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UNIQUENESS IN THE INVERSE CONDUCTIVITY PROBLEM FOR COMPLEX-VALUED LIPSCHITZ CONDUCTIVITIES IN THE PLANE*

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Abstract. We consider the inverse impedance tomography problem in the plane. Using Bukhgeim's scattering data for the Dirac problem, we prove that the conductivity is uniquely determined by the Dirichlet-to-Neuman map.

Key words. Bukhgeim's scattering problem, inverse Dirac problem, inverse conductivity problem, complex conductivity

AMS subject classifications. 35R30, 78A46

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1. Introduction.

Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^2$. The electrical impedance tomography problem (e.g., [6]) concerns determining the impedance in the interior of $\mathcal{O}$, given simultaneous measurements of direct or alternating electric currents and voltages at the boundary $\partial\mathcal{O}$. If the magnetic permeability can be neglected, then the problem can be reduced to the inverse conductivity problem (ICP), i.e., to the problem of reconstructing function $\gamma(z), z = (x, y) \in \mathcal{O}$, from the set of data $(u|_{\partial\mathcal{O}}, \gamma\frac{\partial u}{\partial\nu}|_{\partial\mathcal{O}})$, dense in an adequate topology, where

$$\operatorname{div}(\gamma\nabla u(z)) = 0, \ z \in \mathcal{O}. \tag{1}$$

Here ν is the unit outward normal to $\partial\mathcal{O}$, $\gamma(z) = \sigma(z)+i\omega\epsilon(z)$, where σ is the electric conductivity and ϵ is the electric permittivity. If the frequency ω is negligibly small, then one can assume that γ is a real-valued function; otherwise it is supposed to be a complex-valued function.

An extensive list of references on the tomography problem can be found in the review [6]. Here we will mention only the papers that seem to be particularly related to the present work.

For real γ, the inverse conductivity problem has been reduced to the inverse problem for the Schrödinger equation. The latter was solved by Nachman in [14] in the class of twice differentiable conductivities. Later, Brown and Uhlmann [7] reduced the ICP to the inverse problem for the Dirac equation, which has been solved in [4], [15]. This approach requires the existence of only one derivative of γ. The authors of [7] proved the uniqueness for the ICP. Later, Knudsen and Tamasan [11] extended

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this approach and obtained a method to reconstruct the conductivity. Finally, the ICP has been solved by Astala and Paivarinta in [3] for real conductivities when both $\gamma - 1$ and $1/\gamma - 1$ are in $L^{\infty}_{\text{comp}}(\mathbb{R}^2)$.

If a complex conductivity has at least two derivatives, then one can reduce (1) to the Schrödinger equation and apply the method of Bukhgeim [8] (or some of the works extending this method, such as [2], [5], or [16]). This approach does not work in the case of only one time differentiable complex valued conductivities. On the other hand, the work of Francini [10], where the ideas of [7] were extended to deal with complex conductivities with small imaginary part, are not applicable to general complex conductivities due to possible existence of the so-called exceptional points. In [13], Lakstanov and Vainberg extended the ideas of [12] to apply the $\overline{\partial}$-method in the presence of exceptional points and reconstructed generic conductivities under the assumption that $\gamma - 1 \in W^{1,p}_{\text{comp}}(\mathbb{R}^2), p > 4$, and $\mathcal{F}(\nabla\gamma) \in L^{2-\varepsilon}(\mathbb{R}^2)$. (Here $\mathcal{F}$ is the Fourier transform.)

In this paper, we will prove that complex-valued Lipschitz conductivities are uniquely determined by information on the boundary. Since we use the standard reduction of (1) to the Dirac equation followed by the solution of the inverse problem for the Dirac equation, the condition on γ can be restated in the form $Q \in L^{\infty}_{comp}(\mathbb{R}^2)$, where Q is the potential in the Dirac equation. Our present result is based on a development of the Bukhgeim approach, combined with some of the arguments of Brown and Uhlmann from [7]. The statement of our main theorem is the following.

THEOREM 1.1. *Let $\mathcal{O}$ be a bounded Lipschitz domain in the plane and let γ_1, γ_2 be complex-valued Lipschitz conductivities. Then*

$$\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2,$$

where Λ_{γ_j} is the Dirichlet-to-Neumann (DtN) map for the conductivity γ_j.

