Universidade de Aveiro Departamento de Matemática
2015

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Introdução à Teoria do Controlo Ótimo e sua Aplicação à Diabetes

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Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Mestre em Matemática e Aplicações, realizada sob a orientação científica do Doutor Ricardo Miguel Moreira de Almeida, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro, e do Doutor Delfim Fernando Marado Torres, Professor Catedrático do Departamento de Matemática da Universidade de Aveiro.

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Introduction to Optimal Control Theory and its Application to Diabetes

M.Sc. thesis submitted to the University of Aveiro in fulfilment of the requirements for the degree of Master of Science in Mathematics and Applications, done under the supervision of Professor Ricardo Miguel Moreira de Almeida, Assistant Professor at the Department of Mathematics of University of Aveiro, and of Professor Delfim Fernando Marado Torres, Full Professor at the Department of Mathematics of University of Aveiro.

Dedico este trabalho aos meus pais e aos meus irmãos.

o júri / the jury

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Palavras Chave

Cálculo das Variações, Método de Euler, equação de Euler–Lagrange, Controlo Ótimo, Diabetes *Mellitus*.

Resumo

O Cálculo das Variações e o Controlo Ótimo são dois ramos da Matemática que estão muito interligados entre si e também com outras áreas. Como exemplo, podemos citar a Geometria, a Física, a Mecânica, a Economia, a Biologia, bem como a Medicina. Nesta tese estudamos vários tipos de problemas variacionais e de Controlo Ótimo, estabelecendo a ligação entre alguns destes. Fazemos uma breve introdução sobre a Diabetes *Mellitus*, uma vez que estudamos um modelo matemático que traduz a interação entre a glicose e a insulina no sangue por forma a otimizar o estado de uma pessoa com diabetes tipo 1.

Keywords

Calculus of Variations, Euler's Method, Euler–Lagrange equation, Optimal Control, Diabetes *Mellitus*.

Abstract

The Calculus of Variations and the Optimal Control are two branches of Mathematics that are very interconnected with each other and with other areas. As example, we can mention Geometry, Physics, Mechanics, Economics, Biology and Medicine. In this thesis we study various types of variational problems and of Optimal Control, establishing the connection between some of these. We make a brief introduction to the Diabetes *Mellitus*, because we study a mathematical model that reflects the interaction between glucose and insulin in the blood in order to optimize the state of a person with diabetes type 1.

What we learn with pleasure we never forget.

- Alfred Mercier

Contents

Introduction

In this thesis two important fields of Mathematical Optimization are considered: the Calculus of Variations and the Optimal Control. My first interest in the problems of Mathematical Optimization was due to some courses that I studied in the undergraduation and in the master. Those that had more contribution for this interest were Nonlinear Optimization with Constraints, Numerical Optimization and Mathematical Programming. The goal of these fields is to find the point $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ that maximizes, or minimizes, a real valued objective function and that satisfies a system of equalities or inequalities constraints, where the objective function, or some of the constraints, are nonlinear. As the Calculus of Variations consists to determine the extrema functions that optimize a given functional we can establish a connection between the courses that I have studied and the Calculus of Variations. On the other hand, we can also establish a connection between the Calculus of Variations and the Optimal Control, because this last is a generalization of the first. Therefore, we can consider that the Calculus of Variations is a particular area of the Optimal Control.

In Chapter 1 is studied the basic variational problem with fixed endpoints that consists to find functions $y \in C^2[a, b]$ that optimize a definite integral given by

$$J(y) = \int_a^b f(x, y(x), y'(x))dx \tag{1}$$

and that verify the boundary conditions $y(a) = y_a$ and $y(b) = y_b$, where f is a function assumed to have continuous partial derivatives of the second order with respect to x, y and y' and a, b, y_a and y_b are fixed.

First, it is presented the Euler's approach in order to solve this problem by discretizing it and then the analytical manipulation that Lagrange used to solve the same problem. So, it is studied the necessary and sufficient conditions for that a smooth function to be a solution of a variational problem with fixed endpoints. It is also analysed the variational problem with fixed endpoints for functionals containing second-order derivatives and several dependent variables.

