Multidimensional time fractional diffusion equation*

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Abstract

In this paper we present integral and series representations for the fundamental solution of the time fractional diffusion equation in an arbitrary dimension. The series representation obtained depends on the parity of the dimension. As an application of our results we study the diffusion and stress in the axially symmetric case for plane deformation associated to generalized thermoelasticity theory.

Keywords: Time fractional diffusion operator; Fundamental solutions; Caputo fractional derivative; Thermoelasticity.

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1 Introduction

Time fractional diffusion equations are obtained from the standard diffusion and wave equations by replacing the time derivative by a fractional derivative of order $\beta \in]0, 2]$. When $\beta = 0$ we have the Helmholtz equation, which is associated with localized diffusion. The sub-diffusion case corresponds to the case when $0 < \beta < 1$. When $\beta = 1$ we obtain the ordinary diffusion. The super-diffusion regime happens when $1 < \beta < 2$. If $\beta = 2$ we are dealing with the wave equation (sometimes designated as ballistic diffusion) (see [17]). Some applications of the time-fractional diffusion equation can be found in the study of Brownian motion [15] and in the theory of thermoelasticity [9]. The theory of generalized thermoelasticity appeared due to the use of fractional derivatives with respect to time or space to describe anomalous diffusion processes. The time-fractional diffusion (or heat conduction) equation

$$\frac{d^\beta u}{dt^\beta} = c \Delta u, \quad 0 < \beta \leq 2 \quad (1)$$

where $\frac{d^\beta u}{dt^\beta}$ is a fractional derivative usually considered in the Riemann-Liouville or in the Caputo sense, describes the time non-local dependence between the flux vectors and corresponding gradients with "long-tale" power kernel. Equation (1) is also important in the study of Brownian motion. At the level of individual particle motion the classical diffusion corresponds to the Brownian motion which is characterized by a mean-squared displacement increasing linearly with time $\langle x^2 \rangle \sim at$. In the case of fractional Brownian motion (anomalous diffusion) the mean-squared displacement increases with the power-law time, that is, $\langle x^2 \rangle \sim at^\beta$, $\beta \neq 1$. Therefore, the continuous time random walk theory can be generalized to include variable jump lengths and

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waiting times between successive jumps, giving rise to subdiffusion processes ($0 < \beta < 1$) and superdiffusion processes ($1 < \beta < 2$) (see [16]).

In the one-dimensional case, fundamental solutions and solutions for Cauchy problems related to (1) have been constructed and studied comprehensively in several works (see, for example, [2, 7, 11, 13, 18, 19]) and the references therein indicated). For the multidimensional case there are some works in this direction (see e.g. [5–7, 10, 18]). However, in these works it was not obtained a series representation for the fundamental solution in an arbitrary dimension. A representation of the fundamental solution in the form of an absolute convergent series is important since it enables to use these functions in approximation problems.

In this paper we obtain explicitly integral and series representations for the fundamental solution of the time fractional diffusion equation, for an arbitrary dimension. The series representations obtained depend on the parity of the dimension. As an application of our results we study also diffusion and stress in the axially symmetric case for plane deformation. Integral representations for the solution of this problem were obtained by Povstenko in [17] using Fourier-Laplace techniques.

The structure of the paper reads as follows: in the preliminaries section we recall some basic concepts about fractional calculus and special functions. In Section 3 we construct integral and series representations for the fundamental solution of the time fractional diffusion equation. In Section 4 we compute the fractional moments of arbitrary order $\gamma > 0$ and in Section 5 we present some plots of the fundamental solution for $n = 1$ and $n = 2$. Finally, in Section 6 we study the problem of diffusion and stress in the axially symmetric case for plane deformation.

