Discrete Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On the dominating induced matching problem: Spectral results and sharp bounds

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ARTICLE INFO

Article history:
Received 31 May 2015
Received in revised form 7 November 2015
Accepted 4 January 2016
Available online xxxx

Keywords:
Induced matching
Dominating induced matching
Spectral graph theory

ABSTRACT

A matching M is a dominating induced matching of a graph if every edge is either in M or has a common end-vertex with exactly one edge in M. The extremal graphs on the number of edges with dominating induced matchings are characterized by its Laplacian spectrum and its principal Laplacian eigenvector. Adjacency, Laplacian and signless Laplacian spectral bounds on the cardinality of dominating induced matchings are obtained for arbitrary graphs. Moreover, it is shown that some of these bounds are sharp and examples of graphs attaining the corresponding bounds are given.

1. Introduction

Throughout this paper we consider undirected simple graphs G of order $n > 1$ with a vertex set $V(G)$ and edge set $E(G)$. An element of $E(G)$, which has the vertices i and j as end-vertices, is denoted by ij. A matching M of G is a *dominating induced matching* (say a DIM) of G if every edge of G is either in M or has a common end-vertex with exactly one edge in M. A DIM is also called an *efficient edge domination set* (see for instance [10]). Observe that if M is a DIM of G, then there is a partition of $V(G)$ into two disjoint subsets $V(M)$ and S, where S is an independent set. Conversely, if there exists a graph G such that its vertex set $V(G)$ can be partitioned into two vertex subsets V_1 and V_2, where V_1 induces a matching and V_2 is an independent set, then the subset $M \subset E(G)$ of edges with both ends in V_1 is a DIM. Not all graphs have a DIM, for instance the cycle with four vertices C_4 has no DIM. The *DIM problem* asks whether a given graph has a dominating induced matching.

Dominating induced matchings have been studied, not always under the same designation, in [2,4,5,8,7,14,13,15]. The DIM problem is related with several practical applications. Some of them, as parallel resource allocation of parallel processing systems, encoding theory and network routing, as well as its relation with the 3-colorability problem are referred in [12]. In [12], it is also highlighted that graphs with dominating induced matchings are particular *polar graphs*. Notice that a polar graph is a graph where its vertex set can be partitioned into vertex subsets such that some are disjoint cliques and the others are independent sets with complete links between them [17]. Regarding its theoretical complexity, the DIM problem is NP-complete [10]. However, in [12] it is conjectured that unless $P = NP$, the DIM problem is polynomial-time solvable in the class of M-free graphs (where M is a finite set of graphs) if and only if M contains a graph from the class of graphs such that every connected component corresponds to a long claw, that is, a connected graph with a central vertex of degree three,

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http://dx.doi.org/10.1016/j.dam.2016.01.012
0166-218X/

Please cite this article in press as: E. Andrade, et al., On the dominating induced matching problem: Spectral results and sharp bounds, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.01.012

three vertices of degree one, and all the remaining vertices have degree two (that is, formed by three paths starting from a central vertex). In fact, the sufficient condition was proved in [7], but the necessary one remains open.

This paper is devoted to the study of the DIM problem from the graph spectra point of view. Next, for the reader convenience, we introduce some of the basic concepts and notation used throughout the paper. For the remaining terminology from graph theory, including spectral graph theory, the reader is referred to the book [9].

The adjacency matrix of a graph G of order n is the $n \times n$ symmetric matrix $A(G) = (a_{ij})$ where $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise, respectively. The Laplacian (signless Laplacian) matrix of G is the matrix $L(G) = D(G) - A(G)$ $(Q(G) = D(G) + A(G))$, where $D(G)$ is the $n \times n$ diagonal matrix of vertex degrees of G. The matrices $A(G)$, $L(G)$ and $Q(G)$ are all real and symmetric. From Geršgorin's theorem, it follows that the eigenvalues of $L(G)$ and $Q(G)$ are nonnegative real numbers. The spectrum of $A(G)$, $L(G)$ and $Q(G)$ is denoted by $\sigma_A(G)$, $\sigma_L(G)$ and $\sigma_Q(G)$, respectively. In this text, $\sigma_A(G) = \{\lambda_1^{[i_1]}, \ldots, \lambda_p^{[i_p]}\}$, $\sigma_L(G) = \{\mu_1^{[j_1]}, \ldots, \mu_q^{[j_q]}\}$ and $\sigma_Q(G) = \{q_1^{[k_1]}, \ldots, q_r^{[k_r]}\}$ mean that λ_s, μ_s and q_s are an adjacency, Laplacian and signless Laplacian eigenvalue with multiplicity i_s, j_s or k_s. As usually, we denote the eigenvalues of $A(G)$, $L(G)$ and $Q(G)$ in nonincreasing order, that is, $\lambda_1(G) \geq \cdots \geq \lambda_n(G)$, $\mu_1(G) \geq \cdots \geq \mu_n(G)$ and $q_1(G) \geq \cdots \geq q_n(G)$. Considering a graph G, the largest eigenvalue of $A(G)$, $L(G)$ and $Q(G)$ will be denoted, respectively, by $\rho(A(G))$, $\rho(L(G))$ and $\rho(Q(G))$. As usually, $\rho(A(G))$ is called the index of G and it is also denoted $\rho(G)$. The associated eigenvectors are called the principal eigenvectors of $A(G)$, $L(G)$ or $Q(G)$, respectively. For an arbitrary square matrix C the ith eigenvalue and its trace are denoted by $\lambda_i(C)$ and $\mathrm{tr}(C)$, respectively. Throughout this paper, $\mathbf{j}_k$ denotes the all one vector with k entries and $t + \sigma(C)$ means that we add t to each eigenvalue in $\sigma(C)$.

