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Hyers-Ulam and Hyers-Ulam-Rassias Stability of a Class of Integral Equations on Finite Intervals

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Abstract

The purpose of this work is to study different kinds of stability for a class of integral equations defined on a finite interval. Sufficient conditions are derived in view to obtain Hyers-Ulam stability and Hyers-Ulam-Rassias stability by using fixed point techniques and the Bielecki metric.

Key words: Hyers-Ulam stability, Hyers-Ulam-Rassias stability, Banach fixed point theorem, integral equation
MSC 2000: 45M10, 34K20, 47H10

1 Introduction

During the last seven decades the concepts of Hyers-Ulam stability and Hyers-Ulam-Rassias stability for different kinds of functional equations, differential equations, integral equations and others has been studied in a quite extensive way due to their great number of applications e.g. in elasticity, semiconductors, heat conduction, fluid flow, scattering theory, chemical reactions and population dynamic, among others (see [1, 2, 3, 4, 5, 6, 7, 8]). Originated in 1940 from a famous question raised by S. M. Ulam, the first results of stability of this type were about to discover when a solution of an equation differing "slightly" from a given one must be somehow near to the solution of the given equation. A first parcial answer to this question was given by D. H. Hyers, introducing therefore the so-called Hyers-Ulam

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stability. New directions were introduced by Th. M. Rassias, see [9], introducing therefore the so-called Hyers-Ulam-Rassias stability.

In this work, we will be devoted to analyse Hyers-Ulam and Hyers-Ulam-Rassias stability for the following class of integral equations:

$$y(x) = f\left(x, y(x), \int_a^b k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right), \qquad x \in [a,b], \tag{1}$$

and

$$y(x) = f\left(x, y(x), \int_a^x k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right), \qquad x \in [a,b], \tag{2}$$

where a and b are fixed real numbers, $f : [a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ and $k : [a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ are continuous functions, and $\alpha : [a,b] \to [a,b]$ is a continuous delay function which therefore fulfills $\alpha(\tau) \leq \tau$ for all $\tau \in [a,b]$.

The formal definition of the above mentioned Hyers-Ulam-Rassias stability and Hyers-Ulam stability are now introduced for the integral equation (1).

If for each function y satisfying

$$\left|y(x) - f\left(x, y(x), \int_a^b k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right)\right| \leq \sigma(x), \qquad x \in [a,b], \tag{3}$$

where σ is a non-negative function, there is a solution y_0 of the integral equation and a constant $C > 0$ independent of y and y_0 such that $|y(x) - y_0(x)| \leq C\sigma(x)$, for all $x \in [a,b]$, then we say that the integral equation (1) has the Hyers-Ulam-Rassias stability.

If for each function y satisfying

$$\left|y(x) - f\left(x, y(x), \int_a^b k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b], \tag{4}$$

where $\theta \geq 0$, there is a solution y_0 of the integral equation and a constant $C > 0$ independent of y and y_0 such that $|y(x) - y_0(x)| \leq C\theta$, for all $x \in [a,b]$, then we say that the integral equation has the Hyers-Ulam stability.

Some of the present techniques to study the stability of functional equations use a combination of the following well-known Banach Fixed Point Theorem with a generalized metric in appropriate settings.

Theorem 1 *Let (X,d) be a generalized complete metric space and $T : X \to X$ a strictly contractive operator with a Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d(T^{k+1}x, T^k x) < \infty$ for some $x \in X$, then the following three propositions hold true:*

i) the sequence $(T^n x)_{n\in\mathbb{N}}$ converges to a fixed point x^* of T;

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ii) x^* *is the unique fixed point of* T *in*

$$X^* = \{y \in X : d(T^k x, y) < \infty\}; \tag{5}$$

iii) *if* $y \in X^*$, *then*

$$d(y, x^*) \leq \frac{1}{1-L} d(Ty, y). \tag{6}$$

Let $p > 0$ be a constant, we will be using the space $C_p([a,b])$ of continuous functions $u : [a,b] \to \mathbb{C}$ endowed with the generalized Bielecki metric

$$d_p(u,v) = \sup_{x \in [a,b]} \frac{|u(x) - v(x)|}{e^{p(x-a)}}. \tag{7}$$

We recall that $(C_p([a,b]), d_p)$ is a complete metric spaces (cf., [10]).

2 Hyers-Ulam-Rassias Stability

The present section is devoted to present sufficient conditions for the Hyers-Ulam-Rassias stability of the integral equations (1) and (2).

