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RESEARCH ARTICLE

A study of one class of NLP problems arising in parametric Semi-Infinite Programming Preprint

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The paper deals with a Nonlinear Programming (NLP) problem that depends on a finite number of integers (parameters). This problem has a special form, and arises as an auxiliary problem in study of solutions' properties of parametric Semi-infinite Programming (SIP) problems with finitely representable compact index sets. Therefore it is important to provide a deep study of this NLP problem and its properties w.r.t. the values of the parameters. We are especially interested in the case when optimal solutions of the NLP problem satisfy certain properties due to some specific requirements arising in parametric SIP. We establish the values of the parameters for which optimal solutions of the corresponding NLP problem fulfill the needed properties, and suggest an algorithm that determines the *right* values of the parameters. An example is proposed to illustrate the application of the algorithm.

Keywords: Nonlinear Programming (NLP); Semi-infinite Programming (SIP); Quadratic Programming (QP); parametric problems; optimality conditions

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1. Introduction

Semi-infinite Programming (SIP) deals with extremal problems that involve infinitely many constraints in a finite dimensional space. Semi-infinite optimization has always been a topic of a special interest due to the numerous theoretical and practical applications such as robotic, classical engineering, optimal design, the Chebyshev approximations etc. (see [8, 10, 11], and the references therein). Nowadays, SIP models are efficiently used in dynamic processes, biomedical and chemical engineering, biology, tissue engineering, polymer reaction engineering, etc. (see [1, 25], and others).

Generally, a Semi-infinite Programming (SIP) problem can be formulated as follows:

$$\min_{\varkappa\in\mathbb{R}^n} c(\varkappa) \qquad \text{s.t.} \quad f(\varkappa,\tau)\leq 0 \;\; \forall\, \tau\in T,$$

where $\varkappa\in\mathbb{R}^n$ is a decision variable, τ is a constraint index, $T\subset\mathbb{R}^p$ is an infinite index

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set. When, additionally, the index set T depends on the decision variable $\varkappa$, one gets a problem of the *generalized* SIP (see [12]).

The use of the SIP models for real processes systems is often associated with global parametric identifiability of a dynamic model and a robust design of dynamic experiments where certain parameters arise (see [1, 3, 4], et al). In such situations, the objective function and the functions defining the feasible set depend additionally on so called *perturbational parameters* and problems of *parametric* Semi-infinite Programming arise. Many fundamental and especial results concerning parametric Semi-infinite Programming are due to H. Th. Jongen, J.-J. Rückmann, and G.- W. Weber ([15–17]) as well as to F.J. Bonnans, A. Shapiro, G. Still, O. Stein and others (see e.g. [3, 13, 24]). For applications of the parametric SIP see [1, 23–25] et al.

When one deals with the parametric SIP problems, then even small perturbations of the parameters can seriously change the properties of solutions. Hence the study of the dependence of solutions on parameters is a topical issue in parametric SIP (see e.g. [3, 15, 16, 24]).

In our study, we are interested in properties of auxiliary NLP problems that arise when the following parametric SIP problem is being studied:

$$(SIP(\varepsilon)): \quad \min_{\varkappa} c(\varkappa,\varepsilon), \quad \text{s.t.} \quad f(\varkappa,\tau,\varepsilon)\leq 0,\ \tau\in T=\{\tau\in\mathbb{R}^p : g_i(\tau)\leq 0, i\in\bar I\},$$

where the index set $T\in\mathbb{R}^p$ is compact, functions

$$c(\varkappa,\varepsilon),\ f(\varkappa,\ \tau,\ \varepsilon),\ g_i(\tau), i\in\bar I, \tag{1}$$

are sufficiently smooth w.r.t. all their arguments, and $\varepsilon>0$ is a parameter, $\varepsilon\in\mathcal{E}(\varepsilon_0)=[\varepsilon_0,\ \varepsilon_0+\delta]$, with sufficiently small $\delta>0$, ε_0 being the unperturbed parameter value.

When considering a problem $(SIP(\varepsilon))$, with a fixed ε, one is interested in finding an optimal solution $\varkappa(\varepsilon)$ of this problem, the corresponding active index set $T_a(\varkappa(\varepsilon)):=\{\tau\in T : f(\varkappa(\varepsilon),\tau,\varepsilon)=0\}$ and the Lagrange multipliers vector $\eta(\varepsilon)=(\eta(\tau),\tau\in T_a(\varkappa(\varepsilon)))$ satisfying the first order optimality conditions (see e.g. [10, 11], et al.).

Suppose that for the problem $(SIP(\varepsilon_0))$ (corresponding to an unperturbed parameter value $\varepsilon:=\varepsilon_0$) we have found an optimal solution $\varkappa(\varepsilon_0)$ with a finite active index set $T_a(\varkappa(\varepsilon_0))=\{\tau_j(\varepsilon_0),\ j\in I\}$, and the Lagrange multipliers vector $\eta(\varepsilon_0)=(\eta_j(\varepsilon_0), j\in I)$. Then under rather "nonrestrictive" assumptions one can show that the perturbed problem $(SIP(\varepsilon))$ with $\varepsilon\in\mathcal{E}(\varepsilon_0)$, has an optimal solution $\varkappa(\varepsilon)\in\mathbb{R}^n$ such that $\varkappa(\varepsilon)\to\varkappa(\varepsilon_0)$ as $\varepsilon\to\varepsilon_0$. Moreover the corresponding active index set $T_a(\varkappa(\varepsilon))$ and the Lagrange multipliers vector $\eta(\varepsilon)$ admit the presentations

$$T_a(\varkappa(\varepsilon))=\{\tau_{kj}(\varepsilon), k=1,\dots,p_j,\ j\in\bar J\},\ \eta(\varepsilon)=(\eta_{kj}(\varepsilon), k=1,\dots,p_j,\ j\in\bar J),$$

such that $\tau_{kj}(\varepsilon)\to\tau_j(\varepsilon_0), k=1,\dots,p_j,\ j\in\bar J,\ \sum_{k=1}^{p_j}\eta_{kj}(\varepsilon)\to\eta_j(\varepsilon_0), j\in J$, with some integer parameters $p_j\geq 1, j\in J,\ p_j=1, j\in\bar J\setminus J$, and a set $\bar J,\ J\subset\bar J\subset I,\ J:=\{j\in I : \eta_j(\varepsilon_0)>0\}$.

In other words this means the following:

• for each $j\in\bar J$, the active index $\tau_j(\varepsilon)$ of the unperturbed $(SIP(\varepsilon_0))$ problem generates p_j active indices $\tau_{kj}(\varepsilon), k=1,\dots,p_j$, of the perturbed problem $(SIP(\varepsilon))$,

• for each $j\in I\setminus\bar J$, the active index $\tau_j(\varepsilon)$ of the unperturbed $(SIP(\varepsilon_0))$ problem does not generate any active index of the perturbed problem $(SIP(\varepsilon))$.

Notice that the integer parameters $p_j, j \in J$ and the set $\bar{J}$, are unknown *a priori* and can not be evidently found on the basis of a known solution of the unperturbed problem $(SIP(\varepsilon_0))$.

One of the main goals of study of the parametric SIP problem $(SIP(\varepsilon))$ consists in the following: based on the known solution $\varkappa(\varepsilon_0),\ \tau_j(\varepsilon_0),\ \eta_j(\varepsilon_0),\ j \in I$, of the unperturbed problem $SIP(\varepsilon_0)$ and the derivatives of functions (1) w.r.t. their arguments calculated at $\varkappa(\varepsilon_0),\ \tau_j(\varepsilon_0),\ \ j \in I$, and ε_0, to predict the behavior of optimal solutions of the problem $(SIP(\varepsilon))$ under small perturbations of ε. For example, it is interesting to know

a) the integer parameters $p_j, j \in J$ and the set $\bar{J}$;
b) the initial values $\eta_{ki}(\varepsilon_0 + 0), k = 1, \dots, p_j,\ i \in \bar{J}$;
c) the derivatives $\dot{\varkappa}(\varepsilon_0 + 0),\ \dot{\tau}_{kj}(\varepsilon_0 + 0), k = 1, \dots, p_j, j \in J$, and $\dot{\eta}_{ki}(\varepsilon_0 + 0), k = 1, \dots, p_j,\ i \in \bar{J}$.

In our subsequent paper dedicated to study of the parametric SIP problem $(SIP(\varepsilon))$, we will show that all of these data can be found on the basis of an optimal solution of some auxiliary NLP problem $P(p_j, j \in J)$ that depends on the integers $p_j, j \in J$, mentioned above and has the following decision variables vector: $\xi = (x,\ t_{kj}, y_{kj}, k = 1, ..., p_j,\ j \in J;\ y_i,\ i \in I)$. Namely, if the "right" values of integer parameters $p_j, j \in J$, are found, and $\xi^0 = (x^0,\ t^0_{kj}, y^0_{kj}, k = 1, ..., p_j,\ j \in J;\ y^0_i,\ i \in I)$ is an optimal solution of the corresponding problem $P(p_j, j \in J)$, then $\dot{\varkappa}(\varepsilon_0 + 0) = x^0$, $\dot{\tau}_{kj}(\varepsilon_0 + 0) = t^0_{kj}, k = 1, \dots, p_j, j \in J$, $\eta_{kj}(\varepsilon_0 + 0) = y^0_{kj}, k = 1, \dots, p_j,\ i \in J$; $\eta_{ki}(\varepsilon_0 + 0) = 0, k = 1, \dots, p_j, j \in \bar{J} \setminus J$; and $\sum\limits_{k=1}^{p_i} \dot{\eta}_{ki}(\varepsilon_0 + 0) = y^0_i,\ i \in \bar{J}$.

Hence, to obtain the data a) - b), we have to find the "right" values of the parameters $p_j, j \in J$, and solve the NLP problem $P(p_j, j \in J)$. In its term, the "right choice" of the parameters $p_j, j \in J$, is characterized by the fact that the optimal solutions of the corresponding problem $P(p_j, j \in J)$ possess some additional properties. Therefore it is important to provide a deep study of this auxiliary NLP problem and its properties w.r.t. the values of the parameters.

As well as most NLP problems arising in applications (see for example [2, 18]), the problem $P(p_j, j \in J)$ has a special form. It is well-known that a detailed study of NLP problems taking in respect their specific structure permits one to get more strong theoretical results and to create efficient numerical methods [7, 14].

This paper is dedicated to study of the properties of the problem $P(p_j, j \in J)$ w.r.t. the parameters $p_j \geq 1, j \in J$. We will justify the existence of the parameters' values for which the problem $P(p_j, j \in J)$ admits an optimal solution possessing certain properties and describe a procedure that permits to find these parameters values.

As far as we know, in literature there is no detailed study of NLP problems in the form $P(p_j, j \in J)$ in respect of the above mentioned aspects.

The rest of the paper is organized as follows. In section 2, we formulate problem $P(p_j, j \in J)$ and present some of its properties that will be used in our subsequent paper dedicated to parametric SIP. The main result of section 3 consists in formulation and proof of optimality conditions for problem $P(p_j, j \in J)$. In section 4, we provide a detailed study of properties of optimal solutions of $P(p_j, j \in J)$ for different values of the parameters (lemmas 4.1-4.6) and on the basis of the obtained results, formulate conditions that guarantee the existence of the values of the integers $p_j \geq 1, j \in J$, such that optimal solutions of the corresponding problem $P(p_j, j \in J)$ possess the properties formulated in section 2. In section 5, we present the conditions that guarantee solvability of the problem $P(p_j, j \in J)$. A conceptual algorithm that determines the set of integers $p_j, j \in J$ such that an optimal solution of the problem $P(p_j, j \in J)$ satisfies the properties from section 2, is described in section 6, and an example illustrating application of this algorithm is presented in section 7. The final section 8 contains some conclusions. We have included in the paper three appendices containing some technical proofs and constructive procedures which can contribute to a better understanding of some of the allegations and numerical rules for testing assumptions.