Consider a graph G of order n with a DIM $M \subset E(G)$ such that $|M| = m$, where (as above) $V(G) = V_1 \cup V_2$, with $V_1 = V(M)$ and $V_2 = V(G) \setminus V_1$ is an independent set. The property of having a DIM does not change whether we add edges linking the vertices of V_1 with the vertices of V_2. The extremal graph G', obtained from G adding $m(2(n-2m)+1) - |E(G)|$ edges (which is the maximum as possible) between V_1 and V_2, that is, such that $E(G') = M \cup \{xy : x \in V(M), y \in V(G) \setminus V(M)\}$ is herein called a *complete dominating induced matching*, say a CDIM. These graphs are particular cases of cographs [3].

The paper is organized as follows. In Section 2, the extremal graphs CDIM, are characterized by its Laplacian spectrum and by its principal Laplacian eigenvector. Notice that this characterization is important since in general, as it is well know, co-spectral graphs (relatively to adjacency, Laplacian or signless Laplacian matrices) are not necessarily isomorphic. The principal adjacency and signless Laplacian eigenvectors are deduced. Additionally, the adjacency and signless Laplacian spectra of graphs with a CDIM are presented. In Section 3, adjacency, Laplacian and signless Laplacian spectral bounds on the cardinality of dominating induced matchings are obtained for arbitrary graphs. Moreover, it is shown that some of these bounds are sharp and examples of graphs attaining the corresponding bounds are given.

2. Adjacency, Laplacian and signless Laplacian spectra of graphs with a CDIM

Given a graph H of order n with a CDIM, M such that $|M| = m$, we may define H using the join graph operation as follows. Let $H_r = mK_2$, with $r = 2m$ and $H_s = G[V(G) \setminus V(M)]$, with $s = n - r$, a null graph of order s (that is, a graph formed by s isolated vertices). Then $H = H_r \vee H_s$, that is, H is the join of the graphs H_r and H_s.

Consider the two above vertex disjoint graphs H_r and H_s and label the vertices of $H = H_r \vee H_s$, with the labels $1, 2, \ldots, r$ for the vertices of H_r and with the labels $r+1, \ldots, r+s$, for the vertices of H_s. Let $C(H)$ be a matrix on $H = H_r \vee H_s$. If $C(H) = L(H)$ or $C(H) = A(H)$ or $C(H) = Q(H)$ then, using the above mentioned labeling for the vertices of H, we obtain

$$C(H) = \begin{bmatrix} C_1 & \delta \mathbf{j}_r \mathbf{j}_s^T \\ \delta \mathbf{j}_s \mathbf{j}_r^T & C_2 \end{bmatrix}, \tag{1}$$

where δ is a scalar parameter, $C_1 = A(H_r)$ and $C_2 = A(H_s)$ or $C_1 = L(H_r) + sI_r$ and $C_2 = L(H_s) + rI_s$ or $C_1 = Q(H_r) + sI_r$ and $C_2 = Q(H_s) + rI_s$, when $C(H)$ is the adjacency, Laplacian or signless Laplacian matrix of H, respectively. In any case, in (1) we have $\delta \in \{1, -1\}$. Notice that

$$C_1 \mathbf{j}_r = \gamma_1 \mathbf{j}_r \quad \text{and} \quad C_2 \mathbf{j}_s = \gamma_2 \mathbf{j}_s,$$

with $\gamma_1 = 1$ and $\gamma_2 = 0$ (when $C(H)$ is the adjacency matrix) or $\gamma_1 = s$ and $\gamma_2 = r$ (when $C(H)$ is the Laplacian matrix) or $\gamma_1 = 2 + s$ and $\gamma_2 = r$ (when $C(H)$ is the signless Laplacian matrix).

Let us consider the matrix

$$B = \begin{bmatrix} \gamma_1 & \delta\sqrt{rs} \\ \delta\sqrt{rs} & \gamma_2 \end{bmatrix}, \tag{2}$$

where $\delta = \pm 1$, and its eigenvalues

$$\theta_1 = \frac{1}{2}\left(\gamma_1 + \gamma_2 + \sqrt{(\gamma_1 - \gamma_2)^2 + 4rs}\right) \tag{3}$$

$$\theta_2 = \frac{1}{2}\left(\gamma_1 + \gamma_2 - \sqrt{(\gamma_1 - \gamma_2)^2 + 4rs}\right). \tag{4}$$

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Lemma 2.1. *If B is the matrix in (2) and $\theta \in \sigma(B)$ has an associated eigenvector $\begin{bmatrix} 1 \\ x \end{bmatrix}$, then $\theta \in \sigma(C(H))$ and $\begin{bmatrix} \mathbf{j}_r \\ \sqrt{\frac{r}{s}} x \mathbf{j}_s \end{bmatrix}$ is an associated eigenvector.*

Proof. The result follows after some algebraic computations. ■

Now, it is worth to recall the Theorem 3 in [1], which can be written in the following version. The proof of this version is very similar to the one used in [1].

Theorem 2.2. *Let G be a graph with a set S of k pairwise co-neighbor vertices sharing the same set of l common neighbors, that is, an l-cluster of order k. Assume that G_k is an arbitrary graph such that $V(G_k) = S$, $G^k = G + G_k$ is the graph obtained from G adding the edges of G_k and*

$$\Lambda = \{l + \beta : \beta \in \sigma_L(G_k) \setminus \{0\}\}$$

is a multiset. Then $\sigma_L(G^k)$ overlaps $\sigma_L(G)$ in $n - k + 1$ places and the elements of Λ are the remaining eigenvalues in $\sigma_L(G^k)$.

The next theorem characterizes the graphs with a CDIM by its Laplacian spectrum and its principal Laplacian eigenvector.

