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# A SHORT NOTE ON COPS AND ROBBERS PLAYING ON TOTAL GRAPHS

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## AUTHORS' CONTRIBUTIONS

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**ABSTRACT**

Cop Robber game is a two player game played on an undirected graph. In this game, the cops try to capture a robber moving on the vertices of the graph. The cop number of a graph is the least number of cops needed to guarantee that the robber will be captured. The total graph $T(G)$ of a graph $G$ has a vertex for each edge and vertex of $G$ and an edge in $T(G)$ for every edge-edge, vertex-edge, and vertex-vertex adjacency in $G$. In this paper, we play the game on the total graph $T(G)$, showing in particular that $c(T(G)) \leq 3$ for every planar graph $G$.

**Keywords:** Cops and robbers; vertex-pursuit games.

**AMS Subject Classification:** 05C57, 05C80,05C10.

## 1 Introduction

The Game of Cops and Robbers is a vertex pursuit game played on a graph $G$ introduced by Nowakowski and Winkler [1]. In this game we consider two players, first who controls a set of $k$ cops (or searchers) $C$, where $k > 0$ is a fixed integer, and second who controls the single robber $R$. Initially Cops places $k$ cops on vertices of the graph and then Robbers places the robber on some vertex of the graph. Then Cops and Robbers play alternately. In Cops turn, he moves arbitrarily many cops; in Robbers turn, he moves the robber. A move is placing itself on vertex adjacent to the currently occupied or staying in the same place. Every vertex is allowed to be occupied by more than one cop. The players know each others actual position (that is, the game is played with complete information). The game ends with a cop win if one of the cops moves to the vertex occupied by the robber; if robber has a strategy to continually avoid cops, then $R$ wins. The minimum number of cops needed to capture the robber (regardless of robber's strategy) is called the *cop number* of $G$, and is denoted by $c(G)$. This parameter is well studied in literature (see [2, 3, 4, 5]).

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Throughout this paper we consider only simple graphs (that is, graphs without loops and parallel edges). For additional definitions on graphs we refer to the book [6] and for the game we refer to the book [4].

The total graph $T(G)$ of a graph $G$ is the graph obtained by taking as the vertex the set $V(T(G)) = V(G) \cup E(G)$ and two distinct vertices $x$ and $y$ are adjacent in $T(G)$ if $x$ and $y$ are adjacent vertices of $G$, adjacent edges of $G$,or an incident vertex and edge of $G$ (see [7, 8, 9]). In this paper, we play the game of Cops and Robbers on the total graph $T(G)$ of $G$. We define the *total cop number* as $c^t(G) = c(T(G))$.

In the next section we will characterize the total cop number of a special families of graphs. Our work extend the work of [10] which considered the cop number of line graphs. We showed that with some modifications their methods can be applied in this slightly more general setting. At the end we show that the total cop number of a planar graph is at most equal 3 and this result is best possible.

# 2 Total Cop Number of a Graph

The total graph can also be defined as follows: $T(G)$ is the total graph of $G$ if it is obtained by taking a copy of $G$ and a copy of its line graph $L(G)$ and one vertex in $G$ is joined to one vertex in $L(G)$ if the corresponding elements are incident in the original graph. If $G$ is a simple graph of order $n$, then the following properties are immediate:

- If $G$ is $K_2$, then $c^t(G) = 1$.
- If $G$ is $K_3$, then $c^t(G) = 2$.
- If $G$ is the path $P_n$ of order $n \geq 4$, then $c^t(P_n) = 1$.
- If $G$ is the cycle $C_n$ of order $n \geq 3$, then $c^t(C_n) = 2$.
- If $G$ is the star $K_{1,n}$ , then $c^t(K_{1,n}) = 1$.

