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GLEASON'S PROBLEM ASSOCIATED TO THE FRACTIONAL CAUCHY-RIEMANN OPERATOR, FUETER SERIES, DRURY-ARVESON SPACE AND RELATED TOPICS

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Abstract. In this paper we present the building blocks for a function theory based on fractional Cauchy-Riemann operators. We are going to construct basic monogenic powers and Fueter series. With these tools we are going to study Gleason's problem and reproducing kernel spaces, like the Drury-Arveson space and de Branges-Rovnyak spaces. We end with a statement on Schur multipliers in this setting.

1. Introduction

The concept of a monogenic function as a null solution of a Dirac operator not only allows for a generalization of complex analysis to higher dimensions but it is also incredibly useful in a variety of situations. First, there is Dirac's original equation describing an electron. There is also the concept of monogenic signal as a higher dimensional equivalent to the concept of analytic signal with its applications to image processing [7]. Furthermore, its close relation with reproducing kernel Hilbert spaces leads to many applications in interpolation, sampling, and systems theory, some of which has only recently been explored. But there exist circumstances in which null-solutions of the classical Dirac operator are not good enough. For instance, monogenic functions as null-solutions of the Dirac operator do not allow for a description of higher symmetries than SU(2) which would require a Dirac operator which provides a n-fold factorization of the Laplacian [13]. Furthermore, many applications in mathematical optics require a fractional Fourier transform which is in general only indirectly related to the Dirac operator [11, 18]. To say it more directly, the introduction of fractional derivatives allows for a more accurate description of physical processes by introducing a memory mechanism in the process [6].

Working with fractional derivatives has a major disadvantage. Most of the tools which we are used to in the case of classic derivatives are not available in the fractional case. This includes such important tools like Leibniz formula, chain rule, translation invariance, or spherical coordinates [21]. From our point of view the

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most difficult one to overcome is the lack of a Leibniz formula which in the classic case gives rise to a Heisenberg algebra over the space of polynomials generated by the derivative and the multiplication operator. Yet, as we are going to show one can construct a similar approach on the space of suitably defined homogeneous polynomials. Later this can be extended to the whole space by introducing the so-called CK-extension and CK-product. The CK-extension provides an isomorphism between the space of fractional monogenics of a given degree and the corresponding space of homogeneous polynomials of higher codimension. The later provides a suitable product between series expansions which still preserves homogeneity. This construction will allow us to follow ideas from [3,4] to construct reproducing kernel Hilbert spaces of monogenic functions, in particular, the Drury-Arveson space and de Branges-Rovnyak spaces associated to Schur multipliers. Furthermore, we are going to consider Gleason's problem in this context and its link with Leibenson's shift operators.

There exists quite a zoo of fractional derivatives, like Riemann-Liouville, Weyl, Caputo, and Riesz-Feller. To avoid studying each case individually, we opt here to work with the so-called Gelfond-Leontiev (or G-L-) derivatives [10,17]. These derivatives are based on a representation of fractional derivatives via its action on power series and contain the above mentioned derivatives as special cases.

2. Preliminaries

2.1. **Clifford analysis.** Let $\{e_1,\ldots,e_n\}$ be the standard basis of the Euclidean vector space in $\mathbb{R}^n$. The associated Clifford algebra $\mathbb{R}_{0,n}$ is the free algebra generated by $\mathbb{R}^n$ modulo $x^2=-|x|^2$. The defining relation induces the multiplication rules

$$e_ie_j+e_je_i=-2\delta_{i,j},\quad i,j=1,\ldots,n,$$

where $\delta_{i,j}$ denotes the Kronecker symbol. In particular, as we have $e_i^2=-1$, the standard basis vectors operate as imaginary units.

A vector space basis for $\mathbb{R}_{0,n}$ is given by the set

$$\{e_0=1,e_A=e_{l_1}e_{l_2}\ldots e_{l_r}:\ A=\{l_1,l_2,\ldots,l_r\}\,,1\leq l_1<\cdots<l_r\leq n\}.$$

Each $a\in\mathbb{R}_{0,n}$ can be written in the form $a=\sum_A a_A\,e_A$, with $a_A\in\mathbb{R}$. Now, we introduce the complexified Clifford algebra $\mathbb{C}_n$ as the tensor product

$$\mathbb{C}\otimes\mathbb{R}_{0,n}=\left\{w=\sum_A w_Ae_A,\ w_A\in\mathbb{C},A\subseteq M=\{1,\ldots,n\}\right\},$$

where the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e., $ie_j=e_ji$ for all $j=1,\ldots,n$.

The conjugation in the Clifford algebra $\mathbb{C}_n$ is defined as the automorphism

$$w\mapsto\overline{w}=\sum_A\overline{w}_A\,\overline{e}_A,$$

where $\overline{w}_A$ denotes the usual complex conjugation and $\overline{e}_A=\overline{e}_{l_r}\,\overline{e}_{l_{r-1}}\ldots\overline{e}_{l_1}$, where $\overline{e}_\emptyset=1$ and $\overline{e}_j=-e_j$ for $j=1,\ldots,n$. For a vector $w=\sum_{j=1}^n w_je_j$ we have $w\overline{w}=|w|^2:=\sum_{j=1}^n|w_j|^2$. Hence, each non-zero vector $w=\sum_{j=1}^n w_je_j$ has a unique multiplicative inverse given by $w^{-1}=\frac{\overline{w}}{|w|^2}$.

A $\mathbb{C}_n$-valued function f over a non-empty domain $\Omega \subset \mathbb{R}^n$ is written as $f = \sum_A f_A e_A$, with components $f_A : \Omega \to \mathbb{C}$. Properties such as continuity are understood componentwisely. For example, $f = \sum_A f_A e_A$ is continuous if and only if all components f_A are continuous. Next, we recall the Euclidean-Dirac operator $D = \sum_{j=1}^n e_j \, \partial_{x_j}$, which factorizes the n-dimensional Euclidean-Laplacian, i.e., $D^2 = -\Delta = -\sum_{j=1}^n \partial_{x_j}^2$. A $\mathbb{C}_n$-valued function f is said to be *left-monogenic* if it satisfies $Df = 0$ on Ω (resp. *right-monogenic* if it satisfies $fD = 0$ on Ω).

