Chapter 1
On optimal extended row distance profile

P. Almeida and D. Napp and R. Pinto

Abstract In this paper, we investigate extended row distances of Unit Memory (UM) convolutional codes. In particular, we derive upper and lower bounds for these distances and moreover present a concrete construction of a UM convolutional code that almost achieves the derived upper bounds. The generator matrix of these codes is built by means of a particular class of matrices, called superregular matrices. We actually conjecture that the construction presented is optimal with respect to the extended row distances as it achieves the maximum extended row distances possible. This in particular implies that the upper bound derived is not completely tight. The results presented in this paper further develop the line of research devoted to the distance properties of convolutional codes which has been mainly focused on the notions of free distance and column distance. Some open problems are left for further research.

1.1 Introduction

During the last two decades, renewed efforts were made to investigate the distance properties of convolutional codes, mainly, their free (Hamming) distance and their column distance. In [20] a Singleton bound for convolutional codes was derived (called generalized Singleton bound) and the codes achieving such a bound were called maximum distance separable (MDS). In [23] the first concrete construction of an MDS convolutional code (over the finite field $\mathbb{F}$) of rate $\frac{k}{n}$ and degree δ was presented for every given set of parameters (n,k,δ), (with the characteristic of the finite field $\mathbb{F}$ and the length n of the code being coprime). Bounds and fundamental properties of the column distances of convolutional codes have also been thoroughly investigated, see for instance [7, 8, 11, 18]. Convolutional codes having the largest

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columns distances for a given rate $\frac{k}{n}$ and degree δ are called maximum distance profile (MDP). Their existence was proven in [8] and concrete constructions were given in [7] when $(n-k)|\delta$ and in [17] for every set of given parameters (n,k,δ).

In contrast to the column distances, the extended row distances grow beyond the free distance and therefore provide additional information about the performance of the code. Hence, the notion of (extended) row distance is often used when more detailed knowledge of the distance structure of a convolutional code is needed [11]. One of the advantages of the row distance is that it is easy to calculate and serves as an excellent rejection rule when encoders are tested in search for convolutional code with large free distance. As opposed to the free distance and column distance the notion of (extended) row distance has not been fully investigated in the literature.

In this paper we shall focus on Unit Memory (UM) convolutional codes [14]. These codes may be an interesting alternative to the usual convolutional codes as their block length can be chosen to coincide with the word length of microprocessors, see [14, 24] for details. Binary (partial) UM convolutional codes were investigated in the literature by Lauer [13] and Justensen [12, 24] who showed that unit memory codes can perform better in some situations than codes having the same rate and degree but with memory larger than 1.

It is the aim of this work to analyze the row distances of Unit Memory (UM) convolutional codes with finite support. In particular we derive *upper* bounds for extended row distances of UM convolutional codes for a given rate $\frac{k}{n}$ and degree δ. Moreover, we show that such a bounds are tight by presenting concrete constructions of convolutional codes achieving this bound. The encoder matrices of these codes are built by means of a very particular type of matrices called superregular matrices.

The paper is organized as follows. In Section 1.2, we introduce the basic material for the development of the paper: it includes the necessary introductory material on UM convolutional codes and on the class of superregular matrices. In Section 1.3, we include the main results of the paper. In particular we establish upper and lower bounds on the extended row distances and moreover show how to construct (n,k,δ) UM convolutional codes that have (nearly) optimal profile of extended row distances. We conclude the paper in Section 1.4 where we resume the results of the paper and point out some aspects of this construction that can be improved in order to make it more attractive for applications. Finally some interesting avenues for research in this direction are indicated.

1.2 Distances of convolutional codes

This section contains the mathematical background needed for the development of our results. First we introduce convolutional codes with finite support and in particular unit memory codes. We conclude this section by recalling the notion superregular matrices [2]. Such matrices have some similarities with the ones introduced in [3, 7]. They have similar entries and, therefore, some properties are the same but the structure of these new matrices may be different.

Let $\mathbb{F}$ be a finite field and $\mathbb{F}[D]$ be the ring of polynomials with coefficients in $\mathbb{F}$.