Theorem 2.3. *A graph H of order n has a CDIM, $M \subset E(H)$ such that $|M| = m$, if and only if*

$$\sigma_L(H) = \{n, (n - 2m + 2)^{[m]}, (n - 2m)^{[m-1]}, (2m)^{[n-2m-1]}, 0\} \tag{5}$$

and a principal eigenvector:

$$\mathbf{v} = \begin{bmatrix} \mathbf{j}_{2m} \\ -\frac{2m}{n-2m} \mathbf{j}_{n-2m} \end{bmatrix}. \tag{6}$$

Proof. Let us assume that H has the CDIM $M \subset E(H)$. Taking into account that $H = H_r \vee H_s$, with $H_r = mK_2$ and $H_s = H[V(H) \setminus V(M)]$, where $r = 2m$ and $s = n - 2m$, applying Theorem 8 in [6] we obtain $\sigma_L(H_r \vee H_s) = (s + \sigma_L(H_r) \setminus \{0\}) \cup (r + \sigma_L(H_s) \setminus \{0\}) \cup \sigma(\widetilde{C})$, where

$$\widetilde{C} = \begin{bmatrix} s & -\sqrt{rs} \\ -\sqrt{rs} & r \end{bmatrix}.$$

Therefore, $\sigma_L(H_r \vee H_s) = \{r + s, (s+2)^{[m]}, s^{[m-1]}, r^{[n-2m-1]}, 0\}$. Furthermore, assuming $\gamma = -1$, $C_1 = L(H_{2m}) + (n-2m)I_{2m}$ and $C_2 = L(H_{n-2m}) + 2mI_{n-2m}$, the matrix B in (2) becomes

$$B = \begin{bmatrix} n - 2m & -\sqrt{2m(n-2m)} \\ -\sqrt{2m(n-2m)} & 2m \end{bmatrix}.$$

Therefore, according to (3)–(4), $\sigma(B) = \{n, 0\}$. Let us assume that $\mathbf{x} = \begin{bmatrix} 1 \\ x \end{bmatrix}$ is a principal eigenvector of B. From the eigenvalue equation

$$B\mathbf{x} = \begin{bmatrix} n - 2m & -\sqrt{2m(n-2m)} \\ -\sqrt{2m(n-2m)} & 2m \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = n \begin{bmatrix} 1 \\ x \end{bmatrix},$$

it follows that $n - 2m - x\sqrt{2m(n-2m)} = n \Leftrightarrow x = \frac{-2m}{\sqrt{2m(n-2m)}}$. Using Lemma 2.1,

$$\mathbf{v} = \begin{bmatrix} \mathbf{j}_{2m} \\ \sqrt{\frac{2m}{n-2m}} x \mathbf{j}_{n-2m} \end{bmatrix} = \begin{bmatrix} \mathbf{j}_{2m} \\ \frac{-2m}{n-2m} \mathbf{j}_{n-2m} \end{bmatrix},$$

is a principal eigenvector of $L(H)$.

Conversely, let us assume that the graph H has the Laplacian spectrum (5) and a principal eigenvector (6) and let $R = \{i \in V(H) : \mathbf{v}_i = 1\}$ and $S = V(H) \setminus R$. Then, for all $i \in V(H)$, $d_H(i) = p_i + q_i$, where $p_i = |N_H(i) \cap R|$ and $q_i = |N_H(i) \cap S|$. Thus $n\mathbf{v}_i = (p_i + q_i)\mathbf{v}_i - \sum_{j \in N_H(i) \cap R} \mathbf{v}_j - \sum_{j \in N_H(i) \cap S} \mathbf{v}_j$.

1. If $i \in R$, then $\mathbf{v}_i = 1$ and $(n - p_i - q_i) = -p_i + \frac{2m}{n-2m} q_i \Leftrightarrow q_i = n - 2m$.
2. If $i \in S$, then $\mathbf{v}_i = -\frac{2m}{n-2m} \Leftrightarrow p_i = 2m$.

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Since every vertex in R has $q = n - 2m$ neighbors in S and every vertex in S has $p = 2m$ neighbors in R, it follows that $|R| = 2m$ and $|S| = n - 2m$. Therefore, H contains $K_{2m,n-2m}$ as a spanning subgraph and thus the graph H can be obtained from the bipartite complete graph $G = K_{2m,n-2m}$ applying twice Theorem 2.2, that is,

$$H = \left(G^{2m}\right)^{n-2m} = G^{2m} + G_{n-2m} = (G + G_{2m}) + G_{n-2m}$$

and

- $\sigma_L\left(G^{2m}\right) = \sigma_L(G + G_{2m})$ overlaps $\sigma_L(G)$ in $n-2m+1$ places and the elements of $\Lambda = \{(n-2m)+\beta : \beta \in \sigma_L(G_{2m}) \setminus \{0\}\}$ are the remaining eigenvalues of $G + G_{2m}$, that is,
$$\begin{aligned}\sigma_L(G + G_{2m}) &= \Lambda \cup \left(\sigma_L(K_{2m,n-2m}) \setminus \{(n-2m)^{[2m-1]}\}\right) \\ &= \Lambda \cup \{0, (2m)^{[n-2m-1]}, n\}.\end{aligned}$$
- $\sigma_L(H) = \sigma_L(G^{2m} + G_{n-2m})$ overlaps $\sigma_L(G^{2m}) = \sigma_L(G + G_{2m})$ in $n - (n-2m) + 1 = 2m + 1$ places and the elements of
$$\Lambda' = \{2m + \beta' : \beta' \in \sigma_L(G_{n-2m}) \setminus \{0\}\}$$
are the remaining eigenvalues of H, that is,
$$\begin{aligned}\sigma_L(H) &= \Lambda' \cup \left(\sigma_L\left(G^{2m}\right) \cap X\right) \\ &= \Lambda' \cup \left(\left(\Lambda \cup \{0, (2m)^{[n-2m-1]}, n\}\right) \cap X\right),\end{aligned}$$
such that $X \subseteq \sigma_L(H)$ and $|\left(\Lambda \cup \{0, (2m)^{[n-2m-1]}, n\}\right) \cap X| = 2m + 1$.