2. Problem statement

Suppose that the following index sets:

$$I = I_1 \cup I_2,\ I_1 \cap I_2 = \emptyset,\ |I_1| \le n;\ J,\ |J| \le n, \tag{2}$$

matrices, vectors and numbers

$$\begin{aligned} &D \in \mathbb{R}^{n\times n}, c \in \mathbb{R}^n,\ D_j \in \mathbb{R}^{p\times p}, A_j \in \mathbb{R}^{n\times p}, B_j \in \mathbb{R}^{s_j\times p},\\ &c_j \in \mathbb{R}^p, m_j \in \mathbb{R}, m_j > 0, j \in J,\ q_i \in \mathbb{R}^n, \omega_i \in \mathbb{R}, i \in I, \end{aligned} \tag{3}$$

are given and fixed. These data are uniquely generated by data of unperturbed parametric SIP problem $(SIP(\varepsilon_0))$ and its optimal solution, for example

$$\begin{aligned} D_j := \tfrac{\partial^2 f(\varkappa^0,\tau_i^0,\varepsilon_0)}{\partial t^2},\ A_j := \tfrac{\partial^2 f(\varkappa^0,\tau_i^0,\varepsilon_0)}{\partial t \partial x},\ c_j := \tfrac{\partial^2 f(\varkappa^0,\tau_i^0,\varepsilon_0)}{\partial t \partial \varepsilon},\ m_j := \eta_j(\varepsilon_0) j \in J,\\ q_i := \tfrac{\partial f(\varkappa^0,\tau_i^0,\varepsilon_0)}{\partial x},\ \omega_i = \tfrac{\partial f(\varkappa^0,\tau_i^0,\varepsilon_0)}{\partial \varepsilon},\ i \in I. \end{aligned} \tag{4}$$

Denote $K(j) := \{l \in \mathbb{R}^p : B_j l \le 0\},\ j \in J$, and suppose that

$$D = D^T,\ \ x^T D x \ge 0\ \ \forall x \in \mathbb{R}^n,\ \ D_j = D_j^T,\ t^T D_j t\ \le\ 0\ \ \forall t \in K(j), j \in J. \tag{5}$$

Let relations

$$\sum_{i\in I} q_i \Delta y_i = 0, \Delta y_i \ge 0, i \in I_2, \tag{6}$$

imply the inequality

$$-\sum_{i\in I} \omega_i \Delta y_i \ge 0. \tag{7}$$

We omit here a detailed explanation of the origin of conditions (5) and implication (6) $\Longrightarrow$ (7), just mention that this is a property of the data of the parametric SIP problem $(SIP(\varepsilon_0))$ and is supposed to be satisfied in our study. The importance of this implication (6) $\Longrightarrow$ (7) will be explain is what follows.

For any fixed set of the integers $p_j \geq 1, j \in J$, consider problem in the form

$$\begin{aligned} &\min F(\xi) := \frac{1}{2}x^T Dx - \sum_{j\in J}\sum_{k=1}^{p_j} y_{kj}\Big(\frac{1}{2}t_{kj}^T D_j t_{kj} + c_j^T t_{kj}\Big) - \sum_{i\in I}\omega_i y_i, \\ &\text{s.t. } \mathcal{F}(\xi) := Dx + \sum_{j\in J} A_j \sum_{k=1}^{p_j} y_{kj}t_{kj} + \sum_{i\in I} q_i y_i + c = 0, \\ &y_i \geq 0,\ i \in I_2;\ \sum_{k=1}^{p_j} y_{kj} = m_j,\ y_{kj} \geq 0,\ t_{kj} \in K(j),\ k = 1, ..., p_j;\ j \in J, \end{aligned} \tag{8}$$

where

$$\xi = \xi(p_j, j \in J) = (x,\ t_{kj}, y_{kj}, k = 1, ..., p_j,\ j \in J;\ y_i,\ i \in I) \tag{9}$$

is a vector of decision variables. In what follows, we denote problem (8) by $P(p_j, j \in J)$.

It can be shown that that the fulfillment of the implication (6) $\Longrightarrow$ (7) is a necessary condition for boundedness from below of the cost function of the problem $P(p_j, j \in J)$. Moreover, due to this implication , for the problem $P(p_j, j \in J)$ introduced above, without loss of generality we may consider that $\text{rank}(q_i, i \in I_1) = |I_1|$ (see Lemma A.1 in Appendix A).

The problem $P(p_j, j \in J)$ is a parameterized NLP problem in a special form. When values $y_{kj}, k = 1, ..., p_j,\ j \in J$ are fixed (in particular, when $p_j = 1,\ j \in J$), this problem becomes a nonconvex Quadratic Programming (QP) problem. Hence the problem $P(p_j, j \in J)$ can be considered as *a weighted QP problem* that incorporates additional nonlinearities.

Motivated by the ultimate aim of our study in parametric SIP, we are particularly interested in determination of the values of the parameters $p_j \geq 1, j \in J$, for which the problem $P(p_j, j \in J)$ admits an optimal solution

$$\xi^0 = \xi^0(p_j, j \in J) = (x^0,\ t^0_{kj}, y^0_{kj}, k = 1, ..., p_j,\ j \in J;\ y^0_i,\ i \in I), \tag{10}$$

possessing certain properties that are listed next.

Property 1: The following inequalities take place:

$$y^0_{kj} > 0, k = 1, ..., p_j,\ j \in J. \tag{11}$$

Property 2: The following rank condition is satisfied:

$$\text{rank}\left(A_j(t^0_{kj} - t^0_{1j}),\ k = 2, ..., p_j, j \in J_*, q_i, i \in I_1 \cup I^a_2\right) = |I^a_2| + \sum_{j\in J} p_j + \gamma_*,$$

where $|K|$ denotes a number of elements of a set K, $\gamma_* := |I_1| - |J|$, $J_* = \{j \in J : p_j \geq 2\}$, $I^a_2 = \{i \in I_2 :\ y^0_i > 0\}$.

Notice that the above equality implies $|I^a_2| + \sum\limits_{j\in J} p_j + \gamma_* \leq n$.

Property 3: For any $j \in J$, vectors

$$t^0_{kj}, \ k \in \{s \in \{1, ..., p_j\} : y^0_{sj} > 0\}, \tag{12}$$

are global optimal solutions in the problem

$$\min \ \Psi_j(t) := \Big(-\frac{1}{2}t^T D_j t - (c_j + A_j^T x^0)^T t\Big), \ \text{ s.t. } \ t \in K(j). \tag{13}$$

Notice that problem (13) is quadratic but not convex.

The aim of this paper is to study the properties of the class of the NLP problems $P(p_j, j \in J)$ with different values of the parameters $p_j \geq 1, j \in J$, and on the base of the obtained results to

- prove that there exist the values of the integers $p_j \geq 1, j \in J$, such that the corresponding problem $P(p_j, j \in J)$ possesses the Properties 1) - 3) mentioned above,
- propose an algorithm that allows to find such *right* values of the integers.

3. Optimality conditions for the problem $P(p_j, j \in J)$

Let us recall here some known results of the NLP theory that we will use in what follows.

Consider a general nonlinear problem

$$\min c(x), \ \text{ s.t. } f_i(x) = 0, i \in S_1; \ f_i(x) \leq 0, i \in S_2. \tag{14}$$

Let x^0 be a feasible solution of problem (14). Denote by $S_2(x^0) = \{i \in S_2 : \ f_i(x^0) = 0\}$ the set of the inequality constraints of this problem that are strongly satisfied at x^0.

Definition 1 The Relaxed Constant Rank constraint qualification (RCRCQ) is said to be satisfied at a feasible solution x^0 of problem (14) if there exists a neighborhood $V(x^0)$ of x^0 such that for any index set $S \subset S_2(x^0)$, the set of vectors $\{\nabla f_i(x), i \in S_1 \cup S\}$ has constant rank in $V(x^0)$.

The following statement can be formulated on the basis of [22].

Proposition 3.1 *Let x^0 be an optimal solution of problem (14) and let (RCRCQ) be satisfied at x^0. Then there exist numbers $\lambda_i, i \in S_1 \cup S_2(x^0)$ such that*

$$\nabla c(x^0) + \sum_{i \in S_1 \cup S_2(x^0)} \lambda_i \nabla f_i(x^0) = 0, \ \lambda_i \geq 0, i \in S_2(x^0).$$

Notice that the problem $P(p_j, j \in J)$ is a particular case of problem (14). Let us show that any feasible solution ξ of $P(p_j, j \in J)$ satisfies the condition ($RCRCQ$). Consider the matrix

$$\frac{\partial \mathcal{F}(\xi)}{\partial \xi} = (D, A_j y_{kj}, A_j t_{kj}, k = 1, ..., p_j, \ j \in J; \ q_i, \ i \in I),$$

where the function $\mathcal{F}(\xi)$ is defined in (8). Since ξ is feasible in (8), then from the constraints of this problem we conclude that for any $j \in J$ there exists $k(j) \in \{1, ..., p_j\}$ such that $y_{k(j)j} \neq 0$. Consequently rank $(A_j y_{kj}, A_j t_{kj}, k = 1, ..., p_j) =$ rank $A_j,\ j \in J$ and hence

$$\text{rank } (D, A_j y_{kj}, A_j t_{kj}, k = 1, ..., p_j,\ j \in J;\ q_i,\ i \in I) = \text{ rank } (D, A_j,\ j \in J;\ q_i,\ i \in I).$$

Therefore the gradients of the function $\mathcal{F}(\xi)$ defining the equality constraints $\mathcal{F}(\xi) = 0$ in the problem $P(p_j, j \in J)$, have the constant rank at any feasible solution. Notice that in problem (8) all the constraints in the form of inequalities as well as the equality constraints $\sum\limits_{k=1}^{p_j} y_{kj} = m_j$, $j \in J$, are linear.

Hence we have shown that the constraint qualification $(RCRCQ)$ is satisfied for any feasible solution ξ of the problem $P(p_j, j \in J)$. Thus, it follows from Proposition 3.1 that the necessary optimality conditions for this problem take the form of the following proposition.

Proposition 3.2 *Let*

$$\bar{\xi}^0 = (\bar{x}^0,\ t^0_{kj}, y^0_{kj}, k = 1, ..., p_j,\ j \in J;\ y^0_i,\ i \in I) \tag{15}$$

be an optimal solution of the problem $P(p_j, j \in J)$. Then there exists a vector of the Lagrange multipliers

$$(z \in \mathbb{R}^n, \lambda(j) \in \mathbb{R},\ \mu(k, j) \in \mathbb{R}^{s_j},\ k = 1, ..., p_j, j \in J), \tag{16}$$

such that

$$D\bar{x}^0 + Dz = 0, \tag{17}$$

$$-q_i^T z + \omega_i = 0,\ i \in I_1;\ \ -q_i^T z + \omega_i \leq 0,\ y^0_i(q_i^T z - \omega_i) = 0, i \in I_2, \tag{18}$$

$$A_j^T z - D_j t^0_{kj} - c_j + B_j^T \mu(k, j) = 0, \mu(k, j) \geq 0, \mu^T(k, j) B_j t^0_{kj} = 0, k \in P^*_j, j \in J; \tag{19}$$

$$\begin{aligned} -\tfrac{1}{2} t^{0T}_{kj} D_j t^0_{kj} - c_j^T t^0_{kj} + z^T A_j t^0_{kj} + \lambda(j) = 0,\ k \in P^*_j, \\ -\tfrac{1}{2} t^{0T}_{kj} D_j t^0_{kj} - c_j^T t^0_{kj} + z^T A_j t^0_{kj} + \lambda(j) \geq 0,\ k \in P^0_j,\ \ j \in J, \end{aligned} \tag{20}$$

where

$$P^*_j := \{k \in \{1, ..., p_j\} : y^0_{kj} > 0\},\ P^0_j := \{1, ..., p_j\} \setminus P^*_j,\ j \in J. \tag{21}$$

Taking into account the definition of the problem $P(p_j, j \in J)$, one can notice that if vector (15) is its optimal solution, then any vector $(x,\ t^0_{kj}, y^0_{kj}, k = 1, ..., p_j,\ j \in J;\ y^0_i,\ i \in I)$ with x satisfying the equality $D(\bar{x}^0 - x) = 0$ is an optimal solution as well.

To reduce such an ambiguity in optimal solutions of the problem $P(p_j, j \in J)$, in what follows, without loss of generality, we will consider optimal solutions in the form

$$\xi^0 = (x^0,\ t^0_{kj}, y^0_{kj}, k = 1, ..., p_j,\ j \in J;\ y^0_i,\ i \in I), \tag{22}$$

where $x^0 = -z$, z being the vector of the first components of the Lagrange multiplier vector (16).

The necessary optimality conditions for the solution ξ^0 defined in (22) can be rewritten as follows:

$$-A_j^T x^0 - D_j t_{kj}^0 - c_j + B_j^T \mu(k,j) = 0, \mu(k,j) \geq 0, \mu^T(k,j) B_j t_{kj}^0 = 0, k \in P_j^*, j \in J; \quad (23)$$

$$-\frac{1}{2} t_{kj}^{0T} D_j t_{kj}^0 - c_j^T t_{kj}^0 - x^{0T} A_j t_{kj}^0 + \lambda(j) \begin{cases} = 0, & k \in P_j^*, \\ \geq 0, & k \in P_j^0, \end{cases} \quad j \in J; \quad (24)$$

$$q_i^T x^0 + \omega_i = 0, \ i \in I_1; \ \ q_i^T x^0 + \omega_i \leq 0, \ y_i^0 (q_i^T x^0 + \omega_i) = 0, i \in I_2. \quad (25)$$

Hence we have proved the following theorem.

THEOREM 3.3 *(The first order necessary optimality conditions) Let vector (15) be an optimal solution of the problem $P(p_j, j \in J)$. Then there exist vector x^0 and multipliers $(\lambda(j), \mu(k,j),\ k = 1, ..., p_j, j \in J)$ such that vector (22) is an optimal solution in $P(p_j, j \in J)$ and relations (23)- (25) are satisfied.*

It is easy to verify that from (23), (24), it follows $\lambda(j) = -\frac{1}{2} t_{kj}^{0T} D_j t_{kj}^0$, $k \in P_j^*, j \in J$.
Let us make the following assumption.