The DtN map $\Lambda_\gamma : H^{1/2}(\partial\mathcal{O}) \to H^{-1/2}(\partial\mathcal{O})$ is defined by

$$\Lambda_\gamma[u|_{\partial\mathcal{O}}] = \gamma \frac{\partial u}{\partial \nu}\Big|_{\partial\mathcal{O}},$$

where u is a solution to (1) and $\frac{\partial u}{\partial \nu}$ is the normal derivative of u at the boundary of $\mathcal{O}$. Function $\gamma\frac{\partial u}{\partial \nu} \in H^{-1/2}(\partial\mathcal{O})$ is defined as such an element of the space dual to $H^{1/2}(\partial\mathcal{O})$ that

$$\left\langle \gamma \frac{\partial u}{\partial \nu}, v \right\rangle = \int_{\mathcal{O}} \gamma \nabla u \cdot \nabla v dx dy$$

for each $v \in H^1(\mathcal{O})$.

In section 2, we will describe our approach, stating the most relevant results. All the proofs will be given in section 3.

2. Main steps.

2.1. Reduction to the Dirac equation. From now on, we will consider z as a point of a complex plane, $z = x + iy \in \mathbb{C}$, and $\mathcal{O}$ will be considered as a domain in $\mathbb{C}$. The following observation made in [7] plays an important role. Let u be a solution of (1) and let $\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$. Then the pair $\phi = \gamma^{1/2}(\partial u, \overline{\partial} u)^t$ satisfies the Dirac equation

$$\begin{pmatrix} \overline{\partial} & 0 \\ 0 & \partial \end{pmatrix} \phi = q\phi, \quad z \in \mathcal{O}, \tag{2}$$

where

$$q(z) = \begin{pmatrix} 0 & q_{12}(z) \\ q_{21}(z) & 0 \end{pmatrix}, \quad q_{12} = -\frac{1}{2}\partial \log \gamma, \quad q_{21} = -\frac{1}{2}\overline{\partial} \log \gamma. \tag{3}$$

Thus the inverse Dirac scattering problem is closely related to the ICP. If q is found and the conductivity γ is known at one point $z_0 \in \overline{\mathcal{O}}$, then γ in $\mathcal{O}$ can be immediately found from (3).

From now on, we will use a different form of (2): instead of Beals–Coifmann notation $\phi = (\phi_1, \phi_2)^t$, we will rewrite the equation in Sung notation: $\psi_1 = \phi_1, \psi_2 = \overline{\phi_2}$. We will consider the equation in the whole plane by extending the potential q outside $\mathcal{O}$ by zero. Then the *vector* $\psi = (\psi_1, \psi_2)^t$ is a solution of the following system:

$$\overline{\partial}\psi = Q\overline{\psi}, \quad z \in \mathbb{C}, \tag{4}$$

where

$$Q(z) = \begin{pmatrix} 0 & Q_{12}(z) \\ Q_{21}(z) & 0 \end{pmatrix}, \quad Q_{12} = q_{12}, \quad Q_{21} = \overline{q_{21}}. \tag{5}$$

2.2. Solving the Dirac equation for large $|\boldsymbol{\lambda}|$. Let ψ be a *matrix* solution of (4) that depends on parameter $\lambda \in \mathbb{C}$ and has the following behavior at infinity:

$$\psi(z, w, \lambda)e^{-\lambda(z-w)^2/4} \to I, \ z \to \infty. \tag{6}$$

Note that the unperturbed wave

$$\varphi_0(z, \lambda, w) := e^{\lambda(z-w)^2/4}, \quad w, \lambda \in \mathbb{C}, \tag{7}$$

depends on the spacial parameter w and the spectral parameter λ and grows at infinity exponentially in some directions. The same is true for the elements of the matrix $\psi(z, \lambda, w)$. Let us stress that, contrary to the standard practice, we consider function ψ (and other functions defined by ψ) for all complex values of λ, not just for $i\lambda, \ \lambda > 0$. This allows us to generalize the Bukhgeim method to the case of potentials in $L^\infty_{com}(\mathbb{R}^2)$. From the technical point of view, this allows us to use the Hausdorff–Young inequality.