Since $\sigma_L(H) = \{n, (n-2m+2)^{[m]}, (n-2m)^{[m-1]}, (2m)^{[n-2m-1]}, 0\}$, we may conclude the following:

- From the eigenvalues in $\Lambda' = \{2m + \beta' : \beta' \in \sigma_L(G_{n-2m}) \setminus \{0\}\}$ and taking into account that $2m$ is a Laplacian eigenvalue of H with multiplicity $n - 2m - 1$, the graph induced by the vertices of G_{n-2m} has the eigenvalue 0 with multiplicity $n - 2m$, that is, S is an independent vertex set.
- From the Laplacian eigenvalues $n - 2m + 2$ and $n - 2m$ of H, with multiplicity m and $m - 1$, respectively, it follows that they belong to $\Lambda = \{n - 2m + \beta : \beta \in \sigma_L(G_{2m}) \setminus \{0\}\}$. Then 0 and 2 are Laplacian eigenvalues of G_{2m} each one with multiplicity m.
  - From the multiplicity of 0, it follows that G_{2m} has m components, each one with largest Laplacian eigenvalue less than or equal 2.
  - Since, as it is well known, the largest Laplacian eigenvalue of a graph F of order n is not less than $\Delta(F)+1$ and it is equal to this value if and only if $\Delta(F) = n - 1$ [11], considering a component C_i of G_{2m} with largest Laplacian eigenvalue 2 it follows that $\Delta(C_i) \le 1$, and then C_i is equal to K_2. If the multiplicity of 2 is greater than one, then there is another component which has 2 in its Laplacian spectrum and using the same argument we conclude that such a component is also K_2 and so on.

  Therefore, R induces a subgraph with m components equal to K_2. ■

So far we have not found a pair of Laplacian co-spectral graphs where only one of them has a CDIM. Therefore, it remains as an open problem to know whether the graphs with a CDIM can be characterized just by their Laplacian spectra.

It is immediate that a graph has at most one CDIM. Recently, in [16] several sharp upper bounds on the number of DIMs in graphs were introduced.

Regarding the adjacency and signless Laplacian case, we obtain the following results.

Theorem 2.4. *Let H be a graph of order n with a CDIM, $M \subset E(H)$, such that $|M| = m$. Then, the adjacency and signless Laplacian spectra of H are given by:*

1. $\sigma_A(H) = \{\frac{1\pm\sqrt{1+8m(n-2m)}}{2}, 1^{[m-1]}, 0^{[n-2m-1]}, (-1)^{[m]}\}$.
2. $\sigma_Q(H) = \{\frac{2+n\pm\sqrt{(2+n)^2-16m}}{2}, (n-2m+2)^{[m-1]}, (n-2m)^{[m]}, (2m)^{[n-2m-1]}\}$.

Proof. Taking into account that $H = H_r \vee H_s$, with $H_r = mK_2$ and $H_s = G[V(G) \setminus V(M)]$, where $r = 2m$ and $s = n - 2m$, we may apply the results obtained in [6] as follows.

1. **The adjacency spectrum**: Applying Theorem 5 in [6], it follows that $\sigma_A(H_r \vee H_s) = \sigma_A(H_r) \setminus \{1\} \cup \sigma_A(H_s) \setminus \{0\} \cup \sigma(\widetilde{C})$, where
$$\widetilde{C} = \begin{bmatrix} 1 & \sqrt{rs} \\ \sqrt{rs} & 0 \end{bmatrix}.$$
Therefore, $\sigma_A(H_r \vee H_s) = \{\frac{1+\sqrt{1+4rs}}{2}, 1^{[m-1]}, 0^{[n-2m-1]}, (-1)^{[m]}, \frac{1-\sqrt{1+4rs}}{2}\}$.

2. **The signless Laplacian spectrum**: Applying Theorem 3 in [6], it follows that $\sigma_Q(H_r \vee H_s) = (s + \sigma_Q(H_r) \setminus \{2\}) \cup (r + \sigma_Q(H_s) \setminus \{0\}) \cup \sigma(\widetilde{C})$, where

$$\widetilde{C} = \begin{bmatrix} 2+s & \sqrt{rs} \\ \sqrt{rs} & r \end{bmatrix}.$$

Therefore, $\sigma_Q(H_r \vee H_s) = \{\frac{2+r+s\pm\sqrt{(2+r+s)^2-8r}}{2}, (s+2)^{[m-1]}, s^{[m]}, r^{[n-2m-1]}\}$. ■

From this theorem, it is immediate that $\rho(A(G)) = \frac{1+\sqrt{1+8m(n-2m)}}{2}$ and $\rho(Q(G)) = \frac{2+n+\sqrt{(2+n)^2-16m}}{2}$. Since when an edge is deleted the spectral radius decreases, the following corollary can be stated.

Corollary 2.4.1. *Let G be a graph of order n with a DIM, $M \subset E(G)$ such that $|M| = m$. Then the spectral radius of the adjacency and signless Laplacian matrix of G has the following upper bounds.*

1. $\rho(A(G)) \leq \frac{1+\sqrt{1+8m(n-2m)}}{2}$;
2. $\rho(Q(G)) \leq \frac{2+n+\sqrt{(2+n)^2-16m}}{2}$.