Theorem 2 *Let* $\alpha : [a,b] \to [a,b]$ *a continuous delay function with* $\alpha(t) \leq t$ *for all* $t \in [a,b]$ *and* $\sigma : [a,b] \to (0, \infty)$ *a non-negative function. Moreover, suppose that* $f : [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *is a continuous function satisfying the Lipschitz condition*

$$|f(x, u(x), g(x)) - f(x, v(x), h(x))| \leq M \left(|u(x) - v(x)| + |g(x) - h(x)| \right) \tag{8}$$

with $M > 0$ *and the kernel* $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ *is a continuous kernel function satisfying the Lipschitz condition*

$$|k(x,t,u(t),u(\alpha(t))) - k(x,t,v(t),v(\alpha(t)))| \leq L|u(t) - v(t)| \tag{9}$$

with $L > 0$.

If $y \in C_p([a,b])$ *is such that*

$$\left| y(x) - f\left(x, y(x), \int_a^b k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right) \right| \leq \sigma(x), \qquad x \in [a,b], \tag{10}$$

and $M\left(1 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$, *then there is a unique function* $y_0 \in C_p([a,b])$ *such that*

$$y_0(x) = f\left(x, y_0(x), \int_a^b k(x,\tau,y_0(\tau),y_0(\alpha(\tau)))d\tau\right) \tag{11}$$

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and

$$|u(x) - y_0(x)| \leq \frac{p\sigma(x)}{p - Mp - ML(e^{p(b-a)} - 1)} \tag{12}$$

for all $x \in [a, b]$.

This means that under the above conditions, the integral equation (1) has the Hyers-Ulam-Rassias stability.

Proof. We will consider the operator $T : C_p([a, b]) \to C_p([a, b])$, defined by

$$(Tu)(x) = f\left(x, u(x), \int_a^b k(x, \tau, u(\tau), u(\alpha(\tau)))d\tau\right), \tag{13}$$

for all $x \in [a, b]$ and $u \in C_p([a, b])$.

Under the present conditions, we will deduce that the operator T is strictly contractive with respect to the metric (7). Indeed, for all $u, v \in C_p([a, b])$, we have,

$$\begin{aligned}
d_p(Tu, Tv) &= \sup_{x\in[a,b]} \frac{|(Tu)(x) - (Tv)(x)|}{e^{p(x-a)}} \\
&\leq M \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \{|u(x) - v(x)| + \\
&\qquad \left|\int_a^b k(x, \tau, u(\tau), u(\alpha(\tau)))d\tau - \int_a^b k(x, \tau, v(\tau), v(\alpha(\tau)))d\tau\right|\Big\} \\
&\leq M \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \{|u(x) - v(x)| + \\
&\qquad \int_a^b |k(x, \tau, u(\tau), u(\alpha(\tau))) - k(x, \tau, v(\tau), v(\alpha(\tau)))|\, d\tau\Big\} \\
&\leq M \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \left\{|u(x) - v(x)| + L\int_a^b |u(\tau) - v(\tau)|\, d\tau\right\} \\
&\leq M \left\{\sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{e^{p(x-a)}} + L \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \int_a^b |u(\tau) - v(\tau)|\, d\tau\right\} \\
&= M \left\{\sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{e^{p(x-a)}} + L \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \int_a^b e^{p(\tau-a)} \frac{|u(\tau) - v(\tau)|}{e^{p(\tau-a)}} d\tau\right\} \\
&\leq M \left\{\sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{e^{p(x-a)}} + L \sup_{\tau\in[a,b]} \frac{|u(\tau) - v(\tau)|}{e^{p(\tau-a)}} \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \int_a^b e^{p(\tau-a)} d\tau\right\} \\
&= M \left\{d_p(u, v) + Ld_p(u, v) \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \frac{e^{p(b-a)} - 1}{p}\right\}
\end{aligned}$$

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$$= M\left(1+\frac{L}{p}\left(e^{p(b-a)}-1\right)\right)d_p(u,v). \tag{14}$$

Due to the fact that $M\left(1+\frac{L}{p}\left(e^{p(b-a)}-1\right)\right) < 1$ it follows that T is strictly contractive. Thus, we can apply the above mentioned Banach Fixed Point Theorem, which ensures that we have the Hyers-Ulam-Rassias stability for the integral equation (1). Additionally, (12) follows from (6) and (10).

For the Volterra integral equation (2) we have the following result.