Problem (4)–(6) can be rewritten using a bounded function

$$\mu(z, w, \lambda) := \psi(z, w, \lambda)e^{-\lambda(z-w)^2/4}, \tag{8}$$

i.e., (4)–(6) is equivalent to

$$\overline{\partial}\mu(z, w, \lambda) = Q\overline{\mu}e^{[\overline{\lambda(z-w)^2} - \lambda(z-w)^2]/4}, \quad z \in \mathbb{C}; \quad \mu \to I, \ z \to \infty. \tag{9}$$

Using the fact that $\overline{\partial}\frac{1}{\pi z} = \delta(0)$, (9) can be reduced to the Lippmann–Schwinger equation

$$\mu(z, \lambda, w) = I + \frac{1}{\pi}\int_{\mathbb{C}} Q(z')\frac{e^{-i\Im[\lambda(z'-w)^2]/2}}{z - z'}\overline{\mu}(z', \lambda, w)\, d\sigma_{z'}, \tag{10}$$

where $d\sigma_{z'} = dx'dy'$ and $\mu \to I$ as $z \to \infty$.

Denote

$$\mathcal{L}_\lambda\varphi(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{e^{-i\Im[\lambda(z'-w)^2]/2}}{z - z'}\varphi(z')\, d\sigma_{z'}. \tag{11}$$

Then (10) implies that

$$\mu = I + \mathcal{L}_\lambda Q(I + \overline{\mathcal{L}_\lambda Q}\mu). \tag{12}$$

In particular, for the component μ_{11} of the matrix μ, we have $\mu_{11} = 1 + M\mu_{11}$ with $M = \mathcal{L}_\lambda Q_{12}\overline{\mathcal{L}_\lambda Q_{21}}$, leading to

$$(I - M)(\mu_{11} - 1) = M1. \tag{13}$$

By inverting $I - M$, we can obtain μ_{11}. Other components of μ can be found similarly.

Denote by $L^\infty_{z,w}(B)$ the space of bounded functions of $z, w \in \mathbb{C}$ with values in a Banach space B. The following two lemmas show that M is a contractive operator in the space $L^\infty_{z,w}(L^p_\lambda(\lambda : |\lambda| > R))$ if R is large enough and that $M1$ also belongs to this space. After these lemmas are proved, one can find the solution μ of (10) (using, for example, the Neumann series for the inversion of $I - M$). Then formula (8) provides the solution ψ of (4)–(6).

Lemma 2.1. *Let $p > 2$. Then*

$$\lim_{R\to\infty} \|M\|_{L^\infty_{z,w}(L^p_\lambda(\lambda:|\lambda|>R))} = 0.$$

Lemma 2.2. *Let $p > 2$. Then there exists $R > 0$ such that*

$$M1 \in L^\infty_{z,w}(L^p_\lambda(\lambda : |\lambda| > R)).$$

Note that (13) together with Lemmas 2.1 and 2.2 allows one to solve the direct but not the inverse problem, since operator M depends on Q. The following inclusion is an immediate consequence of (13) and Lemmas 2.1 and 2.2:

$$\mu_{11} - 1 \in L^\infty_{z,w}(L^p_\lambda(\lambda : |\lambda| > R)), \quad p > 2, \tag{14}$$

for large enough R.

2.3. Determination of the potential. Let the matrix h be the *(generalized) scattering data*, given by the formula

$$h(\lambda, w) = \int_{\mathbb{C}} e^{-i\Im[\lambda(z-w)^2]/2} Q(z)\overline{\mu}(z, \lambda, w)\, d\sigma_z. \tag{15}$$

One can use Green's formula

$$\int_{\partial\mathcal{O}} f\, dz = 2i \int_{\mathcal{O}} \overline{\partial} f\, d\sigma_z$$

to rewrite h as

$$h(\lambda, w) = \frac{1}{2i} \int_{\partial\mathcal{O}} \mu(z, \lambda, w)\, dz. \tag{16}$$

Thus, one does not need to know the potential Q in order to find h. Function h can be evaluated if the Dirichlet data $\psi|_{\partial\mathcal{O}}$ is known for (4), since $\mu|_{\partial\mathcal{O}}$ in (16) can be expressed via $\psi|_{\partial\mathcal{O}}$ using (8).

The spectral parameter $i\lambda$ with real λ was used in the standard approach to recover the potential from scattering data (15), and the potential was recovered by the limit of the scattering data as $\lambda \to \infty$. Instead, in the present work, we have $\lambda \in \mathbb{C}$, and the potential is determined by integrating the scattering data over a large annulus in the complex λ-plane.

Let T^λ be the operator defined by

$$T^\lambda[G] = \int_{\mathcal{O}} e^{-i\Im[\lambda(z-w)^2]/2} Q(z)G(z)\, d\sigma_z, \tag{17}$$

where G can be a matrix- or scalar-valued function. Then

$$h(\lambda, w) = T^\lambda[\mu] = T^\lambda[I] + T^\lambda[\mu - I]. \tag{18}$$

We will show that the following statement is valid.