A left (unitary) module over $\mathbb{C}_n$ (left $\mathbb{C}_n$-module for short) is a vector space V together with an algebra morphism $L : \mathbb{C}_n \mapsto \mathrm{End}(V)$, or to say it more explicitly, there exists a linear transformation (also called left multiplication) $L(a)$ of V such that

$$L(ab + c) = L(a)L(b) + L(c)$$

for all $a \in \mathbb{C}_n$, and $L(1)$ is the identity operator. In the same way we have a right (unitary) module if there is a so-called right multiplication $R(a) \in \mathrm{End}(V)$ such that

$$R(ab + c) = R(b)R(a) + R(c).$$

Given either a left or a right multiplication we can always construct a right or a left multiplication by using any anti-automorphism of the algebra, for instance,

$$R(a) = L(\overline{a}).$$

A bi-module is a module which is both a left- and a right-module, or in other words, a module where left and right multiplication commute, i.e.,

$$L(a)R(b) = R(b)L(a), \text{ for all } a, b \in \mathbb{C}_n.$$

If V is a vector space of a $\mathbb{C}_n$-valued function we consider the left (right) multiplication defined by pointwise multiplication

$$(L(a)f)(x) = a(f(x)) \text{ and } (R(a)f)(x) = a(f(x)).$$

Also a mapping K between two right modules V and W is called a $\mathbb{C}_n$-linear mapping if

$$K(fa + g) = K(f)a + K(g).$$

We should also mention that in this paper we understand by a (left or right) Clifford-Banach module (see [20] for example). We say that X is a left Banach $\mathbb{C}_n$-module if X is a left $\mathbb{C}_n$-module and X is also a real Banach space such that for any $a \in \mathbb{C}_n$ and $x \in X$,

$$\|ax\|_X \leq C \, |a| \|x\|_X, \tag{2.1}$$

for some $C > 0$. In particular, equality occurs in (2.1) if $a \in \mathbb{C}$. Similarly, one can define a right Banach $\mathbb{C}_n$-module. These considerations give rise to the adequate right modules of $\mathbb{C}_n$-valued functions defined over any suitable subset E of $\mathbb{R}^n$.

Now, consider H to be a complex Hilbert space. Then $V := H \otimes \mathbb{C}_n$ defines a right Clifford-Hilbert module (right Hilbert module for short in this paper). Furthermore, the inner product $\langle \cdot, \cdot \rangle$ in H gives rise to two inner products in V:

$$\begin{aligned} (x, y) &:= \sum_A \langle x_A, y_A \rangle, \\ \langle x, y \rangle &:= \sum_{A,B} \langle x_A, y_B \rangle \overline{e}_A e_B. \end{aligned}$$

While the first inner product gives rise to a norm the second provides a generalization of Riesz's representation theorem in the sense that a linear functional ϕ is continuous if and only if it can be represented by an element $f_\phi \in V$ such that

$$\phi(g) = \langle f_\phi, g \rangle.$$

Many facts from classic Hilbert spaces carry over to the notion of a Hilbert module. For more details we refer to [12].

2.2. **Generalized fractional derivatives.** In this section we recall some basic facts about generalized fractional calculus. We start by presenting the following definition of generalized differentiation and integration operators.

Let the function

$$\varphi(\lambda) = \sum_{k=0}^{\infty} \varphi_k \, \lambda^k, \tag{2.2}$$

be an entire function with order $\rho > 0$ and degree $\sigma > 0$, that is, such that $\lim_{k\to\infty} k^{\frac{1}{\rho}} \sqrt[k]{|\varphi_k|} = (\sigma e \rho)^{\frac{1}{\rho}}$.

Definition 2.1. Let $\varphi(\lambda)$ be as in (2.2). We define the Gelfond-Leontiev (G-L) operator of generalized differentiation with respect to the function φ, and its corresponding G-L integration operator, as acting on an analytic function $f(z) = \sum_{k=0}^{\infty} a_k z^k, |z| < 1$, as

$$f \mapsto D_\varphi f(z) = \sum_{k=1}^{\infty} a_k \frac{\varphi_{k-1}}{\varphi_k} \, z^{k-1}, \qquad f \mapsto I_\varphi f(z) = \sum_{k=0}^{\infty} a_k \, \frac{\varphi_{k+1}}{\varphi_k} \, z^{k+1}. \tag{2.3}$$

Hence, under the condition on φ that $\limsup_{k\to\infty} \sqrt[k]{\left|\frac{\varphi_{k-1}}{\varphi_k}\right|} = 1$ by the Cauchy-Hadamard formula, we have that both series in (2.3) inherit the same radius of convergence $R > 0$ of the original series f.

More important, we remark that the function φ acts as the exponential function for the Gelfond-Leontiev operator of generalized differentiation since it holds that

$$D_\varphi \varphi(z) = \varphi(z).$$

Example 2.1. Let φ be the Mittag-Leffler function

$$E_{\frac{1}{\rho},\mu}(\lambda) = \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma\left(\mu + \frac{k}{\rho}\right)}, \qquad \rho > 0, \ \mu \in \mathbb{C}, \mathrm{Re}(\mu) > 0. \tag{2.4}$$

Now one has $\varphi_k = \frac{1}{\Gamma\left(\mu + \frac{k}{\rho}\right)}$ and the operators (2.3) turn into the so-called Dzrbashjan-Gelfond-Leontiev (D-G-L) operators (a particular case of (2.3)) of differentiation and integration:

$$D_{\rho,\mu} f(z) = \sum_{k=1}^{\infty} a_k \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k-1}{\rho}\right)} \, z^{k-1}, \qquad I_{\rho,\mu} f(z) = \sum_{k=0}^{\infty} a_k \, \frac{\Gamma\left(\mu + \frac{k}{\rho}\right)}{\Gamma\left(\mu + \frac{k+1}{\rho}\right)} \, z^{k+1}.$$

We now establish the fractional variable $x^\alpha \in \mathbb{C}$, which is a fractional complex power of a real variable to be understood as

$$x^\alpha := \begin{cases} \exp(\alpha \ \ln|x|), & x>0, \\ 0, & x=0, \\ \exp(\alpha \ \ln|x| + i\alpha\pi), & x<0, \end{cases} \tag{2.5}$$

with $0<\alpha<1$. The authors remark that in the present manuscript they restrict themselves to the case of $\alpha \in]0,1[$. Indeed, for values of α outside this range one can always reduce it to the previous case via $\alpha = [\alpha] + \tilde{\alpha}$, where $[\alpha]$ denotes its integer part and $\tilde{\alpha} \in]0,1[$.