Theorem 3 *Let $\alpha : [a,b] \to [a,b]$ a continuous delay function with $\alpha(t) \leq t$ for all $t \in [a,b]$ and $\sigma : [a,b] \to (0,\infty)$ a non-negative function. Moreover, suppose that $f : [a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),g(x)) - f(x,v(x),h(x))| \leq M\left(|u(x)-v(x)| + |g(x)-h(x)|\right) \tag{15}$$

with $M > 0$ and the kernel $k : [a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\alpha(t))) - k(x,t,v(t),v(\alpha(t)))| \leq L|u(t)-v(t)| \tag{16}$$

with $L > 0$.

If $y \in C_p([a,b])$ is such that

$$\left|y(x) - f\left(x, y(x), \int_a^x k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right)\right| \leq \sigma(x), \qquad x \in [a,b], \tag{17}$$

and $M\left(1+\frac{L}{p}\left(\frac{e^{p(b-a)}-1}{e^{p(b-a)}}\right)\right) < 1$, then there is a unique function $y_0 \in C_p([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), \int_a^x k(x,\tau,y_0(\tau),y_0(\alpha(\tau)))d\tau\right) \tag{18}$$

and

$$|u(x) - y_0(x)| \leq \frac{pe^{p(b-a)}\sigma(x)}{e^{p(b-a)}(p-Mp) - ML(e^{p(b-a)}-1)} \tag{19}$$

for all $x \in [a,b]$.

This means that under the above conditions, the Volterra integral equation (2) has the Hyers-Ulam-Rassias stability.

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Proof. We will consider the operator $T : C_p([a,b]) \to C_p([a,b])$, defined by

$$(Tu)(x) = f\left(x, u(x), \int_a^x k(x,\tau,u(\tau),u(\alpha(\tau)))d\tau\right), \tag{20}$$

for all $x \in [a,b]$ and $u \in C_p([a,b])$.

Under the present conditions, we will deduce that the operator T is strictly contractive (with respect to the metric under consideration). Indeed, for all $u, v \in C_p([a,b])$, we have,

$$\begin{aligned}
d_p(Tu,Tv) &= \sup_{x\in[a,b]} \frac{|(Tu)(x) - (Tv)(x)|}{e^{p(x-a)}} \\
&\le M \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \Big\{ |u(x) - v(x)| \\
&\qquad + \left| \int_a^x k(x,\tau,u(\tau),u(\alpha(\tau)))d\tau - \int_a^x k(x,\tau,v(\tau),v(\alpha(\tau)))d\tau \right| \Big\} \\
&\le M \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \Big\{ |u(x) - v(x)| \\
&\qquad + \int_a^x |k(x,\tau,u(\tau),u(\alpha(\tau))) - k(x,\tau,v(\tau),v(\alpha(\tau)))|\, d\tau \Big\} \\
&\le M \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \left\{ |u(x) - v(x)| + L \int_a^x |u(\tau) - v(\tau)|\, d\tau \right\} \\
&\le M \left\{ \sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{e^{p(x-a)}} + L \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \int_a^x |u(\tau) - v(\tau)|\, d\tau \right\} \\
&= M \left\{ \sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{e^{p(x-a)}} + L \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \int_a^x e^{p(\tau-a)} \frac{|u(\tau) - v(\tau)|}{e^{p(\tau-a)}} d\tau \right\} \\
&\le M \left\{ \sup_{x\in[a,b]} \frac{|u(x) - v(x)|}{e^{p(x-a)}} + L \sup_{\tau\in[a,b]} \frac{|u(\tau) - v(\tau)|}{e^{p(\tau-a)}} \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \int_a^x e^{p(\tau-a)} d\tau \right\} \\
&= M \left\{ d_p(u,v) + L d_p(u,v) \sup_{x\in[a,b]} \frac{1}{e^{p(x-a)}} \frac{e^{p(b-a)} - 1}{p} \right\} \\
&= M \left(1 + \frac{L}{p} \left(\frac{e^{p(b-a)} - 1}{e^{p(b-a)}} \right) \right) d_p(u,v).
\end{aligned} \tag{21}$$

Due to the fact that $M\left(1 + \frac{L}{p}\left(\frac{e^{p(b-a)}-1}{e^{p(b-a)}}\right)\right) < 1$ it follows that T is strictly contractive. Thus, we can apply the above mentioned Banach Fixed Point Theorem, which ensures that we have the Hyers-Ulam-Rassias stability for the Volterra integral equation (2). Additionally, (19) follows from (6) and (17).

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3 Hyers-Ulam Stability

The present section is devoted to present sufficient conditions for the Hyers-Ulam stability of the integral equations (1) and (2).