THEOREM 2.3. *Let Q be a complex-valued bounded potential. Then*

$$\sup_{w\in\mathcal{O}} \left| \int_{R<|\lambda|<2R} |\lambda|^{-1}\, T^\lambda[\mu - I]\, d\sigma_\lambda \right| \to 0, \quad \text{as} \quad R \to \infty, \tag{19}$$

and

$$\int_{\mathcal{O}} g(w) \int_{R<|\lambda|<2R} |\lambda|^{-1} T^\lambda[I]\, d\sigma_\lambda\, d\sigma_w \to 4\pi^2 \ln 2 \int_{\mathcal{O}} g(z)Q(z)\, d\sigma_z, \quad \text{as } R \to \infty, \tag{20}$$

for every smooth g with a compact support in $\mathcal{O}$. Thus

$$\int_{\mathcal{O}} g(z)Q(z) d\sigma_z = \frac{1}{4\pi^2 \ln 2} \lim_{R\to\infty} \int_{R<|\lambda|<2R} |\lambda|^{-1} \int_{\mathbb{C}} g(w)h(\lambda, w)\, d\sigma_w d\sigma_\lambda.$$

Therefore, if the scattering data is uniquely determined by the DtN map, then so is the potential Q.

In order to prove (19), we use the two lemmas stated below and (13) rewritten as follows:

$$\mu_{11} - 1 = M(\mu_{11} - 1) + M1. \tag{21}$$

(Other entries of the matrix $\mu - I$ can be handled in a similar way.) Relation (20) follows from the stationary phase approximation.

LEMMA 2.4. *Let $p > 1$. Then there exists $R > 0$ such that*

$$T^\lambda M1 \in L^\infty_w(L^p_\lambda(\lambda : |\lambda| > R)).$$

LEMMA 2.5. *Let $p > 1$. Then there exists $R > 0$ such that*

$$T^\lambda M(\mu_{11} - 1) \in L^\infty_w(L^p_\lambda(\lambda : |\lambda| > R)).$$

3. Proofs. In order to make the calculations more compact, we introduce the following notation for the L^p-space on the complement of the ball:

$$L^p_{|\lambda|>R} = L^p_\lambda(\lambda : |\lambda| > R).$$

We will also use the real-valued function

$$\rho_{\lambda,w}(z) = \Im\left[\lambda(z - w)^2\right]/2,$$

where the dependence on λ and w will be omitted in some cases.

3.1. Preliminary results.

LEMMA 3.1. *Let $1 \le p < 2$. Then the following estimate is valid for an arbitrary $0 \ne a \in \mathbb{C}$ and some constants $C = C(p, R)$ and $\delta = \delta(p) > 0$:*

$$\left\| \frac{1}{u(\sqrt{u} - a)} \right\|_{L^p(u \in \mathbb{C}: |u| < R)} \le C \left(1 + |a|^{-1+\delta}\right).$$

Remark. A more accurate estimate will be proved below with $\delta = \frac{4}{p} - 2$ if $1 \le p < 4/3$ and with the right-hand side replaced by $C(1 + |\ln |a||^{1/p})$ when $p = 4/3$ or by a constant when $4/3 < p < 2$.

Proof. The statement is obvious if $|a| \ge 1$. If $|a| < 1$, then the left-hand side L in the inequality above takes the following form after the substitution $u = |a|^2 v$:

$$L = |a|^{\frac{4}{p}-3} \left\| \frac{1}{v(\sqrt{v} - \dot{a})} \right\|_{L^p(v \in \mathbb{C}: |v| < R/|a|^2)}, \quad \dot{a} = a/|a|. \tag{22}$$

Without loss of the generality, one can assume that $R > 2$. We split the function $f := \frac{1}{v(\sqrt{v} - \dot{a})}$ into two terms $f_1 + f_2$ obtained by multiplying f by α and $1 - \alpha$, respectively, where α is the indicator function of the disk of radius two. The norm of f_1 can be estimated from above by an a-independent constant. The second function can be estimated from above by $\frac{2}{|v|^{3/2}}$. The norm of the latter function can be easily evaluated, and it does not exceed a constant if $p > 4/3$. It does not exceed $C(1 + |\ln |a||^{1/p})$ if $p = 4/3$, and it does not exceed $C|a|^{3-\frac{4}{p}}$ if $p < 4/3$. Since $\|f_1\| \le C\|f_2\|$, we can replace f in (22) by Cf_2, and this implies the statement of the lemma. ∎

LEMMA 3.2. *Let $z_1, w \in \mathbb{C}$, $p > 2$, and $\varphi \in L^\infty_{\mathrm{comp}}$. Then*

$$\left\| \int_{\mathbb{C}} \varphi(z) \frac{e^{i\rho_{\lambda,w}(z)}}{z - z_1} \, d\sigma_z \right\|_{L^p_\lambda(\mathbb{C})} \le C \frac{\|\varphi\|_{L^\infty}}{|z_1 - w|^{1-\delta}},$$

where constant C depends only on the support of φ and on $\delta = \delta(p) > 0$.