1.2.1 Unit memory convolutional codes

A (finite suppport) *convolutional code* $\mathscr{C}$ of rate k/n is an $\mathbb{F}[D]$-submodule of $\mathbb{F}[D]^n$ of rank k given by a *basic* and *minimal* full-rank polynomial *encoder matrix* $G(D) \in \mathbb{F}[D]^{k\times n}$,

$$\mathscr{C} = \operatorname{Im}_{\mathbb{F}[D]} G(D) = \left\{ u(D)G(D) : u(D) \in \mathbb{F}^k[D] \right\},$$

where *basic* means that $G(D)$ has a polynomial right inverse, and *minimal* means that the sum of the row degrees of $G(D)$ attains its minimal possible value δ, called the *degree* of $\mathscr{C}$.[1] The largest row degree of $G(D)$ is called the *memory*. Note that since $G(D)$ is basic the resulting convolutional code if noncatastrophic, and hence we assume that only noncatastrophic codes are of interest [17, 19].

Although this is the general definition of convolutional codes with finite support, in this paper we will focus on a particular subclass of these codes, namely, Unit Memory (UM), i.e., when the encoder matrix $G(D)$ is described by $G(D) = G_0 + G_1 D$, $G_1 \neq 0$ or equivalently when the memory is equal to 1. Following the notation used in [16] a rate k/n UM convolutional code $\mathscr{C}$ of degree δ is called an (n,k,δ)-convolutional code. Note that in this case $1 \le \delta \le k$.

If $u(D) \in \mathbb{F}[D]^k$ has degree $j \ge 0$, $u(D) = u_0 + u_1 D + \ldots + u_{j-1} D^{j-1}$, and

$$G(D) = G_0 + G_1 D,$$

the above representation of $u(D)G(D) = v(D)$ can be expanded as

$$\begin{bmatrix} u_0 & u_1 & \cdots & u_{j-1} \end{bmatrix} \underbrace{\begin{bmatrix} G_0 & G_1 & & & \\ & G_0 & G_1 & & \\ & & \ddots & \ddots & \\ & & & G_0 & G_1 \end{bmatrix}}_{=G_j^r} = \begin{bmatrix} v_0 & v_1 & \cdots & v_j \end{bmatrix}, \tag{1.1}$$

where $G_j^r \in \mathbb{F}^{jk\times(j+1)n}$ is called the *sliding generator matrix*.

An important distance measure for a convolutional code $\mathscr{C}$ is its *free distance* defined as

$$d_{\text{free}}(\mathscr{C}) = \min\left\{ \operatorname{wt}(v(D)) \mid v(D) \in \mathscr{C} \quad \text{and} \quad v(D) \neq 0 \right\},$$

[1] Therefore, the *degree* δ of a convolutional code $\mathscr{C}$ is the sum of the row degrees of one, and hence any, minimal basic encoder.

where $\mathrm{wt}(v(D))$ is the Hamming weight of a polynomial vector

$$v(D) = \sum_{i \in \mathbb{N}} v_i D^i \in \mathbb{F}[D]^n,$$

defined as

$$\mathrm{wt}(v(D)) = \sum_{i \in \mathbb{N}} \mathrm{wt}(v_i),$$

where $\mathrm{wt}(v_i)$ is the number of the nonzero components of v_i.

The *extended row distance* d_j^r is defined [11, 24] to be the minimum Hamming weight of all paths in the minimal code trellis that diverge from the zero state and then return for the first time back in the zero state *only* after j branches. An UM code can be represented by a trellis [4–6] where the state at time t is u_{t-1}. The number of states is $|\mathbb{F}|^k$ and for UM codes the zero state can always be achieved in one step with input $u_t = 0$. Moreover, a path in the trellis is unmerged with the zero path if and only if each information sub-block is nonzero.