ASSUMPTION 1 *For an optimal solution (15) of the problem $P(p_j, j \in J)$, there do not exist two vectors of the Lagrange multipliers (16) satisfying (17)-(20) with different components z.*

Remark 1 It should be noticed that if

$$l^T D l > 0, \ \ \forall \ l \in \{l \in \mathbb{R}^n : q_i^T l = 0, \ i \in I_1; \ \ q_i^T l \leq 0, i \in I_2, l^T A_j t_{kj}^0 = 0, k \in P_j^*, j \in J\} \backslash \{0\},$$

then Assumption 1 is satisfied. Other necessary and sufficient conditions guaranteeing the fulfillment of this assumption (as well as constructive rules for its verification) are presented in the Appendix.

Note that if $y_{kj}^0 = 0$ for some $k \in P_j^*, j \in J$, then any vector from $K(j)$ can be chosen as the component t_{kj}^0 in the optimal solution (22). Therefore, under Assumption 1 the condition (24) can be rewritten in the form

$$\Psi_j(t_{kj}^0) = -\lambda(j), \ k \in P_j^*; \ \ \Psi_j(t) \geq -\lambda(j) \ \forall t \in K(j), \ k \in P_j^0, \ j \in J, \quad (26)$$

the functions $\Psi_j, \ j \in J$, being defined as in (13). Hence we can formulate the following corollary of Theorem 3.3.

COROLLARY 3.4 *Let ξ^0 be an optimal solution of problem $P(p_j, j \in J)$ satisfying Assumption 1 and suppose that $P_j^0 \neq \emptyset$, $j \in J$, in (21). Then ξ^0 satisfies Property 3).*

LEMMA 3.5 *Given $j \in J$, let numbers y_{kj}^0, $k = 1, ..., p_j$, satisfy the conditions*

$$\sum_{k=1}^{p_j} y_{kj}^0 = m_j, \ y_{kj}^0 \geq 0, \ k = 1, ..., p_j,$$

vectors t_{kj}^0, $k \in P_j^*$, be global optimal solutions in problem (13) and $t_{kj}^0 \in K(j), k \in P_j^0$, with P_j^*, P_j^0 defined in (21). Then the vector

$$\nu_j^0 := (t_{kj}^0, \ y_{kj}^0, \ k = 1, ..., p_j) \quad (27)$$

is a global optimal solution of the problem

$$
\begin{aligned}
&\min \Phi_j(\nu_j) := -\sum_{k=1}^{p_j} y_{kj}\Big(\frac{1}{2}t_{kj}^T D_j t_{kj} + (c_j + A_j^T x^0)^T t_{kj}\Big),\\
&\text{s.t.} \sum_{k=1}^{p_j} y_{kj} = m_j,\ y_{kj} \geq 0,\ t_{kj} \in K(j),\ k = 1, ..., p_j.
\end{aligned} \tag{28}
$$

Proof. Let $\nu_j := (t_{kj},\ y_{kj},\ k = 1, ..., p_j)$ be a feasible solution of problem (28).
Since vectors $t_{kj}^0,\ k \in P_j^*$, are the global optimal solutions in problem (13), then having denoted $\Psi_j(t_{kj}^0) := const(j),\ k \in P_j^*$, we get $const(j) \leq \Psi_j(t)\ \forall\ t \in K(j)$. Consequently, for any vector ν_j that is feasible in (28), we have $\Phi_j(\nu_j) = \sum_{k=1}^{p_j} y_{kj}\Psi_j\big(t_{kj}\big) \geq m_j const(j)$. On the other hand, vector ν_j^0 is also feasible in (28) and $\Phi_j(\nu_j^0) = \sum_{k=1}^{p_j} y_{kj}^0 \Psi_j\big(t_{kj}^0\big) =$ $= m_j const(j)$. Hence ν_j^0 is a global optimal solution of problem (28). The lemma is proved. ■

Fix $j \in J$, and let a vector $\nu_j^0 = (t_{kj}^0,\ y_{kj}^0,\ k = 1, ..., p_j)$ be a global optimal solution of problem (28). Hence for all vectors in the form

$$
\nu_j := (t_{kj} := t_{kj}^0 + \Delta t_{kj},\ y_{kj} := y_{kj}^0 + \Delta y_{kj},\ k = 1, ..., p_j),\ j \in J, \tag{29}
$$

such that

$$
\sum_{k=1}^{p_j} \Delta y_{kj} = 0,\ y_{kj}^0 + \Delta y_{kj} \geq 0,\ t_{kj}^0 + \Delta t_{kj} \in K(j),\ k = 1, ..., p_j, j \in J, \tag{30}
$$

we evidently have

$$
\Phi_j(\nu_j) - \Phi_j(\nu_j^0) \geq 0,\ j \in J. \tag{31}
$$

Now we can prove sufficient optimality conditions for the problem $P(p_j, j \in J)$.

THEOREM 3.6 *(Sufficient optimality conditions) Let vector ξ^0 in the form (22) be a feasible solution of problem (8) and let the following conditions be satisfied:*

(1) ξ^0 satisfies (25) and
(2) for $j \in J$, vectors (12) are global optimal solutions of problem (13).

Then the vector ξ^0 is a global optimal solution of problem (8).

Proof. Let us consider any feasible solution ξ of problem (8). This vector admits a presentation

$$
\xi := (x := x^0 + \Delta x, t_{kj} := t_{kj}^0 + \Delta t_{kj},\ y_{kj} := y_{kj}^0 + \Delta y_{kj},\ k = 1, ..., p_j, j \in J, y_i := y_i^0 + \Delta y_i, i \in I).
$$

From feasibility of ξ in (8), it follows that

$$D\Delta x + \sum_{j\in J} A_j \sum_{k=1}^{p_j} (\Delta t_{kj} y^0_{kj} + \Delta y_{kj} t^0_{kj} + \Delta t_{kj} \Delta y_{kj}) + \sum_{i\in I} q_i \Delta y_i = 0, \tag{32}$$

$$\Delta y_i \geq 0, \text{ if } i \in I_2 \text{ and } y^0_i = 0, \tag{33}$$

and relations (30) take place.

Taking into account equalities (32), let us calculate

$$F(\xi) - F(\xi^0) = \frac{1}{2}\Delta x^T D \Delta x - \sum_{i\in I}(x^{0T} q_i + \omega_i)\Delta y_i + \sum_{j\in J}(\Phi_j(\nu_j) - \Phi_j(\nu^0_j)),$$

where vector ν^0_j, ν_j are defined in (27), (29).

Due to the assumption of the positive semi-definitiveness of the matrix D, we have $\Delta x^T D \Delta x \geq 0$ for all $\Delta x \in \mathbb{R}^n$.

Conditions (25) and (33) imply the inequality $-\sum\limits_{i\in I}(x^{0T} q_i + \omega_i)\Delta y_i \geq 0$.

Taking into account condition 2) of this theorem and applying Lemma 3.5, we conclude that inequalities (31) take place when conditions (30) are satisfied.

Consequently, $F(\xi) - F(\xi^0) \geq 0$ for any feasible ξ in problem (8). This means that ξ^0 is a global optimal solution of problem (8). The theorem is proved. ■

4. Properties of the problem $P(p_j, j \in J)$

In the previous sections, we have formulated the Properties 1) - 3) that can be satisfied by the optimal solutions of the problem $P(p_j, j \in J)$ in the form (8). In this section, we establish some additional properties of this problem. In the following lemmas 4.1-4.4 we will study how the change of the parameters in problem (8) affects its optimal value.

LEMMA 4.1 *Suppose that there is an optimal solution ξ^0 (see (22)) of problem (8) such that $y^0_{k_0 j_0} = 0$ for some $1 \leq k_0 \leq p_{j_0}$, $j_0 \in J$. Then*

$$val(P(\bar{p}_j, j \in J)) = val(P(p_j, j \in J)), \tag{34}$$

where $\bar{p}_j = p_j, j \in J \setminus \{j_0\}$, $\bar{p}_{j_0} = p_{j_0} - 1$.

Here and in what follows, $val(P)$ denotes the optimal value of the cost function in an optimization problem P.

Proof. Without loss of generality, we can consider that $k_0 = p_{j_0}$. It is easy to show that vector $\xi^0(\bar{p}_j, j \in J) := (x^0,\ t^0_{kj}, y^0_{kj}, k = 1, ..., \bar{p}_j,\ j \in J;\ y^0_i,\ i \in I)$, is an optimal solution of the problem $P(\bar{p}_j, j \in J)$ and equality (34) takes place. ■

LEMMA 4.2 *Let integers $\bar{p}_j, p_j, j \in J$, satisfy the inequalities $\bar{p}_j \geq p_j,\ j \in J$. Then $val(P(\bar{p}_j, j \in J)) \leq val(P(p_j, j \in J))$.*

Proof. Let vector $\xi^0(p_j, j \in J)$ in the form (10) be an optimal solution of the problem $P(p_j, j \in J)$. It is evident that vector $\xi(\bar{p}_j, j \in J) := (x^0,\ t^0_{kj}, y^0_{kj}, k = 1, ..., \bar{p}_j,\ j \in J;\ y^0_i,\ i \in I)$ with $t^0_{kj} = 0, y^0_{kj} = 0, k = p_j + 1, ..., \bar{p}_j$, is a feasible solution of the problem $P(\bar{p}_j, j \in J)$ and $F(\xi(\bar{p}_j, j \in J)) = F(\xi^0(p_j, j \in J)) = val(P(p_j, j \in J))$, where $F(\xi)$ is the objective function of the problem $P(p_j, j \in J)$ for a feasible ξ.

The last equalities and the inequality $val(P(\bar{p}_j, j \in J)) \leq F(\xi(\bar{p}_j, j \in J))$ imply the inequality $val(P(\bar{p}_j, j \in J)) \leq val(P(p_j, j \in J))$. ∎

Lemma 4.3 *Let a feasible solution (22) of the problem $P(p_j, j \in J)$ satisfy conditions (25) and Property 3). Then for all integers $\bar{p}_j \geq p_j, j \in J$, the following equality holds:*

$$val(P(\bar{p}_j, j \in J)) = val(P(p_j, j \in J)). \tag{35}$$

Proof. It follows from the assumptions of this lemma and from Theorem 3.6 that the vector ξ^0 defined in (22) is a global optimal solution of the problem $P(p_j, j \in J)$.

Consider vector

$$\bar{\xi} = \bar{\xi}(\bar{p}_j, j \in J) = (\tilde{x},\ \bar{t}_{kj}, \bar{y}_{kj}, k = 1, ..., \bar{p}_j,\ j \in J;\ \bar{y}_i,\ i \in I), \tag{36}$$

whose components are defined using that of the vector ξ^0 as follows:

$$\bar{x} = x^0; \bar{t}_{kj} = t^0_{kj}, \bar{y}_{kj} = y^0_{kj}, k = 1, ..., p_j; \bar{t}_{kj} = t^0_{p_j j}, \bar{y}_{kj} = 0, k = p_j + 1, ..., \bar{p}_j, j \in J; \bar{y}_i = y^0_i, i \in I.$$

It is easy to check that

i) vector $\bar{\xi}$ is a feasible solution of the problem $P(\bar{p}_j, j \in J)$,

ii) the value of the cost function of the problem $P(\bar{p}_j, j \in J)$ at $\bar{\xi}$ is equal to the value of the cost function of the problem $P(p_j, j \in J)$ at its feasible solution ξ^0,

iii) all the conditions of Theorem 3.6 are satisfied for vector $\bar{\xi}$ and consequently, vector $\bar{\xi}$ is a global optimal solution of this problem.

The conditions **i)-iii)** imply the equality (35) and the lemma is proved. ∎

Lemma 4.4 *Let problem (8) admit an optimal solution ξ^0 (see (22)) that satisfies Assumption 1 but does not satisfy Property 3). Then $val(P(\bar{p}_j, j \in J)) < val(P(p_j, j \in J))$, where $\bar{p}_j = p_j + 1,\ j \in J$.*

Proof. Suppose that, on the contrary, the equality

$$val(P(\bar{p}_j, j \in J)) = val(P(p_j, j \in J)) \tag{37}$$

takes place. Consider an optimal solution $\xi^0(p_j, j \in J)$ of the problem $P(p_j, j \in J)$. Let $\xi^0(p_j, j \in J)$ have the form (22). It follows from (37) that vector $\xi^0(\bar{p}_j, j \in J)$ defined as

$$\xi^0(\bar{p}_j, j \in J) := (x^0, t^0_{kj}, y^0_{kj}, k = 1, ..., p_j; t^0_{p_j+1,j}, y^0_{p_j+1,j} = 0, j \in J; y^0_i, i \in I)$$

with any $t^0_{p_j+1,j}, j \in J$, satisfying the conditions $t^0_{p_j+1,j} \in K(j), j \in J$, is an optimal solution of the problem $P(\bar{p}_j, j \in J)$. Hence it follows from Assumption 1 and Theorem 3.3 that for the optimal solution $\xi^0(\bar{p}_j, j \in J)$, conditions (26) with $p_j, j \in J$, replaced by $\bar{p}_j = p_j + 1, j \in J$, are satisfied.