Proof. Denote by $F = F(\lambda, w, z_1)$ the integral in the left-hand side of the inequality above. We change variables $u = (z - w)^2$ in F and take into account that $d\sigma_u = 4|z - w|^2 d\sigma_z$. Then

$$F = \frac{1}{4} \sum_{\pm} \int_{\mathbb{C}} \varphi(w \pm \sqrt{u}) \frac{e^{i\Im(\lambda u)/2}}{|u|(\pm\sqrt{u} - (z_1 - w))} \, d\sigma_u.$$

Using the Hausdorff–Young inequality with $p' = p/(p-1)$ and Lemma 3.1, we obtain that

$$\|F\|_{L^p_\lambda} \le \frac{1}{2} \sum_{\pm} \left\| \frac{\varphi(w \pm \sqrt{u})}{|u|(\pm\sqrt{u} - (z_1 - w))} \right\|_{L^{p'}_u} \le C \frac{\|\varphi\|_{L^\infty}}{|z_1 - w|^{1-\delta}}.$$

∎

3.2. Proof of Lemma 2.1. Let

$$A(z, z_2, \lambda, w) = \pi^{-2} \int_{\mathcal{O}} \frac{e^{-i\rho_{\lambda,w}(z_1)}}{z - z_1} Q_{12}(z_1) \frac{e^{i\rho_{\lambda,w}(z_2)}}{\overline{z_1} - \overline{z_2}} \overline{Q}_{21}(z_2) \, d\sigma_{z_1}, \tag{23}$$

so that

$$Mg(z) = \int_{\mathcal{O}} A(z, z_2, \lambda, w) g(z_2) \, d\sigma_{z_2}.$$

Then, from the Minkowski's integral inequality, we have

$$\begin{aligned}\|Mg(z,\cdot)\|_{L^p_{|\lambda|>R}} &\le \int_{\mathcal{O}} \|A(z,z_2,\lambda,w)g(z_2,\cdot)\|_{L^p_{|\lambda|>R}}\, d\sigma_{z_2}\\ &\le \int_{\mathcal{O}} \sup_{\lambda:|\lambda|>R} |A(z,z_2,\lambda,w)|\, d\sigma_{z_2} \sup_{z_2} \|g(z_2,\cdot)\|_{L^p_{|\lambda|>R}}.\end{aligned}$$

Thus it remains to show that, uniformly in $z \in \mathbb{C}$ and $w \in \mathcal{O}$, we have

$$\int_{\mathcal{O}} |A(z,z_2,\lambda,w)|\, d\sigma_{z_2} \to 0 \quad \text{as} \quad |\lambda| \to \infty.$$

Let A^s be given by (23) with the extra factor $\alpha(s|z-z_1|)\alpha(s|z_1-z_2|))$ in the integrand, where $\alpha \in C^\infty$, $\alpha = 1$, outside of a neighborhood of the origin, and α vanishes in a smaller neighborhood of the origin. Since

$$\int_{B_1(0)} \int_{B_1(0)} \frac{1}{|z_1|} \frac{1}{|z_1-z_2|}\, d\sigma_{z_1}\, d\sigma_{z_2} < \infty,$$

for each ε there exists $s = s_0(\varepsilon)$ such that

$$\int_{\mathcal{O}} |A - A^{s_0}|\, d\sigma_{z_2} < \varepsilon$$

for all the values of z, w, λ. Denote by $A^{s_0,n}$ the function A^{s_0} with potentials Q_{12}, Q_{21} replaced by their L_1-approximations $Q_{12}^n, Q_{21}^n \in C_0^\infty$. Since the other factors in the integrand of A^{s_0} are bounded (they are infinitely smooth), we can choose these approximations in such a way that

$$\int_{\mathcal{O}} |A^{s_0} - A^{s_0,n}|\, d\sigma_{z_2} < \varepsilon$$

for all the values of z, w, λ. Now it is enough to show that

$$|A^{s_0,n}(z,z_2,\lambda,w)| \to 0 \quad \text{as} \quad |\lambda| \to \infty$$

uniformly in z, z_2, w. The latter relation follows immediately from the stationary phase method, since the amplitude function in the integral $A^{s_0,n}$ and all the derivatives in z_1 of the amplitude function are uniformly bounded with respect to all the arguments.

