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Combined ship routing and inventory management in the salmon farming industry *

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Abstract We consider a maritime inventory routing problem for Norway's largest salmon farmer both producing the feed at a production factory and being responsible for fish farms located along the Norwegian coast. The company has bought two new ships to transport the feed from the factory to the fish farms and is responsible for the routing and scheduling of the ships. In addition, the company has to ensure that the feed at the production factory as well as at the fish farms is within the inventory limits. A mathematical model of the problem is presented, and this model is reformulated to improve the efficiency of the branch-and-bound algorithm and tightened with valid inequalities. To derive good solutions quickly, several practical aspects of the problem are utilized and two matheuristics developed. Computational results are reported for instances based on the real problem of the salmon farmer.

Keywords Ship routing and scheduling · Maritime transportation · Inventory Routing · valid inequalities

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1 Introduction

Norway is the world's largest producer of farmed salmon and the industry is growing. Continued growth is, among other things, dependent on sustainable solutions to environmental challenges, change of regulations, increased demand and innovations in feed production (Olafsen et al. [21]). If this is achieved, salmon farming will become an increasingly important part of Norwegian exports in the future.

Norway's largest salmon farmer, Marine Harvest Norway, is a vertically integrated salmon farmer with business ranging from egg production to processing, distribution and sales of finished products. Salmon farmers are not traditionally involved in feed production, but Marine Harvest is now aiming to integrate both feed production and delivery. They will be the only salmon farmer in Norway controlling the entire value chain. This allows for implementation of vendor managed inventory (VMI), as opposed to the order-based feed delivery common in the industry. VMI is a popular business practice in supply chain management, and can give benefits such as lower inventory and transportation costs and reduced risk of empty inventories (Simchi-Levi et al. [28]). Costs related to feed account for approximately half of the salmon production costs. Another large cost driver is lost feed days. Marine Harvest aims to lower costs and secure stable deliveries of feed by gaining control over this critical part of the value chain.

Today, Marine Harvest has around 115 fish farms located along the Norwegian coast, as illustrated in Figure 1. The figure also presents the location of the new feed factory. The coast is divided into four regions; North, Mid, West and South. At the moment, Marine Harvest has invested in two LNG-fueled ships to deliver feed in bulk.

When an actor in a maritime supply chain has the responsibility for both transportation of the cargoes and inventories at the ports, the underlying planning problem is called a maritime inventory routing problem (MIRP) (Christiansen et al. [11]). In maritime transportation, inventories are often located at both loading and unloading ports, so the MIRPs are common problems within the maritime business and are in general very complex problems to solve. Therefore, MIRPs have attracted considerable attention during the last fifteen years. The research and resulting publications have formed the basis of several surveys: Papageorgiou et al. [22], Christiansen et al. [11], and Christiansen and Fagerholt [9], [10]. In addition, Coelho et al. [12] and Andersson et al. [6] have been surveying both land-based and maritime inventory routing problems.

The objective of the research presented in this paper has been to develop a mathematical model and solution approach which can support the process of planning feed deliveries to fish farms from the factory based on a VMI principle. The model minimizes transportation costs and avoids inventories below safety stock levels, resulting in a robust transportation plan. The model decides on volumes loaded at the factory and discharged at each fish farm, as well as the order of visits. Production capacity is not sufficient to supply all fish

Fig. 1 Marine Harvest's fish farms located along the Norwegian coast. The big circle represents the feed factory [20].

farms and as demands vary, the model gives an indication of how many fish farms it is realistic to serve at a given time of the year. The model is flexible to allow for future changes in Marine Harvest's planning situation, such as an increased number of factories, fish farms and ships. As opposed to many maritime inventory routing problems, Marine Harvest's problem includes several consecutive deliveries and no tight time windows. Another complicating characteristic is that they do not have capacity to supply all fish farms internally. The real problem is very complex, and the real instances are big. Therefore, the mathematical model is reformulated to either improve the efficiency of the branch-and-bound algorithm or to strengthen the formulation. Furthermore, the formulations are explored based on practical aspects of the problem to propose matheuristics that derive good solutions quickly.

Similarly to the problem studied here, many of the MIRPs described in the literature are concerned with the transportation of a single product and no allocation of different products to compartments needs to be considered. For single product MIRP studies, see for instance Agra et al. [2], Engineer et al. [14], Furman et al. [15], Goel et al. [16], Grønhaug et al. [17], Hewitt et al. [18], Rocha et al. [23], Shen et al. [25], Sherali and Al-Yakoob [26], [27], Song and Furman [30], and Uggen et al. [32]. We will assume constant and fixed production and consumption rates during the planning horizon. In the literature, a mathematical model based on continuous time is often used under these assumptions. See for instance Christiansen [8], Al-Khayyal and Hwang [5], and Siswanto et al. [29]. The model formulated in this paper is general with regard to the number of production factories and fish farms. However, the real case has just one production factory at the moment. Even though the general MIRP concerns a network structure consisting of many production and consumption ports, some real MIRPs studied in the literature include just one central producer or one central customer. Such networks correspond to a classical vehicle routing structure with one depot and a set of customers. Stålhane et al. [31] present a real LNG MIRP for one central producer with inventory considerations and many customers with contract requirements instead of inventories. Furthermore, Dauzère-Pérès et al. [13] describe a MIRP for distribution of calcium carbonate slurry for a central producer and many customers with inventory considerations. In the literature, the solution approaches are both exact methods such as branch-and-cut and branch-and-price and different types of metaheuristics (e.g. genetic algorithms and large neighborhood search). In addition, several approaches are mathheuristics combining exact methods and heuristics (e.g. rolling horizon heuristics and various fix and relax heuristics). Another study from the salmon farming industry is described by Romero et al. [24]. They have developed a GRASP-based algorithm for designing routes and schedules for a fleet of ships distributing salmon feed for a salmon feed supplier in southern Chile. However, no inventory considerations are taken into account.

Our contribution relies on the mathematical formulation of the problem and on the discussion of several improvements, including the addition of valid inequalities and the use of extended formulations. Although many of the techniques used in the paper have been used in the past for related problems, the aggregated version of the subtour elimination constraints is new according to our knowledge. The model improvements take into account the particularities of the practical problem. Such particular aspects are also used to derive two matheuristics which are used to produce feasible solutions for all instances, which include those instances where the solver fails to find feasible solutions within the run time limit.

The remainder of the paper is organized as follows: Section 2 describes the planning problem considered, while in Section 3 the formulation of the problem is presented. Furthermore, several valid inequalities are developed to strengthen the proposed formulation as well as other formulation improvements, and these are described in Section 4. In Section 5, we discuss several practical aspects of the problem that can be utilized when solving the problem as well as suggest two matheuristics. Section 6 presents the computational study. Finally, some concluding remarks follow in Section 7.

2 Problem description

In this section, we will describe the maritime inventory routing problem considered and formulated in Section 3. The problem is based on Marine Harvest's planning problem. We will include a discussion of the assumptions and generalizations made.

The company has at the moment one factory for producing fish feed, but in order to give the possibility of expanding the production capacity, we consider a given set of factories. The company produces several fish feed products, but we aggregate all products and consider a single product only. The product is called fish feed or just feed in the following. This assumption is taken based on insufficient information from the company regarding storage limits of the separate products, how the different products are allocated on board the ships, and the production rates of the various products. In Ivarsøy and Solhaug [19], a multi-product model of the problem is presented. At each production factory, there is a given capacity of the silos storing the feed, and the average production rate is assumed constant during the planning horizon. If production capacity is insufficient, feed must be bought externally to fulfill the demand at one or several fish farms during the planning horizon.

The fish feed is transported by a heterogeneous fixed fleet of ships to fish farms. The ships have a given capacity, speed and cost structure. The ships have highly specialized equipment, so we assume that it is not possible to hire capacity from elsewhere. It is assumed that the capacity of the ships are greater or equal to the capacity of the storages of feed. A ship visiting the factory plant will load feed such that the storage at the factory stock is cleared each time the factory is visited. The loading rate at a factory depends on the factory and the ship visiting the factory. A ship's voyage is defined as a visit to a factory for loading, followed by consecutive visits to fish farms before returning to a factory.