The variational problem that results of adding an isoperimetric constraint to the problem mentioned before is also analysed. This constraint is given by

$$I(y) = \int_a^b g(x, y(x), y'(x))dx = L,$$

where $I : C^2[a, b] \longrightarrow \mathbb{R}$ is a functional, g is a smooth function of x, y and y' and L is a specified constant. This problem is called Isoperimetric Problem.

Even if no boundary conditions are imposed and the endpoints are fixed, the analytical procedure suggested by Lagrange, mentioned before, supplies the right number of boundary conditions that we need to optimize the functional given by (1) and it is proved in this chapter.

Finally, Chapter 1 ends with the variational problem with a variable endpoint. Therefore, it is presented how to find the solution $(\overline{x}, y) \in]a, b] \times C^2[a, b]$ that optimizes a functional given by

$$J(\overline{x}, y) = \int_a^{\overline{x}} f(x, y(x), y'(x))dx$$

subject to the boundary condition $y(a) = y_a$, where f is a function defined as previously and a and y_a are fixed. Thus, in these problems the endpoint $\overline{x}$ of the integral is a variable of the problem, which isn't fixed.

Chapter 2 begins with the study of the Optimal Control theory. It is stated and proved a version of the Pontryagin Maximum Principle that provides a set of necessary conditions for that the pair $(\boldsymbol{y}, \boldsymbol{u})$ solves the basic problem of Optimal Control given by

$$\begin{aligned} (P_{OC}) \quad \max \quad & J(\boldsymbol{y}, \boldsymbol{u}) = \int_a^b f(x, \boldsymbol{y}(x), \boldsymbol{u}(x))dx \\ \text{s.t.} \quad & \boldsymbol{y}'(x) = g(x, \boldsymbol{y}(x), \boldsymbol{u}(x)), \ \forall\, x \in]a, b[\\ & \boldsymbol{y}(a) = \boldsymbol{y}_a, \end{aligned} \tag{2}$$

where a, $b \in \mathbb{R}$ such that $a < b$, $f \in C^1([a, b] \times \mathbb{R}^{k+m}, \mathbb{R})$, $g \in C^1([a, b] \times \mathbb{R}^{k+m}, \mathbb{R}^k)$, $\boldsymbol{y} \in PC^1([a, b], \mathbb{R}^k)$ and the control $\boldsymbol{u} \in PC([a, b], \mathbb{R}^m)$ with $k, m \in \mathbb{N}$.

There are some problems of Optimal Control that can be written as problems of Calculus of Variations and therefore these can have two possible resolutions. This situation is illustrated in this chapter by examples.

The problem that was stated previously has a free control, but in the real applications of Optimal Control the control is usually bounded. Then, it is also analysed in this chapter a problem like (2), but with $m = 1$ and with the constraint

$$u(x) \in U,$$

where $U = [c, d] \subseteq \mathbb{R}$ and $c < d$.

In Chapter 3 we study a real application of Optimal Control to Diabetes *Mellitus*. First, we do a brief explanation of this disease in order to do a correct discussion of the solution that we intend to determine. The goal is to minimize a specified objective functional subject to a mathematical model that translates the interaction between the glucose and the insulin in the blood. We can solve this problem numerically by two methods. First, we obtain a numerical solution of the necessary conditions and then, we discretize the problem using the software IPOPT. Finally, we compare the two numerical solutions with the exact solution and we interpret the results.

Chapter 1

The Calculus of Variations

1.1 Introduction

In the Calculus of Variations we want to find the extrema functions that maximize, or minimize, a given functional. Thus, this area is considered a branch of optimization.

Generally, the functionals are given by definite integrals and the set of admissible functions are defined by boundary conditions and smoothness requirements, as we will see.

The Calculus of Variations and the Calculus were developed somewhat in parallel. In 1927, Forsyth said that the Calculus of Variations "attracted a rather fickle attention at more or less isolated intervals in its growth" [41, p. 1].