We now consider $\mathbf{x} = \sum_{i=1}^n \mathbf{x}_i e_i \in \mathbb{C}^n$, where each component $\mathbf{x}_i = x_i^\alpha$ is a fractional power defined as in (2.5). These variables depend on $\alpha \in]0,1[$ but the index will be omitted to avoid overloading notation in the text. Based on the above statements we introduce the fractional Cauchy-Riemann operator

$$\mathbb{D} = \sum_{j=0}^n e_j \partial_j^\alpha = \partial_0^\alpha + e_1 \partial_1^\alpha + \cdots + e_n \partial_n^\alpha, \tag{2.6}$$

where ∂_j^α represents the G-L generalized derivative (2.3) with respect to the j-coordinate and the Mittag-Leffler function with $\alpha = 1/\rho$ and $\mu = 1$. This choice corresponds to the most important case equivalent to the Caputo derivative, but does not affect the generality of the results. Again, for simplifying the notation we omit the dependence on α since it is always a fixed parameter. Analogous to the Euclidean case a $\mathbb{C}_n$-valued function u is called *fractional left-monogenic* if it satisfies $\mathbb{D}u = 0$ on Ω (resp. *fractional right-monogenic* if it satisfies $u\mathbb{D} = 0$ on Ω). Moreover, we have

$$(\partial_j^\alpha)^s (\mathbf{x}_k)^m = \begin{cases} \varphi(m, m-s)(\mathbf{x}_k)^{m-s} \delta_{j,k}, & s \leq m, \\ 0, \quad \text{otherwise} \end{cases} \tag{2.7}$$

where $m, s \in \mathbb{N}$, $\delta_{j,k}$ denotes the Kronecker delta and $\varphi(a,b) := \frac{\varphi_a}{\varphi_b} = \frac{\Gamma(a\alpha+1)}{\Gamma(b\alpha+1)}$. This last term can be viewed as a (non-constant) deformation factor of the standard derivative.

3. Fractional monogenic functions

We now address the problem of construction of a basis for the space of homogeneous monogenic fractional polynomials. Following the ideas in [19], we consider the vector variable $z_j = \mathbf{x}_j - e_j \mathbf{x}_0$, $j = 1, \ldots, n$. It is easy to check that these vector variables are building blocks for a future construction of monogenic functions, i.e.,

$$\mathbb{D} z_j = \sum_{k=0}^n e_k \partial_k^\alpha (\mathbf{x}_j - e_j \mathbf{x}_0) = (e_j - e_j)\varphi(1,0) = 0.$$

Obviously, the choice of these variables is not unique. For instance, let $\tilde{z}_1 = e_2\mathbf{x}_1 - e_1\mathbf{x}_2$, $\tilde{z}_2 = e_3\mathbf{x}_1 - e_1\mathbf{x}_3$. Then $\frac{1}{2}(\tilde{z}_1\tilde{z}_2 + \tilde{z}_2\tilde{z}_1)$ is also a homogeneous monogenic fractional polynomial. This can be easily checked by straightforward calculations. However, the deformation mentioned in (2.7) now has an important crippling role. In order to construct homogeneous monogenic fractional polynomials it is required to compute for each basis element a correction factor. Although this can be done it is a troublesome and clumsy procedure, thus an indicator that such powers (or their modifications) are not convenient building blocks. Therefore, we turn our attention to the Cauchy-Kovalevskaya (CK-) extension (see [3, 9, 22]).

Lemma 3.1. *Given a homogeneous product $P_{\boldsymbol{\nu}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) := \mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}$, $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in \mathbb{N}_0^n$, its Cauchy-Kovalevskaya extension*

$$CK_\alpha[P_{\boldsymbol{\nu}}](\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) := [E_{\alpha,1}(-\mathbf{x}_0 \underline{\mathbb{D}})]\, \mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}, \tag{3.1}$$

is a monogenic polynomial homogeneous of degree $|\boldsymbol{\nu}| := \nu_1 + \cdots + \nu_n$. Hereby, $E_{\alpha,1}$ denotes the Mittag-Leffler function and $\underline{\mathbb{D}} := \sum_{j=1}^n e_j \partial_j^\alpha$.

Proof. Indeed, we have $\mathbb{D} = \partial_0^\alpha + \underline{\mathbb{D}}$ so that

$$\begin{aligned}
\mathbb{D} CK_\alpha[P_{\boldsymbol{\nu}}](\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) &= \mathbb{D}\,[E_{\alpha,1}(-\mathbf{x}_0 \underline{\mathbb{D}})]\, \mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n} \\
&= (\partial_0^\alpha + \underline{\mathbb{D}}) \sum_{k=0}^\infty \frac{(-1)^k (\mathbf{x}_0)^k}{\Gamma(k\alpha+1)} \underline{\mathbb{D}}^k (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}) \\
&= \sum_{k=1}^\infty \frac{(-1)^k \varphi(k, k-1) (\mathbf{x}_0)^{k-1}}{\Gamma(k\alpha+1)} \underline{\mathbb{D}}^k (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}) \\
&\quad + \sum_{k=0}^\infty \frac{(-1)^k (\mathbf{x}_0)^k}{\Gamma(k\alpha+1)} \underline{\mathbb{D}}^{k+1} (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}) \\
&= \sum_{k=0}^\infty \frac{(-1)^{k+1} \varphi(k+1, k) (\mathbf{x}_0)^k}{\Gamma((k+1)\alpha+1)} \underline{\mathbb{D}}^{k+1} (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}) \\
&\quad + \sum_{k=0}^\infty \frac{(-1)^k (\mathbf{x}_0)^k}{\Gamma(k\alpha+1)} \underline{\mathbb{D}}^{k+1} (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}) \\
&= \sum_{k=0}^\infty \left[\underbrace{-\frac{\varphi(k+1, k)}{\Gamma((k+1)\alpha+1)} + \frac{1}{\Gamma(k\alpha+1)}}_{=0} \right] (-1)^k (\mathbf{x}_0)^k \underline{\mathbb{D}}^{k+1} (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}) = 0. \qquad \square
\end{aligned}$$

The importance of the CK-extension lies in the fact that it provides an isomorphism between the space of fractional monogenics generated by $(\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ with coefficients in the Clifford algebra and the space of homogeneous polynomials in the variables $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ with Clifford-valued coefficients.

First, we consider the monogenic powers

$$\zeta^{\boldsymbol{\nu}}(\mathbf{x}) = CK_\alpha[P_{\boldsymbol{\nu}}](\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) := \sum_{j=0}^{|\boldsymbol{\nu}|} \frac{(-1)^j (\mathbf{x}_0)^j}{\Gamma(j\alpha+1)} \underline{\mathbb{D}}^j (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}), \tag{3.2}$$

where $|\boldsymbol{\nu}| = \nu_1 + \cdots + \nu_n$.

Example 3.1. We have $\zeta^0(\mathbf{x}) = 1$, while

$$\begin{aligned}
\zeta_j(\mathbf{x}) := \zeta^{e_j}(\mathbf{x}) = CK_\alpha[\mathbf{x}_j](\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) &= \frac{\mathbf{x}_j}{\Gamma(1)} - \frac{\mathbf{x}_0}{\Gamma(\alpha+1)} \underline{\mathbb{D}} \mathbf{x}_j \\
&= \mathbf{x}_j - \frac{\mathbf{x}_0}{\Gamma(\alpha+1)} \varphi(1, 0) e_j = \mathbf{x}_j - \mathbf{x}_0 e_j = z_j, \quad j = 1, \ldots, n.
\end{aligned}$$

Second, we consider the right Clifford module of power series.