The feed is unloaded at fish farms. It is assumed that the consumption of fish feed is continuous. The ships are not dependent on visiting the factory or fish farms during working hours. However, at many of the fish farms, maximum silo capacity can only be utilized when there are people at work who can control the unloading. Outside normal working hours, these silos can only be filled up to a certain percentage of the storage capacity. The silos at the fish farms should never run empty and to ensure this the company maintains a safety stock equal to one day of feed. Hence, both lower and upper limits on the feed storages at the fish farms have to be considered. However, the inventory level can be below the lower limit at a penalty cost. For fish farms with a high demand rate, it might be beneficial to deliver feed to a fish farm more than once during a ship's voyage.

The ships are allowed to wait before starting loading at the factory or unloading at the fish farms. Due to limited production capacity, there must be a certain number of days between departures if the ships are to leave factories with close to full ship loads. Therefore, we also assume that ships cannot be loaded simultaneously, even though this is physically possible. This is equivalent to a berth capacity of one ship. To ensure evenly distributed deliveries to fish farms, there should also be a certain number of days between feed deliveries to a particular fish farm.

To summarize, this problem is to design ship routes and schedules where each ship's voyage starts by leaving the factory fully loaded or has as much load on board as there are feed in the factory storage. Then the ship visits a set of fish farms and unloads feed before returning empty to a factory for a new loading operation. Figure 2 illustrates the current planning situation of Marine Harvest with one production factory and two ships. Each fish farm should either be supplied from the factory or externally during the entire planning horizon. For the internal supplied fish farms, the unloading quantity has to be determined at each visit. Finally, the inventories at the factories and fish farms must be within the lower and upper limits. The typical planning horizon in this business is two weeks.

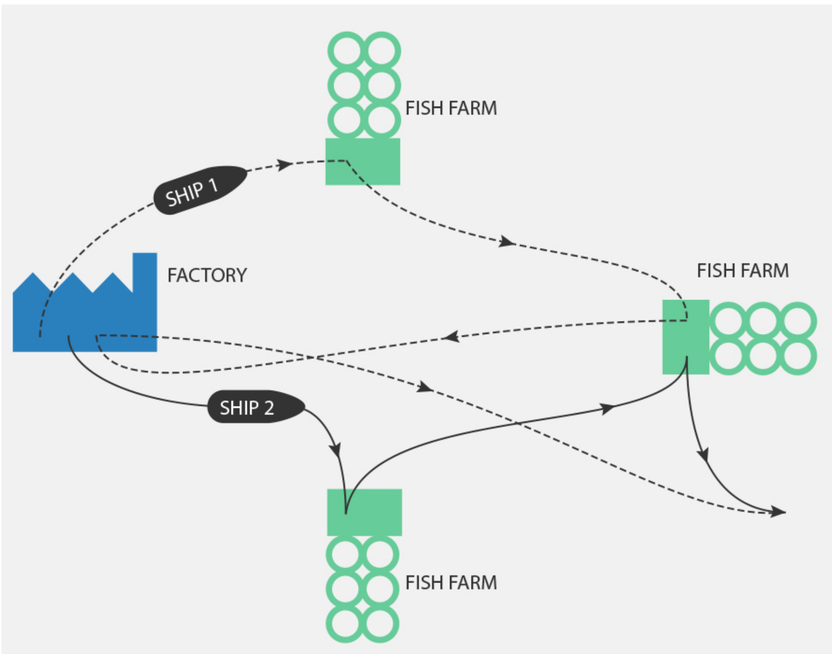

Fig. 2 Illustration of Marine Harvest's planning situation.

The objective is to minimise costs, while ensuring in-time deliveries. Costs related to both transportation and external supply must be considered. Transportation costs include fuel consumption and other costs of operation that do not incur when a ship is idle. Since we are dealing with a vertically integrated supply chain, transportation will not affect inventory costs so these costs are disregarded. External feed costs consist of transportation costs plus the supplier's profit margin.

3 Mathematical formulation

In this section, the notation and model of the problem described in Section 2 is presented.

3.1 Notation

The notation is based on the use of lower-case letters to represent subscripts and decision variables, and capital letters to represent sets and parameters. Capital letters are also used as literal superscripts to define mnemonic composite letters defining either variables, sets or parameters.

3.1.1 Indices

i, j	Locations (factories or fish farms)
$o(v)$	Start node of ship v (factory or ship farm)
$d(v)$	Dummy end node of ship v
m, n	Visit numbers
v	Ships
d	Days

3.1.2 Sets

$\mathcal{N}^P$	Set of factories
$\mathcal{N}^C$	Set of fish farms
$\mathcal{M}_i$	Set of visit numbers for location i
$\mathcal{V}$	Set of ships
$\mathcal{S}^P$	Set of factory visits (i, m) where $i \in \mathcal{N}^P$ and $m \in \mathcal{M}_i$ (includes all relevant $o(v)$)
$\mathcal{S}^C$	Set of fish farm visits (i, m) where $i \in \mathcal{N}^C$ and $m \in \mathcal{M}_i$ (includes all relevant $o(v)$)
$\mathcal{S}$	Set of visits, $\mathcal{S} = \mathcal{S}^P \cup \mathcal{S}^C$
$\mathcal{S}_v$	Set of feasible visits for ship v, $\mathcal{S}_v = \mathcal{S} \cup \{d(v)\}$
$\mathcal{D}$	Set of days within the planning horizon

3.1.3 Parameters

C^P	Penalty cost for stock levels below the safety stock level
C^T_{ijv}	Transportation cost for sailing from location i to location j with ship v
C^{EQ}	Unit cost of buying feed externally
C^{ET}_i	Transportation cost for external feed delivery to fish farm i
L^0_v	Initial load of feed on ship v
$\overline{K}_v$	Maximum capacity for ship v
$\underline{Q}_i$	Minimum unloading quantity of feed at fish farm i
S^0_i	Initial inventory level at location i
$\underline{S}^C_i$	Minimum inventory level at location i
$\overline{S}_i$	Maximum inventory level at location i
A_i	Reduction rate in storage capacity at fish farm i outside working hours
J_i	Location type for location i, 1 for factories and -1 for fish farms
R_i	Production rate at factory i or consumption rate at fish farm i
$\overline{T}$	Length of planning period
T_{ijv}	Sailing time from location i to location j with ship v
T^Q_{iv}	Loading or unloading time per ton of feed for location i by ship v
$\underline{T}_d$	Start of service hours for day d
$\overline{T}_d$	End of service hours for day d
T^B_i	Time between visits to location i

3.1.4 Decision variables

x_{imjnv}	1 if ship v sails from visit (i, m) to visit (j, n), else 0
y_{im}	1 if visit (i, m) is not made by any ship, else 0
w_{imv}	1 if visit (i, m) is made by ship v, else 0
u_i	1 if fish farm i is supplied internally, 0 if supplied externally
σ_{imd}	1 if visit $(i, m) \in \mathcal{S}^C$ is within service hours on day d, else 0
q_{imv}	Amount of feed loaded/unloaded by ship v during visit (i, m)
l_{imjnv}	Amount of feed on board ship v when traveling on arc (i, m, j, n)
s_{im}	Amount of feed in stock at the start of visit $(i, m) \in \mathcal{S}$
s^E_{im}	Amount of feed in stock at the end of visit $(i, m) \in \mathcal{S}$
d_{im}	Amount of feed below $\underline{S}_i$ at the start of visit $(i, m) \in \mathcal{S}^C$
t_{im}	Time for start of service for visit (i, m)
t^E_{im}	Time for end of service for visit (i, m)

Parts of the notation are illustrated in Figure 3. The model will be called an Arc-Flow formulation due to the x_{imjnv} and l_{imjnv} flow variables.

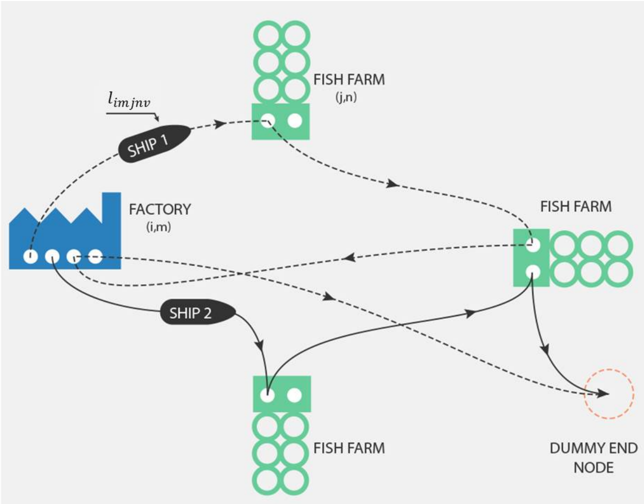

Fig. 3 Illustration of the Arc-Flow formulation.