Theorem 4 *Let $\alpha : [a,b] \to [a,b]$ a continuous delay function with $\alpha(t) \le t$ for all $t \in [a,b]$. Moreover, suppose that $f : [a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),g(x)) - f(x,v(x),h(x))| \le M\left(|u(x)-v(x)| + |g(x)-h(x)|\right) \tag{22}$$

with $M > 0$ and the kernel $k : [a,b]\times[a,b]\times\mathbb{C}\times\mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\alpha(t))) - k(x,t,v(t),v(\alpha(t)))| \le L|u(t)-v(t)| \tag{23}$$

with $L > 0$.

If $y \in C_p([a,b])$ is such that

$$\left|y(x) - f\left(x, y(x), \int_a^b k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right)\right| \le \theta, \qquad x \in [a,b], \tag{24}$$

where $\theta > 0$ and $M\left(1 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$, then there is a unique function $y_0 \in C_p([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), \int_a^b k(x,t,y_0(t),y_0(\alpha(t)))dt\right) \tag{25}$$

and

$$|u(x) - y_0(x)| \le \frac{p\theta}{p - Mp - ML(e^{p(b-a)} - 1)} \tag{26}$$

for all $x \in [a,b]$

This means that under the above conditions, the integral equation (1) has the Hyers-Ulam stability.

Proof. We will consider the operator $T : C_p([a,b]) \to C_p([a,b])$, defined by

$$(Tu)(x) = f\left(x, u(x), \int_a^b k(x,\tau,u(\tau),u(\alpha(\tau)))d\tau\right), \tag{27}$$

for all $x \in [a,b]$ and $u \in C_p([a,b])$.

By the same above procedure we have T strictly contractive with respect to the metric (7) due to the fact that $M\left(1 + \frac{L}{p}\left(e^{p(b-a)} - 1\right)\right) < 1$. Thus, we can again apply the Banach Fixed Point Theorem, which ensures that we have the Hyers-Ulam stability for the integral equation with (26) being obtained by using (6) and (24).

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Now, we consider the Volterra integral equation (2).

Theorem 5 *Let $\alpha : [a,b] \to [a,b]$ a continuous delay function with $\alpha(t) \leq t$ for all $t \in [a,b]$. Moreover, suppose that $f : [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous function satisfying the Lipschitz condition*

$$|f(x,u(x),g(x)) - f(x,v(x),h(x))| \leq M\left(|u(x)-v(x)| + |g(x)-h(x)|\right) \tag{28}$$

with $M > 0$ and the kernel $k : [a,b] \times [a,b] \times \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is a continuous kernel function satisfying the Lipschitz condition

$$|k(x,t,u(t),u(\alpha(t))) - k(x,t,v(t),v(\alpha(t)))| \leq L|u(t)-v(t)| \tag{29}$$

with $L > 0$.

If $y \in C_p([a,b])$ is such that

$$\left|y(x) - f\left(x, y(x), \int_a^x k(x,\tau,y(\tau),y(\alpha(\tau)))d\tau\right)\right| \leq \theta, \qquad x \in [a,b], \tag{30}$$

where $\theta > 0$ and $M\left(1 + \frac{L}{p}\left(\frac{e^{p(b-a)}-1}{e^{p(b-a)}}\right)\right) < 1$, then there is a unique function $y_0 \in C_p([a,b])$ such that

$$y_0(x) = f\left(x, y_0(x), \int_a^x k(x,t,y_0(t),y_0(\alpha(t)))dt\right) \tag{31}$$

and

$$|u(x) - y_0(x)| \leq \frac{pe^{p(b-a)}\theta}{e^{p(b-a)}(p - Mp) - ML(e^{p(b-a)}-1)} \tag{32}$$

for all $x \in [a,b]$

This means that under the above conditions, the Volterra integral equation (2) has the Hyers-Ulam stability.

Proof. We will consider the operator $T : C_p([a,b]) \to C_p([a,b])$, defined by

$$(Tu)(x) = f\left(x, u(x), \int_a^x k(x,\tau,u(\tau),u(\alpha(\tau)))d\tau\right), \tag{33}$$

for all $x \in [a,b]$ and $u \in C_p([a,b])$.

By the same above procedure we have T strictly contractive (with respect to the metric under consideration) due to the fact that $M\left(1 + \frac{L}{p}\left(\frac{e^{p(b-a)}-1}{e^{p(b-a)}}\right)\right) < 1$. Thus, we can again apply the Banach Fixed Point Theorem, which ensures that we have the Hyers-Ulam stability for the integral equation with (32) being obtained by using (6) and (30).

Remark 6 *Is possible analyse the Hyers-Ulam-Rassias stability of the integral equation but defined on infinite intervals. These results will be presented in a future work.*

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