For $j \geq 1$, let I_j denote the set of all $u(D)$ such that $u_\lambda \neq 0$ for $\lambda = 0, 1, \ldots, j-1$ and $u_j = 0$. We formally define the extended row distance d_j^r as

$$d_j^r = \min_{u(D) \in I_j} \mathrm{wt}(u(D)G(D))$$

Thus we are considering the minimum weight of subcodewords corresponding to paths in the trellis from the zero state which reach the zero state again for the first time after exactly $j+1$ time instances. Note that $d_{\text{free}} \leq d_{j+1}^r \leq d_j^r$ and moreover for noncatastrophic codes it holds that $d_{\text{free}} = d_\infty^r = \min_{j=0,1,2,\ldots} d_j^r$ and $\alpha = \lim_{j \to +\infty} \frac{d_j^r}{j}$ gives the average linear *slope* of d_j^r.

1.2.2 Superregular matrices

Let $A = [\mu_{i\ell}]$ be a square matrix of order m over $\mathbb{F}$ and S_m the symmetric group of order m. The determinant of A is given by

$$| A | = \sum_{\sigma \in S_m} (-1)^{\mathrm{sgn}(\sigma)} \mu_{1\sigma(1)} \cdots \mu_{m\sigma(m)}.$$

A *trivial term* of the determinant is a term $\mu_\sigma = \mu_{1\sigma(1)} \cdots \mu_{m\sigma(m)}$, with at least one component $\mu_{i\sigma(i)}$ equal to zero. If A is a square submatrix of a matrix B with entries in $\mathbb{F}$, and all the terms of the determinant of A are trivial, we say that $| A |$ is a *trivial minor* of B (if $B = A$ we simply say that $| A |$ is a trivial minor). We say that a matrix B is *superregular* if all its nontrivial minors are different from zero.

The next results were derived in [3] and they will be very useful for our purposes in the next section

Theorem 1. *Let $\mathbb{F}$ be a field and $a, b \in \mathbb{N}$, such that $a \geq b$ and $B \in \mathbb{F}^{a\times b}$. Suppose that $u = [u_i] \in \mathbb{F}^{b\times 1}$ is a row matrix such that $u_i \neq 0$ for all $1 \leq i \leq b$. If B is a superregular matrix and every column of B has at least one nonzero entry then* $\mathrm{wt}(uB) \geq b - a + 1$.

Theorem 2. *Let α be a primitive element of a finite field $\mathbb{F} = \mathbb{F}_{p^N}$ and $B = [\nu_{i\ell}]$ be a matrix over $\mathbb{F}$ with the following properties*

1. *if $\nu_{i\ell} \neq 0$ then $\nu_{i\ell} = \alpha^{\beta_{i\ell}}$ for a positive integer $\beta_{i\ell}$;*
2. *If $\nu_{i\ell} = 0$ then $\nu_{i'\ell} = 0$, for any $i' > i$ or $\nu_{i\ell'} = 0$, for any $\ell' < \ell$;*
3. *if $\ell < \ell'$, $\nu_{i\ell} \neq 0$ and $\nu_{i\ell'} \neq 0$ then $2\beta_{i\ell} \leq \beta_{i\ell'}$;*
4. *if $i < i'$, $\nu_{i\ell} \neq 0$ and $\nu_{i'\ell} \neq 0$ then $2\beta_{i\ell} \leq \beta_{i'\ell}$.*

Suppose N is greater than any exponent of α appearing as a nontrivial term of any minor of B. Then B is superregular.

We note that there exist several notions of superregular matrices in the literature. The definition given above generalizes all these notions. Frequently, see for instance [21], a superregular matrix is defined to be a matrix for which every square submatrix is nonsingular. Obviously all the entries of these matrices must be nonzero. Also, in [1, 22], several examples of triangular matrices were constructed in such a way that all submatrices inside this triangular configuration were nonsingular. However, all these notions do not apply to our case as they do not consider submatrices that contain zeros. The more recent contributions [7, 9, 10, 25, 26] consider the same notion of superregularity as us, but defined only for lower triangular matrices. Hence, many examples can be found in these references. In the following section we will adapt this general notion of superregularity to the case of interest in this paper, namely, the sliding generator matrices G_j^r.