3.2 Model Formulation

The problem is formulated as a mixed integer linear programming problem. First, the objective function is presented. Then, the routing constraints are given before the loading and unloading constraints. We continue with the inventory constraints at both the production factories and fish farms, and, finally, present the time aspects.

3.2.1 Objective function

$$\begin{aligned} \min z = & \sum_{(i,m)\in\mathcal{S}} \sum_{(j,n)\in\mathcal{S}} \sum_{v\in\mathcal{V}} C^T_{ijv} x_{imjnv} + \sum_{i\in\mathcal{N}^C} (C^{EQ} R_i \overline{T} + C^{ET}_i)(1-u_i) \\ & + \sum_{(i,m)\in\mathcal{S}^C} C^P \frac{d_{im}}{R_i} \end{aligned} \quad (1)$$

The objective function (1) minimizes transportation costs, costs of buying feed externally and penalty costs related to low stock levels. A transportation cost C^T_{ijv} is included for each arc used. External feed costs consist of a margin per unit of feed required, C^{EQ}, and a transportation cost, C^{ET}_i. The penalty cost, C^P, is added for each unit of time that a fish farm is below the safety stock level. The time is given by the fraction of underage over the rate of feed depletion from stock.

3.2.2 Routing constraints

$$\sum_{(j,n)\in\mathcal{S}_v} x_{o(v)jnv} = 1 \qquad v\in\mathcal{V} \quad (2)$$

$$w_{imv} - \sum_{(j,n)\in\mathcal{S}_v} x_{imjnv} = 0 \qquad (i,m) \in \mathcal{S}_v \backslash \{o(v)\}, v \in \mathcal{V} \tag{3}$$

$$w_{imv} - \sum_{(j,n)\in\mathcal{S}_v} x_{jnimv} = 0 \qquad (i,m) \in \mathcal{S}_v \backslash \{o(v)\}, v \in \mathcal{V} \tag{4}$$

$$\sum_{(j,n)\in\mathcal{S}_v} x_{jnd(v)v} = 1 \qquad v \in \mathcal{V} \tag{5}$$

$$\sum_{v\in\mathcal{V}} w_{imv} + y_{im} = 1 \qquad (i,m) \in \mathcal{S} \tag{6}$$

$$y_{im} - y_{im-1} \geq 0 \qquad (i,m) \in \mathcal{S}|m > 1 \tag{7}$$

$$u_i + y_{i1} = 1 \qquad i \in \mathcal{N}^C \tag{8}$$

$$x_{imjnv} \in \{0,1\} \qquad (i,m) \in \mathcal{S}, (j,n) \in \mathcal{S}_v, v \in \mathcal{V} \tag{9}$$

$$w_{imv} \in \{0,1\} \qquad (i,m) \in \mathcal{S}_v, v \in \mathcal{V} \tag{10}$$

$$y_{im} \in \{0,1\} \qquad (i,m) \in \mathcal{S} \tag{11}$$

$$u_i \in \{0,1\} \qquad i \in \mathcal{N}^C \tag{12}$$

Constraints (2) ensure that each ship leaves its initial position and sails towards another location or its dummy node, meaning the ship is not used. The flow conservation constraints are given by (3) and (4), while constraints (5) make sure that each ship ends in its designated end node.

Constraints (6) guarantee that at most one ship can make visit (i,m), and the constraints also set the value of y_{im} to one if the visit is not made. The number of symmetrical solutions in the integer solution method is reduced by including constraints (7), by ensuring that only the smallest subsequent visit numbers are used. Furthermore, constraints (8) state that a fish farm is either served by the internal factory ($u_i = 1$) or supplied externally ($y_{i1} = 1$) during the planning horizon. The variable u_i is included in the presentation of the model to facilitate the reading.

Finally, the binary restrictions for the variables are given in (9)-(12).

3.2.3 Loading and unloading constraints

$$\sum_{(j,n)\in\mathcal{S}_v} l_{o(v)jnv} - J_i q_{o(v)v} = L_v^0 \qquad v \in \mathcal{V} \tag{13}$$

$$\sum_{(j,n)\in\mathcal{S}_v} l_{jnimv} + J_i q_{imv} - \sum_{(j,n)\in\mathcal{S}_v} l_{imjnv} = 0 \qquad (i,m) \in \mathcal{S}_v, v \in \mathcal{V} \tag{14}$$

$$l_{imjnv} \leq \overline{K}_v x_{imjnv} \qquad (i,m) \in \mathcal{S}, (j,n) \in \mathcal{S}_v, v \in \mathcal{V} \tag{15}$$

$$\underline{Q}_i w_{imv} \leq q_{imv} \leq \overline{S}_i w_{imv} \qquad (i,m) \in \mathcal{S}^C, v \in \mathcal{V} \tag{16}$$

$$s_{im} - \overline{S}_i y_{im} \leq \sum_{v\in\mathcal{V}} q_{imv} \leq s_{im} + \overline{S}_i y_{im} \qquad (i,m) \in \mathcal{S}^P \tag{17}$$

$$q_{imv} \geq 0 \qquad (i,m) \in \mathcal{S}_v, v \in \mathcal{V} \tag{18}$$

$$l_{imjnv} \geq 0 \qquad (i,m) \in \mathcal{S}, (j,n) \in \mathcal{S}_v, v \in \mathcal{V} \tag{19}$$

$$s_{im} \geq 0 \qquad (i,m) \in \mathcal{S} \tag{20}$$

Constraints (13) set the load on board each ship after its initial visit equal to the sum of its initial load and the quantity loaded or unloaded. For all subsequent visits, ship loads are updated according to constraints (14). Constraints (15) ensure that the load on board a ship cannot exceed the ship capacity. If arc (i, m, j, n) is not used by ship v, the corresponding load-variable is forced to zero.

The quantity unloaded at a fish farm has to be within certain limits and these restrictions are given by (16). The quantity cannot exceed the fish farm's maximum storage capacity. The storage capacity is used as the bound, because the ship capacity is much larger than the storage capacity at any fish farm. If a visit is not made by ship v, the corresponding quantity-variable is forced to zero. Visits to fish farms are required to be a certain amount of time apart, ensured later by constraints (40). We can calculate the amount of feed consumed during the smallest spread between visits, and use it as a lower bound on the unloading quantity.

The company has specified that a ship visiting a factory should always clear the stock, so constraints (17) ensure that the loaded quantity equals the current stock level, if the visit is made. Here it is assumed that the ship capacity is at least as large as the storage capacities, so the entire stock can be loaded.

Finally, the non-negative requirements for the variables are given in (18)-(20).

3.2.4 Inventory constraints

$$s_{im} - J_i R_i t_{im} = S_i^0 \qquad (i,m) \in \mathcal{S} | m = 1 \tag{21}$$

$$s_{im} + J_i R_i (t_{im}^E - t_{im}) - J_i \sum_{v \in \mathcal{V}} q_{imv} - s_{im}^E = 0 \qquad (i,m) \in \mathcal{S} \tag{22}$$

$$s_{i(m-1)}^E + J_i R_i (t_{im} - t_{i(m-1)}^E) - s_{im} = 0 \qquad (i,m) \in \mathcal{S} | m > 1 \tag{23}$$

$$s_{im} \leq \overline{S}_i \qquad (i,m) \in \mathcal{S}^P \tag{24}$$

$$s_{im}^E \leq (1 - A_i)\overline{S}_i + A_i \overline{S}_i \sum_{d \in \mathcal{D}} \sigma_{imd} + A_i \overline{S}_i y_{im} \qquad (i,m) \in \mathcal{S}^C \tag{25}$$

$$s_{im} + d_{im} \geq \underline{S}_i u_i \qquad (i,m) \in \mathcal{S}^C \tag{26}$$

$$d_{im} \leq \underline{S}_i u_i \qquad (i,m) \in \mathcal{S}^C \tag{27}$$

$$s_{im}^E + R_i (\overline{T} - t_{im}^E) \leq \overline{S}_i \qquad (i,m) \in \mathcal{S}^P | m = |\mathcal{M}_i| \tag{28}$$

$$s_{im}^E - R_i (\overline{T} u_i - t_{im}^E) \geq \underline{S}_i u_i \qquad (i,m) \in \mathcal{S}^C | m = |\mathcal{M}_i| \tag{29}$$

$$\sigma_{imd} \in \{0, 1\} \qquad (i,m) \in \mathcal{S}^C, d \in \mathcal{D} \tag{30}$$

$$d_{im} \geq 0 \qquad (i,m) \in \mathcal{S}^C \tag{31}$$

$$s_{im}^E, t_{im}, t_{im}^E \geq 0 \qquad (i,m) \in \mathcal{S} \tag{32}$$

Equations (21) set the stock level at the start of the first visit to factories and fish farms as the initial stock plus the amount produced or consumed before the first visit.