Definition 3.1. We define the right Clifford module $\mathcal{M}$ as the space of monogenic powers

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(\mathbf{x}) f_{\boldsymbol{\nu}} = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} CK_\alpha[P_{\boldsymbol{\nu}}](\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_n) f_{\boldsymbol{\nu}}, \tag{3.3}$$

where $\sum_{\boldsymbol{\nu}} |f_{\boldsymbol{\nu}}|^2 < \infty$.

The series in (3.3) is called Fueter series since for $\alpha = 1$ it corresponds to the classical Fueter series.

From the construction it is clear that $\mathbb{D} f = 0$ for all $f \in \mathcal{M}$. Furthermore, it is easy to see that for $(\partial^\alpha)^{\boldsymbol{\mu}} := (\partial_1^\alpha)^{\mu_1} \cdots (\partial_n^\alpha)^{\mu_n}$ we get

$$\begin{aligned}
(\partial^\alpha)^{\boldsymbol{\mu}} f(\mathbf{x}) &= \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} (\partial^\alpha)^{\boldsymbol{\mu}} \zeta^{\boldsymbol{\nu}}(\mathbf{x}) f_{\boldsymbol{\nu}} \\
&:= \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \sum_{l=0}^{|\boldsymbol{\nu}|} \frac{(-\mathbf{x}_0 \underline{\mathbb{D}})^l}{\Gamma(\alpha l + 1)} (\partial_1^\alpha)^{\mu_1} \cdots (\partial_n^\alpha)^{\mu_n} (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}) f_{\boldsymbol{\nu}} \\
&= \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \Pi_{j=1}^{k} \varphi(\nu_j, \nu_j - \mu_j) \sum_{l=0}^{|\boldsymbol{\nu}|} \frac{(-\mathbf{x}_0 \underline{\mathbb{D}})^l}{\Gamma(\alpha l + 1)} (\mathbf{x}_1^{\nu_1} \cdots \mathbf{x}_n^{\nu_n}) f_{\boldsymbol{\nu}} \\
&= \sum_{l=0}^{\infty} \sum_{|\boldsymbol{\nu}|=l} \varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - \boldsymbol{\mu}) \zeta^{\boldsymbol{\nu} - \boldsymbol{\mu}}(\mathbf{x}) f_{\boldsymbol{\nu}}.
\end{aligned}$$

Hereby, we set $\varphi(\boldsymbol{\nu}, \boldsymbol{\nu} - \boldsymbol{\mu}) = \Pi_{j=1}^{k} \varphi(\nu_j, \nu_j - \mu_j)$.

Here, to obtain the Clifford-valued coefficients $f_{\boldsymbol{\nu}}$ we take (up to a constant) $(\partial^\alpha)^{\boldsymbol{\nu}} f(\mathbf{x})|_{\mathbf{x}=0}$. This leads us to the formula

$$f_{\boldsymbol{\nu}} := \frac{1}{\varphi(\boldsymbol{\nu}, 0)} \, (\partial^\alpha)^{\boldsymbol{\nu}} f(\mathbf{x})|_{\mathbf{x}=0}.$$

Here, we have to emphasize that we deal with G-L derivatives in which case the ground state, that is to say, the function [1] which is annihilated via $\partial^\alpha[1](\mathbf{x}) = 0$, is $[1](\mathbf{x}) = 1$. However, and in general, the ground state is not a constant but an analytic function with inverse in a neighborhood of zero. Thus, to overcome this problem it is required to first multiply with the inverse of the ground state function $[1](\mathbf{x})^{-1}$.

Finally, we define a product between monogenic power series. Let $f, g \in \mathcal{M}$ have the series expansion

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(\mathbf{x}) f_{\boldsymbol{\nu}}, \qquad g(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\mu}|=k} \zeta^{\boldsymbol{\mu}}(\mathbf{x}) g_{\boldsymbol{\mu}}.$$

Then we consider the following Cauchy product between both series:

$$(f \otimes g)(\mathbf{x}) := \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(\mathbf{x}) \left(\sum_{0 \le |\boldsymbol{\mu}| \le |\boldsymbol{\nu}|} f_{\boldsymbol{\mu}} g_{\boldsymbol{\nu} - \boldsymbol{\mu}} \right).$$

The above considerations about our series expansion allow us to study our problems in the setting of the ring of monogenic power series (cf. [3] for the case of $\alpha = 1$).

4. A Gleason-type problem

We now aim to express a radial difference $f(\mathbf{x})-f(0)$ as a sum of linear monogenic powers (our building blocks) times certain fixed bounded operators. Since the corresponding term in the fractional case is $f(\mathbf{x}) - [1](\mathbf{x})f_0$, or in our particular case $f(\mathbf{x}) - f_0$, we are going to express it in terms of reproducing kernels. To this end we are going to need some basic concepts on reproducing kernel Hilbert modules in the case of Clifford-valued functions.

We address the question of reproducing kernel the right-Hilbert module (RKHS) arising from our monogenic formal powers. We begin by fixing the weights. Given a sequence $c = (c_{\boldsymbol{\nu}}),\ c_{\boldsymbol{\nu}} \geq 0$, for all $\boldsymbol{\nu} \in \mathbb{N}^n$, we define its support as

$$\operatorname{supp}(c) := \{\boldsymbol{\nu} \in \mathbb{N}^n : c_{\boldsymbol{\nu}} \neq 0\}. \tag{4.1}$$

We can define the kernel

$$k_c(\mathbf{x}, \mathbf{y}) = \sum_{k=0}^{\infty} \sum_{\boldsymbol{\nu}\in\operatorname{supp}(c), |\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}} \zeta^{\boldsymbol{\nu}}(\mathbf{x}) \overline{\zeta^{\boldsymbol{\nu}}(\mathbf{y})} \tag{4.2}$$

associated to the domain

$$\Omega_c = \left\{ x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^{n+1} : \sum_{k=0}^{\infty} \sum_{\boldsymbol{\nu}\in\operatorname{supp}(c), |\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}} |\zeta^{\boldsymbol{\nu}}(\mathbf{x})|^2 < \infty \right\}. \tag{4.3}$$

To simplify our notation we shall write $\mathbf{x} \in \Omega_c$ whenever $x_0+x_1e_1+\cdots+x_ne_n \in \Omega_c$.