Based on these conditions and the equalities $y^0_{\bar{p}_j,j} = 0, j \in J$, we conclude that, for any $j \in J$, vectors $t^0_{kj}, k = 1, ..., p_j$, are optimal in problem (13). This means that the optimal solution ξ^0 of the problem $P(p_j, j \in J)$ possesses Property 3). But this contradicts the assumptions of the lemma. The obtained contradiction completes the proof. ■

In the final part of this section, we present the conditions that guarantee that the problem $P(p_j, j \in J)$ admits optimal solutions satisfying Properties 1) - 3).

LEMMA 4.5 *Suppose that the problem $P(p_j, j \in J)$ with $p_j = n+2, j \in J$, admits an optimal solution satisfying Assumption 1. Then the optimal solution satisfies Property 3).*

Proof. Suppose that the problem $P(p_j, j \in J)$ with $p_j = n+2, j \in J$, admits an optimal solution $\xi^0 = (x^0,\ t^0_{kj}, y^0_{kj}, k = 1, ..., p_j,\ j \in J;\ y^0_i,\ i \in I)$. Consider the sets P^0_j, $j \in J$, defined in (21).

It follows from Assumption 1 and Theorem 3.3 that conditions (26) are satisfied. Hence, if for all $j \in J$, we have $P^0_j \neq \emptyset$, then the optimal solution ξ^0 satisfies Property 3) and the lemma is proved.

Suppose now that for some $j \in J$ it holds $P^0_j = \emptyset$, i.e. $y^0_{kj} > 0$ for all $k = 1, ..., p_j$. It follows from (26) that for these $j \in J$ and ξ^0, there exists a multiplier $\lambda(j)$ such that

$$-\frac{1}{2} t^{0T}_{kj} D_j t^0_{kj} - c^T_j t^0_{kj} = x^{0T} A_j t^0_{kj} - \lambda(j),\ k = 1, ..., p_j. \tag{38}$$

Hence

$$-\sum_{k=1}^{p_j} y^0_{kj} (\frac{1}{2} t^{0T}_{kj} D_j t^0_{kj} + c^T_j t^0_{kj}) = \sum_{k=1}^{p_j} y^0_{kj} (x^{0T} A_j t^0_{kj} - \lambda(j)) = \sum_{k=1}^{p_j} y^0_{kj} x^{0T} A_j t^0_{kj} - \lambda(j) m_j. \tag{39}$$

Consider the $(n+1)-$vectors $\begin{pmatrix} A_j t^0_{kj} \\ 1 \end{pmatrix}$, $k = 1, ..., p_j$, where $p_j = n+2$.

It is evident that these vectors are linearly dependent, hence there exists a vector $\Delta y := (\Delta y_k, k = 1, ..., p_j) \neq 0$ such that $\sum\limits_{k=1}^{p_j} \Delta y_k A_j t^0_{kj} = 0$, $\sum\limits_{k=1}^{p_j} \Delta y_k = 0$.

Set: $\lambda_k := \infty$ if $\Delta y_k \geq 0$, $\lambda_k := -y^0_{kj}/\Delta y_k$ if $\Delta y_k < 0$, $k = 1, ..., p_j$, and calculate $\lambda := \min\limits_{k=1,...,p_j} \lambda_k > 0$. Consider the numbers

$$\bar{y}^0_{kj} := y^0_{kj} + \lambda \Delta y_k, k = 1, ..., p_j. \tag{40}$$

By construction, we have

$$\sum_{k=1}^{p_j} \bar{y}^0_{kj} = m_j, \bar{y}^0_{kj} \geq 0, k = 1, ..., p_j, \exists\ 1 \leq k_0 \leq p_j \text{ such that } \bar{y}^0_{k_0 j} = 0,$$
$$\sum_{k=1}^{p_j} \bar{y}^0_{kj} A_j t^0_{kj} = \sum_{k=1}^{p_j} y^0_{kj} A_j t^0_{kj},$$

$$-\sum_{k=1}^{p_j} \bar{y}^0_{kj}(\frac{1}{2}t^{0T}_{kj}D_j t^0_{kj} + c^T_j t^0_{kj}) = \sum_{k=1}^{p_j} \bar{y}^0_{kj}(x^{0T}A_j t^0_{kj} - \lambda(j)) = \sum_{k=1}^{p_j} \bar{y}^0_{kj}x^{0T}A_j t^0_{kj} - \lambda(j)m_j$$
$$= \sum_{k=1}^{p_j} y^0_{kj}x^{0T}A_j t^0_{kj} - \lambda(j)m_j = -\sum_{k=1}^{p_j} y^0_{kj}(\frac{1}{2}t^{0T}_{kj}D_j t^0_{kj} + c^T_j t^0_{kj}).$$

Consequently the vector $\xi^* := (x^0,\ t^0_{kj}, \bar{y}^0_{kj}, k = 1, ..., p_j,\ j \in J;\ y^0_i,\ i \in I)$, with the components $\bar{y}^0_{kj}$, $k = 1, ..., p_j,\ j \in J$, defined by the rule:

- $\bar{y}^0_{kj} = y^0_{kj}$, $k = 1, ..., p_j$, if $P^0_j \neq \emptyset$;
- $\bar{y}^0_{kj}$, $k = 1, ..., p_j$, are given by formulae (40), if $P^0_j = \emptyset$,

is an optimal solution in $P(p^*_j, j \in J)$ as well.

The vector ξ^* satisfies Assumption 1 and, by construction, $\min\{\bar{y}^0_{kj}, k = 1, ..., p_j\} = 0,\ j \in J$. Hence it follows from Corollary 3.4 that ξ^* satisfies Property 3).

Taking into account the rules for constructing the components $\bar{y}^0_{kj}$, $k = 1, ..., p_j$, $j \in J$, and the fact that the vectors ξ^0 and ξ^* have the same components x^0 and t^0_{kj}, $k = 1, ..., p_j$, $j \in J$, we conclude that ξ^0 satisfies Property 3) as well. The lemma is proved. ■

Lemma 4.6 *Suppose that the problem $P(p_j, j \in J)$ has an optimal solution satisfying Property 3). Then there exist integers $1 \leq \bar{p}_j \leq p_j, j \in J$, such that problem $P(\bar{p}_j, j \in J)$ has an optimal solution satisfying Properties 1)- 3).*

Proof. Suppose that the problem $P(p_j, j \in J)$ has an optimal solution satisfying Property 3). If this solution does not satisfy Property 1),then following the proof of Lemma 4.1, we can easily find numbers $\tilde{p}_j \leq p_j, j \in J$, such that the problem $P(\tilde{p}_j, j \in J)$ has an optimal solution

$$\xi^0(\tilde{p}_j, j \in J) := (x^0,\ t^0_{kj}, y^0_{kj}, k = 1, ..., \tilde{p}_j,\ j \in J;\ y^0_i,\ i \in I) \tag{41}$$

satisfying Property 3) and, additionally, Property 1):

$$y^0_{kj} > 0,\ k = 1, ..., \tilde{p}_j, j \in J. \tag{42}$$

Consider the sets $J_* := \{j \in J : \tilde{p}_j \geq 2\}$ and $I^a_2 = \{i \in I_2 : y^0_i > 0\}$. Suppose that Property 2) is not satisfied for $\xi^0(\tilde{p}_j, j \in J)$, i.e.

$$m(\xi^0(\tilde{p}_j, j \in J)) < |I^a_2| + \sum_{j \in J} \tilde{p}_j + \gamma_*, \tag{43}$$

where $m(\xi^0(\tilde{p}_j, j \in J)) := \text{rank}\left(A_j(t^0_{kj} - t^0_{1j}), k = 2, ..., \tilde{p}_j, j \in J_*, q_i, i \in I_1 \cup I^a_2\right)$. Hence vectors $\begin{pmatrix} A_j t^0_{kj} \\ e_j \end{pmatrix}, k = 1, ..., \tilde{p}_j, j \in J,$ and $\begin{pmatrix} q_i \\ \mathbf{0} \end{pmatrix}, i \in I_1 \cup I^a_2$, where $e_j = (e_{ij}, i \in J)^T$, $e_{ij} = 0$ if $i \neq j$, $e_{ij} = 1$ if $i = j$, $i \in J$, $j \in J$, $\mathbf{0} = (0, 0, ..., 0)^T \in \mathbb{R}^{|J|}$, are linearly dependent. Consequently, there exist numbers $\Delta y_{kj}, k = 1, ..., \tilde{p}_j, j \in J$, $\Delta y_i, i \in I_1 \cup I^a_2$, such that

$$(\Delta y_{kj}, k=1,...,\tilde{p}_j, j\in J, \Delta y_i, i\in I_2^a)\neq 0,$$
$$\sum_{j\in J}\sum_{k=1}^{\tilde{p}_j}\Delta y_{kj}A_j t^0_{kj}+\sum_{i\in I_1\cup I_2^a} q_i\Delta y_i=0,\ \sum_{k=1}^{\tilde{p}_j}\Delta y_{kj}=0,\ j\in J. \tag{44}$$

Let us set

$$\begin{gathered}\lambda_{kj}:=\infty\ \text{ if } \Delta y_{kj}\geq 0,\ \lambda_{kj}:=-y^0_{kj}/\Delta y_{kj}\ \text{ if } \Delta y_{kj}<0,\ k=1,...,\tilde{p}_j, j\in J;\\ \lambda_i:=\infty\ \text{ if } \Delta y_i\geq 0,\ \lambda_i:=-y^0_i/\Delta y_i\ \text{ if } \Delta y_i<0,\ i\in I_2^a;\\ \lambda:=\min\{\lambda_{kj}, k=1,...,\tilde{p}_j, j\in J; \lambda_i, i\in I_2^a\}>0,\end{gathered} \tag{45}$$

and

$$\begin{gathered}\bar{y}^0_{kj}:=y^0_{kj}+\lambda\Delta y_{kj}, k=1,...,\tilde{p}_j, j\in J;\\ \bar{y}^0_i:=y^0_i+\lambda\Delta y_i,\ i\in I_1\cup I_2^a,\ \bar{y}^0_i:=y^0_i,\ i\in I\setminus(I_1\cup I_2^a).\end{gathered} \tag{46}$$

By construction, we have $\sum\limits_{k=1}^{\tilde{p}_j}\bar{y}^0_{kj}=m_j,\ \bar{y}^0_{kj}\geq 0, k=1,...,\tilde{p}_j, j\in J;\ \bar{y}^0_i\geq 0, i\in I_2.$

Due to inequalities (42), it is easy to show that for all $j\in J$, relations (38) take place with p_j replaced by $\tilde{p}_j$. Hence

$$-\sum_{k=1}^{\tilde{p}_j}y^0_{kj}(\frac{1}{2}t^{0T}_{kj}D_j t^0_{kj}+c_j^T t^0_{kj})=\sum_{k=1}^{\tilde{p}_j}y^0_{kj}(x^{0T}A_j t^0_{kj}-\lambda(j))=\sum_{k=1}^{\tilde{p}_j}y^0_{kj}x^{0T}A_j t^0_{kj}-\lambda(j)m_j,$$

$$\begin{aligned}-\sum_{k=1}^{\tilde{p}_j}\bar{y}^0_{kj}(\frac{1}{2}t^{0T}_{kj}D_j t^0_{kj}+c_j^T t^0_{kj})&=\sum_{k=1}^{\tilde{p}_j}\bar{y}^0_{kj}x^{0T}A_j t^0_{kj}-\lambda(j)m_j\\ &=-\sum_{k=1}^{\tilde{p}_j}y^0_{kj}(\frac{1}{2}t^{0T}_{kj}D_j t^0_{kj}+c_j^T t^0_{kj})+\sum_{k=1}^{\tilde{p}_j}\Delta y_{kj}x^{0T}A_j t^0_{kj}.\end{aligned} \tag{47}$$

Recall that $F(\xi)$ stays for the objective function of the problem $P(p_j, j\in J)$ in ξ. Taking into account the last relations and (44), it is easy to verify that

$$F(\xi^0(\tilde{p}_j, j\in J))=F(\bar{\xi}^0(\tilde{p}_j, j\in J)),$$

where $\xi^0(\tilde{p}_j, j\in J)$ is defined in (41) and

$$\bar{\xi}^0(\tilde{p}_j, j\in J):=(x^0,\ t^0_{kj}, \bar{y}^0_{kj}, k=1,...,\tilde{p}_j,\ j\in J;\ \bar{y}^0_i,\ i\in I).$$

From the considerations above, it follows that $\xi^0(\tilde{p}_j, j\in J)$ is an optimal solution of the problem $P(\tilde{p}_j, j\in J)$.