2 Preliminaries

For a locally integrable function f on $\mathbb{R}^n$, the multidimensional Fourier transform of f is the function defined by the integral

$$(\mathcal{F}f)(\kappa) = \widehat{f}(k) = \int_{\mathbb{R}^n} e^{i\kappa \cdot x} f(x) dx,$$

where $x, \kappa \in \mathbb{R}^n$ and $\kappa \cdot x$ denotes the usual inner product in $\mathbb{R}^n$. For the Laplace operator $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ we have $\widehat{\Delta f}(\kappa) = -|\kappa|^2 \widehat{f}(\kappa)$. Finally, we recall the definition of the Caputo fractional derivative ${}^C\partial_t^\beta$ of order $\beta > 0$ (see [7])

$$\left({}^C\partial_t^\beta f\right)(x, t) = \frac{1}{\Gamma(m - \beta)} \int_0^t \frac{1}{(t - w)^{\beta - m + 1}} \frac{\partial^m f}{\partial w^m}(x, w) dw,$$

where $m = [\beta] + 1$, $t > 0$, and $[\beta]$ means the integer part of β . For $\beta = m \in \mathbb{N}$, the Caputo fractional derivative coincides with the standard derivative of order m .

Some special functions naturally arise in the study of fractional calculus. For our work we need the one-parametric Mittag-Leffler function E_α (see [3]), which is defined in terms of the power series by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad z \in \mathbb{C}, \quad \Re(\alpha) \in \mathbb{R}^+.$$

Another special function with an important role in fractional calculus is the Wright function $W_{\alpha, \beta}$ (see [4]) which is defined as the convergent series

$$W_{\alpha, \beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + \beta) n!}, \quad \alpha > -1, \quad \beta \in \mathbb{C}.$$

The Wright function is related with the Bessel function of first kind with index ν by the formula

$$J_\nu(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{\Gamma(n + \nu + 1) k!} \left(\frac{z}{2}\right)^{2k + \nu} = \left(\frac{z}{2}\right)^\nu W_{1, \nu+1} \left(-\frac{z^2}{4}\right).$$

The fundamental solution obtained in this paper can be represented in terms of the Fox H-function $H_{p, q}^{m, n}$, which is defined by a Mellin-Barnes type integral in the form (see [8])

$$H_{p, q}^{m, n} \left[z \left| \begin{array}{c} (a_1, \alpha_1), \dots, (a_p, \alpha_p) \\ (b_1, \beta_1), \dots, (b_q, \beta_q) \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{i=1}^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^p \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - \beta_j s)} z^{-s} ds,$$

where $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}^+$, for $i = 1, \dots, p$ and $j = 1, \dots, q$, and $\mathcal{L}$ is a suitable contour in the complex plane separating the poles of the two factors in the numerator (see [8]).

3 Fundamental solution of the multidimensional time fractional diffusion-wave equation

In [1] we have considered the following Cauchy problem

$$\begin{cases} \left({}^C\partial_t^\beta - c^2 \Delta \right) G_n^\beta(x, t) = 0 \\ G_n^\beta(x, 0) = \prod_{i=1}^n \delta(x_i), & 0 < \beta \leq 2 \\ \frac{\partial G_n^\beta}{\partial t}(x, 0) = 0, & 1 < \beta \leq 2 \end{cases}$$

where for $0 < \beta \leq 1$ we only consider the first initial condition and for $1 < \beta \leq 2$ we need to consider the two initial conditions, and δ is the delta Dirac function. Using the Fourier transform with respect to space we obtain the following unique solution in Fourier domain given in terms of the Mittag-Leffler function (see [1])

$$\widehat{G}_n^\beta(\kappa, t) = E_\beta(-c^2|\kappa|^2 t^\beta), \quad \beta \in]0, 2].$$

Applying the inverse Fourier transform we get the following integral representation for the fundamental solution:

$$G_n^\beta(x, t) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-i\kappa \cdot x} E_\beta(-c^2|\kappa|^2 t^\beta) d\kappa, \quad x \in \mathbb{R}^n, \quad t > 0. \quad (2)$$