Leonhard Euler (1707–1783) was a Swiss mathematician and physicist. He introduced a general mathematical procedure to find the general solution of variational problems in his pioneering work *The method of finding plane curves that show some property of maximum and minimum*, in 1744. Along the way, he formulated the variational principle for mechanics (Euler's version of the principle of least action). Mathematicians consider that this event was the beginning of the Calculus of Variations. It is not known when he became seriously attracted by variational problems, but we know that Euler was first influenced by Jacob and Johann Bernoulli and after by Newton and Leibniz. The first version of the Calculus of Variations that Euler developed was intuitive and required elementary mathematics and a geometrical insight of the variational problem. We will study this approach in the section Euler's Method of Finite Differences.

Joseph Louis Lagrange (1736–1813) was an Italian mathematician. In 1755, he wrote a letter to Euler where he showed that the resolution of each variational problem can be reduced to a quite general and powerful analytical manipulation. One point of this study consists in the Euler's solution to the isoperimetric problem. This problem was present in Euler's work of 1744.

Forthwith, he adopted the formal algebraic method of Lagrange that was more rigorous. Euler renamed the subject Calculus of Variations and the elegant techniques of Lagrange eliminated the intuitive approach and the geometrical insight that Euler used.

Later, in 1900, David Hilbert presented 23 (now famous) problems, in the International Congress of Mathematicians, and the 23rd was entitled *Further development of the methods of the calculus of variations.* Before the description of the problem he remarked [41, p. 1]:

" — ... I should like to close with a general problem, namely with the indication of a branch of mathematics repeatedly mentioned in this lecture – which, in spite of the considerable advancement lately given it by Weierstrass, does not receive the general appreciation which in my opinion it is due – I mean the calculus of variations."

After, there was a further development in this area and mathematicians like David Hilbert, Emmy Noether, Leonida Tonelli, Henri Lebesgue and Jacques Hadamard, among others, dedicated significantly to the Calculus of Variations. In the eighteenth and nineteenth centuries its development was motivated especially by problems in mechanics.

Nowadays, this subject continues to cause concern, because it has applications in several areas: physics (particularly mechanics), economics and urban planning, among others.

In this chapter we will study the variational problem with fixed endpoints (first by the prospect of Euler and after by the prospect of Lagrange), the isoperimetric problem and the variational problem with an endpoint variable.

1.2 Variational Problem with Fixed Endpoints

Throughout the text, to refer a variational problem with fixed endpoints we will just write VPFE. Before presenting this particular problem we are going to recall some definitions and a fundamental result. We are going to follow the approach used by van Brunt in [41].

We say that a function f is **smooth** if it has as many continuous derivatives as are necessary to perform whatever operations that are required.

Theorem 1.2.1 (Optimality condition of first order) *Let X be an open subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ a function. If f is differentiable at $\widehat{x}$ and if $\widehat{x}$ is a local extremizer of f, then $\nabla f(\widehat{x}) = 0$.*

Definition 1.2.1 (Functional) *Let X be a vector space of functions. A functional J is a function with domain X and range $\mathbb{R}$:*

$$J : X \to \mathbb{R}.$$

Consider the vector space $X = C^n[a, b]$, for some $n \in \mathbb{N}_0$ endowed with a norm $\|\cdot\|$.

Definition 1.2.2 (Local maximizer of a functional) *Let $S \subseteq X$ be a normed space with norm $||\cdot||$. We say that $y \in S$ is a local maximizer of the functional J if there exists some $\epsilon > 0$ such that $J(\widehat{y}) - J(y) \leqslant 0$ for all $\widehat{y} \in S$ such that $||\widehat{y} - y|| < \epsilon$.*

Remark 1.2.1 We say that $y \in S$ is a local minimizer of the functional J if y is a local maximizer of the functional $-J$.

To simplify the writing we are going to say maximizer (minimizer) instead of local maximizer (minimizer).

Problem Statement (VPFE): The basic variational problem with fixed endpoints consists of finding the functions $y \in C^2[a, b]$ that solves the problem

$$\begin{aligned} (P_{CV1}) \quad \max \quad & J(y) = \int_a^b f(x, y(x), y'(x))dx \\ \text{s.t.} \quad & y(a) = y_a \\ & y(b) = y_b, \end{aligned} \tag{1.1}$$

where $J : C^2[a, b] \longrightarrow \mathbb{R}$ is a functional, f (usually called Lagrangian) is a function assumed to have, at least, continuous partial derivatives of the second order with respect to x, y and y' and a, b, y_a and y_b are fixed.