We denote by $\mathcal{H}(c)$ the associated reproducing kernel right-Hilbert module. Here $f(\mathbf{x}) = \sum_{\boldsymbol{\nu}\in\operatorname{supp}(c)} \zeta^{\boldsymbol{\nu}}(\mathbf{x}) f_{\boldsymbol{\nu}}$ belongs to $\mathcal{H}(c)$ iff

$$\|f\|_c^2 := \sum_{\boldsymbol{\nu}\in\operatorname{supp}(c)} \frac{|f_{\boldsymbol{\nu}}|^2}{c_{\boldsymbol{\nu}}} < \infty$$

and we have the reproducing formula

$$f(\mathbf{x}) = \int_{\Omega_c} k_c(\mathbf{x}, \mathbf{y}) f(\mathbf{y}) dy$$

which preserves right-linearity with respect to Clifford-valued constants.

We can introduce the Leibenson's shift operator

$$R_j f = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}-e_j} \frac{\varphi(\nu_j, \nu_j - 1)}{\sum_{r=0}^n \varphi(\nu_r, \nu_r - 1)} f_{\boldsymbol{\nu}}.$$

This operator is bounded in $\mathcal{H}(c)$ iff the set $\operatorname{supp}(c)$ is lower inclusive and it holds that

$$\sup_j \left(\frac{\varphi(\nu_j, \nu_j - 1)}{\sum_{k=0}^n \varphi(\nu_k, \nu_k - 1)} \right)^2 \frac{c_{\boldsymbol{\nu}}}{c_{\boldsymbol{\nu}-e_j}}. \tag{4.4}$$

In particular, we can easily see that these operators commute

$$R_j R_l f = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}-e_j-e_l} \frac{\varphi(\nu_j, \nu_j - 1)}{\sum_{r=0}^n \varphi(\nu_r, \nu_r - 1)} \frac{\varphi(\nu_l, \nu_l - 1)}{\sum_{r=0}^n \varphi(\nu_r, \nu_r - 1)} f_{\boldsymbol{\nu}} = R_l R_j f.$$

This leads us to the following version of Gleason's problem:

Let $\mathcal{M}$ be a set of functions which are fractional monogenic in a neighborhood of the origin. Given $f \in \mathcal{M}$ we want to find functions $p_1, \ldots, p_n \in \mathcal{M}$ such that

$$f(\mathbf{x}) - f(0) = \sum_{j=1}^{n} (\zeta_j \otimes p_j)(\mathbf{x}).$$

Theorem 4.1. *Gleason's problem is solvable in the reproducing kernel right-Hilbert module $\mathcal{H}(c)$ where the weights c satisfy (4.4) and our Leibenson's shift operators provide the only commutative solution to the problem.*

The proof of solvability is straightforward:

$$\begin{aligned}\sum_{j=1}^{n} \zeta_j \otimes R_j f &= \sum_{k=0}^{\infty} \sum_{|\nu|=k} \sum_{j=1}^{n} \zeta_j \otimes \zeta^{\boldsymbol{\nu}-e_j} \frac{\varphi(\nu_j, \nu_j - 1)}{\sum_{l=0}^{n} \varphi(\nu_l, \nu_l - 1)} f_{\boldsymbol{\nu}} \\ &= \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}} \sum_{j=1}^{n} \frac{\varphi(\nu_j, \nu_j - 1)}{\sum_{l=0}^{n} \varphi(\nu_l, \nu_l - 1)} f_{\boldsymbol{\nu}} = f - f_0.\end{aligned}$$

Hereby, we use that $\sum_{j=1}^{n} \frac{\varphi(\nu_j, \nu_j - 1)}{\sum_{l=0}^{n} \varphi(\nu_l, \nu_l - 1)} = 1$.

Let us now assume that $T_1, \ldots, T_n$ are commuting bounded operators on $\mathcal{H}(c)$ which solve Gleason's problem. Then we have for $f \in \mathcal{H}(c)$,

$$\begin{aligned}f(\mathbf{x}) &= f_0 + \sum_{j=1}^{n} (\zeta_j \otimes T_j f)(\mathbf{x}) \\ &= f_0 + \sum_{j=1}^{n} \zeta_j(\mathbf{x}) T_j f + \sum_{j,l=1}^{n} (\zeta_j \otimes \zeta_l \otimes T_l T_j f)(\mathbf{x}).\end{aligned}$$

Comparison with the power series expansion of f leads to

$$\begin{aligned}f(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} & \frac{\sum_{l_1=0}^{n} \varphi(\nu_{l_1}, \nu_{l_1} - 1)}{\varphi(\boldsymbol{\nu}, 0)} \\ & \times \frac{\sum_{l_2=0}^{n} \varphi(\nu_{l_2} - 1, \nu_{l_2} - 2) \ldots \sum_{l_k=0}^{n} \varphi(\nu_{l_k} - k + 1, \nu_{l_k} - k)}{\varphi(\boldsymbol{\nu}, 0)} \zeta^{\boldsymbol{\nu}}(\mathbf{x})(T^{\boldsymbol{\nu}} f),\end{aligned}$$

where $\varphi(\boldsymbol{\nu}, 0) = \Pi_{l=1}^{n} \varphi(\nu_l, 0)$. This means that we have

$$\begin{aligned}T^{\boldsymbol{\nu}} f = & \frac{\sum_{l_1=0}^{n} \varphi(\nu_{l_1}, \nu_{l_1} - 1)}{\varphi(\boldsymbol{\nu}, 0)} \\ & \times \frac{\sum_{l_2=0}^{n} \varphi(\nu_{l_2} - 1, \nu_{l_2} - 2) \ldots \sum_{l_k=0}^{n} \varphi(\nu_{l_k} - k + 1, \nu_{l_k} - k)}{\varphi(\boldsymbol{\nu}, 0)} f_{\boldsymbol{\nu}}\end{aligned}$$

since the coefficients are complex-valued and commute with the basic polynomial powers $\zeta^{\boldsymbol{\nu}}$.

5. Fractional monogenic operator-valued functions

Let us consider $\mathcal{H}$ as a right Clifford-Banach module. Under the sesquilinear form $\langle f, g \rangle = \int_{\mathbb{R}^{n+1}} f(\mathbf{x}) \overline{g(\mathbf{x})} dx$ its dual space $\mathcal{H}^*$ is also a right-linear Clifford Banach module. Let $\Omega \subset \mathbb{R}^{n+1}$ be a domain containing the origin and $f : \Omega \mapsto \mathcal{H}^*$ be a mapping such that for all $h \in \mathcal{H}$ we have $f(\cdot)h$ being a (left-)monogenic function in Ω. Such a mapping is called an $\mathcal{H}^*$-valued (left-)monogenic function in Ω.

As in the classical case [3] we can state the following theorem.