Since $E_\beta(-c^2|\kappa|^2 t^\beta)$ is a radial function in κ , (2) can be rewritten as

$$G_n^\beta(x, t) = \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^{+\infty} \tau^{\frac{n}{2}} E_\beta(-c^2 \tau^2 t^\beta) J_{\frac{n}{2}-1}(\tau|x|) d\tau. \quad (3)$$

where $J_{\frac{n}{2}-1}$ is the Bessel function of first kind with index $\frac{n}{2}-1$. Using the Mellin transform and the Mellin convolution theorem it is possible to rewrite (3) as a Mellin convolution, i.e., $\mathcal{M}\{G_n^\beta\}(s) = \mathcal{M}\left\{(f *_{\mathcal{M}} g)\left(\frac{1}{|x|}\right)\right\}(s)$, where $g(\tau) = E_\beta(-c^2 \tau^2 t^\beta)$ and $f(\tau) = \frac{1}{(2\pi)^{\frac{n}{2}} |x|^{\frac{n}{2}} \tau^{\frac{n}{2}+1}} J_{\frac{n}{2}-1}\left(\frac{1}{\tau}\right)$. Applying the inverse Mellin transform we obtain a representation of (3) as a Mellin-Barnes integral and, consequently, as a Fox H-function (for more details see [1]):

$$G_n^\beta(x, t) = \frac{1}{2\pi^{\frac{n}{2}} |x|^n} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(1-\frac{s}{2}\right) \Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(1-\frac{\beta s}{2}\right)} \left(\frac{2c t^{\frac{\beta}{2}}}{|x|}\right)^{-s} ds \quad (4)$$

$$= \frac{1}{2\pi^{\frac{n}{2}} |x|^n} H_{2,1}^{0,2} \left[\frac{2c t^{\frac{\beta}{2}}}{|x|} \left| \begin{matrix} (0, \frac{1}{2}), (1-\frac{n}{2}, \frac{1}{2}) \\ (0, \frac{\beta}{2}) \end{matrix} \right. \right]. \quad (5)$$

Remark 3.1 The representation (5) of G_n^β in terms of the H-function coincides with the one given in [7, Sec. 6.2, (6.3.33)] under a suitable change of variables in the Mellin-Barnes integral.

The integral (4) can be computed explicitly applying the Residue Theorem where the contour of integration must be transformed to the loop $\mathcal{L}_{+\infty}$ (see [8]) starting and ending at infinity and encircling all the poles of the functions $\Gamma\left(1-\frac{s}{2}\right)$ and $\Gamma\left(\frac{n-s}{2}\right)$ in the numerator. Since the gamma function $\Gamma(s)$ has simple poles at $s = -n$, with $n \in \mathbb{N}_0$, we have that $\Gamma\left(1-\frac{s}{2}\right)$ has poles at $s = 2k+2$, with $k \in \mathbb{N}_0$, and $\Gamma\left(\frac{n-s}{2}\right)$ has poles at $s = 2k+n$, with $k \in \mathbb{N}_0$. Therefore, in the odd case we have two non-coincident sequences of simple poles, $s = 2k+2$, $k \in \mathbb{N}_0$, and $s = 2k+n$, $k \in \mathbb{N}_0$, while in the even case we have a finite sequence of simple poles coming from $\Gamma\left(1-\frac{s}{2}\right)$ at the points $s = 2k+2$, for $k = 0, 1, \dots, \frac{n}{2}-2$, and an infinite sequence of double poles coming from $\Gamma\left(1-\frac{s}{2}\right) \Gamma\left(\frac{n-s}{2}\right)$ at the points $s = 2k+2$, for $k \geq \frac{n}{2}-1$. Applying the Residue Theorem and after some manipulations (see [1]) we obtain the following result.

Theorem 3.2 For n odd and $0 < \beta \leq 2$ the FS of the time fractional diffusion-wave operator is given by

$$G_n^\beta(x, t) = \frac{1}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2} t^\beta} \sum_{k=0}^{\frac{n-3}{2}} \frac{\Gamma(-1-k+\frac{n}{2})}{\Gamma(1-\beta(k+1)) k!} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^k \\ + \frac{(-1)^{\frac{n-1}{2}} \sqrt{\pi}}{(4\pi c^2 t^\beta)^{\frac{n}{2}}} \sum_{p=0}^{+\infty} \frac{1}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} \Gamma\left(1-\frac{\beta(p+n)}{2}\right) p!} \left(-\frac{|x|}{c t^{\frac{\beta}{2}}}\right)^p. \quad (6)$$