Sometimes, to simplify the notation, we can write $f(x, y, y')$ instead of $f(x, y(x), y'(x))$, or simply f instead of $f(x, y, y')$.

1.2.1 Euler's Method of Finite Differences

In this section we will show how Euler solved the variational problem (P_{CV1}) before Lagrange write to him (see [27, p. 28–32]).

First, we define

$$\Delta x := \frac{b-a}{n+1}$$

and we take in the interval $[a, b]$ the points $x_0 = a$, $x_i = x_0 + i\Delta x$ for $i = 1, \dots, n$ and $x_{n+1} = b$. Note that $n \in \mathbb{N}$. So, we divide the interval $[a, b]$ into $n + 1$ equal parts. Consider the following definition.

Definition 1.2.3 (Finite Forward Difference) *Let $f : [a, b] \to \mathbb{R}$ be a function and consider $x_i = x_0 + i\Delta x$, for all $i = 1, \dots, n$, such that $x_0 = a$, $x_{n+1} = b$ and $\Delta x = \frac{b-a}{n+1}$. The finite forward difference of first order of f is given by*

$$\Delta f(x_i) := f(x_{i+1}) - f(x_i).$$

Define $y_i := y(x_i)$ for all $i = 0, \dots, n+1$. For $i = 1, \dots, n$ we don't know the values y_i, because the function which solves the problem is unknown yet.

As we know, the integral (1.1) is the limit of a summation and y'_i can be approximated by

$$\frac{y_{i+1} - y_i}{\Delta x}.$$

Consequently, we may approximate the integral (1.1) by the following function $\phi(y_1, \dots, y_n)$:

$$\phi(y_1, \dots, y_n) = \sum_{i=0}^{n} f\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right) \Delta x.$$

We can determinate the quantities y_i, $i = 1, \dots, n$, using the following equations:

$$\frac{\partial \phi}{\partial y_i}(y_1, \dots, y_n) = 0, \ \forall i = 1, \dots, n.$$

Thus, for $i = 1, \dots, n$ we have that

$$\begin{aligned}
\frac{\partial \phi}{\partial y_i}(y_1, \dots, y_n) &= \frac{\partial}{\partial y}\left(f\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right) \Delta x\right) \\
&\quad + \frac{\partial}{\partial y}\left(f\left(x_{i-1}, y_{i-1}, \frac{y_i - y_{i-1}}{\Delta x}\right) \Delta x\right) \\
&= \frac{\partial f}{\partial y}\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right) \Delta x - \frac{\partial f}{\partial y'}\left(x_i, y_i, \frac{y_{i+1} - y_i}{\Delta x}\right) \\
&\quad + \frac{\partial f}{\partial y'}\left(x_{i-1}, y_{i-1}, \frac{y_i - y_{i-1}}{\Delta x}\right) = 0.
\end{aligned}$$

Assuming that $\Delta y_i = y_{i+1} - y_i$, we have that

$$\begin{aligned} &\frac{\partial f}{\partial y}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\frac{\partial f}{\partial y'}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\partial f}{\partial y'}\left(x_{i-1}, y_{i-1}, \frac{\Delta y_{i-1}}{\Delta x}\right)}{\Delta x} = 0 \\ \Rightarrow &\frac{\partial f}{\partial y}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) - \frac{\Delta \frac{\partial f}{\partial y'}}{\Delta x}\left(x_i, y_i, \frac{\Delta y_i}{\Delta x}\right) = 0. \end{aligned} \tag{1.2}$$

The above equation is the finite difference version of the Euler–Lagrange equation (see (1.5) on page 15).

Example 1.2.1 Consider the problem

$$\begin{aligned} \max \quad & J(y) = \int_0^1 (y'(x))^2 - y^2(x) - 2xy(x)\ dx \\ \text{s.t.} \quad & y(0) = 1 \\ & y(1) = 2. \end{aligned} \tag{1.3}$$

We will find an approximation to the solution of this problem by Euler's Method for $n = 1$, $n = 2$, $n = 3$ and $n = 4$.