3.3. Proof of Lemma 2.2. Recall that

$$M1 = \pi^{-2} \int_{\mathcal{O}} \int_{\mathcal{O}} \frac{e^{-i\rho_\lambda(z_1)}}{z-z_1} Q_{12}(z_1) \frac{e^{i\rho_\lambda(z_2)}}{\overline{z_1}-\overline{z_2}} \overline{Q_{21}}(z_2)\, d\sigma_{z_2}\, d\sigma_{z_1}.$$

Let C be a constant that may depend on $\|Q\|_{L^\infty}$ and $\mathcal{O}$. Then, by Minkowski's integral inequality and Lemma 3.2, we have

$$\begin{aligned}\|M1\|_{L^p_{|\lambda|>R}} &\le \int_{\mathcal{O}} \left\| \frac{e^{-i\rho_\lambda(z_1)}}{z-z_1} Q_{12}(z_1) \int_{\mathcal{O}} \frac{e^{i\rho_\lambda(z_2)}}{\overline{z_1}-\overline{z_2}} \overline{Q_{21}}(z_2)\, d\sigma_{z_2} \right\|_{L^p_{|\lambda|>R}} d\sigma_{z_1}\\ &\le \int_{\mathcal{O}} \left| \frac{Q_{12}(z_1)}{z-z_1} \right| \left\| \int_{\mathcal{O}} \frac{e^{i\rho_\lambda(z_2)}}{\overline{z_1}-\overline{z_2}} \overline{Q_{21}}(z_2)\, d\sigma_{z_2} \right\|_{L^p_{|\lambda|>R}} d\sigma_{z_1}\\ &\le C \int_{\mathcal{O}} \frac{1}{|z-z_1||z_1-w|^{1-\delta}}\, d\sigma_{z_1} < \infty,\end{aligned}$$

since $\delta > 0$.

3.4. Proof of Lemma 2.4. Let C be a constant that may depend on $\|Q\|_{L^\infty}$ and $\mathcal{O}$. Then, applying successively Minkowski's integral inequality, Holder's inequality, and Lemma 3.2, we see that

$$\begin{aligned}
&\|T^\lambda[M1]\|_{L^p_{|\lambda|>R}} \\
&\quad \le \int_{\mathcal{O}} \left\| \int_{\mathcal{O}} \frac{e^{-i(\rho(z_1)+\rho(z))}}{z-z_1} Q(z)\, d\sigma_z \int_{\mathcal{O}} \frac{e^{i\rho(z_2)}}{\overline{z_1}-\overline{z_2}} \overline{Q_{21}}(z_2)\, d\sigma_{z_2} \right\|_{L^p_{|\lambda|>R}} |Q_{12}(z_1)| d\sigma_{z_1} \\
&\quad \le C \int_{\mathcal{O}} \left\| \int_{\mathcal{O}} \frac{e^{-i\rho(z)}}{z-z_1} Q(z)\, d\sigma_z \right\|_{L^{2p}_{|\lambda|>R}} \left\| \int_{\mathcal{O}} \frac{e^{i\rho(z_2)}}{\overline{z_1}-\overline{z_2}} \overline{Q_{21}}(z_2)\, d\sigma_{z_2} \right\|_{L^{2p}_{|\lambda|>R}} d\sigma_{z_1} \\
&\quad \le C \int_{\mathcal{O}} \frac{1}{|z_1-w|^{1-\delta}} \frac{1}{|z_1-w|^{1-\delta}}\, d\sigma_{z_1} < \infty,
\end{aligned}$$

as $\delta > 0$.