1.3 Bounds and constructions

In this section we present results of upper and lower bounds on extended row distances of UM convolutional codes. Moreover, we show how we can use the notion of superregular matrices to construct codes that achieve these bounds. We also provide a concrete class of superregular matrices that can be used to build UM convolutional codes with good design row extended distance. We point out some of the advantages and disadvantages of this construction in terms of the size of the field $\mathbb{F}$.

Given a generator matrix $G(D) = G_0 + G_1 D$ of $\mathscr{C}$ we shall assume without loss of generality that the zero rows of G_1 are at the top, i.e.,

$$G_0 = \begin{bmatrix} G_0^{(1)} \\ G_0^{(2)} \end{bmatrix} \qquad G_1 = \begin{bmatrix} 0 \\ G_1^{(2)} \end{bmatrix} \tag{1.2}$$

with $G_i^{(1)} \in \mathbb{F}^{k-\delta \times n}$ and $G_i^{(2)} \in \mathbb{F}^{\delta \times n}$, where δ is the degree of $\mathscr{C}$. We write $u = [u^{(1)} \quad u^{(2)}]$ accordingly. Note that since $G(D)$ is basic and minimal G_0 and $\begin{bmatrix} G_0^{(1)} \\ G_1^{(2)} \end{bmatrix}$ have full row rank.

The following result establishes an upper bound for the extended row distances.

Theorem 3. *Let $\mathscr{C}$ be a UM (n,k,δ)-convolutional code with generator matrix given by $G(D) = G_0 + G_1 D$ as above. Then,*

$$d_j^r \leq (n-k+1)j + n \tag{1.3}$$

Proof. We want to estimate

$$\min_{u(D)\in I_j} \mathrm{wt}(u(D)G(D)) = \min_{u_i \neq 0} \mathrm{wt}([u_0\ u_1\ \cdots\ u_{j-1}]G_j^r) \tag{1.4}$$

where G_j^r is the sliding generator matrix defined in (1.1). Clearly

$$\min_{u_0 \neq 0} \mathrm{wt}(v_0) = \min_{u_0 \neq 0} \mathrm{wt}(u_0 G_0) \leq n-k+1$$

as $n-k+1$ is the Singleton bound for (n,k)-block codes.

If $u_0^{(2)} \neq 0$ then $u_0 G_1 \neq 0$ and therefore $\begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$ has at least $k+1$ rows. Thus, exists u_1 such that

$$\mathrm{wt}(v_1) = \mathrm{wt}([u_0\ u_1]\begin{bmatrix} G_1 \\ G_0 \end{bmatrix}) \leq n-k. \tag{1.5}$$

However we may have $u_1 = 0$ which contradicts $u_i \neq 0$, for all i, and $u_0^{(2)} = 0$ which implies $u_0 G_1 = 0$ and therefore

$$\mathrm{wt}(v_1) \leq n-k+1. \tag{1.6}$$

Hence, in any case

$$\min_{u_0 \neq 0} \mathrm{wt}(v_1) \leq n-k+1. \tag{1.7}$$

Following the same reasoning, for any u_{i-1} there exists u_i such that

$$\min_{u_0 \neq 0} \mathrm{wt}(v_i) = \min_{u_0 \neq 0} \mathrm{wt}([u_{i-1}\ u_i]\begin{bmatrix} G_1 \\ G_0 \end{bmatrix}) \leq n-k+1.$$

for $i = 1, \ldots, j-1$, since, if with $u_{i-1}^{(2)} = 0$ then $\mathrm{wt}(v_i) = n-k+1$. Obviously $\mathrm{wt}(v_j) = \mathrm{wt}(u_{j-1}G_1) \leq n$ and hence for $[v_0\ v_1\ \cdots\ v_j] = [u_0\ u_1\ \cdots\ u_{j-1}]G_j^r$ with $u_i \neq 0$, we have that

$$\begin{aligned}\min_{u_i \neq 0} \mathrm{wt}([v_0\ v_1\ \cdots\ v_j]) &= \min_{u_i \neq 0} (\mathrm{wt}(v_0) + \sum_{i=1}^{j-1} \mathrm{wt}(v_i) + \mathrm{wt}(v_j)) \\ &\leq (n-k+1)j + n\end{aligned}$$