For n even and $0 < \beta \leq 2$ the FS of the time fractional diffusion-wave operator is given by

$$G_n^\beta(x, t) = \frac{1}{4c^2 \pi^{\frac{n}{2}} |x|^{n-2} t^\beta} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma(\frac{n}{2}-k-1)}{\Gamma(1-\beta(k+1)) k!} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^k \\ + \frac{(-1)^{\frac{n}{2}+1}}{(4\pi c^2 t^\beta)^{\frac{n}{2}}} \sum_{k=0}^{+\infty} \frac{\psi(k+\frac{n}{2}) - \beta \psi(1-\beta(k+\frac{n}{2})) + \psi(k+1) + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)}{\Gamma(k+\frac{n}{2}) \Gamma(1-\beta(k+\frac{n}{2})) k!} \left(\frac{|x|^2}{4c^2 t^\beta}\right)^k. \quad (7)$$

4 Fractional Moments

For each $n \in \mathbb{N}$ the fractional moments $\mathbf{M}_n^\gamma(t)$ of order $\gamma > 0$ of the fundamental solution $G_n^\beta(x, t)$ can be computed explicitly using the Mellin transform (see [1])

$$\mathbf{M}_n^\gamma(t) = \int_0^{+\infty} r^\gamma G_n(r, t) dr = \frac{(4c^2 t^\beta)^{\frac{\gamma-n+1}{2}} \Gamma\left(\frac{3+\gamma-n}{2}\right) \Gamma\left(\frac{\gamma+1}{2}\right)}{2 \pi^{\frac{n}{2}} \Gamma\left(1+\frac{\beta(\gamma-n+1)}{2}\right)}.$$

In Table 1 we present the expression of the moments for some particular values of n and γ .

	$n = 1$	$n = 2$	$n = 3$	$n \in \mathbb{N}$
$\gamma = 1$ (mean value)	$\frac{c t^{\frac{\beta}{2}}}{2 \Gamma(1+\frac{\beta}{2})}$	$\frac{1}{2\pi}$	$\frac{1}{4\pi c t^{\frac{\beta}{2}} \Gamma(1-\frac{\beta}{2})}$	$\frac{(4c^2 t^\beta)^{\frac{2-n}{2}} \Gamma(2-\frac{n}{2})}{2 \pi^{\frac{n}{2}} \Gamma(1+\frac{\beta(2-n)}{2})}$
$\gamma = 2$ (variance)	$\frac{c^2 t^\beta}{\Gamma(1+\beta)}$	$\frac{c t^{\frac{\beta}{2}}}{4 \Gamma(1+\frac{\beta}{2})}$	$\frac{1}{4\pi}$	$\frac{(4c^2 t^\beta)^{\frac{3-n}{2}} \Gamma(\frac{5-n}{2})}{4 \pi^{\frac{n-1}{2}} \Gamma(1+\frac{\beta(3-n)}{2})}$
$\gamma = 3$ (3rd moment)	$\frac{3(c^2 t^\beta)^{\frac{3}{2}}}{\Gamma(1+\frac{3\beta}{2})}$	$\frac{2c^2 t^\beta}{\pi \Gamma(1+\beta)}$	$\frac{c t^{\frac{\beta}{2}}}{\pi \Gamma(1+\frac{\beta}{2})}$	$\frac{(4c^2 t^\beta)^{\frac{4-n}{2}} \Gamma(\frac{6-n}{2})}{2 \pi^{\frac{n}{2}} \Gamma(1+\frac{\beta(4-n)}{2})}$

Table 1 - Particular moments of G_n^β

5 Graphical representations of the fundamental solution

Now we present some plots of the fundamental solution for $c = 1$, $n = 1, 2$, and some values of the fractional parameter β . For the one-dimensional case we use the series representation of G_1^β since the fundamental solution reduces to a Wright function and we can use the algorithm developed by Luchko in [12].