3.5. Proof of Lemma 2.5. Let $f = \mu_{11} - 1$ and let C be a constant that may depend on $\|Q\|_{L^\infty}$ and $\mathcal{O}$. Then the same arguments as in the proof of Lemma 2.4 imply that

$$\begin{aligned}
&\|T^\lambda[Mf]\|_{L^p_{|\lambda|>R}} \\
&\quad \le C \int_{\mathcal{O}} \left\| \int_{\mathcal{O}} \frac{e^{-i\rho(z)}}{z-z_1} Q(z) d\sigma_z \right\|_{L^{2p}_{|\lambda|>R}} \left\| \int_{\mathcal{O}} \frac{e^{i\rho(z_2)}}{\overline{z_1}-\overline{z_2}} \overline{Q_{21}}(z_2) f(z_2) d\sigma_{z_2} \right\|_{L^{2p}_{|\lambda|>R}} d\sigma_{z_1} \\
&\quad \le C \int_{\mathcal{O}} \left\| \int_{\mathcal{O}} \frac{e^{-i\rho(z)}}{z-z_1} Q(z) d\sigma_z \right\|_{L^{2p}_{|\lambda|>R}} \int_{\mathcal{O}} \left| \frac{\overline{Q_{21}}(z_2)}{\overline{z_1}-\overline{z_2}} \right| \|f(z_2)\|_{L^{2p}_{|\lambda|>R}} d\sigma_{z_2} d\sigma_{z_1} \\
&\quad \le C \|f\|_{L^\infty_{z,w}\left(L^{2p}_{|\lambda|>R}\right)} \int_{\mathcal{O}} \frac{1}{|z_1-w|^{1-\delta}}\, d\sigma_{z_1} < \infty,
\end{aligned}$$

since $\delta > 0$ and (14) holds for $f = \mu_{11} - 1$.

3.6. Proof of Theorem 2.3. Let us prove (19). We fix $p \in (1,2)$. From (21) and Lemmas 2.4 and 2.5, it follows that there exists $R > 0$ such that $T^\lambda[\mu_{11} - 1] \in L^\infty_w(L^p_{|\lambda|>R})$. Other entries of matrix $\mu - I$ can be treated similarly, i.e.,

$$T^\lambda[\mu - I] \in L^\infty_w \left(L^p_{|\lambda|>R} \right).$$

Since $q = \frac{p}{p-1} > 2$, Holder's inequality implies that

$$\left| \int_{R<|\lambda|<2R} |\lambda|^{-1}\, T^\lambda[\mu - I]\, d\sigma_\lambda \right| \le \left[\int_{R<|\lambda|<2R} |\lambda|^{-q} d\sigma_\lambda \right]^{\frac{1}{q}} \left\| T^\lambda[\mu - I] \right\|_{L^\infty_w(L^p_{|\lambda|>R})} \to 0$$

as $R \to \infty$. Relation (19) is proved.

The stationary phase approximation implies that

$$\begin{aligned}
\int_{\mathcal{O}} T^\lambda[1] g(w) d\sigma_w &= \int_{\mathcal{O}} \int_{\mathcal{O}} e^{-i\Im[\lambda(z-w)^2]/2} g(w) d\sigma_w Q(z)\, d\sigma_z \\
&= \int_{\mathcal{O}} \left[\frac{2\pi}{|\lambda|} g(z) + O\left(|\lambda|^{\frac{-3}{2}} \right) \right] Q(z) d\sigma_z.
\end{aligned}$$

This immediately justifies (20). The last statement of the theorem follows from (18)–(20).

3.7. Proof of Theorem 1.1. Due to Theorem 2.3, one only needs to show that the scattering data h for $|\lambda| \gg 1$ is uniquely determined by the DtN operator Λ_γ. This will be done by repeating the arguments used in [7, Theorem 4.1] and [10, Theorem 5.1].

Let $\gamma_j, j = 1, 2$, be two Lipshitz conductivities in $\mathcal{O}$ such that $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$. Since γ_j is Lipschitz continuous, it is differentiable almost everywhere, and the derivatives are bounded [9]. Since $\Lambda_{\gamma_1} = \Lambda_{\gamma_2}$ and $\gamma_1, \gamma_2 \in W^{1,\infty}(\mathcal{O})$, we have $\gamma_1|_{\partial\mathcal{O}} = \gamma_2|_{\partial\mathcal{O}}$ (see [1]). We extend γ_j outside $\mathcal{O}$ in such a way that $\gamma_1 = \gamma_2$ in $\mathbb{C}\backslash\mathcal{O}$ and $1-\gamma_j \in W^{1,\infty}_{comp}(\mathbb{C})$. Let $\widetilde{\mathcal{O}}$ be a bounded domain with a smooth boundary that contains supports of functions $1 - \gamma_j$. All the previous results will be used below with $\mathcal{O}$ replaced by $\widetilde{\mathcal{O}}$ and γ extended as described above. Let $Q_j, \psi_j, \mu_j, h_j,\ j = 1, 2$, be the potential and the solution in (4), the function in (8), and the scattering data in (15) associated with the extended conductivity γ_j. Let us note that functions $\psi_j, \mu_j, h_j,\ j = 1, 2$, defined by the conductivity problem in $\widetilde{\mathcal{O}}$ are not extensions of the functions defined by the problem in $\mathcal{O}$.