□

Remark 1. Taking a closer look at the proof of the previous lemma we see that between the two upper bounds (1.5) and (1.6) we had to consider the largest one (1.6) in order to prove (1.3). However we believe that (1.5) will hold for a $[u_0\, u_1\, \cdots\, u_{j-1}]$ minimizing (1.4). Since we failed to come up with a formal proof for this we leave it for future research and conjecture that the actual upper bound in (1.3) should be slightly smaller, namely,

$$d_j^r \leq (n-k)j + n + 1. \tag{1.8}$$

In the next section, we will construct a code that achieves the upper bound in (1.8).

If $\mathscr{C}$ has its extended row distances achieving the bound (1.8) for every $j \in \mathbb{N}$ we say that $\mathscr{C}$ has an *almost optimal extended row distances profile* (AOEDP). Note that this upper-bound does not depend on the degree δ of $\mathscr{C}$ in contrast to the generalized Singleton bound for the free distance [20]. Also note that the bound given in (1.8) grows infinitely and in practice one is interested in knowing the values of d_j^r, $1 \leq j \leq J$ for same given integer J.

The assumption that the zero rows of G_1 are at the top implies that the matrix $\begin{bmatrix} G_1 \\ G_0 \end{bmatrix}$ cannot have zero rows between two nonzero rows.

We will construct UM convolutional codes with designed extended row distances and for that we will require the sliding generator matrix G_j^r to be superregular. Next result characterizes and simplifies the conditions such a G_j^r to be superregular.

Lemma 1. *Let G_j^r be a sliding generator matrix as defined above. Then, G_j^r is superregular if and only if every square submatrix of G_j^r that does not contain zeros in the diagonal is invertible.*

Proof. The proof amounts to showing that the unique nontrivial minors of G_j^r are exactly the ones that do not contain zeros in their diagonal. Let $A = [a_{ij}] \in \mathbb{F}^{t \times t}$ be a square submatrix of G_j^r. Obviously, if all the elements in the diagonal of A are nonzero then the corresponding minor is nontrivial. Thus, it is left to prove that if contains a zero in the diagonal, say a_{ss}, then the corresponding minor is trivial. In fact only two possibilities can happen due to the particular structure of blocks of zeros of G_j^r. Or there exists a block of zeros in the upper right corner of A, namely, $a_{ij} = 0$ for $0 \leq i \leq s$ and $s \leq j \leq t$ or otherwise there exists a block o zeros in the left bottom corner of A, namely, $a_{ij} = 0$ for $s \leq i \leq t$ and $0 \leq j \leq s$. It is easy to verify that all terms of $|A|$ have components in both blocks which concludes the proof. □

The next result shows how superregular matrices are related to UM convolutional codes that have an AOEDP.

Theorem 4. *Let $\mathscr{C}$ be a UM (n,k,δ)-convolutional code generated by $G(D) = G_0 + G_1 D$. If all the entries of G_0 and $G_1^{(2)}$ are nonzero and the sliding generator matrix G_j^r is superregular then*

$$d_j^r \geq (n-k)j + n + 1,$$

i.e., $\mathscr{C}$ has an AOEDP.

Proof. For $j \geq 1$, let $u(D) \in I_j$. Suppose that the weight of $[u_0\ u_1\ \cdots\ u_{j-1}]$ is t and let $\bar{u}$ be the vector formed by the nonzero components of $[u_0\ u_1\ \cdots\ u_{j-1}]$ and B be the matrix formed by the t rows of G^r_j corresponding to $\bar{u}$. Thus B has $(j+1)n$ columns and t rows. Since $u_\lambda \neq 0$ for $\lambda = 0,1,\ldots,j-1$ then the $(j+1)n$ columns of B are nonzero. The matrix B is superregular as it is assumed that G^r_j is superregular and any submatrix of a superregular matrix is superregular. Then we can apply theorem 1 to obtain,

$$\mathrm{wt}(\bar{u}B) = \mathrm{wt}(v(D)) \geq (j+1)n - t + 1.$$

Since $t \leq jk$, we have that

$$\mathrm{wt}(v(D)) \geq (j+1)n - jk + 1 = (n-k)j + n + 1.$$

This concludes the proof. □

For a given $J \geq 1$ and a set of parameters (n,k,δ), with $\delta \leq k < n$ we propose a concrete construction of UM convolutional code constructed via the following class of superregular regular matrices.