5.1 Case $n = 1$

For $n = 1$, the fundamental solution can be written as a Wright function (see [13]). In fact, putting $n = 1$ in (6) we obtain

$$G_1^\beta(x, t) = \frac{1}{2c t^{\frac{\beta}{2}}} W_{-\frac{\beta}{2}, 1-\frac{\beta}{2}}\left(-\frac{|x|}{c t^{\frac{\beta}{2}}}\right).$$

Figure 1: Plots of G_1^β for $c = 1$, $\beta = 0.2, 0.5, 0.75, 0.9$ (1st line, from left to right), $\beta = 1.0, 1.2, 1.5, 1.7$ (2nd line, from left to right), and $t = 0.2, 0.4, 0.6, 0.8$.

It is known that G_1^β is a probability density function corresponding to a slow diffusion in the case $0 < \beta < 1$ and a fast diffusion when $1 < \beta < 2$ (see [14]). In the slow diffusion case the fundamental solution attains its maximum value at $x = 0$.

5.2 Case $n = 2$

Considering $n = 2$ in (7) the fundamental solution simplifies to

$$G_2^\beta(x, t) = \frac{1}{4\pi c^2 t^\beta} \sum_{k=0}^{+\infty} \frac{2\psi(k+1) - \beta\psi(1-\beta(k+1)) + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)}{\Gamma(k+1)\Gamma(1-\beta(k+1))k!} \left(\frac{|x|^2}{4c^2 t^\beta}\right)^k.$$

In this case we use the integral representation (3) to make the plots since the series for these cases doesn't involve exclusively Wright functions. Moreover, the integral representation (3) allows us to evaluate the fundamental solution with a better accuracy.

In Figure 2 we show some plots of the fundamental solution for $c = 1$ and some fixed values of β and t .

Figure 2: Plots of G_2^β for $c = 1$, $\beta = 0.2, 0.5, 0.75, 0.9$ (1st line, from left to right), and $\beta = 1.0, 1.2, 1.5, 1.7$ (2nd line, from left to right), with $|x| \in [0.1, 2]$ and $t \in [0.1, 1]$.

From the plots we observe that for $\beta \in]0, 1[$ the fundamental solution represents a slow diffusion process both in space and time. The decay is more pronounced in space than in time due to the term $|x|^2$ that appears

in the series representation of the fundamental solution. For $\beta \in]1, 2[$ we have a fast diffusion process and its behaviour can be interpreted as propagation of damped waves whose amplitude decreases with time. For $\beta = 1$ the plot represents the classical solution of the heat equation. In the limit case $\beta \rightarrow 2^-$, it reduces to the fundamental solution of the wave equation.

6 Application to diffusion in thermoelasticity

From [17] we know that the quasi-static uncoupled theory of diffusive stress results from the equilibrium equation in terms of displacements

$$\mu \Delta \mathbf{v} + (\lambda + \mu) \text{grad div } \mathbf{v} = \eta_u \mathbf{K}_u \text{grad } u,$$

the stress-strain-concentration relation

$$\sigma = 2\mu \mathbf{e} + (\lambda \text{tr } \mathbf{e} - \eta_u \mathbf{K}_u u) \mathbf{I},$$

and the time fractional diffusion equation

$$\left({}^C \partial_t^\beta - c^2 \Delta \right) u = Q, \quad (8)$$

where $\mathbf{v}$ is the displacement vector, σ the stress tensor, $\mathbf{e}$ the linear strain tensor, u the concentration, Q the mass source, c^2 the diffusivity coefficient, λ and μ are Lamé constants, $\mathbf{K}_u = \lambda + \frac{2\mu}{3}$, η_u is the diffusion coefficient of volumetric expansion, and $\mathbf{I}$ denotes the unit tensor. Taking into account Parkus (1959) and Nowacki (1986), the author in [17] used the representation of the non-zero components of the stress tensor in terms of displacements potential ϕ

$$\sigma = 2\mu (\nabla \nabla \phi - \mathbf{I} \Delta \phi), \quad (9)$$

where ∇ is the gradient operator, and ϕ results from the equation

$$\Delta \phi = m c, \quad m = \frac{1 + \nu}{1 - \nu} \frac{\eta_u}{3},$$

where ν is the Poisson ratio. In [17], equation (8) was subject to the following initial conditions:

$$u(x, 0) = P(x), \quad 0 < \beta \leq 2, \quad \text{and} \quad \frac{\partial u}{\partial t}(x, 0) = W(x), \quad 1 < \beta \leq 2. \quad (10)$$