Theorem 5.1. *Let f be an $\mathcal{H}^*$-valued monogenic function in a ball $B(0,R)$ centered at the origin with radius R. Then f has a representation via*

$$f(\mathbf{x}) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(\mathbf{x}) f_{\boldsymbol{\nu}}$$

where $f_{\boldsymbol{\nu}} \in \mathcal{H}^*$ are linear functionals over $\mathcal{H}$. Hereby, the series converges normally in $B(0,R)$.

Proof. Let us start with the remark that the set of functionals $f(\mathbf{x}) : |\mathbf{x}| \leq R'$ is uniformly bounded for each $R' < R$, i.e., $\sup_{|\mathbf{x}|\leq R'} |f(\mathbf{x})| < \infty$. We choose $h \in \mathcal{H}$ arbitrarily and we consider

$$f(\mathbf{x})h = \sum_{k=0}^{\infty} P_k(\mathbf{x}, h)$$

as the expansion of $f(\mathbf{x})h$ into a series of fractional homogeneous polynomials of $\mathbf{x}$. For the terms $P_k(\mathbf{x},h)$ we have $|P_k(\mathbf{x},h)| \leq C\|h\| \sup_{|\mathbf{x}|\leq R'} \|f(\mathbf{x})\|$. Furthermore, we have that $P_k(\mathbf{x},h)$ is linear in h, i.e., $P_k(\mathbf{x},h) = P_k(\mathbf{x})h$ with $P_k(\mathbf{x}) \in \mathcal{H}^*$ and the series $\sum_k P_k(\mathbf{x})$ converges normally in $B(0,R)$ with respect to the operator norm. Since $P_k(\mathbf{x})h$ is fractional monogenic in $\mathbf{x}$ we can write it as $P_k(\mathbf{x})h = \sum_{|\boldsymbol{\nu}|=k} \zeta^{\boldsymbol{\nu}}(\mathbf{x}) f_{\boldsymbol{\nu}}(h)$ with $f_{\boldsymbol{\nu}}(h)$ being linear and bounded in h. This leads to our statement. □

The following corollary is a straightforward consequence of the previous theorem.

Corollary 5.1.1. *A reproducing kernel $k(\mathbf{x},\mathbf{y})$ can be represented as*

$$k(\mathbf{x},\mathbf{y}) = g(\mathbf{x})g(\mathbf{y})^*,$$

where $g(\mathbf{x})$ is an $\mathcal{H}(k)^*$-valued monogenic function and $\mathcal{H}(k)^*$ denotes the dual of the RKHS associated to $k = k(\cdot,\cdot)$.

Since $k(\mathbf{x},\mathbf{y})$ is a reproducing kernel we have $k(\mathbf{x},\mathbf{y}) = \langle k(\cdot,\mathbf{y}), k(\cdot,\mathbf{x})\rangle$. Introducing the operator $g(\mathbf{x})$ such that $g(\mathbf{x})^*1 = k(\cdot,\mathbf{x})$ is the operator of point evaluation we get, immediately, the above corollary.

6. Reproducing kernel spaces

Let us call the RKHS with the reproducing kernel $k_c(\mathbf{x},\mathbf{y})$ where the coefficients are given by

$$c_{\boldsymbol{\nu}} = \frac{\sum_{l_1=0}^{n} \varphi(\nu_{l_1}, \nu_{l_1}-1) \sum_{l_2=0}^{n} \varphi(\nu_{l_2}-1, \nu_{l_2}-2) \ldots \sum_{l_k=0}^{n} \varphi(\nu_{l_k}-k+1, \nu_{l_k}-k)}{\varphi(\boldsymbol{\nu},0)},$$

where $k = |\boldsymbol{\nu}|$, the fractional Drury-Arveson space (or module) $\mathcal{A}$. For this kernel we can state directly the following theorem.

Theorem 6.1. *Consider*

$$c_{\boldsymbol{\nu}} = \frac{\sum_{l_1=0}^{n} \varphi(\nu_{l_1}, \nu_{l_1}-1) \sum_{l_2=0}^{n} \varphi(\nu_{l_2}-1, \nu_{l_2}-2) \ldots \sum_{l_k=0}^{n} \varphi(\nu_{l_k}-k+1, \nu_{l_k}-k)}{\varphi(\boldsymbol{\nu},0)},$$

where $k = |\boldsymbol{\nu}|$*. Then* Ω_c *is the ellipsoid*

$$\Omega_c = \{x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}^{n+1} \ : \ n|x_0|^{2\alpha} + |x_1|^{2\alpha} + \cdots + |x_n|^{2\alpha} < 1\}$$

and (recall $\mathbf{x} = x_0^\alpha + x_1^\alpha e_1 + \cdots + x_n^\alpha e_n,\ \mathbf{y} = y_0^\alpha + y_1^\alpha e_1 + \cdots + y_n^\alpha e_n$*)*

$$k_{\mathcal{A}}(\mathbf{x},\mathbf{y}) := (1 - \langle \zeta(\mathbf{x}), \zeta(\mathbf{y})\rangle)^{-\otimes}.$$

Hereby, we have

$$(1 - \langle \zeta(\mathbf{x}), \zeta(\mathbf{y})\rangle)^{-\otimes} = \sum_{k=0}^{\infty} \langle \zeta(\mathbf{x}), \zeta(\mathbf{y})\rangle^{\otimes k},$$

where the products are taken in the sense of the CK-product $\otimes$. The ellipsoid comes from the fact that $|\zeta_1|^2 + |\zeta_2|^2 + \cdots + |\zeta_n|^2 = n|x_0|^{2\alpha} + |x_1|^{2\alpha} + \cdots + |x_n|^{2\alpha}$.

Theorem 6.2. *Let us denote by* C *the operator of evaluation at the origin, i.e.,*

$$Cf := \frac{1}{\varphi(\boldsymbol{\nu}, 0)} \left((\partial^\alpha)^{\boldsymbol{\nu}} f(\mathbf{x})\right)\big|_{\mathbf{x}=0}$$

and by M_{ζ_j} *the multiplication operator* $M_{\zeta_j} f = f \otimes \zeta_j$*. Then we have*

$$(I - \sum_{j=1}^{n} M_{\zeta_j} M_{\zeta_j}^*) = C^* C \tag{6.1}$$

if and only if $c_{\boldsymbol{\nu}} = \frac{\sum_{l_1=0}^{n} \varphi(\nu_{l_1}, \nu_{l_1}-1) \sum_{l_2=0}^{n} \varphi(\nu_{l_2}-1, \nu_{l_2}-2) \ldots \sum_{l_k=0}^{n} \varphi(\nu_{l_k}-k+1, \nu_{l_k}-k)}{\varphi(\boldsymbol{\nu},0)}$*, i.e., if and only if* f *belongs to the fractional Drury-Arveson space* $\mathcal{A}$*.*

The proof of this theorem is immediate if we apply the operator identity (6.1) to the kernel k_c and solve the resulting system.