Constraints (22) ensure that the stock at the end of a location visit equals the stock at the start of the visit, adjusted for the amount of feed produced or consumed, and the amount (un)loaded during the visit. The end stock variables s_{im}^E are included to facilitate the readability. Constraints (23) relate the stock at the end of a visit to the stock at the start of the next visit by considering the production or consumption that takes place between the visits.

Constraints (24) and (25) ensure that the stock level at the start of service in a factory and at the end of service at a fish farm does not exceed the storage capacities. Since loading and unloading rates are higher than production and consumption rates, only these variables need to be constrained by upper limits. Then, the stock variable at the end of service in a factory and the stock variable at the start of service in a fish farm will never be above the maximum stock level. Constraints (25) take into account that the storage capacity is reduced by A_i if start of service is outside the service time windows. If the visit is not made (y_{im}=1), the σ-variables are zero and the upper bound on the variable is $\overline{S}_i$. The relationship between the y- and σ-variables will be stated in the next subsection.

Constraints (26) ensure that the sum of the inventory level and the underage stays above the safety stock level. If a fish farm is supplied externally, the constraint is not binding. Note that constraints (26) only apply to fish farms. The lower inventory limit at a factory is zero, ensured by non-negativity requirements on the factory stock variables. For fish farms served internally, the underage cannot be larger than the minimum stock level, as stated by constraints (27). If a fish farm is supplied externally, the underage is forced to zero.

Constraints (28) are added to ensure a feasible inventory level for factories at the end of the planning horizon. This is done by ensuring that the sum of the stock after the last visit and the production during the remaining time is below the storage capacity. Similarly, constraints (29) are added to ensure feasible inventory levels for fish farms at the end of the planning horizon. For fish farms supplied by the internal factories, the inventory level at the end of the last visit cannot be less than the safety stock level until the end of the planning period. If an external supplier is given responsibility for a fish farm's feed supply, the constraint is not binding, because all the time variables for externally supplied fish farms are forced to zero by constraints (36) presented later.

Finally, the binary and non-negative requirements for the variables not declared previously are given in (30)-(32).

3.2.5 Timing constraints

$$t_{im} + \sum_{v \in \mathcal{V}} T^Q_{iv} q_{imv} - t^E_{im} = 0 \qquad (i,m) \in \mathcal{S} \tag{33}$$

$$t^E_{im} + \sum_{v \in \mathcal{V}} T_{ijv} x_{imjnv} - t_{jn} + \overline{T} \sum_{v \in \mathcal{V}} x_{imjnv} \leq \overline{T} \qquad (i,m) \in \mathcal{S}, (j,n) \in \mathcal{S} \tag{34}$$

$$t^E_{im} \leq \overline{T} \qquad (i,m) \in \mathcal{S} \tag{35}$$

$$t_{im} \leq \overline{T} u_i \qquad (i,m) \in \mathcal{S}^C \tag{36}$$

$$\underline{T}^C_d (\sigma_{imd} - y_{im}) \leq t_{im} \leq \overline{T}^C_d + (\overline{T} - \overline{T}^C_d)(1 - \sigma_{imd} + y_{im}) \qquad (i,m) \in \mathcal{S}^C, d \in \mathcal{D} \tag{37}$$

$$\sum_{d\in\mathcal{D}} \sigma_{imd} \leq 1 \qquad (i,m) \in \mathcal{S}^C \tag{38}$$

$$\sum_{d\in\mathcal{D}} \sigma_{imd} \leq 1 - y_{im} \qquad (i,m) \in \mathcal{S}^C \tag{39}$$

$$t_{im} - t^E_{i(m-1)} \geq T^B_i(1 - y_{im}) \qquad (i,m) \in \mathcal{S} \tag{40}$$

In order to relate the start and end time of a visit, constraints (33) are added. They ensure that a visit ends when the loading or unloading operation has finished. If no feed is loaded or unloaded, as when a visit is not made, the end time is set equal to the start time. The end time variables t^E_{im} are included to facilitate the reading.

Constraints (34) ensure consistency in timing of visits. If a ship sails directly between two visits, the start time of the next visit cannot be earlier than the end time of the previous visit plus the sailing time between the two locations. Here, waiting on arrival is allowed. The largest value (Big M) that $t^E_{im} - t_{jn}$ can take is $\overline{T}$. This big - M can be reduced by creating time windows for the time variables, and, hence, the constraints can be tightened. See Section 4.

In constraints (35) it is ensured that visits cannot end later than the end of the planning horizon. Furthermore, constraints (36) set the start time of all visits to a fish farm to zero if this fish farm is supplied externally.

If a fish farm visit starts outside the given service time windows, σ_{imd} is set to 0 in constraints (37). Then, only a percentage of the storage capacity can be utilized in constraints (25). If a visit is not made, the constraints are not binding. For each visit, at most one of the time window variables can be set to 1, ensured by constraints (38). If the extra capacity is not needed, the σ_{imd} variables are not necessarily set to 0, even if the visit is within service hours. Visits that are not made should all have σ_{imd} equal to 1, ensured by constraints (39).

The company wants deliveries to their fish farms to be evenly spread throughout the planning horizon. Due to production constraints, the factories cannot provide full ship loads every day and visits should be separated for factories as well. By adding constraints (40), visits are separated by at least T^B_i hours for both factories and fish farms.

The Arc-Flow formulation (1)-(40) is denoted the AF formulation.

4 Formulation improvements

The Arc-Flow formulation presented in Section 3 is the one that has provided better computational results for related problems, see Agra et al. [3]. Nevertheless this formulation still provides large integrality gaps which is common for continuous time models. Agra et al. [3] compared continuous and discrete time models for a maritime inventory routing problem and found that the discrete time models tended to provide better bounds, but the running times using the discrete time models were in general worse than the running times using the continuous time model. They concluded that, for the constant consumption rate case, a continuous time model with valid inequalities was the best option among all the tested ones to solve small real sized instances.

In this section we discuss some directions that have been tested either to obtain better lower bounds than the one provided by the linear relaxation of the AF formulation or to improve the efficiency of the branch-and-cut algorithm based on such improved formulations.

4.1 Special ordered sets

A time window for working hours is given for each day of the planning horizon. The implementation of soft time windows is done using the binary variables, σ_{imd}. For each fish farm visit (i, m), at most one of the $\mid \mathcal{D} \mid$ time window variables σ_{imd} can be chosen. Therefore, we have modeled the variables as members of a special ordered set of type 1 (SOS1) for each farm visit (i, m). A SOS1 is an ordered set of variables where at most one variable can have a non-zero value. This means that there is one SOS1 for each fish farm visit (i, m) and the set variables are ordered according to days. Generally, branching on SOS1 variables leads to more balanced search trees and is more efficient than branching directly on binary variables, as discussed in Beale and Tomlin [7].

4.2 Tightening timing constraints

In order to tighten the timing constraints we start by creating time windows for the time variables t_{im} and t^E_{im}. We introduce the following notation:

A_{im}	Earliest start time for visit (i, m),
B_{im}	Latest start time for visit (i, m),
A^E_{im}	Earliest end time for visit (i, m),
B^E_{im}	Latest end time for visit (i, m).

The following inequalities limit the value of a visit's start and end time.

$$A_{im}(1 - y_{im}) \leq t_{im} \leq B_{im} \qquad (i, m) \in \mathcal{S} \tag{41}$$

$$A^E_{im}(1 - y_{im}) \leq t^E_{im} \leq B^E_{im} \qquad (i, m) \in \mathcal{S} \tag{42}$$

If a visit is not made, the time variables are set equal to the end of the previous visit and the lower bound should not apply.