Notice that, by construction, $\min\{\bar{y}^0_{kj}, k=1,...,\tilde{p}_j, j\in J; \bar{y}_i, i\in I_2^a\}=0$. Following lemma 4.1 let us find numbers $\bar{\tilde{p}}_j\leq\tilde{p}_j, j\in J$, such that the vector

$$\bar{\xi}^0(\bar{\tilde{p}}_j, j\in J):=(x^0,\ t^0_{kj}, \bar{y}^0_{kj}, k=1,...,\bar{\tilde{p}}_j,\ j\in J;\ \bar{y}^0_i,\ i\in I)$$

is optimal for the problem $P(\bar{\tilde{p}}_j, j \in J)$ and $\bar{y}^0_{kj} > 0, k = 1, ..., \bar{\tilde{p}}_j,\ j \in J$. It is easy to check that

$$m(\bar{\xi}^0(\bar{\tilde{p}}_j, j \in J)) = m(\xi^0(\tilde{p}_j, j \in J)),\ \ |I^a_2| + \sum_{j\in J} \tilde{p}_j > |\bar{I}^a_2| + \sum_{j\in J} \bar{\tilde{p}}_j, \tag{48}$$

where $\bar{I}^a_2 := \{i \in I_2 : \bar{y}^0_i > 0\}$, $\bar{J}_* := \{j \in J : \bar{\tilde{p}}_j \geq 2\}$,

$$m(\bar{\xi}^0(\bar{\tilde{p}}_j, j \in J)) := \text{rank}\left(A_j(t^0_{kj} - t^0_{1j}), k = 2, ..., \bar{\tilde{p}}_j, j \in \bar{J}_*, q_i, i \in I_1 \cup \bar{I}^a_2\right).$$

It follows from (43) and (48) that in a finite number of iterations, one can find the numbers $\bar{p}_j \leq p_j, j \in J$, such that Properties 1)-3) are satisfied for an optimal solution of the problem $P(\bar{p}_j, j \in J)$. The lemma is proved. ■

Based on lemmas 4.5 and 4.6, it is easy to prove the following theorem.

THEOREM 4.7 *Suppose that the problem $P(p^*_j, j \in J)$ with $p^*_j = n + 2, j \in J$, admits an optimal solution satisfying Assumption 1. Then there exist numbers $p_j \geq 1, j \in J$, $\sum_{j\in J} p_j \leq n - \gamma_*$, such that the problem $P(p_j, j \in J)$ has an optimal solution satisfying Properties 1)-3).*

The main result of this section consists in the proof that for the existence of integers $p_j \geq 1, j \in J$, such that the problem $P(p_j, j \in J)$ possesses an optimal solution satisfying Properties 1) - 3), it is sufficient that the problem $P(p^*_j, j \in J)$ with $p^*_j = n + 2, j \in J$, had an optimal solution for which all the Lagrange multiplier vectors in the form (16) have the same first component z.

In section 6, we develop a constructive procedure of determination of the values of the parameters for which the problem $P(p_j, j \in J)$ satisfied Properties 1) - 3).

5. On solvability of the problem $P(p_j, j \in J)$

In section 4, we considered properties of the optimal solutions of the NLP problem $P(p_j, j \in J)$ in the form (8) having supposed that the optimal solutions of this problem exist. Now, we will study in which cases one can guarantee the existence of the optimal solutions of $P(p_j, j \in J)$.

First of all, we should notice that if the feasible set of problem

$$P_{min} := P(p_j = 1, j \in J) \tag{49}$$

is not empty, then the same we can state about the feasible sets of all problems $P(p_j, j \in J)$ with $p_j \geq 1, j \in J$.

In what follows, we will need the following assumption.

ASSUMPTION 2 *In (8), the matrices D_j, $j \in J$, satisfy*

$$t^T D_j t < 0\ \forall\ t \in K(j) \setminus \{0\},\ j \in J. \tag{50}$$

Denote the feasible set of the problem $P(p_j, j \in J)$ by $\mathcal{X}$ and a feasible solution $\xi \in \mathcal{X}$ of problem (8) (see (9)) by

$\xi = (\gamma, y)$, where $\gamma = (x,\ t_{kj}, k = 1, ..., p_j, j \in J;\ y_i, i \in I), y = (y_{kj}, k = 1, ..., p_j, j \in J)$.

LEMMA 5.1 *Given problem $P(p_j, j \in J)$ satisfying Assumption 2, suppose that its feasible set $\mathcal{X}$ is not empty. Then the objective function $F(\xi)$ of this problem is not bounded from below on $\mathcal{X}$ if and only if there exist numbers $\Delta y_i, i \in I$, such that the following conditions are satisfied:*

$$\sum_{i \in I} q_i \Delta y_i = 0,\ \Delta y_i \geq 0, i \in I_2,\ \ -\sum_{i \in I} \omega_i \Delta y_i < 0. \tag{51}$$

Proof. $\Leftarrow$) Evidently, if $\mathcal{X} \neq \emptyset$ and there exist numbers $\Delta y_i, i \in I$, satisfying (51), then the objective function of the problem $P(p_j, j \in J)$ is not bounded from below on the feasible set $\mathcal{X}$.

$\Rightarrow$) Suppose now that in the problem $P(p_j, j \in J)$, the objective function $F(\xi)$ is not bounded on the feasible set. Then there exists a sequence of the feasible solutions $\bar{\xi}^s = (\bar{\gamma}^s, \bar{y}^s)$, $s = 1, 2, ...$, such that $F(\bar{\xi}^s) =: M_s \to -\infty$ as $s \to \infty$.

For each $s \in \mathbb{N}$, consider the following NLP problem:

$$\begin{aligned} &||\gamma||^2 \to \min, \\ \text{s.t. } &F(\xi) \leq M_s, ||\gamma|| \leq P_s := ||\bar{\gamma}^s||;\ \mathcal{F}(\xi) = 0, \\ &y_i \geq 0,\ i \in I_2;\ \sum_{k=1}^{p_j} y_{kj} = m_j,\ y_{kj} \geq 0,\ t_{kj} \in K(j), k = 1, ..., p_j;\ j \in J, \end{aligned} \tag{52}$$

where $\xi = (\gamma, y)$, and the functions $\mathcal{F}(\xi)$, $F(\xi)$ are defined in (8).

Problem (52) has an optimal solution since its feasible set is nonempty, bounded and closed. Let $\xi^s = (\gamma^s, y^s)$ be an optimal solution of problem (52).

Evidently, the sequence $\xi^s = (\gamma^s, y^s)$, $s = 1, 2, ...$, does not possess any convergent subsequence since $F(\xi^s) \leq M_s$, $M_s \to -\infty$ as $s \to \infty$. Therefore, taking into account that $||y^s||_1 = \sum\limits_{j \in J} m_j$, $s = 1, 2, ...$, where $||\cdot||_1$ stays for the l_1 norm, we can conclude that $||\gamma^s|| \to \infty$ as $s \to \infty$ for any norm, including the Euclidean norm $||\cdot||$.

Let us divide both sides of the inequality $F(\xi^s) \leq M_s < 0$ by $||\gamma^s||^2$ and pass to the limit. As a result we obtain

$$\frac{1}{2}\Delta x^T D \Delta x - \sum_{j \in J} \sum_{k=1}^{p_j} y^0_{kj} \frac{1}{2} \Delta t^T_{kj} D_j \Delta t_{kj} \leq 0 \tag{53}$$

where

$$\Delta\gamma = (\Delta x, \Delta t_{kj}, k = 1, ..., p_j,\ j \in J;\ \Delta y_i,\ i \in I) = \lim_{s \to \infty} \tfrac{\gamma^s}{||\gamma^s||},\ \ ||\Delta\gamma|| = 1, \tag{54}$$
$$y^0 = (y^0_{kj}, k = 1, ..., p_j,\ j \in J) = \lim_{s \to \infty} y^s.$$

It follows from (53) that

$$\Delta x^T D \Delta x = 0,\ \Delta t^T_{kj} D_j \Delta t_{kj} = 0, k = 1, ..., p_j,\ j \in J. \tag{55}$$

These equalities together with (50) imply

$$D\Delta x = 0,\ \Delta t_{kj} = 0, k = 1, ..., p_j,\ j \in J. \tag{56}$$

Taking into account that for any $\xi \in \mathcal{X}$, it holds

$$\frac{1}{2}x^T Dx - \sum_{j\in J}\sum_{k=1}^{p_j} y_{kj}\Big(\frac{1}{2}t_{kj}^T D_j t_{kj}\Big) \geq 0, \tag{57}$$

we conclude that $-\sum\limits_{j\in J}\sum\limits_{k=1}^{p_j} y_{kj}^s c_j^T t_{kj}^s - \sum\limits_{i\in I}\omega_i y_i^s \leq M_s < 0,\ s = 1, 2, ...$

Let us divide both sides of the last inequality by $||\gamma^s||$ and pass to the limit, taking into account (56). As a result, we obtain

$$-\sum_{i\in I}\omega_i\Delta y_i \leq 0. \tag{58}$$

Since ξ^s is feasible, the equality $\mathcal{F}(\xi^s) = 0$ holds. Having divided both sides of this equality by $||\gamma^s||$ and passing to the limit, taking into account (56) , we get

$$\sum_{i\in I} q_i\Delta y_i = 0. \tag{59}$$

Notice that the inequalities $y_i^s \geq 0$, $i \in I_2$, imply

$$\Delta y_i \geq 0,\ i \in I_2. \tag{60}$$

Consider the inequality (58). Suppose, first, that it is strictly satisfied:

$$-\sum_{i\in I}\omega_i\Delta y_i < 0. \tag{61}$$

Then the relations (59)-(61) imply the existence of the numbers $\Delta y_i, i \in I$, satisfying (51), and the lemma is proved.

Suppose now that (58) is verified as an equality:

$$-\sum_{i\in I}\omega_i\Delta y_i = 0. \tag{62}$$

By construction, for any $s = 1, 2, \ldots$, the vector ξ^s can be presented in the form

$$\xi^s = \begin{pmatrix} \gamma^s \\ y^s \end{pmatrix} = \begin{pmatrix} (\Delta\gamma + \delta\gamma^s)||\gamma^s|| \\ y^s \end{pmatrix}, \tag{63}$$

where

$$\delta\gamma^s := (\delta x^s, \delta t_{kj}^s, k = 1, ..., p_j,\ j \in J;\ \delta y_i^s,\ i \in I)^T,\ ||\delta\gamma^s|| \to 0 \text{ as } s \to \infty. \tag{64}$$

Notice that by construction,

$$\Delta y_i + \delta y_i^s \geq 0,\ \Delta y_i \geq 0,\ i \in I_2. \tag{65}$$

Hence

$$\text{if } \Delta y_i = 0 \text{ then } \delta y_i^s \geq 0, i \in I_2. \tag{66}$$

Denote

$$\eta^s := \max\{\eta_i^s, i \in I_2\}, \tag{67}$$

where

$$\eta_i^s := \begin{cases} 0, & \text{if } \delta y_i^s \geq 0, \\ -\delta y_i^s/\Delta y_i, & \text{if } \delta y_i^s < 0, \end{cases} \quad i \in I_2. \tag{68}$$

It follows from (64) and (66) that $\eta^s \geq 0$ and $\eta^s \to 0$ as $s \to \infty$. By construction,

$$\Delta t_{kj} + \delta t_{kj}^s \in K(j),\ \Delta t_{kj} = 0 \in K(j),\ k = 1, ..., p_j, j \in J. \tag{69}$$

Then

$$\delta t_{kj}^s \in K(j),\ k = 1, ..., p_j, j \in J. \tag{70}$$

It is evident that

$$\eta^s \Delta y_i + \delta y_i^s \geq 0,\ i \in I_2,\ \ \eta^s \Delta t_{kj} + \delta t_{kj}^s \in K(j),\ k = 1, ..., p_j, j \in J. \tag{71}$$

From (63), (64), it follows that vector ξ^s can be presented in the form

$$\xi^s = \begin{pmatrix} \Delta\gamma\theta^s \\ 0 \end{pmatrix} + \hat{\xi}^s,\ \hat{\xi}^s = \begin{pmatrix} \hat{\gamma}^s \\ y^s \end{pmatrix}, \hat{\gamma}^s := (\Delta\gamma\eta^s + \delta\gamma^s)||\gamma^s||,\ \theta^s := (||\gamma^s|| - \eta^s). \tag{72}$$

Taking into account this presentation and (56), (59), (62), it is easy to show that

$$F(\xi^s) = F(\hat{\xi}^s),\ \ \mathcal{F}(\xi^s) = \mathcal{F}(\hat{\xi}^s).$$

It follows from the last equalities and (71) that the vector $\hat{\xi}^s$ is a feasible solution of problem (52). Taking into account that $(\Delta\gamma\eta^s + \delta\gamma^s) \to 0$ as $s \to \infty$, we obtain the inequality

$$||\gamma^s|| > ||\hat{\gamma}^s|| = ||(\Delta\gamma\eta^s + \delta\gamma^s)|| \cdot ||\gamma^s|| \tag{73}$$

that contradicts the optimality of ξ^s in problem (52). The obtained contradiction proves that equality (62) can not take place and hence inequality (58) is always verified as a strict one. The lemma is proved. ■

Remark 2 In formulating and proving Lemma 5.1 we do not assume that the implication (6) $\Longrightarrow$ (7) takes place.

Corollary 5.2 *Given problem $P(p_j, j \in J)$, suppose that $\mathcal{X} \neq \emptyset$, Assumption 2 and the implication (6) $\Longrightarrow$ (7) are fulfilled. Then the objective function of this problem is bounded from below on the set $\mathcal{X}$.*

Lemma 5.3 *Let Assumption 2 be fulfilled for the problem $P(p_j, j \in J)$ in the form (8). If the problem P_{min} defined in (49) is feasible and the objective function of the problem $P(p_j, j \in J)$ is bounded from below on its feasible set, then for all $\bar{p}_j \le p_j$, $j \in J$, the problems $P(\bar{p}_j, j \in J)$, admit optimal solutions.*

Proof. Let us show, first, that the problem $P(p_j, j \in J)$ admits an optimal solution if it is feasible and its objective function is bounded from below in the feasible set.