In this corollary, the Laplacian case is not considered, since for any graph the largest Laplacian eigenvalue is not greater than the order of the graph.

Theorem 2.5. *Let H be a graph of order n with a CDIM, $M \subset E(H)$ such that $|M| = m$. Then, the principal eigenvectors of the adjacency and signless Laplacian matrix of H are the following:*

1. *The principal eigenvector of $A(H)$, that is, the eigenvector corresponding to the eigenvalue $\rho(A(H)) = \frac{1+\sqrt{1+8m(n-2m)}}{2}$ is*

$$\mathbf{u} = \begin{bmatrix} \mathbf{j}_{2m} \\ \frac{\rho(A(H))-1}{n-2m}\mathbf{j}_{n-2m} \end{bmatrix}.$$

2. *The principal eigenvector of $Q(H)$, that is, the eigenvector corresponding to the eigenvalue $\rho(Q(H)) = \frac{2+n+\sqrt{(2+n)^2-16m}}{2}$ is*

$$\mathbf{w} = \begin{bmatrix} \mathbf{j}_{2m} \\ \frac{\rho(Q(H))-(n-2m+2)}{n-2m}\mathbf{j}_{n-2m} \end{bmatrix}.$$

Proof. As in Lemma 2.1, consider $H = H_r \vee H_s$ with $H_r = mK_2$ and $H_s = H[S]$, where S is an independent set of H of size $n-2m$. Then $r = 2m$, $s = n-2m$ and we may analyze each of the following cases.

1. Assuming $\gamma = 1$, $C_1 = A(H_{2m})$ and $C_2 = A(H_{n-2m})$, then the matrix (2) becomes

$$B = \begin{bmatrix} 1 & \sqrt{2m(n-2m)} \\ \sqrt{2m(n-2m)} & 0 \end{bmatrix}.$$

Therefore, according to (3)–(4),

$$\sigma(B) = \left\{\frac{1+\sqrt{1+8m(n-2m)}}{2}, \frac{1-\sqrt{1+8m(n-2m)}}{2}\right\}.$$

Let $\rho = \frac{1+\sqrt{1+8m(n-2m)}}{2}$ and assume that $\mathbf{x} = \begin{bmatrix} 1 \\ x \end{bmatrix}$ is an eigenvector of B associated to ρ. From the eigenvalue equation

$$B\mathbf{x} = \begin{bmatrix} 1 & \sqrt{2m(n-2m)} \\ \sqrt{2m(n-2m)} & 0 \end{bmatrix}\begin{bmatrix} 1 \\ x \end{bmatrix} = \rho\begin{bmatrix} 1 \\ x \end{bmatrix},$$

it follows that $1 + x\sqrt{2m(n-2m)} = \rho \Leftrightarrow x = \frac{\rho-1}{\sqrt{2m(n-2m)}}$. Therefore, using Lemma 2.1,

$$\mathbf{u} = \begin{bmatrix} \mathbf{j}_{2m} \\ \sqrt{\frac{2m}{n-2m}}x\mathbf{j}_{n-2m} \end{bmatrix} = \begin{bmatrix} \mathbf{j}_{2m} \\ \frac{\rho-1}{n-2m}\mathbf{j}_{n-2m} \end{bmatrix}$$

is an eigenvector of $A(H)$ associated to the eigenvalue ρ.

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2. Assuming $\gamma = 1$, $C_1 = Q(H_{2m}) + (n-2m)I_{2m}$ and $C_2 = Q(H_{n-2m}) + 2mI_{n-2m}$, then the matrix (2) becomes

$$B = \begin{bmatrix} n-2m+2 & \sqrt{2m(n-2m)} \\ \sqrt{2m(n-2m)} & 2m \end{bmatrix}.$$

Therefore, according to (3)–(4), $\sigma(B) = \{\frac{n+2+\sqrt{(n+2)^2-16m}}{2}, \frac{n+2-\sqrt{(n+2)^2-16m}}{2}\}$. Let $\rho = \frac{n+2+\sqrt{(n+2)^2-16m}}{2}$ and let us assume that $\mathbf{x} = \begin{bmatrix} 1 \\ x \end{bmatrix}$ is an eigenvector of B associated to ρ. From the eigenvalue equation

$$B\mathbf{x} = \begin{bmatrix} n-2m+2 & \sqrt{2m(n-2m)} \\ \sqrt{2m(n-2m)} & 2m \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = \rho \begin{bmatrix} 1 \\ x \end{bmatrix},$$

it follows that $n-2m+2+x\sqrt{2m(n-2m)} = \rho \Leftrightarrow x = \frac{\rho-(n-2m+2)}{\sqrt{2m(n-2m)}}$. Therefore, using Lemma 2.1,

$$\mathbf{w} = \begin{bmatrix} \mathbf{j}_{2m} \\ \sqrt{\frac{2m}{n-2m}} \frac{\rho-(n-2m+2)}{\sqrt{2m(n-2m)}} \mathbf{j}_{n-2m} \end{bmatrix} = \begin{bmatrix} \mathbf{j}_{2m} \\ \frac{\rho-(n-2m+2)}{n-2m} \mathbf{j}_{n-2m} \end{bmatrix}$$

is an eigenvector of $Q(H)$ associated to ρ. ■

Example 2.1. Let H be a graph obtained from the graph depicted in Fig. 1 adding the edges 18, 19, 29, 37, 39, 47, 49, 57, 67, 68. Then H is a graph with a CDIM, $M = \{12, 34, 56\}$. Applying Theorems 2.3 and 2.4, we obtain

1. $\sigma_A(H) = \{4.772\ldots, 1^{[2]}, 0^{[2]}, -1^{[3]}, -3.772\ldots\}$;
2. $\sigma_L(H) = \{9, 6^{[2]}, 5^{[3]}, 3^{[2]}, 0\}$;
3. $\sigma_Q(H) = \{9.772\ldots, 6^{[2]}, 5^{[2]}, 3^{[3]}, 1.228\ldots\}$.