As in the classical case we have the following corollary.

Corollary 6.2.1. *The multiplication operator* M_{ζ_j} *is a continuous operator in the fractional Drury-Arveson space and its adjoint is given by the Leibenson's shift operator* R_j*.*

Now, we can consider a function s such that the kernel

$$k_s(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{\infty} \sum_{|\boldsymbol{\nu}|=k} c_{\boldsymbol{\nu}} \left(\zeta^{\boldsymbol{\nu}}(\mathbf{x}) \overline{\zeta^{\boldsymbol{\nu}}(\mathbf{y})} - (\zeta^{\boldsymbol{\nu}} \otimes s)(\mathbf{x}) \overline{(\zeta^{\boldsymbol{\nu}} \otimes s)(\mathbf{y})} \right)$$

is positive. Such a function will be called a Schur multiplier. The reason is that such a function defines an operator M_s acting on a function f associated to the sequence $(f_{\boldsymbol{\nu}})_{\boldsymbol{\nu}}$ as

$$M_s f = \sum_{\boldsymbol{\nu}} \zeta^{\boldsymbol{\nu}} \left(\sum_{|\boldsymbol{\mu}| \le |\boldsymbol{\nu}|} s_{\boldsymbol{\mu}} f_{\boldsymbol{\nu}-\boldsymbol{\mu}} \right)$$

and this operator is a contraction from ℓ^2 into the fractional Drury-Arveson space $\mathcal{A}$. Please, note that M_s is a CK-multiplication of f with s from the left. This allows us write $k_s(\cdot, y) = (I - M_s M_s^*) k_{\mathcal{A}}$ and the right module operator range

$$\mathcal{H}(s) := (I - M_s M_s^*)^{\frac{1}{2}} \mathcal{A}$$

is the reproducing kernel module which is the counterpart to the de Branges-Rovnyak space in our setting. This operator range definition is one of the characterizations of de Brange-Rovnyak spaces [2, 8].

Let $\ell^2(\mathbb{C}_n)$ denote the space $\mathbb{C}_n$-valued sequences $\{(f_{\boldsymbol{\nu}})_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \mathbb{N}^n \text{ and } f_{\boldsymbol{\nu}} \in \mathbb{C}_n\}$ such that $\sum c_{\boldsymbol{\nu}} |f_{\boldsymbol{\nu}}|^2 < \infty$.

Theorem 6.3. *Given a $\mathcal{H}^*$-valued Schur multiplier s, then there exists a co-isometry*

$$V = \begin{pmatrix} T_1 & F_1 \\ T_2 & F_2 \\ \vdots & \vdots \\ T_n & F_n \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{H}(s) \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{H}(s)^n \\ \mathcal{H} \end{pmatrix}$$

such that

$$\begin{aligned} f(\mathbf{x}) - f(0) &= \sum_{j=1}^n (\zeta_j \otimes T_j)(\mathbf{x}), \\ (s(\mathbf{x}) - s(0))\, h &= \sum_{j=1}^n (\zeta_j \otimes F_j)(\mathbf{x}), \\ Gf &= f(0), \\ Hf &= s(0)h. \end{aligned}$$

Furthermore, $s(\mathbf{x})$ admits the representation

$$s(\mathbf{x})h = Hh + \sum_{k=1}^n \sum_{\boldsymbol{\nu} \in \mathbb{N}^n} c_{\boldsymbol{\nu}} (\zeta_j \otimes \zeta^{\boldsymbol{\nu}})(\mathbf{x}) G T^{\boldsymbol{\nu}} F_k h, \quad \mathbf{x} \in \Omega, h \in \mathcal{H},$$

where $T^{\boldsymbol{\nu}} := T_1^{\nu_1} \times \cdots \times T_n^{\nu_n}$.

Proof. We denote by $\mathcal{H}(s)_n$ the closure in $\mathcal{H}(s)^n$ of the linear span of the elements of the form

$$w_{\mathbf{y}} = \begin{pmatrix} R_1 k_s(\cdot, \mathbf{y}) \\ \vdots \\ R_n k_s(\cdot, \mathbf{y}) \end{pmatrix} = \begin{pmatrix} (I - M_s M_s^*) R_1 k_{\mathcal{A}}(\cdot, \mathbf{y}) \\ \vdots \\ (I - M_s M_s^*) R_n k_{\mathcal{A}}(\cdot, \mathbf{y}) \end{pmatrix}, \quad \mathbf{y} \in \Omega.$$

We define

$$(\hat{T} w_{\mathbf{y}} q)(\mathbf{x}) = (k_s(\mathbf{x}, \mathbf{y}) - k_s(\mathbf{x}, 0))q, \quad (\hat{F} w_{\mathbf{y}} q)(\mathbf{x}) = (s(\mathbf{y})^* - s(0)^*)q,$$
$$(\hat{G} q)(\mathbf{x}) = k_s(\mathbf{x}, 0)q, \quad \hat{H} q = s(0)^* q,$$

keeping in mind the isometry

$$\left\langle \begin{pmatrix} \hat{T} w_{\mathbf{y}_1} q_1 + \hat{G} p_1 \\ \hat{F} w_{\mathbf{y}_1} q_1 + \hat{H} p_1 \end{pmatrix}, \begin{pmatrix} \hat{T} w_{\mathbf{y}_2} q_2 + \hat{G} p_2 \\ \hat{F} w_{\mathbf{y}_2} q_2 + \hat{H} p_2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} w_{\mathbf{y}_1} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} w_{\mathbf{y}_2} q_2 \\ p_2 \end{pmatrix} \right\rangle,$$

for any $\mathbf{y}_1, \mathbf{y}_2 \in \Omega$ and $p_1, p_2, q_1, q_2 \in \mathbb{C}_n$. The latter isometry is important for the definition of the operators since a priori a linear combination of $w_{\mathbf{y}_i}$ could be zero and correspond to a non-zero image. The isometry formula overcomes that problem since a densely defined isometric relation between Hilbert modules extends to the graph of an isometry. Hence, the operator matrix $\hat{V} = \left(\begin{smallmatrix} \hat{T} & \hat{G} \\ \hat{F} & \hat{H} \end{smallmatrix}\right)$ can be extended as an isometry from $\left(\begin{smallmatrix} \mathcal{H}(s) \\ \mathcal{H} \end{smallmatrix}\right)$ into $\left(\begin{smallmatrix} \mathcal{H}(s)^n \\ \mathcal{H} \end{smallmatrix}\right)$. Let us set $V = \left(\begin{smallmatrix} T & G \\ F & H \end{smallmatrix}\right) = \hat{V}^*$. Then the previous relations imply $f(\mathbf{x}) - f(0) = \sum_{j=1}^n (\zeta_j \otimes T_j)(\mathbf{x}), (s(\mathbf{x}) - s(0))\, h = \sum_{j=1}^n (\zeta_j \otimes F_j)(\mathbf{x}), Gf = f(0)$ and $Hf = s(0)h$. Now, iterating $f(\mathbf{x}) - f(0) = \sum_{j=1}^n (\zeta_j \otimes T_j)(\mathbf{x})$ as before leads to the representation for $s(\mathbf{x})h$. $\square$