Let $G(D) = G_0 + G_1 D$, where G_i, with $i = 1,2$, are described by

$$G_i = [\gamma_{rs}] \text{ for } \gamma_{rs} = \begin{cases} \alpha^{2^{n+r+s-2}} & \text{if } i = 0 \\ \alpha^{2^{r+s-2}} & \text{if } i = 1 \quad \text{and} \quad r > k - \delta \\ 0 & \text{if } i = 1 \quad \text{and} \quad r \leq k - \delta \end{cases} \tag{1.9}$$

where α is a primitive element of the finite field $\mathbb{F} = \mathbb{F}_{p^N}$.

Lemma 2. *Let $G(D)$ be as in (1.9). Suppose N is greater than any exponent of α appearing as a nontrivial term of any minor of G^r_J. Then assumptions of Theorem 4 hold for $j = 1,\ldots,J$, namely, all the entries of G_0 and $G_1^{(2)}$ are nonzero and the sliding generator matrix G^r_J is superregular.*

Proof. The fact that the entries of G_0 and $G_1^{(2)}$ are nonzero is straightforward. To show that the sliding generator matrix G^r_J is superregular permutate the columns of G^r_J to obtain the matrix

$$A = \begin{bmatrix} & & & G_1 & G_0 \\ & & G_1 & G_0 & \\ & \iddots & \iddots & & \\ G_1 & G_0 & & & \end{bmatrix}. \tag{1.10}$$

One can check that A satisfies the conditions of theorem 2 and therefore it is superregular. Since the minors of A are equal (or symmetric) to the minors of G^r_J this implies that G^r_J is also superregular. □

We are now in a position to present a result that readily follows from theorem 4 and Lemma 2 and states that the construction rendered in (1.9) gives rise to a UM convolutional code with a designed extended row distance and moreover has a AOEDP.

Corollary 1. *Let $\mathscr{C}$ be a UM (n,k,δ)-convolutional code generated by $G(D) = G_0 + G_1 D \in \mathbb{F}^{k\times n}$, where G_0 and G_1, are described above. Assume that $\mathbb{F} = \mathbb{F}_{p^N}$, for p prime and N sufficiently large, then the sliding generator matrix G^r_J is superregular and*

$$d^r_j = (n-k)j + n + 1$$

, for $j =, 0, 1, \ldots J$, i.e., d^r_j reaches the upper-bound given in (1.8) for $j =, 0, 1, \ldots J$.

1.4 Conclusions

A great deal of attention has been devoted in recent years to the study of convolutional codes with good distance properties. In particular, Maximum Distance Profile (MDP) or Maximum Distance Separable (MDS) have been thoroughly investigated. In this paper we have focused our attention to the construction of unit memory convolutional codes with good extended row distance. It turns out that the question of how to construct them can be related to the construction of a class of matrices, called superregular. We have given conditions for the sliding generator matrix of a code to yield UM convolutional codes with nearly optimal extended row distances. A concrete construction have been presented based on a type of superregular matrices that had been recently used for the authors to build MDP [2]. Moreover, it was recently shown [15] that this class of matrices perform very well when considering rank metric instead of the Hamming metric, producing Maximum Sum Rank Distance convolutional codes. It is natural to ask whether also the presented codes have optimal extended row distance with respect to the rank metric (to be formally defined). This opens up a interesting avenue of future research. Finally we remark that one of the disadvantages of the presented constructions is that they require large fields and it would be convenient to come up with new constructions of superregular matrices over smaller fields.

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