The earliest start time for each first visit to a location is calculated as the travel time from the nearest possible initial position. The earliest start time for subsequent visits is set to the earliest start time of the previous plus the minimum number of hours between each visit. The latest start time for the first visit to a fish farm is when its stock level reaches zero, while for the factory it is when the maximum capacity is reached. The latest start time for subsequent visits to fish farms is the latest end time of the previous visit plus the time it takes to reach a stock level of zero, given that it was filled to maximum. For the factory, the latest start time for subsequent visits is given by the latest end time of the previous visit plus the time it takes to again reach maximum capacity, given that the stock was cleared. The earliest end time of a visit to a location is the earliest start time of the visit plus the loading or unloading time of the minimum loading or unloading quantity. The minimum loading or unloading quantity is given by the minimum number of hours between visits to a location. The latest end time is set to the latest start time plus the time it takes to load or unload the maximum quantity, given by the location's storage capacity. If any of the calculations give a value higher than the end time, $\overline{T}$, the value is set to $\overline{T}$ instead.

Creating time windows for each visit, allows us to tighten constraints (34). Instead of using $\overline{T}$ as the Big M, we use the difference between the latest end time of visit (i,m) and the earliest start time of visit (j,n).

$$t^E_{im} + \sum_{v\in\mathcal{V}} (T_{ijv} + B^E_{im} - A_{jn}) x_{imjnv} - t_{jn} \leq B^E_{im} - A_{jn}(1 - y_{jn}), \quad (i,m),(j,n) \in \mathcal{S} \tag{43}$$

4.3 Subtour elimination constraints

Subtour Elimination Constraints (SECs) are commonly used in vehicle routing problems. These inequalities can be written as follows Adulyasak et al. [1].

$$\sum_{(i,m)\in\mathcal{G}} \sum_{(j,n)\in\mathcal{G}} x_{imjnv} \leq \mid \mathcal{G} \mid -1 \qquad \mathcal{G} \subseteq \mathcal{S}_v, v \in \mathcal{V} \tag{44}$$

Another form of SECs has been used for selective vehicle routing problems, that is, problems where there is a binary variable associated with each node indicating whether the node is visited by a vehicle or not. Here that information is given by w_{imv}-variables. The following form of SECs is valid for the set of feasible solutions, see Adulyasak et al. [1] and the references therein.

$$\sum_{(i,m)\in\mathcal{G}} \sum_{(j,n)\in\mathcal{G}} x_{imjnv} \leq \sum_{(i,m)\in\mathcal{G}} w_{imv} - w_{\ell kv} \qquad \mathcal{G} \subseteq \mathcal{S}_v, (\ell,k) \in \mathcal{G}, v \in \mathcal{V} \tag{45}$$

Preliminary tests have shown that w_{imv}-variables tend to be integer in the fractional solutions for the particular instances tested here. So we will focus only on (44).

A stronger variant of SECs is obtained by aggregating on the ships.

$$\sum_{v\in\mathcal{V}} \sum_{(i,m)\in\mathcal{G}} \sum_{(j,n)\in\mathcal{G}} x_{imjnv} \leq \mid \mathcal{G} \mid -1 \qquad \mathcal{G} \subseteq \mathcal{S} \tag{46}$$

A particular interesting case are the SECs involving pairs of fish farm visits, (i,m,j,n) and (j,n,i,m), because preliminary tests have shown that between pairs of fish farms with short travel times the routing variables are often set to a fractional value close to one for both arcs connecting the two nodes.

$$\sum_{v\in\mathcal{V}} x_{imjnv} + \sum_{v\in\mathcal{V}} x_{jnimv} \leq 1 \qquad (i,m),(j,n) \in \mathcal{S} \tag{47}$$

4.4 Dynamic cut generation of clique inequalities

Clique inequalities for maritime inventory routing problems were introduced in Agra et al. [4]. These inequalities are created by finding conflicting routing variables. This means finding a set of x_{imjnv}-variables where at most one of the variables can be one. We have considered the following conflicts:

x_{imjnv} and x_{jnimw} where $(i,m),(j,n) \in \mathcal{S}, v,w \in \mathcal{V}$
x_{imjnv} and x_{imkow} where $(i,m),(j,n),(k,o) \in \mathcal{S}, v,w \in \mathcal{V}$
x_{jnimv} and x_{koimw} where $(i,m),(j,n),(k,o) \in \mathcal{S}, v,w \in \mathcal{V}$
x_{jnimv} and x_{imjow} where $(i,m),(j,n),(j,o) \in \mathcal{S}|n > o, v,w \in \mathcal{V}$
x_{jnimv} and x_{imkow} where $(i,m),(j,n),(k,o) \in \mathcal{S}, v,w \in \mathcal{V}|v \neq w$.

The first conflict coincides with the SECs (47) for two visits. The second says that a ship cannot sail from a location to more than one other location, and the third says that a ship cannot sail to a location from more than one other location. The fourth conflict says that a ship cannot come from a location with a visit number n and then leave to the same location with a visit number lower than n. The last conflict says that the same visit has to be done by a single ship.

These inequalities are added dynamically to the solver cut manager as they are violated during the branch-and-bound search. A current fractional solution is searched for conflicting variables and a conflict matrix is created. Finding the maximum weight clique is an NP-hard problem. So, we use a greedy heuristic approach to find maximal cliques. First, we find the pair of conflicting variables with the largest total fractional value. Then other conflicting variables are added in a greedy fashion. If the resulting clique of conflicting variables is violated (meaning that the clique sums to more than one), a cut is added.

4.5 Multi-Commodity Flow Formulation

In order to further tighten the formulation, we derive a Multi-Commodity Flow (denoted by MCF) formulation by introducing the flow variables $f_{imjnkov}$ which are one if ship v travels from (i,m) to (j,n) in the route to farm visit (k,o), where $(i,m) \in \mathcal{S}$ and $(j,n),(k,o) \in \mathcal{S}_v \setminus \{o(v)\}$. Figure 4 illustrates some of the notation of the MCF formulation.

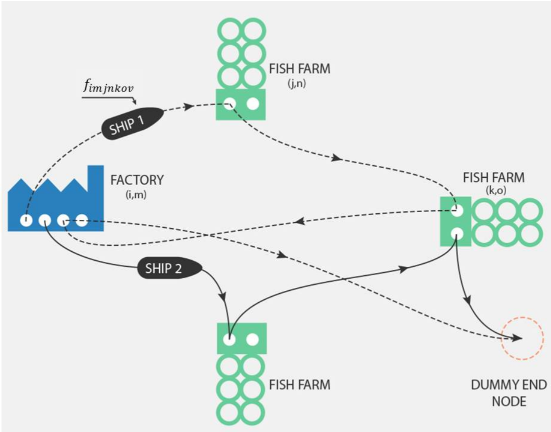

Fig. 4 Illustration of the Multi-Commodity Flow formulation.

The following set of constraints is included.

$$\sum_{(j,n)\mathcal{S}} f_{jnimkov} - \sum_{(j,n)\in\mathcal{S}_v\setminus\{o(v)\}} f_{imjnkov} = 0 \qquad (i,m),(k,o)\in\mathcal{S}_v\setminus\{o(v)\}, v\in\mathcal{V} \tag{48}$$

$$\sum_{(j,n)\in\mathcal{S}} f_{jnkokov} = w_{kov} \qquad (k,o)\in\mathcal{S}_v\setminus\{o(v)\}, v\in\mathcal{V} \tag{49}$$

$$f_{imjnkov} \leq x_{imjnv} \qquad (i,m)\in\mathcal{S}, (j,n),(k,o)\in\mathcal{S}_v\setminus\{o(v)\}, v\in\mathcal{V} \tag{50}$$

Constraints (48) are the flow conservation constraints at node (i,m) and ensure that the flow entering into (i,m) is equal to the flow leaving (i,m) in the path to farm visit (k,o). Constraints (49) state that if the visit (k,o) is made then there must exist a flow of one unit entering into (k,o). Constraints (50) ensure that if the arc (i,m,j,n) is in the path to (k,o), then x_{imjnv} must be one.

The size of the MCF formulation gets too large for many instances. A possible practical approach is to consider only a subset of variables, which is known as an *approximate extended formulation*. For details on approximate extended formulations and their relation to subtour elimination constraints for the TSP, see Van Vyve and Wolsey [33]. Here, for each node $(k,o)\in\mathcal{S}$, we may consider only those variables $f_{imjnkov}$ where i and j are within a given distance from k.

4.6 Route-Assignment Formulation

Currently, only one factory is planned to be operating, so $\mid P \mid= 1$. We denote the factory by p in the formulation given in the following. For the tested instances we assume $o(v)$ coincides with the factory. Thus all the ship routes start at the factory, and each fish farm visit is assigned to a route which is identified by the number of the factory visit, see Figure 5. The model can be easily adapted for the general case.

