let $\mathcal{B}_{(w_1,w_2)}$ be a behavior with hidden behavior $\mathcal{N}_{w_2}\left(\mathcal{B}_{(w_1,w_2)}\right) = \ker R_2$. Then a behavior $\mathcal{V} \subset \mathcal{U}_{w_2}$ described by $\mathcal{V} = \ker V$ is a detectability subspace of $\mathcal{B}_{(w_1,w_2)}$ if and only if $\exists q \in \mathbb{R}[s]$ stable (i.e., with all zeros in $\mathbb{C}^-$) such that $\ker R_2 \subset \ker qV$.

The proof of this proposition in similar to the one of Proposition 25.

7. CONCLUSIONS

In this paper we introduced and characterized the properties of invariance, conditioned invariance and detectability subspaces in the behavioral framework. Our starting point for the definition of the latter two properties were the notions of behavioral tracking observer and of behavioral asymptotic observer from (Valcher and Willems, 1999) and (Trumpf et al., 2011), combined with the notion of behavioral invariance introduced here. We believe that this contribution constitutes a good basis for the extension to the behavioral setting of the geometric approach to state space systems.

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