Due to (16), we have

$$h_j(\lambda, w) = \frac{1}{2i}\int_{\partial\widetilde{\mathcal{O}}} \mu_j(z, \lambda, w)\, dz.$$

Thus it is enough to prove that

$$\mu_1 = \mu_2 \quad \text{on } \partial\widetilde{\mathcal{O}} \quad \text{when } |\lambda| \gg 1. \tag{24}$$

Let $\varphi = (\varphi_1, \varphi_2)^t$ be the first column of ψ_1 and $v = \gamma_1^{-1/2}\varphi_1$, $w = \gamma_1^{-1/2}\overline{\varphi_2}$. Since $\overline{\partial}\varphi = Q_1\overline{\varphi}$, and (2) holds for $\phi^{(1)} = (\varphi_1, \overline{\varphi_2})^t$, it follows that $\overline{\partial}v = \partial w$ in $\mathbb{C}$, and therefore there exists u_1 such that

$$\partial u_1 = v, \quad \overline{\partial}u_1 = w \quad \text{in } \mathbb{C},$$

which is a solution to

$$\operatorname{div}(\gamma_1\nabla u_1) = 0 \quad \text{in } \mathbb{C}.$$

Now we define u_2 by

$$u_2 = \begin{cases} u_1 & \text{in } \mathbb{C}\setminus\mathcal{O}, \\ \widehat{u} & \text{in } \mathcal{O}, \end{cases}$$

where $\widehat{u}$ is the solution to the Dirichlet problem

$$\begin{cases} \operatorname{div}(\gamma_2\nabla\widehat{u}) = 0 & \text{in } \mathcal{O}, \\ \widehat{u} = u_1 & \text{on } \partial\mathcal{O}. \end{cases}$$

Let $g \in C_0^\infty(\mathbb{C})$. Then

$$\begin{aligned}\int_{\mathbb{C}} \gamma_2\nabla u_2\nabla g\, d\sigma_z &= \int_{\mathbb{C}\backslash\mathcal{O}} \gamma_1\nabla u_1\nabla g\, d\sigma_z + \int_{\mathcal{O}} \gamma_2\nabla\widehat{u}\nabla g\, d\sigma_z \\ &= -\int_{\partial\mathcal{O}} \Lambda_{\gamma_1}[u_1|_{\partial\mathcal{O}}]g\, dz + \int_{\partial\mathcal{O}} \Lambda_{\gamma_2}[\widehat{u}|_{\partial\mathcal{O}}]g\, dz \\ &= 0.\end{aligned}$$

Hence $\operatorname{div}(\gamma_2\nabla u_2) = 0$ in $\mathbb{C}$. Then

$$\phi^{(2)} = \gamma_2^{1/2}\left(\partial u_2, \overline{\partial}u_2\right)^t$$

is the solution of (2) with $\gamma = \gamma_2$, and

$$\varphi^{(2)} = (\phi^{(2)}, \overline{\phi^{(2)}})^t$$

is the solution of (4) with $Q = Q_2$.

Lemmas 2.1 and 2.2 imply the unique solvability of the Lippmann–Schwinger equation when $|\lambda| > R$ and R is large enough. Thus, $\varphi^{(2)}$ is equal to the first column of ψ_2 when $|\lambda| > R$. On the other hand, $\varphi^{(2)}$ in $\mathbb{C}\backslash\mathcal{O}$ coincides with the first column φ of ψ_1. Thus the first columns of ψ_1 and ψ_2 are equal on $\mathbb{C}\backslash\mathcal{O}$ when $|\lambda| > R$. Repeating the same steps with the second columns of ψ_1, ψ_2, we obtain that $\psi_1|_{\partial\widetilde{\mathcal{O}}} = \psi_2|_{\partial\widetilde{\mathcal{O}}}$ when $|\lambda| > R$, and therefore (24) holds.

The uniqueness of h and Theorem 2.3 imply that the potential Q in the Dirac equation (4) is defined uniquely, and therefore q is defined uniquely. Now the conductivity γ can be found from (3) uniquely up to an additive constant. Finally, this constant can be defined uniquely since $\gamma|_{\partial\mathcal{O}}$ is defined uniquely by Λ_γ.

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