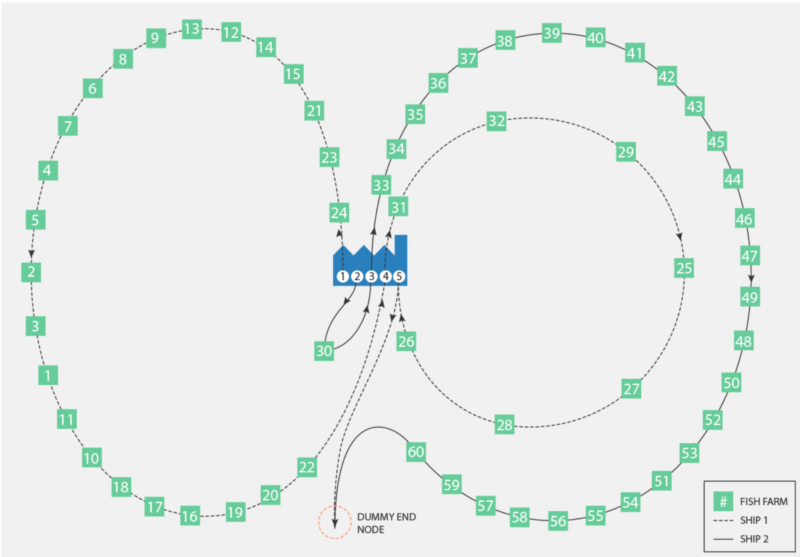

Fig. 5 Illustration of a solution found for the largest instance solved.

The Route-Assignment formulation assigns each possible visit (i, m) to a route, where a route r is defined by a factory visit (p, r). We define the set of possible routes by $\mathcal{R}$. This set coincides with the set of possible visit numbers to the factory.

For this formulation we define the following binary variables ,z_{imrv}, which are 1 if visit $(i, m) \in \mathcal{S}^C$ is associated with route r done by ship v, and 0 otherwise.

These variables are related to the original variables through the following constraints.

$$\sum_{r \in \mathcal{R}} z_{imrv} = w_{imv} \qquad (i, m) \in \mathcal{S}^C, v \in \mathcal{V} \tag{51}$$

$$z_{imrv} \leq w_{prv} \qquad (i, m) \in \mathcal{S}^C, r \in \mathcal{R}, v \in \mathcal{V} \tag{52}$$

$$x_{prjnv} = z_{jnrv} \qquad (j, n) \in \mathcal{S}^C, r \in \mathcal{R}, v \in \mathcal{V} \tag{53}$$

$$x_{imjnv} + z_{imrv} \leq 1 + z_{jnrv} \qquad (i, m), (j, n) \in \mathcal{S}^C, r \in \mathcal{R}, v \in \mathcal{V} \tag{54}$$

$$x_{imjnv} + z_{jnrv} \leq 1 + z_{imrv} \qquad (i, m), (j, n) \in \mathcal{S}^C, r \in \mathcal{R}, v \in \mathcal{V} \tag{55}$$

Constraints (51) ensure that if a visit (i, m) is made by ship v then it must be assigned to a single route of v. Constraints (52) ensure that a farm visit of ship v can be assigned to a route if the corresponding route is done by that ship.

Constraints (53) impose that if the ship leaves the factory after visit number r and directly sails to a farm, then this farm visit belongs to route r. Finally, constraints (54) and (55) ensure that if ship v sails from (i,m) to (j,n) then both visits belong to the same route.

This formulation, which includes all the variables and constraints of the AF formulation and includes additionally the z_{jnrv} variables and constraints (51)-(55), will be denoted the RA formulation (for Route-Assignment).

The RA formulation is not tighter then the original AF formulation but it includes additional information that can and will be exploited when solving the problem. This formulation can be strengthened in different directions using the new Route-Assignment information. Next we introduce a type of valid inequalities which relates the start time of visit (j,n) to the time that ship leaves the factory in that visit.

$$t^E_{pr} + \sum_{v\in\mathcal{V}} (T_{pjv} + B^E_{pr} - A_{jn}) z_{jnrv} - t_{jn} \leq B^E_{pr} - A_{jn}(1-y_{jn}) \qquad (j,n)\in\mathcal{S}^C, r\in\mathcal{R} \tag{56}$$

Another approach to tighten the formulation is to add the multi-commodity reformulation to the RA formulation. In this case one may define an MCF formulation for each route. Thus a variable $f^r_{imjnkov}$ with the same meaning of $f_{imjnkov}$ is created for each $r\in\mathcal{R}$. The set of constraints becomes as follows.

$$\sum_{(j,n)\mathcal{S}} f^r_{jnimkov} - \sum_{(j,n)\in\mathcal{S}_v\setminus\{o(v)\}} f^r_{imjnkov} = 0 \qquad (i,m),(k,o)\in\mathcal{S}_v\setminus\{o(v)\}, v\in\mathcal{V}, r\in R \tag{57}$$

$$\sum_{(j,n)\in\mathcal{S}} f^r_{jnkokov} = z_{korv} \qquad (k,o)\in\mathcal{S}_v\setminus\{o(v)\}, r\in\mathcal{R}, v\in\mathcal{V} \tag{58}$$

$$f^r_{imjnkov} \leq x_{imjnv} \qquad (i,m)\in\mathcal{S}, (j,n),(k,o)\in\mathcal{S}_v\setminus\{o(v)\}, r\in\mathcal{R}, v\in\mathcal{V} \tag{59}$$

$$f^r_{imjnkov} \leq z_{imrv} \qquad (i,m)\in\mathcal{S}_v\setminus\{d(v)\}, (j,n),(k,o)\in\mathcal{S}_v\setminus\{o(v)\}, r\in\mathcal{R}, v\in\mathcal{V} \tag{60}$$

$$f^r_{imjnkov} \leq z_{imrv} \qquad (i,m)\in\mathcal{S}_v\setminus\{d(v)\}, (j,n),(k,o)\in\mathcal{S}_v\setminus\{o(v)\}, r\in\mathcal{R}, v\in\mathcal{V} \tag{61}$$

Constraints (57)-(59) have the same meaning as (48)-(50). Constraints (60) and (61) ensure that the nodes (i,m) and (j,n) are in route r when the arc (i,m,j,n) is used in that route.

5 Practical solution approaches

As we report in Section 6, the tested instances based on the real problem are not solved to (proven) optimality, within reasonable running times, using the branch-and-cut algorithm from a commercial software based on the models discussed in the paper. Thus, the branch-and-cut acts as a heuristic. So it makes sense to explore the formulations derived so far based on practical aspects of the problem in order to propose matheuristics that derive *good* solutions as quickly as possible.

5.1 Practical aspects

Here we discuss some assumptions based on practical reasoning and some preliminary tests not reported here.

5.1.1 Sets of fish farms

In order to reduce the size of the mathematical model, the fish farms have been split into two sets and each of the two ships is fixed to one set. The splitting results in a less flexible model, but it helps reducing the number of symmetric solutions arising from the fact that we have two homogeneous ships. This is also in accordance with the companies view of operation, where each ship most likely will serve partly separate areas.

Two alternative splits, a full split and a partial split, were tested. In the first splitting approach, the fish farms farthest north are served by one ship, while the ones farthest south are served by the other and the two sets are disjunct. The partial split includes the fish farms closest to the factory in both sets, because it will not be a significant detour for a ship sailing in either direction to visit them. Preliminary results showed that the partial split was the best alternative. This is a reasonable simplification, due to the geographical situation of the fish farms along the Norwegian coast, see Figure 1. Moreover, fish farms that are farthest north or farthest south can be assigned to the same ship route.

5.1.2 Reduced number of visits

Preliminary results showed that most fish farms are only visited once within a planning horizon of 10 days. By setting this visit limit tight, the reduction in number of variables is significant. Reducing the number of visits limits the model's flexibility. Therefore we allow two visits per fish farm for the smallest test instances. For the larger test instances, the number of visits is set to one.

5.1.3 EOH

A common characteristic of maritime inventory routing problems is the unwanted End Of Horizon (EOH) effects, that essentially consists in having all the stock levels reaching their extreme values at the end of the planning horizon. To avoid the EOH effects there are several alternatives. One is to create tighter inventory bounds at the EOH. Another alternative is to impose (large) minimum delivering quantities. Here, we opted to include a minimum stock level at the EOH, which implies, a minimum delivery quantity corresponding to the net consumption (the total consumption minus the initial stock level) plus the minimum stock level for those instances where at most one visit is made. Additionally, we test heuristics that include a final step that maximizes the delivered quantity to the fish farms.