Theorem 6.4. *Let $\mathcal{G}, \mathcal{H}$ be right $\mathbb{C}_n$-valued Hilbert modules and let*

$$V = \begin{pmatrix} T_1 & F_1 \\ \vdots & \vdots \\ T_n & F_n \\ G & H \end{pmatrix} : \begin{pmatrix} \mathcal{G} \\ \mathcal{H} \end{pmatrix} \mapsto \begin{pmatrix} \mathcal{G}^n \\ \mathbb{C}_n \end{pmatrix}$$

be a co-isometry. Then

$$s_V(\mathbf{x}) = H + \sum_{k=1}^n \sum_{\boldsymbol{\nu}\in\mathbb{N}^n} c_{\boldsymbol{\nu}} (\zeta_j \otimes \zeta^{\boldsymbol{\nu}})(\mathbf{x}) G T^{\boldsymbol{\nu}} F_k, \quad \mathbf{x} \in \Omega,$$

is an $\mathcal{H}^*$-valued Schur multiplier.

Proof. We define

$$A_{\boldsymbol{\mu}}(\mathbf{x}) = \sum_{\boldsymbol{\nu}\in\mathbb{N}^n} c_{\boldsymbol{\nu}} \ \left(\Pi_{j=1}^n (\zeta_j \otimes \zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}})(\mathbf{x}) G T^{\boldsymbol{\nu}}\right),$$

$$B_{\boldsymbol{\mu}}(\mathbf{x}) = \sum_{\boldsymbol{\nu}\in\mathbb{N}^n} c_{\boldsymbol{\nu}} \ \zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}}(\mathbf{x}) G T^{\boldsymbol{\nu}}, \quad C(\mathbf{x}) = \sum_{\boldsymbol{\nu}\in\mathbb{N}^n} c_{\boldsymbol{\nu}} \ \zeta^{\boldsymbol{\nu}}(\mathbf{x}) G T^{\boldsymbol{\nu}}.$$

We get

$$\begin{aligned} A_{\boldsymbol{\mu}}(\mathbf{x}) A_{\boldsymbol{\mu}}(\mathbf{y})^* + \zeta^{\boldsymbol{\nu}}(\mathbf{x})\overline{\zeta^{\boldsymbol{\nu}}(\mathbf{y})} &= \left(A_{\boldsymbol{\mu}}(\mathbf{x})\zeta^{\boldsymbol{\nu}}(\mathbf{x})\right) V V^* \left(A_{\boldsymbol{\mu}}(\mathbf{y})\zeta^{\boldsymbol{\nu}}(\mathbf{y})\right)^* \\ &= \left[B_{\boldsymbol{\mu}}(\mathbf{x})(\zeta^{\boldsymbol{\nu}} \otimes s_V)(\mathbf{x})\right] \left[B_{\boldsymbol{\mu}}(\mathbf{y})(\zeta^{\boldsymbol{\nu}} \otimes s_V)(\mathbf{y})\right]^* \\ &= B_{\boldsymbol{\mu}}(\mathbf{x}) B_{\boldsymbol{\mu}}(\mathbf{y})^* + (\zeta^{\boldsymbol{\nu}} \otimes s_V)(\mathbf{x}) \left[(\zeta^{\boldsymbol{\nu}} \otimes s_V)(\mathbf{y})\right]^*. \end{aligned}$$

Hence,

$$\begin{aligned} k_{s_V}(\mathbf{x}, \mathbf{y}) &= \sum_{\boldsymbol{\mu}\in\mathbb{N}^n} c_{\boldsymbol{\mu}} \left(\zeta^{\boldsymbol{\mu}}(\mathbf{x})\overline{\zeta^{\boldsymbol{\mu}}(\mathbf{y})} - (\zeta^{\boldsymbol{\nu}} \otimes s_V)(\mathbf{x})(\zeta^{\boldsymbol{\nu}} \otimes s_V)(\mathbf{y})^*\right) \\ &= \sum_{\boldsymbol{\mu}\in\mathbb{N}^n} c_{\boldsymbol{\mu}} \left(B_{\boldsymbol{\mu}}(\mathbf{x}) B_{\boldsymbol{\mu}}(\mathbf{y})^* - A_{\boldsymbol{\mu}}(\mathbf{x}) A_{\boldsymbol{\mu}}(\mathbf{y})^*\right). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\sum_{\boldsymbol{\mu}\in\mathbb{N}^n} c_{\boldsymbol{\mu}} \ \left(\sum_{j=1}^n (\zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}} \otimes \zeta_j)(\mathbf{x}) G T^{\boldsymbol{\nu}} (T^{\boldsymbol{\eta}})^* G^* \overline{\zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}} \otimes \zeta_j)(\mathbf{y})}\right) \\ &\quad = \sum_{j=1}^n \sum_{\boldsymbol{\mu}:\mu_n>0} \frac{\mu_n}{|\boldsymbol{\mu}|} \ c_{\boldsymbol{\mu}} \ \zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}}(\mathbf{x}) G T^{\boldsymbol{\nu}} (T^{\boldsymbol{\eta}})^* G^* \overline{\zeta^{\boldsymbol{\eta}+\boldsymbol{\mu}}(\mathbf{y})} \\ &\quad = \sum_{\boldsymbol{\mu}:|\boldsymbol{\mu}|>0} \zeta^{\boldsymbol{\nu}+\boldsymbol{\mu}}(\mathbf{x}) G T^{\boldsymbol{\nu}} (T^{\boldsymbol{\eta}})^* G^* \overline{\zeta^{\boldsymbol{\eta}+\boldsymbol{\mu}}(\mathbf{y})}, \end{aligned}$$

so that $k_{s_V}(\mathbf{x}, \mathbf{y}) = C(\mathbf{x}) C(\mathbf{y})^*$. □

As a concluding remark we would like to point out that with the machinery which was just developed in this paper one can go further and define Blaschke factors and products, define rational functions and introduce the counterpart of Schur-Agler classes in this setting [1, 5]. These should have applications to fractional linear systems [14, 15].

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