For $n = 1$ we have that $\Delta x = \frac{1}{2}$. Thus, $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$. We know that $y_0 = 1$ and $y_2 = 2$. To determine y_1 we need to write the function ϕ given by

$$\begin{aligned} \phi(y_1) =& \sum_{i=0}^{1} \left[\left(\frac{y_{i+1} - y_i}{\Delta x}\right)^2 - y_i^2 - 2x_i y_i\right] \Delta x \\ =& \frac{7}{2} y_1^2 - \frac{25}{2} y_1 + \frac{19}{2} \end{aligned}$$

and solve

$$\frac{\partial \phi}{\partial y_1} = 0 \Leftrightarrow 7y_1 - \frac{25}{2} = 0 \Leftrightarrow y_1 = \frac{25}{14}.$$

So, we obtain the points $A = (0, 1)$; $B = \left(\frac{1}{2}, \frac{25}{14}\right)$ and $C = (1, 2)$. We can observe the approximation (dashed line), these points and the extremal[1]

$$y(x) = \frac{3 - \cos(1)}{\sin(1)} \sin(x) + \cos(x) - x \tag{1.4}$$

to problem (1.3) (solid line) in Figure 1.1 (see Example 1.2.2 on page 17).

[1] The concept of extremal is introduced later, in Definition 1.2.5.

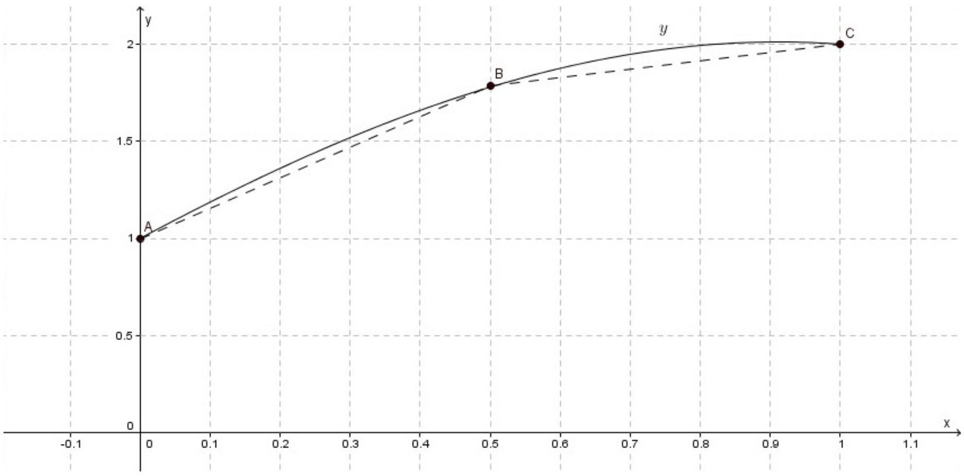

Figure 1.1: Euler's Method for $n = 1$ versus the extremal to (1.3).

For $n = 2$ we have that $\Delta x = \frac{1}{3}$. Thus, $x_0 = 0$, $x_1 = \frac{1}{3}$, $x_2 = \frac{2}{3}$ and $x_3 = 1$. We know that $y_0 = 1$ and $y_3 = 2$. To determine y_1 and y_2 we need to write the function ϕ given by

$$\begin{aligned}\phi(y_1, y_2) =& \sum_{i=0}^{2} \left[\left(\frac{y_{i+1} - y_i}{\Delta x} \right)^2 - y_i^2 - 2x_i y_i \right] \Delta x \\ =& \frac{17}{3} y_1^2 + \frac{17}{3} y_2^2 - \frac{56}{9} y_1 - \frac{112}{9} y_2 - 6 y_1 y_2 + \frac{44}{3}\end{aligned}$$

and solve the system

$$\begin{cases} \frac{\partial \phi}{\partial y_1} = 0 \\ \frac{\partial \phi}{\partial y_2} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{34}{3} y_1 - \frac{56}{9} - 6y_2 = 0 \\ \frac{34}{3} y_2 - \frac{112}{9} - 6y_1 = 0 \end{cases} \Leftrightarrow \begin{cases} y_1 = \frac{245}{156} \\ y_2 = \frac{301}{156}. \end{cases}$$

So, we obtain the points

$$A = (0, 1); \; B = \left(\frac{1}{3}, \frac{245}{156} \right); \; C = \left(\frac{2}{3}, \frac{301}{156} \right) \text{ and } D = (1, 2).$$

We can observe the approximation (dashed line), these points and the extremal to problem (1.3) given by (1.4) (solid line) in Figure 1.2.