Additionally, from Theorems 2.3 and 2.5, we may conclude that the principal eigenvectors of $A(H)$, $L(H)$ and $Q(H)$ are $\mathbf{u} = \begin{bmatrix} \mathbf{j}_6 \\ 1.257\ldots\mathbf{j}_3 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} \mathbf{j}_6 \\ -2\mathbf{j}_3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} \mathbf{j}_6 \\ 1.591\ldots\mathbf{j}_3 \end{bmatrix}$, respectively.

3. Bounds on the size of a DIM, obtained from the adjacency, Laplacian and signless Laplacian spectra

From now on, we denote a graph with a CDIM, M, by $K_{M,S}$, where M is a dominating induced matching, S is an independent set and each vertex of S is connected by an edge to each vertex of $V(M)$.

3.1. Bounds obtained from the adjacency spectra of graphs with a DIM

Lemma 3.1. *Let G be a graph of order n with a DIM $M \subset E(G)$ such that $|M| = m$. Then*

$$\left(\frac{n}{2}\right)^2 \geq \rho(\rho-1), \quad (7)$$

where $\rho = \rho(A(G))$, with equality if and only if $n = 4m$ and $G = K_{M,S}$.

Proof. Assuming that $|M| = m$ and $S = V(G) \setminus V(M)$, since $f(x) = 4x^2 - 4x$ is a strictly increasing function for $x > 1/2$, it follows that

$$4(\rho^2 - \rho) \leq 4(\rho^2(K_{M,S}) - \rho(K_{M,S})) = 8mn - 16m^2. \quad (8)$$

Moreover, $(n-4m)^2 \geq 0 \Leftrightarrow 8mn - 16m^2 \leq n^2$. Therefore, from (8), the inequality (7) follows. It is immediate that (7) holds as equality if and only if $4m = n$ and $G = K_{M,S}$. ■

From Lemma 3.1, taking into account the item 1 of Theorem 2.4, we are able to obtain the following result.

Theorem 3.2. *Let G be a graph of order n with a DIM $M \subset E(M)$ such that $|M| = m$ and let $\rho = \rho(G)$. If $G \neq K_{M,S}$, then*

$$\left\lceil \frac{1}{4}\left(n - \sqrt{n^2 - 4(\rho^2-\rho)}\right) \right\rceil \leq m \leq \left\lfloor \frac{1}{4}\left(n + \sqrt{n^2 - 4(\rho^2-\rho)}\right) \right\rfloor.$$

Proof. Since from Theorem 2.4-1, $\rho(K_{M,S}) = \frac{1}{2}(1 + \sqrt{1+8m(n-2m)})$, then $\rho(G) \leq \frac{1}{2}(1+\sqrt{1+8m(n-2m)})$ and, setting $\rho = \rho(G)$, after some algebraic steps we get

$$4m^2 - 2nm + \rho^2 - \rho \leq 0. \quad (9)$$

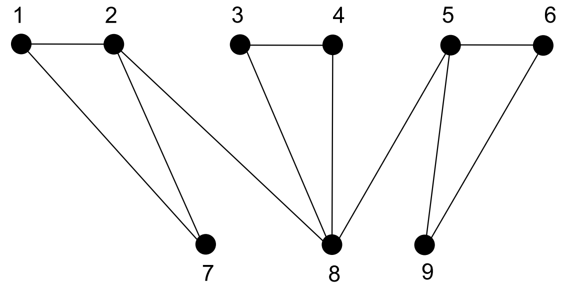

Fig. 1. A graph G with a DIM $M = \{12, 34, 56\}$.

Let $q(m) = 4m^2 - 2nm + \rho^2 - \rho$. Since $G \neq K_{M,S}$, then $n^2 - 4(\rho^2 - \rho) > 0$ and therefore $q(m) = 0$ has two real roots

$$m_1 = \frac{1}{8}\left(2n - \sqrt{4n^2 - 16(\rho^2 - \rho)}\right)$$
$$m_2 = \frac{1}{8}\left(2n + \sqrt{4n^2 - 16(\rho^2 - \rho)}\right).$$

Hence, the inequality (9) holds when $m_1 \leq m \leq m_2$. ■

Example 3.1. Let us consider the graph G depicted in Fig. 1 which has order $n = 9$ and minimum degree $\delta(G) = 2$. Since the index of G is $\rho(G) = 2.636\ldots$, then $m_1 = 0.254\ldots$ and $m_2 = 4.246\ldots$. Therefore, according to Theorem 3.2, $1 \leq m \leq 4$.

Let M be a DIM of G. Considering $|M| = m$ and labeling the vertices of V_1 as $1, \ldots, 2m$, the adjacency matrix of G is as follows:

$$A(G) = \begin{bmatrix} P & R \\ R^T & 0 \end{bmatrix}. \tag{10}$$

One can see that $P^2 = I$.