Indeed, since the objective function $F(\xi)$ in (8) is bounded from below, there exists a sequence $\bar{\xi}^s = (\bar{\gamma}^s, \bar{y}^s)$, $s = 1, 2, ...,$ such that

$$F(\bar{\xi}^s) =: M_s, \; M_s \to M_0, \text{ as } s \to \infty; \;\; M_0 := \inf_{\xi \in \mathcal{X}} F(\xi),$$

where $\mathcal{X}$ is the set of all feasible solutions of the problem $P(p_j, j \in J)$.

For any s, let us consider problem (52). This problem admits an optimal solution $\xi^s = (\gamma^s, y^s)$ since its feasible set is nonempty, bounded and closed. If the sequence $\xi^s = (\gamma^s, y^s)$, $s = 1, 2, ...,$ admits a convergent subsequence ξ^{s_i}, $i = 1, 2, ...$ such that $s_i \to \infty$ as $i \to \infty$ and $\lim\limits_{i \to \infty} \xi^{s_i} = \xi^0$, then it is obvious that ξ^0 should be an optimal solution of the problem $P(p_j, j \in J)$ and the lemma is proved.

Suppose now that all subsequences of $\xi^s = (\gamma^s, y^s)$, $s = 1, 2, ...,$ diverge. In this case we have $||\gamma^s|| \to \infty$ as $s \to \infty$. Here (as before) we took into account that $||y^s||_1 = \sum\limits_{j \in J} m_j$, $s = 1, 2, ...$

Let us divide both sides of the inequality $F(\xi^s) \le M_s$ by $||\gamma^s||^2$ and pass to the limit, taking into account that the numbers M_s are finite. As a result we obtain inequality (53), where $\Delta\gamma$, y^0 are defined in (54).

It follows from (53) that equalities (55) take place. These equalities together with (50) imply the equalities (56). Taking into account that for any $\xi \in \mathcal{X}$, the inequality (57) takes place, we conclude that

$$-\sum_{j \in J} \sum_{k=1}^{p_j} y_{kj} c_j^T t_{kj}^s - \sum_{i \in I} \omega_i y_i^s \le M_s.$$

Divide both sides of the last inequality by $||\gamma^s||$ and pass to the limit, taking into account (56) and finiteness of numbers M_s. As a result we obtain inequality (58).

Since ξ^s is feasible in (52), then $\mathcal{F}(\xi^s) = 0$.

Now divide both sides of the equality $\mathcal{F}(\xi^s) = 0$ by $||\gamma^s||$ and pass to the limit, taking into account (56). As a result we get equality (59).

Notice that inequalities $y_i^s \ge 0$, $i \in I_2$, imply (60).

Let us suppose that inequality (58) is strict: $-\sum\limits_{i \in I} \omega_i \Delta y_i < 0$. Then according to Lemma 5.1, this inequality together with (60) and (59) imply that the cost function $F(\xi)$ in unbounded from below on the feasible set $\mathcal{X}$. But this contradicts the assumptions of the lemma. Hence equality (62) takes place.

By construction, the vector ξ^s can be presented in the form (63), (64). Notice that inequalities (65) take place and hence (66) holds.

Consider the sequence of the numbers η^s defined in (67), (68). It follows from (65), (66) that $\eta^s \geq 0$ and $\eta^s \to 0$ as $s \to \infty$. By construction, inclusions (69) take place, hence inclusions (70) take place as well. Consequently, conditions (71) are fulfilled.

It follows from (63), (64) that for any $s \in \mathbb{N}$, vector ξ^s can be presented in the form (72). Taking into account this presentation and (56), (59), (62), one can show that

$$F(\xi^s) = F(\hat{\xi}^s), \ \ \mathcal{F}(\xi^s) = \mathcal{F}(\hat{\xi}^s).$$

It follows from the last equalities and (71) that vector $\hat{\xi}^s$ is a feasible solution of problem (52). Notice that taking into account that $(\Delta\gamma\eta^s + \delta\gamma^s) \to 0$ as $s \to \infty$, we get inequality (73) that contradicts the optimality of ξ^s in problem (52).

This contradiction proves that the sequence $\xi^s = (\gamma^s, y^s)$, $s = 1, 2, ...,$ has a convergent subsequence $\xi^{s_i}, i = 1, 2, ...,$ such that $s_i \to \infty$, $\lim\limits_{i\to\infty} \xi^{s_i} = \xi^0$, as $i \to \infty$, and hence, ξ^0 is optimal in the problem $P(p_j, j \in J)$.

Thus, we have proved that the problem $P(p_j, j \in J)$ admits an optimal solution if it is feasible and its objective function is bounded from below on the feasible set. To complete the proof of the lemma, let us notice that

• the feasibility of the problem P_{min} implies the feasibility of any problem $P(p_j, j \in J)$ with $p_j \geq 1, j \in J$,

• the boundedness from below of the objective function of the problem $P(p_j, j \in J)$ on its feasible set implies the boundedness from below on the feasible set of the objective function of the problem $P(\bar{p}_j, j \in J)$ when $\bar{p}_j \leq p_j, j \in J$.

The lemma is proved. ∎

Based on the results of this section and the previous one, we can prove the following theorem.

THEOREM 5.4 *Suppose that Assumption 2 is fulfilled, the problem P_{min} is feasible, and there are no numbers $\Delta y_i, i \in I$, satisfying (51). Then problem $P(p_j, j \in J)$ with $p_j \geq 1, j \in J$, has an optimal solution.*

6. Determination of the "right" values of the parameters $p_j, j \in J$, in the problem $P(p_j, j \in J)$

The results of the previous sections, permit one to develop algorithmic procedures that determine integers $p_j, j \in J$, such that the problem $P(p_j, j \in J)$ has an optimal solution satisfying Properties 1)-3).

Below, we describe a conceptual algorithm that is based on theorems and the lemmas proved in the sections 4 and 5.

Algorithm

Step 1. Solve the problem $P(p^*_j, j \in J)$ with $p^*_j = n + 2, j \in J$. If this problem has no solution, then STOP: there are no integers $p_j, j \in J$, such that the problem $P(p_j, j \in J)$ has an optimal solution satisfying Properties 1)-3). Otherwise go to Step 2.

Step 2. Suppose that for the optimal solution found at Step 1, Assumption 1 is fulfilled. (See Remark 1 that gives sufficient conditions for fulfillment of Assumption 1 and the Appendix for the common rules that can be used for verification of this assumption.)

It follows from Lemma 4.5 that the optimal solution found at Step 1, satisfies Property 3). (The rules for testing Property 3) are described in Appendix.) Go to Step 3.

Step 3. At this step, we have an optimal solution

$$\xi^0(p_j^*, j \in J) = (x^0, t_{kj}^0, y_{kj}^0, k = 1, ...p_j^*, j \in J, y_i^0, i \in I)$$

of the problem $P(p_j^*, j \in J)$. This solution satisfies Property 3). If, additionally, $\xi^0(p_j^*, j \in J)$ satisfies Property 1) then set $p_j(1) := p_j^*,\ j \in J$, and go to Step 4.

If $\xi^0(p_j^*, j \in J)$ does not satisfy Property 1), then follow the method used in the proof of Lemma 4.1 to find integers $p_j(1) \leq p_j^*,\ j \in J$, such that $P(p_j(1), j \in J)$ has an optimal solution

$$\xi^0(p_j(1), j \in J) = (x^0, t_{kj}^0, y_{kj}^0, k = 1, ...p_j(1), j \in J, y_i^0, i \in I)$$

that satisfies Property 1). Go to Step 4.

Step 4. At the beginning of this step we have $s \geq 1$ and integers $p_j(s),\ j \in J$, such that $P(p_j(s), j \in J)$ has an optimal solution

$$\xi^0(p_j(s), j \in J) = (x^0, t_{kj}^0, y_{kj}^0, k = 1, ...p_j(s), j \in J, y_i^0, i \in I)$$

satisfying Properties 1) and 3). If this solution satisfies also Property 2) then STOP: we have found the "right" integers $p_j, j \in J$.

Otherwise, following the rules described in the proof of Lemma 4.6, find new integers $\bar{\bar{p}}_j \leq p_j(s), j \in J$, and an optimal solution

$$\bar{\xi}^0(\bar{\bar{p}}_j, j \in J) = (x^0,\ t_{kj}^0, \bar{y}_{kj}^0, k = 1, ..., \bar{\bar{p}}_j,\ j \in J;\ \bar{y}_i^0,\ i \in I)$$

of the problem $P(\bar{\bar{p}}_j, j \in J)$ that satisfies the Properties 1) and 3) and

$$m(\bar{\xi}^0(\bar{\bar{p}}_j, j \in J)) = m(\xi^0(p_j(s), j \in J)),\ \ |I_2^a| + \sum_{j \in J} p_j(s) > |\bar{I}_2^a| + \sum_{j \in J} \bar{\bar{p}}_j, \tag{74}$$

where $m(\bar{\xi}^0(\bar{\bar{p}}_j, j \in J)) := \text{rank}\left(A_j(t_{kj}^0 - t_{1j}^0), k = 2, ..., \bar{\bar{p}}_j, j \in \bar{J}_*, q_i, i \in I_1 \cup \bar{I}_2^a\right)$, $m(\xi^0(p_j(s), j \in J)) := \text{rank}\left(A_j(t_{kj}^0 - t_{1j}^0), k = 2, ..., p_j(s), j \in J_*, q_i, i \in I_1 \cup I_2^a\right)$, and the sets are defined as $\bar{I}_2^a = \{i \in I_2 : \bar{y}_i^0 > 0\}$, $\bar{J}_* = \{j \in J : \bar{\bar{p}}_j \geq 2\}$, $I_2^a = \{i \in I_2 : y_i^0 > 0\}$, $J_* = \{j \in J : p_j(s) \geq 2\}$.

Set $p_j(s+1) = \bar{\bar{p}}_j, j \in J,\ \xi^0(p_j(s+1), j \in J) = \bar{\xi}^0(\bar{\bar{p}}_j, j \in J)$, and repeat Step 4 with s replaced by $s + 1$.

It follows from (74) that in a finite number of iterations we will find parameters $\bar{\bar{p}}_j, j \in J$ such that $m(\bar{\xi}^0(\bar{\bar{p}}_j, j \in J)) = |\bar{I}_2^a| + \sum\limits_{j \in J} \bar{\bar{p}}_j$, i.e. Property 2) is satisfied and according to Step 4 the algorithm finishes its work. Consequently, the described algorithm is finite.

7. Example

Consider the problem $P(p_j, j \in J)$ (see (8)) with the following data:

$$n = 5,\ p = 4,\ J = \{1, 2, 3\},\ I_1 = \{1, 2\},\ I_2 = \{3, 4\},\ D = E \in \mathbb{R}^{n \times n},$$

$$D_1=-\begin{pmatrix}0&1&2&1\\1&0&1&1\\2&1&0&3\\1&1&3&0\end{pmatrix},D_2=-\begin{pmatrix}1&1&0&0\\1&0&1&5\\0&1&1&2\\0&5&2&1\end{pmatrix},D_3=-\begin{pmatrix}1&0&0&1.5\\0&0&-1&-0.5\\0&-1&-1&1.5\\1.5&-0.5&1.5&0\end{pmatrix},$$
$$A_1=A_2=\begin{pmatrix}1&0&0&-2\\4&-1&2&1\\0&1&3&0\\2&-1&0&1\\0&0&1&0\end{pmatrix},A_3=\begin{pmatrix}0&0&0&2\\1&-1&-2&1\\0&1&0&0\\2&-1&0&1\\1&0&1&0\end{pmatrix},B_3=\begin{pmatrix}1&-1&0&-1\\1&1&1&0\\0&1&-1&1\\1&2&1&0\\-1&0&0&0\end{pmatrix},$$

$B_1=B_2=-E\in\mathbb{R}^{p\times p}$, $c_1=(-7,4,5,1)^T$, $c_2=(-6,3,6,2)^T$, $c_3=(-2,1,4,-2)^T$, $\omega_1=5$, $\omega_2=-7$, $\omega_3=8$, $\omega_4=0$, $q_1=(1,0,2,0,-1)^T$, $q_2=(1,2,-1,1,0)^T$, $q_3=(0,-1,2,0,3)^T$, $q_4=(1,-1,0,0,-2)^T$, $c=(-25.5,-37.25,4.5,-27.75,-3.75)^T$, $m_1=1.5$, $m_2=2$, $m_3=3$.

According to the algorithm proposed in section 6, let us find such values of the integers $p_j\geq 1, j\in J$, that a solution of the corresponding problem $P(p_j,j\in J)$ possesses Properties 1)-3).