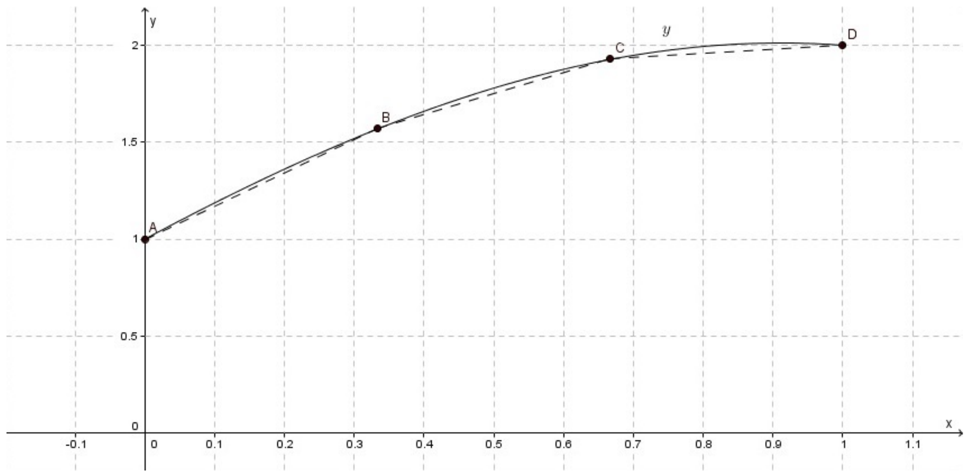

Figure 1.2: Euler's Method for $n = 2$ versus the extremal to (1.3).

For $n = 3$ we have that $\Delta x = \frac{1}{4}$. Thus, $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$ and $x_4 = 1$. We know that $y_0 = 1$ and $y_4 = 2$. To determine y_1, y_2 and y_3 we need to write the function ϕ given by

$$\begin{aligned}\phi(y_1, y_2, y_3) =& \sum_{i=0}^{3} \left[\left(\frac{y_{i+1} - y_i}{\Delta x} \right)^2 - y_i^2 - 2x_i y_i \right] \Delta x \\ =& \frac{31}{4} y_1^2 + \frac{31}{4} y_2^2 + \frac{31}{4} y_3^2 - \frac{65}{8} y_1 - \frac{1}{4} y_2 \\ &- \frac{131}{8} y_3 - 8y_1 y_2 - 8y_2 y_3 + \frac{79}{4}\end{aligned}$$

and solve the system

$$\begin{cases} \frac{\partial \phi}{\partial y_1} = 0 \\ \frac{\partial \phi}{\partial y_2} = 0 \\ \frac{\partial \phi}{\partial y_3} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{31}{2} y_1 - \frac{65}{8} - 8y_2 = 0 \\ \frac{31}{2} y_2 - \frac{1}{4} - 8y_1 - 8y_3 = 0 \\ \frac{31}{2} y_3 - \frac{131}{8} - 8y_2 = 0 \end{cases} \Leftrightarrow \begin{cases} y_1 = \frac{80353}{55676} \\ y_2 = \frac{1599}{898} \\ y_3 = \frac{109987}{55676}. \end{cases}$$

So, we obtain the points

$$A = (0, 1);\ B = \left(\frac{1}{4}, \frac{80353}{55676} \right);\ C = \left(\frac{1}{2}, \frac{1599}{898} \right);\ D = \left(\frac{3}{4}, \frac{109987}{55676} \right) \text{ and }$$
$$E = (1, 2).$$

We can observe the approximation (dashed line), these points and the extremal to problem (1.3) given by (1.4) (solid line) in Figure 1.3.