Theorem 3.3. *Let G be a graph of order n, minimum degree $\delta = \delta(G)$, and index $\rho = \rho(G)$. If G has a DIM $M \subseteq E(G)$, then*

$$|M| \geq \left\lceil \frac{n(2\delta - \rho)}{2(2\delta - 1)} \right\rceil. \tag{11}$$

Proof. Let us assume that the graph G has a DIM $M \subset E(G)$ such that $|M| = m$ and thus $V(G)$ can be partitioned into the vertex subsets $V(M)$ and S, where S is an independent set. Then the adjacency matrix of G can be written as in (10) and it follows:

$$\begin{aligned}
\rho &\geq \frac{1}{\sqrt{n}}[\mathbf{j}_{2m}^T \quad \mathbf{j}_{n-2m}^T]\begin{bmatrix} P & R \\ R^T & 0 \end{bmatrix}\frac{1}{\sqrt{n}}\begin{bmatrix} \mathbf{j}_{2m} \\ \mathbf{j}_{n-2m} \end{bmatrix} \\
&= \frac{1}{n}\left(\mathbf{j}_{2m}^T P \mathbf{j}_{2m} + \mathbf{j}_{n-2m}^T R^T \mathbf{j}_{2m} + \mathbf{j}_{2m}^T R \mathbf{j}_{n-2m}\right) \\
&= \frac{1}{n}\left(2m + 2\sum_{v \in S} d(v)\right) \\
&\geq \frac{1}{n}(2m + 2(n - 2m)\delta).
\end{aligned}$$

Therefore, $\frac{n(\rho - 2\delta)}{2(1 - 2\delta)} \leq m$. ■

Example 3.2. The graph G depicted in Fig. 1 is an example for which the lower bound (11) is sharp. In fact, since the graph G has a DIM $M \subset E(G)$, $n = 9$, $\delta(G) = 2$ and $\rho(G) = 2.636\ldots$, it follows that $\frac{n(2\delta(G) - \rho(G))}{2(2\delta(G) - 1)} = 2.045\ldots$ and therefore $\lceil \frac{n(2\delta(G) - \rho(G))}{2(2\delta(G) - 1)} \rceil = 3 = |M|$.

Before introducing the next result, let us recall the following classical Cauchy interlacing theorem.

Please cite this article in press as: E. Andrade, et al., On the dominating induced matching problem: Spectral results and sharp bounds, Discrete Applied Mathematics (2016), http://dx.doi.org/10.1016/j.dam.2016.01.012

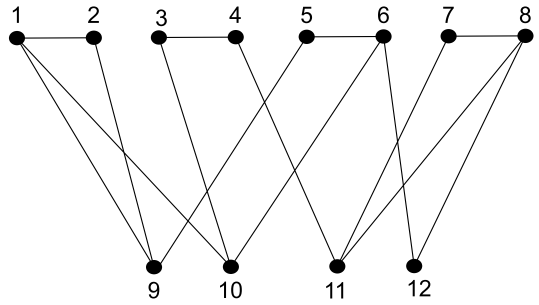

Fig. 2. A graph G with a DIM $M = \{12, 34, 56, 78\}$.

Theorem 3.4 (*Cauchy Interlacing Theorem [9]*). *Let*

$$A = \begin{bmatrix} B & C^* \\ C & D \end{bmatrix}$$

be a $p \times p$ Hermitian matrix and B a $q \times q$ matrix with $q < p$. Then

$$\lambda_k(A) \geq \lambda_k(B) \geq \lambda_{k+p-q}(A) \quad \text{for } k = 1, 2, \ldots, q.$$

Now, applying this theorem to the adjacency matrix of a graph G with an induced matching, we may conclude the following result.

Theorem 3.5. *Let G be a graph and let $M \subseteq E(G)$ be an induced matching such that $|M| = m$. Then $\sigma_A(G)$ includes m eigenvalues not greater than -1 and m eigenvalues not less than 1.*

Proof. Let $A(G) = \begin{bmatrix} P & R \\ R^T & Q \end{bmatrix}$. Since the order of P is $2m$ and the order of $A(G)$ is n, then R is a $2m \times (n-2m)$ matrix. Applying Theorem 3.4 to $A(G)$, setting $k = m$ and $k = m+1$, respectively, we obtain

$$\begin{aligned} \lambda_m(A(G)) &\geq \lambda_m(P) = 1 \\ -1 &= \lambda_{m+1}(P) \\ &\geq \lambda_{m+1+n-2m}(A(G)) = \lambda_{n-m+1}(A(G)). \end{aligned}$$

Then $A(G)$ has m eigenvalues not greater than -1 and m eigenvalues not less than 1. ■

As immediate consequence we have the following corollary.

Corollary 3.5.1. *Let G be a graph where $\Lambda^- = \{\lambda \in \sigma_A(G) : \lambda \leq -1\}$ and $\Lambda^+ = \{\lambda \in \sigma_A(G) : \lambda \geq 1\}$. If $M \subseteq E(G)$ is an induced matching of G, then*

$$|M| \leq \min\left\{\left|\Lambda^-\right|, \left|\Lambda^+\right|\right\}. \tag{12}$$

Example 3.3. Considering the graph G of Example 3.1 and taking into account that $\sigma_A(G) = \{-2.067\ldots, -1^{[4]}, -0.222\ldots, 1.652\ldots, 2, 2.636\ldots\}$, it follows that $\Lambda^- = \{-2.066\ldots, -1^{[4]}\}$ and $\Lambda^+ = \{1.652\ldots, 2, 2.636\ldots\}$. Therefore, according to Corollary 3.5.1, if $M \subset E(G)$ is an induced matching, then

$$|M| \leq 3.$$

In this case, if M is a DIM, combining (11) with (12) we may conclude that $|M| = 3$.

Example 3.4. The graph G depicted in Fig. 2 is an example for which the upper bound (12) is sharp. In fact, since the graph G has an induced matching $M \subset E(G)$, and its adjacency spectrum is equal to $\{-2.156\ldots, -1.870\ldots, -1.597\ldots, -1.311\ldots, -0.897\ldots, -0.547\ldots, 0.034\ldots, 0.579\ldots, 1.386\ldots, 1.481\ldots, 2.308\ldots, 2.590\ldots\}$, it follows that $|\Lambda^-| = 4$ and $|\Lambda^+| = 4$. Therefore, $|M| \leq 4$.