We can state following relation between the cop number and the total cop number

**Theorem 2.1.** *Let $G$ be any connected simple graph of order $n$. Then $c(G) \leq c^t(G) \leq c(G) + 1$.*

*Proof.* First we prove that $c^t(G)$ cops are enough to catch the robber in $G$. Fix a cop winning strategy on $T(G)$ with $c^t(G)$ cops. Now we design a winning strategy for cops on $G$ with $c^t(G)$ cops as follows. Since the robber moves along $V(G)$ in the game on $G$, which is a subset of $V(T(G))$, the winning strategy for cops for the game on $T(G)$ tells us how the cops move. Then in the game on $G$, at any step, for each cop, we move it as follows: if its position in the game on $T(G)$ is a vertex $v$ of $G$, then its position in the game on $G$ is $v$; if its position in the game on $T(G)$ is an edge $e$ of $G$, then its position in the game on $G$ is a vertex of $G$ incident with $e$. Note that it is always possible to reach the designated positions as mentioned above since for every edge $e$ of $G$ and for every endpoint $v$ of $e$, any element in the closed neighborhood of $e$ in $T(G)$ but not in the closed neighborhood of $v$ in $T(G)$ is an edge of $G$ which has an endpoint adjacent to $v$. In addition notice that the robber can not make a shortcut using vertices corresponding to edges $E(G)$, i.e. reach a vertex $u$ which is at distance $k$ from the vertex occupied by the cop in a graph $G$ in less than $k$ steps in the graph $T(G)$ (since in $T(G)$ and in $G$ $d_G(v, v) = d_T(u, v)$ for any two arbitrary vertices $u$ and $v$ in $G$, where $d_G(u, v)$-denote the distance between the vertices $u$ and $v$ in $G$). Therefore capturing the robber on $T(G)$ with $c^t(G)$ cops implies a strategy for capturing him on $G$. Thus the lower bound for $c^t(G)$ follows.

The strategy for capturing the robber with $c(G)+1$ is alike the one presented in [10]. Whenever the robber moves through the graph $T(G)$ on vertices corresponding to the $V(G)$ we follow the optimal strategy to capture the robber on graph $G$. Whenever it moves to a vertex corresponding to $e$ of $E(G)$ we treat it as if he would occupy one endpoint vertex of the edge $e$. The only thing we have to keep trace of is that when the robber moves to a vertex corresponding to an edge $e'$ of $E(G)$ adjacent to $e$, then we continue the game as if it would move to an adjacent vertex incident with $e'$ in $G$. At some point we reach the state in which we capture the robber in $G$ (meaning that the closed neighbourhood of the vertex occupied by the robber is contained in the union of the closed neighbourhoods of cops).

Let us show that despite the next move of the robber we can put one cop at a vertex adjacent to the vertex occupied by the robber. First assume that the robber was at a vertex corresponding to $V(G)$, then the games is finished as there is no safe place for the robber. If he moves to a vertex of $V(G)$, or maintain at his position, then he will be captured. If he moves to a vertex $v_e$ of $E(G)$, then he can not choose the one that is incident to cops position. But despite his choice in the next round cops can occupy both vertices of $V(G)$ which are endpoints of $e$ in $G$. Therefore the robber will be capture in the next round.

In the second case, robber is positioned at the vertex corresponding to the edge in grph $G$. In the final state there has to be a cop which then can be moved to a vertex adjacent to it. In the forthcoming turns this cop can follow the robber and always stay at a vertex $v$ of $V(T(G))$ adjacent to the vertex occupied by the robber (thus preventing him from ever getting back to the vertices from the $V(G)$ part of $T(G)$). See Fig. 1 for details.

Thus now robber can only move along vertices corresponding to $E(G)$ having a cop chasing him on adjacent vertex corresponding to one endpoint of $e \in E(G)$. We continue the game using optimal strategy for cop-robber game on graph $G$ assuming that robber occupy other vertex of $e$. At some point remaining $c(G)$ cops will capture the robber in such game. At this point there is no vertex onto which the robber can move without being captures. Thus he loses the game in the next turn. □

The *edge cop number* was defined by Dudek, Gordinowicz and Prałat [10] as $\overline{c}(G) = c(L(G))$, where $L(G)$ is the line graph of $G$. See their paper for more results on this topic.