For the particular case of the two-dimensional axisymmetric case, using (8), (9) and the initial conditions (10) we obtain the following initial-value problem in polar coordinates (see [17] for more details):

$$\left\{ \begin{array}{ll} {}^C \partial_t^\beta u = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) + Q(r, t) & \\ u(r, 0) = P(r), & 0 < \beta \leq 2 \\ \frac{\partial u}{\partial t}(r, 0) = W(r), & 1 < \beta \leq 2 \end{array} \right. \quad (11)$$

and

$$\sigma_{zz} = \sigma_{rr} + \sigma_{\theta\theta} = -2\mu \Delta \phi, \quad \sigma_{rr} - \sigma_{\theta\theta} = 2\mu \left(\frac{\partial^2 \phi}{\partial r^2} - \frac{1}{r} \frac{\partial \phi}{\partial r} \right).$$

In [17] it is proved that the solution of (11) is given by the formula

$$\begin{aligned} u(r, t) &= 2\pi \int_0^t \int_0^{+\infty} R Q(R, t) E_Q(r, R, t - \tau) dR d\tau + 2\pi \int_0^{+\infty} R P(R) E_P(r, R, t) dR \\ &\quad + 2\pi \int_0^{+\infty} R W(R) E_W(r, R, t) dR, \end{aligned}$$

where $E_Q(r, R, t)$, $E_P(r, R, t)$ and $E_W(r, R, t)$ are three types of fundamental solutions corresponding to

$$Q(r, R, t) = \frac{q}{2\pi R} \delta(r - R) \delta_+(t), \quad P(r, R) = \frac{p}{2\pi R} \delta(r - R),$$

$$W(r, R) = \frac{w}{2\pi R} \delta(r - R),$$

with q , p and w being introduced multipliers in order to obtain non-dimensional quantities. A particular case of (11) is

$$\left\{ \begin{array}{l} {}^C\partial_t^\beta u = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \\ u(r, 0) = \frac{p}{2\pi} \delta(r), \quad 0 < \beta \leq 2 \\ \frac{\partial u}{\partial t}(r, 0) = 0, \quad 1 < \beta \leq 2 \end{array} \right. \quad (12)$$

For the solution of the Cauchy problem (12) we have (see [17])

$$u(r, t) = \frac{p}{2\pi} \int_0^{+\infty} E_\beta(-c^2 \xi^2 t^\beta) J_0(r \xi) \xi d\xi$$

$$\sigma_{rr} = -2\mu m \frac{p}{2\pi r} \int_0^{+\infty} E_\beta(-c^2 \xi^2 t^\beta) J_1(r \xi) d\xi. \quad (13)$$

It is immediate that integral representation (13) coincides with (3) for the case of $n = 2$. Therefore the solution of (13) has the following series representation:

$$u(r, t) = \frac{p}{4\pi c^2 t^\beta} \sum_{k=0}^{+\infty} \frac{2\psi(k+1) - \beta\psi(1-\beta(k+1)) + \ln\left(\frac{4c^2 t^\beta}{r^2}\right)}{\Gamma(k+1) \Gamma(1-\beta(k+1)) k!} \left(\frac{r^2}{4c^2 t^\beta}\right)^k.$$

Applying the same techniques presented in [1] we obtain the following series representation for σ_{rr} :

$$\sigma_{rr} = -\frac{\mu m p}{4\pi c^2 t^\beta} \sum_{k=0}^{+\infty} \frac{\psi(2+k) + \psi(1+k) - \beta\psi(1-\beta(k+1)) + \ln\left(\frac{4c^2 t^\beta}{r^2}\right)}{\Gamma(k+2) \Gamma(1-\beta(k+1)) k!} \left(\frac{r^2}{4c^2 t^\beta}\right)^k.$$

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