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3.2. Lower bounds obtained from the Laplacian spectra of graphs with a DIM

Let us consider a graph G of order n, with a DIM, $M \subset E(G)$, such that $|M| = m$ and thus $V(M)$ and the independent set S is a partition of $V(G)$. Let D_1 and D_2 be the diagonal matrices whose diagonal entries are the degrees of the vertices in $V(M)$ and S, respectively. The Laplacian matrix of G can be written as

$$L(G) = \begin{bmatrix} D_1 - P & -R \\ -R^T & D_2 \end{bmatrix}$$

and then we have

$$\begin{aligned} \mu_1(G) &\geq \frac{1}{\sqrt{n}} \left[\mathbf{j}_{2m}^T \quad -\mathbf{j}_{n-2m}^T\right] \begin{bmatrix} D_1 - P & -R \\ -R^T & D_2 \end{bmatrix} \frac{1}{\sqrt{n}} \begin{bmatrix} \mathbf{j}_{2m} \\ -\mathbf{j}_{n-2m} \end{bmatrix} \\ &= \frac{1}{n} \left(\mathbf{j}_{2m}^T D_1 \mathbf{j}_{2m}^T + \mathbf{j}_{n-2m}^T D_2 \mathbf{j}_{n-2m}^T - \mathbf{j}_{2m}^T P \mathbf{j}_{2m}^T + 2\mathbf{j}_{n-2m}^T R^T \mathbf{j}_{2m}^T\right) \\ &= \frac{1}{n} \left(\mathrm{tr}(L(G)) - 2m + 2\sum_{v \in S} d(v)\right) \\ &\geq \frac{1}{n} \left(\mathrm{tr}(L(G)) - 2m + 2(n-2m)\delta(G)\right). \end{aligned}$$

As a consequence, we may state the following Laplacian spectral lower bound on the cardinality of a DIM.

Theorem 3.6. *Let G be a graph of order n with a DIM $M \subset E(G)$. Then $|M| \geq \frac{tr(L(G)) - n(\mu_1(G) - 2\delta(G))}{2(2\delta(G)+1)}$.*

3.3. Lower bounds obtained from the signless Laplacian spectra of graphs with a DIM

Let G be a graph of order n, with a DIM, $M \subset E(G)$, such that $|M| = m$. As in the previous subsection, let D_1 and D_2 be the diagonal matrices whose diagonal entries are the degrees of the vertices in $V(M)$ and $S = V(G) \setminus V(M)$, respectively. The signless Laplacian matrix of G can be written as follows:

$$Q(G) = \begin{bmatrix} D_1 + P & R \\ R^T & D_2 \end{bmatrix} = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} + \begin{bmatrix} P & R \\ R^T & 0 \end{bmatrix}.$$

Then we have

$$\begin{aligned} q_1(G) &\geq \frac{1}{\sqrt{n}} \left[\mathbf{j}_{2m}^T \ \mathbf{j}_{n-2m}^T\right] Q(G) \frac{1}{\sqrt{n}} \begin{bmatrix} \mathbf{j}_{2m} \\ \mathbf{j}_{n-2m} \end{bmatrix} \\ &= \frac{1}{n} \left(\sum_{v \in V(G)} d(v) + \mathbf{j}_{2m}^T P \mathbf{j}_{2m} + \mathbf{j}_{n-2m}^T R^T \mathbf{j}_{2m}^T + \mathbf{j}_{2m}^T R \mathbf{j}_{n-2m}^T\right) \\ &= \frac{1}{n} \left(\sum_{v \in V(G)} d(v) + \mathbf{j}_{2m}^T P \mathbf{j}_{2m} + 2\mathbf{j}_{2m}^T R \mathbf{j}_{n-2m}^T\right). \end{aligned}$$

Therefore,

$$\begin{aligned} q_1(G) &\geq \frac{1}{n} \left(\mathrm{tr}(Q(G)) + 2m + 2\sum_{v \in S} d(v)\right) \\ &\geq \frac{1}{n} \left(\mathrm{tr}(Q(G)) + 2m + 2(n-2m)\delta(G)\right) \end{aligned}$$

and

$$\frac{n(q_1(G) - 2\delta(G)) - \mathrm{tr}(Q(G))}{2(1 - 2\delta(G))} \leq m. \tag{13}$$

Now, from (13) we may state the following signless Laplacian spectral lower bound on the cardinality of a DIM.

Theorem 3.7. *Let G be a graph of order n with a DIM $M \subset E(G)$. Then $|M| \geq \frac{tr(Q(G)) - n(q_1(G) - 2\delta(G))}{2(2\delta(G)-1)}$.*

Example 3.5. Considering the graph of Fig. 1, since

$$\sigma_Q(G) = \{1^{[4]}, 1.586\ldots, 2.268\ldots, 4, 4.414\ldots, 5.732\ldots\},$$

the lower bound on Theorem 3.7 produces

$$\frac{\mathrm{tr}(Q(G)) - n(q_1 - 2\delta(G))}{2(2\delta(G) - 1)} = \frac{22 - 9(5.7321 - 4)}{2(4 - 1)} = 1.0685$$

and thus $|M| \geq 2$.

Acknowledgments

We are indebted to two anonymous referees for the careful reading and for all their suggestions which improved the text. The research of Enide Andrade and Domingos M. Cardoso was partially supported by the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), through the CIDMA—Center for Research and Development in Mathematics and Applications, within project UID/MAT/04106/2013 and also by the Project PTDC/MAT/112276/2009. Both authors jointly with Oscar Rojo thank the support of Project Fondecyt Regular 1130135. The first two authors also thank the hospitality of the Mathematics Department of Universidad Católica del Norte, Chile, where this paper was finished.

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