**Lemma 2.1.** *If $\overline{c}(G) < c(G)$, then $c^t(G) = c(G)$.*

*Proof.* As showed in the proof of Theorem 2.1 with $c(G)$ cops we can always force the robber to move into the part of the graph corresponding to the $E(G)$ and place a cop on a vertex adjacent to the robber. This cop can prevent robber from moving into the part of the graph corresponding to the $V(G)$, as $c(G)-1$ cops are enough to capture robber on the line graph of $G$, they will eventually win the game. □

As proven in [10] Petersen graph has edge cop number 2 and has cop number 3. The line graph of the dodecahedron is the icosidodecahedral graph. Again as proven in [10] dodecahedron has edge cop number 2 and has cop number 3. In general it is not known for which graphs the inequality $\overline{c}(G) < c(G)$ holds.

**Corollary 2.1.** *If $G$ is the Petersen graph or the dodecahedron graph, then $c(G) = c^t(G) = 3$.*

A graph $G$ is said to be *cop winning* if $c(G) = 1$ and is said to be *total cop winning* if $c^t(G) = 1$. For a vertex $x$ in a graph $G$, the neighborhood of $x$ is denoted by $N(x)$ and is defined as $N(x) = \{y \in V(G) | \{x, y\} \in E(G)\}$. The closed neighborhood of $x$ is defined as $N[x] = \{x\} \cup N(x)$. A vertex $x$ is said to be the *corner*, if there exists a vertex $y$ such that $N[x] \subseteq N[y]$. Corner vertices play important roles in characterizing cop winning graphs (see [4]). A graph $G$ is *dismantlable* if we can reduce $G$ to single vertex by repeatedly removing corner vertices. In case of total graphs we can prove following statement.

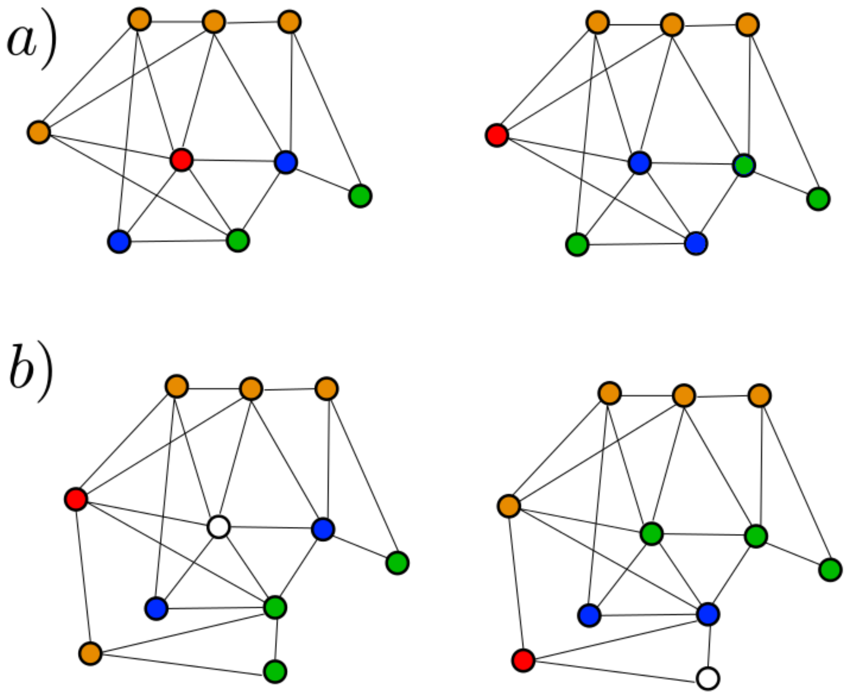


**Fig. 1. Figure presents game on total graph $T(G)$. Green vertices represent vertices corresponding to $V(G)$, orange vertices represent vertices corresponding to $E(G)$, blue vertices are position of the cops, red vertex is the robber position, white vertex is a position which cops treats as robber position in their pursuit on vertices of $V(G)$. In the case a) cops captured the robber on vertex of $V(G)$. He will be then capture in the next two rounds. In the case b) cops "capture" robber on vertex of $V(G)$, while he is moving along the part corresponding to $E(G)$. In the next move one cop can place himself at a vertex adjacent to robber position. In both cases game can be continued as if robber was at the other vertex adjacent to $e$**