5.2 Matheuristics

The practical instances considered here can hardly be solved to optimality using an exact algorithm such as the branch-and-cut. Our computational experiments have shown that, for some instances, the branch-and-cut based on a tight AF formulation could not obtain any feasible solution after five hours of running time. Thus, in this section we present a robust heuristic strategy that uses the mathematical models discussed before to provide feasible solutions.

After running the branch-and-cut for a limited time, two cases can occur, either a feasible solution was found or not. If the feasible solution was found and the branch-and-cut algorithm could not terminate with proven optimality (which happened for all the run tests) then we improve the solution in order to obtain a local optimum. For this case we derive Algorithm 1 that converts the solution from the AF formulation into a solution of the RA formulation. Then, for each route $r \in \mathcal{R}$, we solve the TSP problem with time windows by fixing the delivered quantities and adding the corresponding (un)loading times to the sailing time. The time constrained TSP is denoted $TSPTW(r)$ and results from the inventory routing problem restricted to the set of fish farms served in route r. The time windows are computed from the inventory levels.

Algorithm 1 A local search Matheuristic to improve the routings.

```
Set X to the best feasible solution known
Decompose X into a set of | R | single vehicle inventory routing problems
Fix all (un)load variables
for route r ∈ R do
    Solve the TSPTW(r) for β seconds using branch-and-cut
    Update the solution X
end for
Fix the routing variables and free the (un)load variables
Solve the model in order to maximize the quantity delivered
```

In case the branch-and-cut has terminated with no solutions found we propose a new heuristic, Algorithm 2, that builds up a feasible solution by determining a route in each iteration. The algorithm solves the RA formulation, tightened with an approximate MCF formulation (48)-(50), where only variables corresponding to visits between farms within a distance K are kept. The routing variables x_{imjnv} are relaxed to continuous variables, but the assignment z-variables are kept binary. In each iteration the problem is restricted to the first unsolved route, that is, the problem is restricted to the set of nodes that are visited by the ship leaving the first unexplored factory visit. So the restricted problem has only one ship and a smaller set of nodes to be visited at most once. The end time of the visit to the factory and the load quantity are fixed. Once the restricted problem is solved, the routing variables of that route are fixed and the relaxed problem is solved again, until no further routing problems are left to be solved.

When solving the restricted problem, we may have farm nodes that need to be supplied externally. To select those nodes to be supplied by the factory we first solve the subproblem by maximizing the external feed cost, that is, by identifying those farms whose penalty for external supply is higher. Then, the subproblem is solved again, but restricted to the set of farms served by the factory, identified previously, in order to minimize the transportation cost and the penalty cost for low stock levels. If those nodes are distant from the factory (they belong only to one of the two fish farms set), we fix the corresponding u-variable to zero. If the nodes belong to both sets, that is, if the nodes are close to the factory, and there are more routings to be explored we simply ignore then and leaves them open to the next iteration.

Algorithm 2 A constructive matheuristic.

Consider the RA formulation tighten with an approximate MCF formulation
Relax the routing variables and solve the branch-and-cut for α seconds
Set X to the best solution obtained
for route $r \in \mathcal{R}$ **do**
    Fix the time ship leaves the factory and the load quantities to the values given in X
    Solve the restricted problem with a single ship and the set of nodes in route r for β seconds
    if the subproblem is infeasible or no solution is found **then**
        Solve the restricted problem with selection on the visited nodes to maximize the external feed cost
        Solve the subproblem restricted to the selected nodes
        Fix u_i variables to zero for the far distant farms not visited
    end if
    Fix the routing variables for route r
    Relax the remaining routing variables (for routes $k \in \mathcal{R}, k > r$) and solve the branch-and-cut for α seconds
    Update solution X
end for
Fix the routing variables and free the load/unload variables
Solve the model in order to maximize the quantity delivered

Both algorithms include a final step where the routing variables are fixed and the amount to be delivered is maximized. This step aims to provide better solutions from the practical point of view by minimizing the EOH effect.

6 Computational results

In this section we report the computational results obtained on 9 instances. The formulations described were written in Mosel and implemented in Xpress-IVE Version 1.24.00, 64 bit. We have run all tests on a computer with 16GB RAM and an Intel(R) Core i7 CPU using the Xpress Optimizer Version 25.01.05. A preliminary study that supports some of our modeling choices can be found in [19].

6.1 Test Cases

We consider two homogeneous ships, one factory, and three test cases with 20, 40 and 60 fish farms. We use realistic data provided by Marine Harvest for Region North, Mid and West during high season. For the test cases with 20 and 40 fish farms, the production rate and the storage capacity of the factory and ships are downscaled to one-third and two-thirds, respectively. The test case with 60 fish farms use full-scaled values and includes all fish farms from Region North, Mid and West that have fish during the period from which the demand rates were calculated.

The production rate is an estimation given by Marine Harvest, since the factory was not yet operating during the project period. The rate will vary throughout the year, but kept constant at an estimated maximum level of 45 tons per hour during high season. Maximum storage capacity of the ships, factory and fish farms are known. The ships and the factory have an equal capacity of 3000 tons. The hourly consumption rates for fish farms are calculated from historical monthly data.

The following table summarizes the hourly consumption rates and maximum storage capacities of fish farms.

Number of fish farms + factory		21	41	61
Consumption rates	Minimum	0.18	0.06	0.03
	Maximum	1.56	1.77	1.77
	Average	0.81	0.48	0.75
Storage capacity	Minimum	4	2	1
	Maximum	38	43	43
	Average	20	20	18

Table 1 Statistics concerning hourly consumption rates and maximum storage capacities of fish farms.

The safety stock level is set to one feed day for each fish farm. The service hours at fish farms are from 8:00 a.m. to 4:00 p.m. every day. Outside working hours, storage capacity is reduced with 10% for all fish farms. Visits to the factory and fish farms should be separated by 24 hours. This enables us to calculate minimum unloading quantities for fish farms by finding the amount of feed consumed during this period of time. The loading rate is 270 tons per hour, while the unloading rate is 180 tons per hour.

Travel distances between all locations are estimations provided by the company and travel times are calculated using a constant speed of 13 knots. The travel times range between 1 and 43 hours. The transportation cost is 1600 NOK per hour. The cost is based on LNG consumption per hour and is dependent on LNG prices. The cost of external supply is divided into two parts, a fixed transportation cost of 4400 NOK for each fish farm and a profit margin of 250 NOK per ton of feed. The external transportation cost is parameterized relatively to the internal transportation cost, while the profit margin is an estimation based on the company current cost of standard feed. The penalty cost per hour of being below the safety stock level is parameterized to make cost-effective plans, while avoiding low stock levels. It has been set to 400 NOK per hour and is equal for all fish farms.

The planning period was set to 10 days. This allows to plan for the coming week, while also accounting for three more days of production and consumption. For each test case the different scenarios for the initial stock levels have been considered, which will be denoted by A, B and C. In the first (A), fish farms have initial stock levels of six feed days, in the second (B) and third (C) scenarios the initial stock level is randomly generated from four to eight and from two to ten feed days, respectively. These instances are denoted by n_x where n indicates the number of nodes (factory and farms) and x is the letter A, B, or C identifying the initial stock level. The initial stock level at the factory is set to half capacity. Both ships are assumed to start at the factory and are initially empty.

The average size of the three test cases for the AF and MCF formulations are given in Tables 2 and 3, respectively.

Number of fish farms + factory		21	41	61
# of variables	Before presolve	4130	4302	8464
	After presolve	3946	4011	8092
# of constraints	Before presolve	17170	50384	107436
	After presolve	5170	5111	10078

Table 2 Average size for the AF formulation.

Number of fish farms + factory		21	41	61
# of variables	Before presolve	38698	44300	138596
	After presolve	36946	39754	135215
# of constraints	Before presolve	41977	49685	147557
	After presolve	38773	44301	138695

Table 3 Average size for the MCF formulation.

6.2 Computational experiments and results

In order to solve the instances we first tested the AF model. Preliminary results showed that best results were obtained by defining σ_{imd} as SOS1. However, clique inequalities were ineffective because many of them would be included and the branch-and-cut gets much slower at each branching node. So we discard them. SECs inequalities (47) were included and the timing constraints were strengthen with the time windows. Then we have tested the resulting improved formulation which was run for 5 hours for each instance.