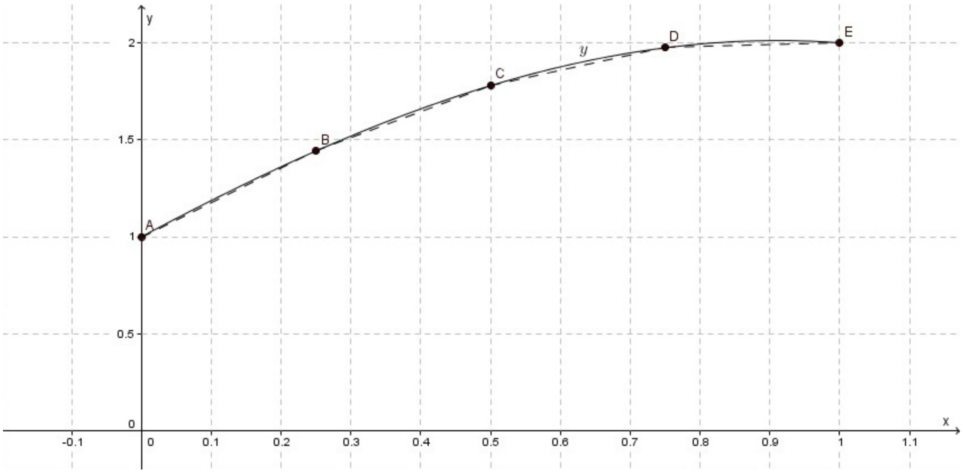

Figure 1.3: Euler's Method for $n = 3$ versus the extremal to (1.3).

For $n = 4$ we have that $\Delta x = \frac{1}{5}$. Thus, $x_0 = 0$, $x_1 = \frac{1}{5}$, $x_2 = \frac{2}{5}$, $x_3 = \frac{3}{5}$, $x_4 = \frac{4}{5}$ and $x_5 = 1$. We know that $y_0 = 1$ and $y_5 = 2$. To determine y_1, y_2, y_3 and y_4 we need to write the function ϕ given by

$$\begin{aligned}\phi(y_1, y_2, y_3, y_4) =& \sum_{i=0}^{4} \left[\left(\frac{y_{i+1} - y_i}{\Delta x} \right)^2 - y_i^2 - 2x_i y_i \right] \Delta x \\ =& \frac{49}{5} y_1^2 + \frac{49}{5} y_2^2 + \frac{49}{5} y_3^2 + \frac{49}{5} y_4^2 - \frac{252}{25} y_1 - \frac{4}{25} y_2 \\ & - \frac{6}{25} y_3 - \frac{508}{25} y_4 - 10 y_1 y_2 - 10 y_2 y_3 - 10 y_3 y_4 + \frac{124}{5}\end{aligned}$$

and solve the system

$$\begin{cases} \frac{\partial \phi}{\partial y_1} = 0 \\ \frac{\partial \phi}{\partial y_2} = 0 \\ \frac{\partial \phi}{\partial y_3} = 0 \\ \frac{\partial \phi}{\partial y_4} = 0 \end{cases} \Leftrightarrow \begin{cases} \frac{98}{5} y_1 - \frac{252}{25} - 10 y_2 = 0 \\ \frac{98}{5} y_2 - \frac{4}{25} - 10 y_1 - 10 y_3 = 0 \\ \frac{98}{5} y_3 - \frac{6}{25} - 10 y_2 - 10 y_4 = 0 \\ \frac{98}{5} y_4 - \frac{508}{25} - 10 y_3 = 0 \end{cases} \Leftrightarrow \begin{cases} y_1 = \frac{11255699}{8267755} \\ y_2 = \frac{13727273}{8267755} \\ y_3 = \frac{15517472}{8267755} \\ y_4 = \frac{16488546}{8267755}. \end{cases}$$

So, we obtain the points

$$\begin{aligned} A = (0, 1);\ B = \left(\frac{1}{5}, \frac{11255699}{8267755} \right);\ C = \left(\frac{2}{5}, \frac{13727273}{8267755} \right); \\ D = \left(\frac{3}{5}, \frac{15517472}{8267755} \right);\ E = \left(\frac{4}{5}, \frac{16488546}{8267755} \right) \text{ and } F = (1, 2). \end{aligned}$$

We can observe the approximation (dashed line), these points and the extremal to problem (1.3) given by (1.4) (solid line) in Figure 1.4.