**Theorem 2.2.** *$T(G)$ is dismantlable if and only if $G$ is a tree.*

*Proof.* Notice that vertices corresponding to vertices of degree one in $G$ are corners in total graph. After deleting those vertices, vertices which correspond to edges incident to them are corners. Thus we can successively remove vertices of degree one in $G$ in $T(G)$ until we get a single vertex, therefore total graph of a tree is dismantlable.

Assume that graph $G$ is not a tree, as mentioned above in the case of vertices of degree one we can remove them iteratively one by one without changing the cop number. Thus we can assume that our graph is a graph with minimal degree two. We show that in this case none of its vertices is a corner. Vertices corresponding to edges in $G$ are not corners since such vertices can not dominate neighbourhood of one another as they are connected to different pairs of vertices. Moreover vertex which correspond to a vertex in $G$ can not dominate their neighbourhood since it is not connected to all edges adjacent to given edge (as minimal degree in a graph is two). On the other hand vertices corresponding to vertices in $G$ can not dominate each others neighbourhoods as we always find an edge which is incident with one of them and not incident with the other. It is clear that a vertex corresponding to an edge can not dominate neighbourhood of a vertex corresponding to a vertex of $G$. □

The cop-win graphs were completely characterized by Nowakowski and Winkler [1].

**Theorem 2.3** ([1]). *A graph $G$ is cop win if and only if it is dismantlable.*

Based on the above results we have the following:

**Corollary 2.2.** *$G$ is total cop win if and only if $G$ is a tree.*

**Corollary 2.3.** *Let $G$ be a dismantlable graph which is not a tree, then $c^t(G) = 2$.*

*Proof.* Let $G$ be a dismantlable graph which is not a tree. Since $G$ is dismantlable, we have $c(G) = 1$. Also $G$ is not a tree, by Theorem 2.2 $c^t(G) \neq 1$. This implies

$$c^t(G) \geq 2. \tag{2.1}$$

Again, by Theorem 2.1 $c^t(G) \leq c(G) + 1$. This yields

$$c^t(G) \leq 2. \tag{2.2}$$

From (1) and (2) the result follows. □

Thus in general

**Corollary 2.4.** *If $G$ is the complete graph $K_n$, then* $c^t(K_n) = \begin{cases} 1, & \text{if } n = 1, 2 \\ 2, & \text{if } n \geq 3 \end{cases}$.

For planar graphs we can derive the following bound.

**Theorem 2.4.** *Let $G$ be any connected planar graph. Then*

$$c^t(G) \leq 3$$

*Proof.* By Theorem 2.1 there exist planar graphs for which total cop number is at least 3 (which are planar graph with cop number exactly 3 [3]). We show that 3 cops are always enough to capture the robber. Assume that we are given a planar representation of the graph $G$ on a plane. Proof follows the strategy invented by Aigner and Fromme [3]. We say that a cop $c$ controls the path if whenever the robber tries to cross over the path $P$ (i.e. in planar representation moves from a vertex on one side of the path to vertex on the other) he is captured by $c$ in the next round. Crucial idea in the proof of Aigner and Fromme is the following lemma

**Lemma 2.2** ([3]). *Let $G$ be any graph, $u, v \in V(G)$ and $P$ be the shortest path between $u$ and $v$. Then a single cop $c$ on $G$ can, after a finite number of moves, prevent the robber from entering $P$. That is, $R$ will be immediately captured if he moves onto $P$.*

We need a little bit different version of this lemma

**Lemma 2.3.** *Let $G$ be a graph and $T(G)$ be its total graph, $u, v \in V(G)$ and $P$ be the shortest path between $u$ and $v$ in $G$. Then a single cop $c$ on $T(G)$ can, after a finite number of moves, prevent the robber from entering or crossing the $P$. That is, given the embedding of $P$ in the plane, $R$ will be immediately captured if he moves onto vertices corresponding to $V(P)$ or $E(P)$ or cross over the path $P$.*