In order to test the lower bound provided by the linear relaxation of the AF formulation, we compare it with the one obtained at the end of the running time limit of 5 hours; the one resulting from the linear relaxation of the MCF formulation; and the one obtained with the RA formulation strengthen with the MFC formulation and with the routing variables relaxed at the root node of the branch-and-cut. The corresponding lower bounds (columns LB) and integrality gaps (columns Gap), are given in Table 4 where the columns identify the corresponding formulations AF, AF(5h), MCF, and RA. The last column BFS gives the value of the best feasible solution known for each instance. All the results reported are values obtained by the solver to solve the problem by branch-and-cut, so they include the preprocessing step with variable and constraints reduction. We can see that the integrality gaps are quite large.

Table 4 Lower bounds and integrality gaps for the tested instances.

	AF		AF(5h)		MCF		RA		
Instance	LB	Gap	LB	Gap	LB	Gap	LB	Gap	BFS
21_A	107.1	23.9	124.3	11.7	108.5	22.7	108.8	22.7	140.7
21_B	108.3	42.6	118.0	37.4	109.9	42.4	108.6	42.4	188.6
21_C	108.4	75.1	137.4	68.5	113.5	72.4	120.4	72.4	436.1
41_A	133.3	41.7	152.9	33.1	136.8	40.1	135.7	40.1	228.6
41_B	138.7	60.1	147.7	57.5	141.1	39.5	244.7	29.5	347.2
41_C	136.0	84.3	256.4	70.4	352.6	84.4	135.5	84.4	866.6
61_A	132.3	41.8	167.5	26.3	157.8	29.7	159.9	29.7	227.4
61_B	132.2	84.4	164.6	80.6	157.8	81.6	156.6	81.6	849.2
61_C	132.0	83.4	159.8	79.8	159.8	79.8	160.0	79.8	792.9

For the small size instance cases with 21 nodes the improvement in the lower bound by using reformulation techniques is not as good as for the remaining cases. A possible explanation is due to the fact that the reformulations used are based on classical vehicle routing problems which are more similar to the larger sets of instances where each farm node is visited only once.

The objective function values obtained from the two different approaches are presented in Table 5. The second column reports the best solution objective function value obtained after a run limited to five hours using the AF formulation. An asterisk means no feasible solution was found. Columns *Time BFS*, *# FS*, and *# Nodes* give the time at which the best value was found, the number of feasible solutions found, and the number of branch and bound nodes, respectively. The last two columns *Value* and *Time BFS* report the objective function and running time of the heuristic procedure. This procedure run the same AF formulation for one hour. If a feasible solution was found we apply Algorithm 1 with $\beta = 600$ seconds, otherwise we apply Algorithm 2 with $\alpha = 500$ and $\beta = 1500$ seconds. The objective function value of the solutions obtained by Algorithm 2 are much higher than the values of the other solutions. Also, the integrality gaps are large. This happens because these solutions include several farms that must be supplied externally which increases the cost greatly.

Table 5 Objective function values for the branch-and-cut and the Matheuristic approach.

	Branch-and-cut				Heuristic Approach	
Instance	Value	Time BFS	# FS	# Nodes	Value	Time
21_A	141.7	597	10	267000	140.7	3602
21_B	196.6	2018	12	313000	188.6	3605
21_C	*	*	0	88000	436.1	5733
41_A	240.1	5273	27	365100	228.6	4202
41_B	386.8	17143	30	313200	394.5	4207
41_C	*	*	0	318000	866.6	8526
61_A	228.4	3023	31	290800	227.4	4421
61_B	*	*	0	323000	849.2	8913
61_C	*	*	0	186000	792.9	13656

The results show that when the initial inventory levels measured in number of feed days vary the instances become harder. That could be justified by the fact that time aspects become more relevant. As inequalities (56) are weak, the fractional solutions have, in general, feasible time assignments for all the farm visits. Thus, in those fractional solutions there is no need for external feed supply and the corresponding variables are set to zero.

In Table 6, we provide the value of the branch-and-cut after 5 hours, 1 hour and 30 minutes represented by columns AF(5h), AF(1h), AF(30min) for those 5 instances for which the branch-and-bound could identify a feasible solution. Columns AF(5h)+I., AF(1h)+I. and AF(30min) + I. give the value obtained by applying the improving matheuristic to the corresponding solution. The asterisk means that no feasible solution was found within the given time limit.

Table 6 Objective function values for different run times of the branch-and-cut with and without routing improvements.

Instance	AF(5h)	AF(5h)+I.	AF(1h)	AF(1h)+I.	AF(0.5h)	AF(0.5h) + I.
21_A	141.7	140.7	141.7	140.7	141.7	140.7
21_B	196.6	188.6	196.6	188.6	196.6	188.6
41_A	240.1	230.2	243.3	228.6	252.3	241.1
41_B	386.8	347.2	442.1	394.5	*	*
61_A	228.4	227.4	228.4	227.4	234.6	234.1

Finally in Table 7, we present details of the best feasible solution obtained for each instance. The objective function value given in column BFS is decomposed into its components: transportation cost (TC), external transportation cost (ETC), external feed cost (EFC), and penalty cost (PC) in the following four columns. Columns Ship 1 and Ship 2 give the number of loads at the production factory (columns *loads*) and list the number of unloads at the fish farms in each route (columns *unloads*) by ship 1 and ship 2, respectively. In the list, unloads of zero means that the ship has returned to the production factory to load in order to ensure the farm storing capacity is obeyed.

Table 7 Characterization of the best feasible solution found for each instance.

						Ship 1		Ship 2	
Inst.	BFS	TC	ETC	EFC	PC	loads	unloads	loads	unloads
21_A	140.7	139	0	0	1.7	3	[10, 1, 0]	1	[10]
21_B	188.6	173	0	0	15.6	3	[5, 6, 2]	1	[10]
21_C	436.1	172	22.1	242	0	3	[7, 1, 1]	2	[6, 0]
41_A	228.6	210	0	0	18.6	2	[8, 9]	3	[2, 21, 0]
41_B	347.2	255	4.3	56.1	31.7	2	[10, 10]	3	[12, 6, 0]
41_C	866.6	235	34.8	595.2	1.6	3	[14, 1, 0]	2	[10, 7]
61_A	227.4	186	0	0	41.4	3	[24, 8, 0]	1	[28]
61_B	849.2	223	59.8	562.7	3.7	3	[19, 1, 0]	2	[20, 6]
61_C	792.9	256	34.1	450.2	52.5	3	[15, 7, 0]	2	[19, 11]

7 Concluding remarks

Norway's largest salmon farmer is involved in the value chain from egg production to sales of finished salmon products. Recently, they have been building a factory for feed production and bought two specialized ships to take the responsibility for the feed production and feed delivery as well. This allows for implementation of vendor managed inventory (VMI) as opposed to the order based feed delivery common in the industry.

We have developed a mathematical model and solution approach for a combined ship routing and inventory management problem including several characteristics from the salmon farming industry. Compared to most existing maritime inventory routing problems described in the literature, the problem studied here includes many fish farms compared to production factories resulting in relatively long routes. The time windows derived from the network, loading quantity- and inventory limits are wide. Another complicating characteristic is that the company has not sufficient production and ship capacity to supply all fish farms internally, so some of the farms have to be supplied externally. Finally, the storages at the fish farms cannot be fully loaded if the farm is visited outside working hours. Our study shows that the resulting problem is very complex to solve for large real instances.

The mathematical formulation clearly defines the problem, and the process of developing the model has been important for the company and the project team in order to understand the complex issues of the problem. The mathematical model is strengthened by tightening the timing constraints and including subtour elimination constraints. To further tighten the model, extended formulations, such as a Multi-Commodity Flow formulation and a Route-Assignment formulation, are developed. The mathematical models and the extended formulations have in addition been the basis for the development of matheuristics to solve larger instances of the problem. The matheuristics are based on improving an existing solution (or building a feasible solution if no solution is found) by using the mathematical formulations with partially fixing variables in an iterative procedure.

The computational results show that feasible solutions can be found so the company has the possibility of introducing VMI with their new plant and two ships. The improved mathematical formulation can find feasible solutions for most of the instances, but the first matheuristic can find better solutions in a limited number of time. The second matheuristic that builds a solution in case none is found by the mathematical model is useful for some of the instances. This heuristic constructs a solution by solving smaller routing subproblems in sequence. This approach can be easily extended to other maritime inventory routing problems. However, the quality of the heuristic could not be properly evaluated given the poor quality of the lower bounds derived.

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