Step 1. Solve the problem $P(p^*_j,j\in J)$ with $p^*_j=n+2=7, j\in J$. It has an optimal solution $\xi^0(p^*_j,j\in J)=\left(x^0,t^0_{kj},y^0_{kj},k=1,...,p^*_j,\ j\in J,\ y^0_i,i\in I\right)$, where

$$\begin{gathered}x^0=(1,2,-3,-1,0)^T,\ \ y^0_1=1,y^0_2=-1,y^0_3=1,y^0_4=0,\\ t^0_{11}=(0,0,0,1)^T,\ t^0_{21}=(0,1,0,0)^T,t^0_{31}=(0,0,1,0)^T,t^0_{i1}=(1,0,0,0)^T,i=4,...,7,\\ y^0_{11}=0.5,\ y^0_{21}=0.25,\ y^0_{31}=0.25,\ y^0_{41}=0.5,\ y^0_{51}=y^0_{61}=y^0_{71}=0;\\ t^0_{12}=(1,0,1,0)^T,t^0_{i2}=(1,0,0,1)^T,i=2,...,7;\ y^0_{12}=1.5,\ y^0_{22}=0.5,\ y^0_{i2}=0,i=3,...,7;\\ t^0_{i3}=(0,-4,0,4)^T,i=1,...,7;\ y^0_{13}=3,\ y^0_{i3}=0,i=2,...,7.\end{gathered}\tag{75}$$

Go to Step 2.

Step 2. Assumption 1 is fulfilled since matrix $D=E$ is positive definite (see Remark 1). Go to Step 3.

Step 3. The optimal solution $\xi^0(p^*_j,j\in J)$ satisfies Property 3) but does not satisfy Property 1). Then following the method used in the proof of Lemma 4.1 we find the integers $p_1(1)=4,p_2(1)=2,p_3(1)=1$, such that $P(p_j(1),j\in J)$ has an optimal solution $\xi^0(p_j(1),j\in J)=(x^0,t^0_{kj},y^0_{kj},k=1,...p_j(1),j\in J,y^0_i,i\in I)$ with the data given in (75) and this solution satisfies Property 1). Go to Step 4.

Step 4. Calculate $\gamma_*=|I_1|-|J|=2-3=-1$, $I^a_2=\{3\}$ and verify that the solution $\xi^0(p_j(1),j\in J)$ of the problem $P(p_j(1),j\in J)$ found on the previous step, does not satisfy Property 2) since

$$m(\xi^0(p_j(1),j\in J))=5\ <\ |I^a_2|+\sum_{j\in J}p_j(1)+\gamma_*=1+4+2+1-1=7.$$

According to the proof of Lemma 4.6, find the vector $(\Delta y_{kj},k=1,...,p_j(1),j\in J,\Delta y_i,i\in I_1\cup I^a_2)=$ $(0.73,\ 0.27,\ 0,\ -1,\ 0.18,\ -0.18,\ 0,\ 0.39,\ 1.72,\ 0.07)$ satisfying (44) and the numbers

$$\begin{gathered}\lambda=\min\{\lambda_{41}=0.5,\lambda_{22}=2.83\}=0.5,\\ \bar y^0_{11}=0.86,\ \bar y^0_{21}=0.39,\ \ \bar y^0_{31}=0.25,\ \ \bar y^0_{41}=0,\ \bar y^0_{12}=1.59,\ \ \bar y^0_{22}=0.41,\ \bar y^0_{13}=3,\end{gathered}$$

$$\bar{y}_1^0 = 1.19, \quad \bar{y}_2^0 = -0.14, \quad \bar{y}_3^0 = 1.04, \quad \bar{y}_4^0 = 0$$

by the rules (45), (46). Since $\bar{y}_{41}^0 = 0$, then, following the proof of Lemma 4.2, define new numbers $\bar{\bar{p}}_1 = p_1(1) - 1 = 3$, $\bar{\bar{p}}_2 = p_2(1) = 2$, $\bar{\bar{p}}_3 = p_3(1) = 1$. The vector $\bar{\xi}^0(\bar{\bar{p}}_j, j \in J) = (x^0,\ t_{kj}^0, \bar{y}_{kj}^0, k = 1, ..., \bar{\bar{p}}_j,\ j \in J;\ \bar{y}_i^0,\ i \in I)$ is an optimal solution of the problem $P(\bar{\bar{p}}_j, j \in J)$ that satisfies the Properties 1) and 3). Set $p_1(2) = \bar{\bar{p}}_1 = 3,\ p_2(2) = \bar{\bar{p}}_2 = 2,\ p_3(2) = \bar{\bar{p}}_3 = 1,\ \xi^0(p_j(2), j \in J) = \bar{\xi}^0(\bar{\bar{p}}_j, j \in J)$ and repeat Step 4 with $s = 2$.

Step 4. Solution $\xi^0(p_j(2), j \in J)$ of problem $P(p_j(2), j \in J)$ does not satisfy Property 2) since $m(\xi^0(p_j(2), j \in J)) = 5 \ < \ |I_2^a| + \sum\limits_{j \in J} p_j(2) + \gamma_* = 1 + 3 + 2 + 1 - 1 = 6.$

Following the rules described in the proof of Lemma 4.6, find vector

$$(\Delta y_{kj}, k = 1, ..., p_j(2), j \in J, \Delta y_i, i \in I_1 \cup I_2^a) = (1,\ 0,\ -1,\ 1,\ -1,\ 0,\ 0,\ 0,\ 0),$$

satisfying (44) and numbers $\lambda = \min\{\lambda_{31} = 0.25,\ \lambda_{22} = 0.41\} = 0.25$,

$$\bar{y}_{11}^0 = 1.11, \quad \bar{y}_{21}^0 = 0.39, \quad \bar{y}_{31}^0 = 0, \tag{76}$$
$$\bar{y}_{12}^0 = 1.84, \quad \bar{y}_{22}^0 = 0.16, \ \bar{y}_{13}^0 = 3, \ \bar{y}_1^0 = 1.19, \quad \bar{y}_2^0 = -0.14, \quad \bar{y}_3^0 = 1.04, \quad \bar{y}_4^0 = 0,$$

by formulae (45), (46). Since $\bar{y}_{31}^0 = 0$, then following Lemma 4.2, define new numbers $\bar{\bar{p}}_1 = p_1(2) - 1 = 2$, $\bar{\bar{p}}_2 = p_2(2) = 2$, $\bar{\bar{p}}_3 = p_3(2) = 1$. The optimal solution $\bar{\xi}^0(2, 2, 1)$ of the problem $P(2, 2, 1)$ satisfies the Properties 1), 3).

Set $p_1(3) = 2,\ p_2(3) = 2,\ p_3(3) = 1,\ \xi^0(p_j(3), j \in J) = \bar{\xi}^0(\bar{\bar{p}}_j, j \in J)$ and repeat Step 4 again with $s = 3$.

Step 4. Solution $\xi^0(p_j(3), j \in J)$ of problem $P(p_j(3), j \in J)$ satisfies Property 2): $m(\xi^0(p_j(3), j \in J)) = 5 \ = \ |I_2^a| + \sum\limits_{j \in J} p_j(3) + \gamma_* = 1 + 2 + 2 + 1 - 1 = 5$, then STOP.

As a result of applying of the algorithm to the example, we have found integers $p_1 = 2, p_2 = 2, p_3 = 1$ and an optimal solution $\bar{\xi}^0(p_1, p_2, p_3)$ of the corresponding problem $P(p_1, p_2, p_3)$ that satisfies the Properties 1)–3). Here $\bar{\xi}^0(p_1, p_2, p_3) = (x^0, t_{kj}^0, \bar{y}_{kj}^0, k = 1, ..., p_j, j \in J; \bar{y}_i^0, i \in I)$ with components $x^0, t_{kj}^0, k = 1, ..., p_j, j \in J$, defined in (75) and components $\bar{y}_{kj}^0, k = 1, ..., p_j, j \in J; \bar{y}_i^0, i \in I$, defined in (76).

8. Conclusions

In this paper, given a finite set J, $|J| \leq n$, and a finite number of integers $p_j, j \in J$, we have considered the NLP problem $P(p_j, j \in J)$ in the form (8).

This problem appears as an auxiliary problem in our study of the parametric problems of SIP and may have different values of parameters $p_j, j \in J$. When the differential properties of solutions of the parametric SIP problems are being studied, we are especially interested in such values of parameters $p_j \geq 1, j \in J$, that the corresponding problem $P(p_j, j \in J)$ has an optimal solution possessing Properties 1)-3).

Use of the specificity of the problems $P(p_j, j \in J)$ and in-depth analysis of their properties allowed us to get the following results.

- We have shown that all the feasible solutions of the problem $P(p_j, j \in J)$ are regular, in the sense that they satisfy the Relaxed Constant Rank CQ. This has permitted us to formulate and prove the first order necessary and sufficient optimality conditions.

- Taking into account the obtained optimality conditions, in Section 4, we have studied in details how the change of the parameters in the problem $P(p_j, j \in J)$ affects the optimal value of its cost function.
- We have shown that if the problem $P(p^*_j, j \in J)$ with $p^*_j = n+2, j \in J$, admits an optimal solution satisfying Assumption 1, then for some values of the parameters $p_j \geq 1, j \in J, \sum\limits_{j\in J} p_j \leq n - \gamma_*$, the corresponding problem $P(p_j, j \in J)$ satisfies Properties 1) - 3). We have also proposed conditions that guarantee the solvability of the problem $P(p^*_j, j \in J)$.
- Finally, we have constructed an algorithm that in a finite number of iterations either finds the values of the parameters for which the corresponding problem $P(p_j, j \in J)$ has optimal solutions satisfying Properties 1) - 3) or proves that such parameters do not exist.

The results of the paper will be used in the forthcoming paper devoted to study of the parametric SIP problems with finitely representable index sets.

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Appendix A. Proof of Lemma A.1

LEMMA A.1 *Let the implication (6) $\Longrightarrow$ (7) take place and*

$$\operatorname{rank}(q_i, i \in I_1) = \operatorname{rank}(q_i, i \in \bar{I}_1) = |\bar{I}_1| < |I_1|, \quad \bar{I}_1 \subset I_1. \tag{A1}$$

Then in the problem $P(p_j, j \in J)$, without loss of generality we can exclude from consideration variables $y_i, i \in I_1 \setminus \bar{I}_1$, having replaced I_1 by $\bar{I}_1$.

Proof. From the implication (6) $\Longrightarrow$ (7), one can deduce the following one:

$$\sum_{i \in I_1} q_i \Delta y_i = 0 \implies \sum_{i \in I_1} \omega_i \Delta y_i = 0. \tag{A2}$$

Indeed, let us consider two sets of parameters

$$(\Delta y_i^*, i \in I_1,\ \Delta y_i = 0, i \in I_2)\ \ and(-\Delta y_i^*, i \in I_1,\ \Delta y_i = 0, i \in I_2)$$

with $\Delta y_i^*, i \in I_1$, such that $\sum_{i\in I_1} q_i \Delta y_i^* = 0$. Both these sets satisfy conditions (6) that imply (see (7)) the inequalities $-\sum_{i\in I_1} \omega_i \Delta y_i^* \geq 0$ and $-\sum_{i\in I_1} \omega_i(-\Delta y_i^*) \geq 0$. The implication (A2) is proved.

It follows from (A1) that

$$\text{for any } \ \Delta y_i, i \in I_1 \setminus \bar{I}_1,\ \exists \Delta y_i, i \in \bar{I}_1, \text{ such that } \sum_{i\in I_1} q_i \Delta y_i = 0. \tag{A3}$$

Taking into account (A2), (A3), it is easy to see that for any feasible solution ξ (see (9)) of the problem $P(p_j, j \in J)$, there exists a feasible solution

$$\bar{\xi} = (x,\ t_{kj}, y_{kj}, k = 1, ..., p_j,\ j \in J;\ \bar{y}_i, i \in I_1,\ y_i,\ i \in I_2)$$

such that $\bar{y}_i = 0, i \in I_1 \setminus \bar{I}_1$ and $F(\xi) = F(\bar{\xi})$ where $F(\xi)$ denotes the cost function in the problem $P(p_j, j \in J)$. Hence, without loss of generality, in the problem $P(p_j, j \in J)$ we can exclude from consideration variables $y_i, i \in I_1 \setminus \bar{I}_1$, having replaced I_1 by $\bar{I}_1$. ■

Appendix B. Verification of Assumption 1

It is evident that system (17)-(20) can be written in the form

$$Az + B\mu = b,\ \mu \geq 0, \tag{B1}$$

where $A \in \mathbb{R}^{m\times n}, B = (b_i, i \in I) \in \mathbb{R}^{m\times|I|}$, and $b \in \mathbb{R}^m$ are given matrices and vector that are constructed on the base of initial data (3). It is known that the system has a solution. Then Assumption 1 takes the form of the following one.

Assumption 3 *Given a solution (z^*, μ^*) of system (B1), there is no another solution $(\bar{z}, \bar{\mu})$ of this system such that $\bar{z} \neq z^*$.*

Denote $\mathcal{M}^* := \{\mu \in \mathbb{R}^{|I|} : Az^* + B\mu = b,\ \mu \geq 0\}$, $I_* := \{i \in I : \exists \mu = (\mu_i, i \in I) \in \mathcal{M}^*,\ \mu_i > 0\}$.