*Proof.* We take any planar embedding of $G$ and put vertices corresponding to $E(G)$ on top of the edges of this embedding. Denote such embedding of $T(G)$ as $R$. Denote by $d(x, y)$ the distance (=length of a shortest path) between $x$ and $y$. In [3] Aigner and Fromme proved that a cop $c$ placed on the path $P$ once obtained, can preserve the property

$$d(r, z) \geq d(c, z) \text{ for all } z \in V(P)$$

where $r$ stands for the position of the robber. This property is true due to distance in a graph preserving the triangle inequality. Strategy is as follows: If robber stays put, so does the cop. If robber moves toward the path $P$ then cop moves accordingly preserving the distances.

Notice that above property is true for any graph, in particular also for total graph. It remains to prove that the stated lemma is true when robber moves onto vertices corresponding to $E(G)$.

First assume that robber moves onto vertex $v$ corresponding to $(x, y) \in E(G)$ from a vertex $u$ of $V(G)$. This can only happen if he was at either the vertex $x$ or $y$, as both of them are vertices of the path $P$ he would be captured in the previous step.

In the second case robber moves onto $v$ from a vertex corresponding to some edge $k$ adjacent to $(x, y)$ in $G$. Without loss of generality we can assume that vertex $x$ is adjacent to $k$ in $G$. In order to preserve the distance property when robber moved onto $k$ cop had to move onto vertex $x$. Thus if robber moves onto $v$ he will be captured by the cop in the next move. The only possibility for robber to cross over the path $P$ in embedding $R$ without moving onto vertices corresponding to $V(P)$ or $E(G)$ is to move from $k$ to another vertex corresponding to edge $q$ corresponding to edge $(x, p)$ in $E(G)$, such that $p$ and $y$ are on different sides of $P$ in $R$. As in the previous case if robber moves onto $q$ he will be captured by the cop in the next move. □

W.l.o.g we can assume that $G$ has a 2-connected subgraph. The idea of the proof of Theorem 2.4 is to secure by a single cop the shortest path between two vertices $u$ and $v$ in $G$ (denote it by $P$). Then second cop will secure the path $P'$ - shortest path between $u$ and $v$ in $G \backslash P$ (after removing from $G$ vertices of $P$). Both paths $P$ and $P'$ create a cycle and robber is placed either outside or inside of it. It is important that shortest path between two vertices of $V(G)$ in total graph contains only vertices of $V(G)$.

We use the embedding of the graph $G$ into the plane and the property that the robber can not move from outside of the cycle to inside of it without being captured (and conversely). As we showed in Lemma 2.3 this property holds also in the case of total graphs of planar graphs (if in the embedding we put vertices corresponding to edges on top of the edges of the graph $G$). Then in next rounds we can extend the cycle by other vertices (using third cop) reducing the subgraph outside (or inside) of it. In general in stage $i$, we assign to the robber a certain subgraph $G_i$, the robber's territory, which contains all edges which $r$ may still safely enter, and we show that, after a finite number of moves, $G_i$ is reduced to $G_{i+1} \subsetneq G_i$. Game ends as eventually there is no safe place for the robber to move. See [1] for the details of the proof. □

# 3 Conclusions

In the paper we addressed the problem of determining the cop number for total graphs, which we called a total cop number of a graph. In particular we showed that methods from [10] and [3] can be extended to total graphs. This allows us to characterize graphs with total cop number equal to one, give an upper bound for a cop number in general graphs and proof that for planar graphs the total cop number is at most 3 (and this result is sharp).

There are few questions that remains open, for example characterization of graphs with $c^t(G) = c(L(G))$. We know that this equality is true for the paths, cycles, trees and complete graphs. Characterization of all families of graphs with this property is an open problem. Another challenging question which we left unsolved is to characterize the graphs with $c(G) = c^t(G) + 1$.


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# Competing Interests

Authors have declared that no competing interests exist.

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