Proposition B.1 *Assumption 3 is fulfilled if and only if the following two conditions are satisfied:*

$$1)\ \ \operatorname{rank}(A, b_i, i \in I_*) = n + \operatorname{rank}(b_i, i \in I_*),\ \ 2)\ val(LP_*) = 0,$$

where $val(LP_*)$ denotes the optimal value of the cost function of the following Linear Programming (LP) problem:

$$LP_* : \qquad \max \sum_{i\in I_0} \Delta\mu_i,$$
$$\text{s.t. } A\Delta z + B\Delta\mu = 0,\ \sum_{i\in I_0} \Delta\mu_i \leq 1,\ \Delta\mu_i \geq 0, i \in I_0 = I \setminus I_*.$$

Proof. Notice that by construction, there exists a vector $\tilde{\mu} \in \mathcal{M}^*$ such that $\tilde{\mu}_i > 0, i \in I_*$, and $(z^*, \tilde{\mu})$ is a solution of system (B1).

$\Rightarrow$) Suppose first, that Assumption 3 is fulfilled, but condition 1) is not satisfied: rank $(A, b_i, i \in I_*) < n +$ rank $(b_i, i \in I_*)$. Let $\tilde{I}_* \subset I_*$ be such a set that

$$\text{rank}\,(b_i, i \in I_*) = \text{rank}\,(b_i, i \in \tilde{I}_*) = |\tilde{I}_*|.$$

Hence, there exists a vector $(\Delta z, \Delta \mu_i, i \in \tilde{I}_*) \neq 0$ with the componentes satisfying the condition $A\Delta z + \sum\limits_{i \in \tilde{I}_*} b_i \Delta \mu_i = 0$, $\Delta z \neq 0$. It is evident that for sufficiently small $\epsilon > 0$ the vector $(\bar{z} = z^* + \epsilon \Delta z, \bar{\mu}_i = \tilde{\mu}_i + \epsilon \Delta \mu_i, i \in \tilde{I}_*, \bar{\mu}_i = \tilde{\mu}_i, i \in I \setminus \tilde{I}_*)$ is a solution of system (B1) and $z^* \neq \bar{z}$. But this contradicts Assumption 3. Hence under Assumption 3, condition 1) should be satisfied.

Now suppose that Assumption 3 is fulfilled, but condition 2) is not satisfied. Hence there exists a vector (an optimal solution of problem (LP_*)) $(\Delta z^0, \Delta \mu_i^0, i \in I)$ with $\Delta \mu_i^0 \geq 0, i \in I_0$, $\sum\limits_{i \in I_0} \Delta \mu_i^0 = 1$. If suppose that in this vector $\Delta z^0 = 0$, then it is easy to show that for a sufficiently small $\epsilon > 0$, the vector $(\bar{\mu}_i = \tilde{\mu}_i + \epsilon \Delta \mu_i^0, i \in I)$ belongs to the set $\mathcal{M}^*$ defined above, and there exists $i \in I_0$ such that $\bar{\mu}_i > 0$. But this contradicts the rules for constructing the set I_*. Hence $\Delta z^0 \neq 0$. As before, it is easy to show that in this case the vector $(\bar{z} = z^* + \epsilon \Delta z^0, \bar{\mu}_i = \tilde{\mu}_i + \epsilon \Delta \mu_i^0, i \in I)$ is a solution of system (B1) and $z^* \neq \bar{z}$ that contradicts Assumption 3.

Hence we have proved that if Assumption 3 is fulfilled, then conditions 1) and 2) are satisfied.

$\Leftarrow$) Now suppose that conditions 1) and 2) are satisfied but system (B1) admits an another solution $(\bar{z}, \bar{y})$ such that $z^* \neq \bar{z}$. Denote $\Delta z^* = \bar{z} - z^* \neq 0$, $\Delta \mu^* = \bar{\mu} - \tilde{\mu}$. Notice that by construction, $\Delta \mu_i^* = \bar{\mu}_i \geq 0, i \in I_0$. If suppose that at least one of the last inequalities is strictly satisfied, then $\sum\limits_{i \in I_0} \Delta \mu_i^* > 0$, and we obtain a contradiction with condition 2). Hence $\Delta \mu_i^* = 0, i \in I_0$. Therefore $A\Delta z^* + \sum\limits_{i \in I_*} b_i \Delta \mu_i^* = 0$, wherefrom, taking into account the definition of the set $\tilde{I}_*$, one can conclude that there exist $\Delta \bar{\mu}_i^*, i \in \tilde{I}_*$, such that $A\Delta z^* + \sum\limits_{i \in \tilde{I}_*} b_i \Delta \bar{\mu}_i^* = 0$. Since $\Delta z^* \neq 0$, then from the last equality it follows: rank $(A, b_i, i \in \tilde{I}_*) < n + |\tilde{I}_*| = n +$ rank $(b_i, i \in \tilde{I}_*)$. Taking into account that rank $(A, b_i, i \in \tilde{I}_*) =$ rank $(A, b_i, i \in I_*)$ we obtain a contradiction with condition 1). Hence conditions 1) and 2) imply Assumption 3. The proposition is proved. ■

It follows from Proposition B.1, that to verify the fulfillment of Assumption 3 one needs to find the set I_*. This set can be constructed by a procedure described below.

Recall here that $val(P)$ denotes the optimal value of the cost function in an optimization problem (P).

A procedure of constructing the set I_*.

Let (z^*, μ^*) be a known solution of system (B1).

Initialization. Set $I_*^{(1)} := \{i \in I : \mu_i^* > 0\}$, $I_0^{(1)} := \emptyset$, $s := 1$.

Step s. If $I_*^{(s)} \cup I_0^{(s)} = I$, then set $I_* := I_*^{(s)}$ and STOP.

Suppose that $I^{(s)} = I \setminus (I_*^{(s)} \cup I_0^{(s)}) \neq \emptyset$. Solve the following LP problem:

$$LP_0 : \quad \max \sum_{i \in I^{(s)}} \Delta\mu_i, \ \text{ s.t. } \sum_{i \in I_*^{(s)} \cup I^{(s)}} b_i \Delta\mu_i = 0, \sum_{i \in I^{(s)}} \Delta\mu_i \leq 1, \Delta\mu_i \geq 0, \ i \in I^{(s)}.$$

This problem has an optimal solution. Let $(\Delta\mu_i^0, i \in I_*^{(s)} \cup I^{(s)})$ be a primal and $\lambda \in \mathbb{R}^m$ be a dual optimal solutions of problem (LP_0).

There are two possibilities here: either $val(LP_0) = 0$ or $val(LP_0) = 1$.

If $val(LP_0) = 0$ then set $I_* := I_*^{(s)}$ and STOP.

Suppose now that $val(LP_0) = 1$. From the LP optimality conditions we have

$$\lambda^T b_i = 0, i \in I_*^{(s)}, \ \lambda^T b_i \geq 0, \ \Delta\mu_i^0 \lambda^T b_i = 0, i \in I^{(s)}.$$

Set $I_*^{(s+1)} := I_*^{(s)} \cup \{i \in I^{(s)} : \Delta\mu_i^0 > 0\}$, $I_0^{(s+1)} := I_0^{(s)} \cup \{i \in I^{(s)} : \lambda^T b_i > 0\}$, and go to the next step having set $s := s + 1$.

Since by construction $|I_*^{(s+1)}| \geq |I_*^{(s)}| + 1$ and $|I_0^{(s+1)}| \geq |I_0^{(s)}|$, it is evident that the described procedure constructs the set I_* in a finite number of steps.

Notice that the matrices A and B were introduced to rewrite system (17)-(20) in the form (B1) and therefore have special structure. Accounting of this structure considerably simplifies the described above procedure and verification of the conditions 1), 2).

Appendix C. Testing optimality of feasible solutions of the QP problem (13)

Any problem (13) can be written in the following form:

$$QP^* : \quad \min \frac{1}{2} t^T Q t + a^T t, \ \text{ s.t. } t \in K = \{t \in \mathbb{R}^p : b_s^T t \leq 0, \ s \in S\},$$

where matrix Q satisfies the condition $t^T Q t \geq 0, \ \forall t \in K$.

Problem (QP^*) is a nonconvex QP problem with the unbounded feasible set. It is known that such problems are NP-hard and over the past decades, much effort has been applied to the search for solutions of these problems. Some computational algorithms for solving special classes of nonconvex QP problems can be found in [5, 6, 23].

In what follows, we present three theorems (see [5, 23]) that can be used to test the optimality of a given feasible solution of problem (QP^*).

THEOREM C.1 *(Necessary and sufficient optimality conditions for local optimality) A vector $t^0 \in K$ is a local minimizer of problem (QP^*) iff there exists a vector $(\mu_s, s \in S_a(t^0))$ with $S_a(t^0) := \{s \in S : \ b_s^T t^0 = 0\}$ such that*

$$Qt^0 + a + \sum_{s \in S_a(t^0)} b_s \mu_s = 0, \ \mu_s \geq 0, s \in S_a(t^0), \tag{C1}$$

and $l^T Q l \geq 0, \ \forall l \in \{l \in \mathbb{R}^p : b_s^T l \leq 0, s \in S_a(t^0), \ (Qt^0 + a)^T l \leq 0\}$.

THEOREM C.2 *(Sufficient optimality conditions for global optimality) A vector $t^0 \in K$ is a global minimizer of problem (QP^*) if there exists a vector $(\mu_s, s \in S_a(t^0))$ such that conditions (C1) are satisfied and $l^T Q l \geq 0, \ \forall l \in \{l \in \mathbb{R}^p : b_s^T l \leq 0, s \in S_a(t^0)\}$.*

THEOREM C.3 *If in problem (QP^*), matrix Q has s negative eigenvalues and t^0 is any local (global) optimal solution of this problem, then $|S_a(t^0)| \geq s$.*

In the cases when application of Theorems C.1- C.3 does not permit to check the optimality of a given feasible solution t^0 of problem (QP^*), the following procedure can be applied.

Denote by $\mathcal{S}$ be the set of all subsets S^* of the index set S. For $S^* \in \mathcal{S}$, solve the following LP problem:

$$LP(S^*): \quad \begin{aligned} &\max(\lambda - a^T t), \\ &\lambda a + Qt + \sum_{s \in S^*} b_s \mu_s = 0, \ 0 \leq \lambda \leq 1; \\ &b_s^T t = 0, s \in S^*; \ b_s^T t \leq 0, s \in S \setminus S^*. \end{aligned}$$

This problem possesses a feasible solution: $\lambda = 0, t = 0, \mu_s = 0, s \in S^*$. Hence $val(LP(S^*)) \geq 0$.

If $val(LP(S^*)) = \infty$, then $val(QP^*) = -\infty$ and problem (QP^*) does not admit optimal solutions since its cost function is not bounded from below on the set of its feasible solutions.

Suppose that the problem $(LP(S^*))$ has an optimal solution that we denote here by $\lambda^*, t^*, \mu_s^*, s \in S^*$.

a) If $val(LP(S^*)) = 0$, set $v(S^*) := 0$. Notice that the equality $val(LP(S^*)) = 0$ implies the relations $\lambda^* = 0, a^T t^* = 0$.

b) If $0 < val(LP(S^*)) < \infty$, set $v(S^*) := -\frac{1}{2} t^{*T} Q t^* \leq 0$. In this case $\lambda^* = 1$ and $a^T t^* = -t^{*T} Q t^* = a^T t = -t^T Q t$ for all $t \in K(S^*)$, where

$$K(S^*) = \{t \in \mathbb{R}^p : \exists \mu_s, s \in S^*, a + Qt + \sum_{s \in S^*} b_s \mu_s = 0, \ b_s^T t = 0, s \in S^*; \ b_s^T t \leq 0, s \in S \setminus S^*\}.$$

Therefore in the case b), we have $v(S^*) = \frac{1}{2} t^{*T} Q t^* + a^T t^* = \frac{1}{2} t^T Q t + a^T t$ for all $t \in K(S^*)$.

Suppose that $val(LP(S^*)) < \infty$ for all $S^* \in \mathcal{S}$. Then $val(QP^*) = \min\limits_{S^* \in \mathcal{S}} v(S^*)$.

To test the optimality of a given $t^0 \in K$, one has to compare two values: $f(t^0) := \frac{1}{2} t^{0T} Q t^0 + a^T t^0$ and $val(QP^*)$. If $f(t^0) = val(QP^*)$, then t^0 is an optimal solution of problem (QP^*). If $f(t^0) > val(QP^*)$, then t^0 is not optimal in (QP^*).

Remark 3 The described procedure, additionally, provides the following information about the set K^0 of all optimal solutions of problem (QP^*):

- if there exists S^* such that $val(LP(S^*)) = \infty$, then $val(QP^*) = -\infty$ and $K^0 = \emptyset$;
- if $val(QP^*) = 0$ then $K^0 = \{t \in K : t^T Q t = 0, a^T t = 0\}$;
- if $0 > val(QP^*) > -\infty$ then $K^0 = \bigcup\limits_{S^* \in \mathcal{S}^0} K(S^*)$, where $\mathcal{S}^0 = \{S^* \in \mathcal{S} : val(QP^*) = v(S^*)\}$.

Remark 4 Applying the approach described above, using Theorems C.1-C.3, the branch and bound method and the duality theory, one can develop more efficient procedures permitting in many cases to avoid iterating through all the existing options.