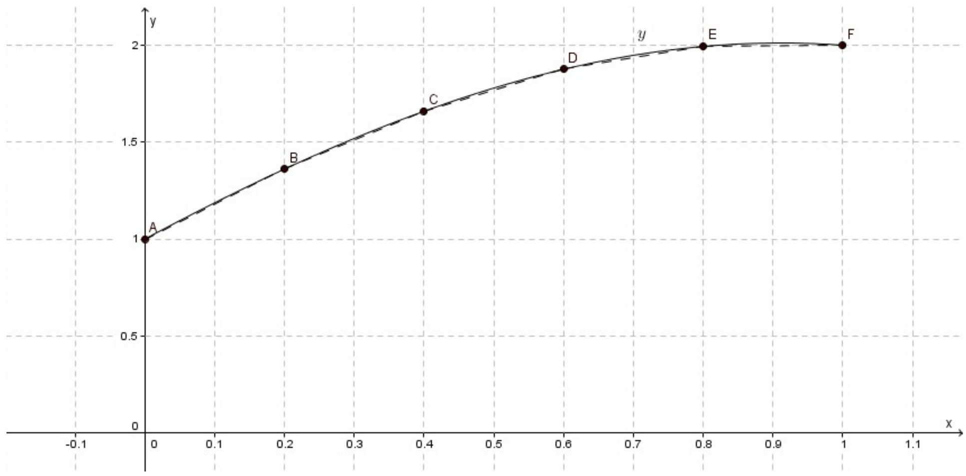

Figure 1.4: Euler's Method for $n = 4$ versus the extremal to (1.3).

Graphically, we observe that the approximations converge to the extremal y given by (1.4). We remark that, for this extremal, $J(y) \simeq -3.38$ and that $\phi(y_1) \simeq -1.66$, $\phi(y_1, y_2) \simeq -2.23$, $\phi(y_1, y_2, y_3) \simeq -2.51$ and $\phi(y_1, y_2, y_3, y_4) \simeq -2.68$. As the value of n increases, the values of the approximation approach to $J(y)$.

Note that for the four cases studied previously, we determine the points x_i and y_i, for $i = 0, \ldots, n+1$, the function $\phi(y_1, \ldots, y_n)$ and the graphics with the help of the routines developed in MATLAB that are in Appendix A.

1.2.2 Lagrange's Method

Now we will study the Lagrange's approach to solve the problem (P_{CV1}), but before we will recall some definitions and prove some lemmas which we will need later.

Lemma 1.2.2 *Let α and β be two real numbers such that $\alpha < \beta$. Then, there exists a smooth function υ such that $\upsilon(x) > 0$ for all $x \in]\alpha, \beta[$ and $\upsilon(x) = 0$ for all $x \in \mathbb{R} \backslash]\alpha, \beta[$.*

Proof: Consider the function $\theta : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$\theta(x) = \begin{cases} e^{-\frac{1}{x}} & , \ x > 0 \\ 0 & , \ x \leq 0. \end{cases}$$

Let us prove, by mathematical induction, that for all $m \in \mathbb{N}_0$, $\theta \in C^m$ and $\theta^{(m)}(0) = 0$. For $m = 0$ it is obvious. Suppose that $\theta \in C^m$ and $\theta^{(m)}(0) = 0$. Now we will prove that $\theta \in C^{m+1}$ and $\theta^{(m+1)}(0) = 0$. Clearly, for all $x \in \mathbb{R}^+$

$$\theta^{(m)}(x) = \frac{e^{-\frac{1}{x}}P(x)}{Q(x)}$$

and for all $x \in \mathbb{R}^-$ we have that $\theta^{(m)}(x) = 0$, where $P(x)$ and $Q(x)$ are polynomials. So,

$$\begin{aligned}\theta^{(m+1)}(0^+) &= \lim_{x\to 0^+} \frac{\theta^{(m)}(x) - \theta^{(m)}(0)}{x-0} = \lim_{x\to 0^+} \frac{\theta^{(m)}(x)}{x} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}\frac{P(x)}{Q(x)}}{x} \\ &= \lim_{x\to 0^+} \frac{e^{-\frac{1}{x}}P(x)}{xQ(x)} = 0\end{aligned}$$

and

$$\theta^{(m+1)}(0^-) = \lim_{x\to 0^-} \frac{\theta^{(m)}(x) - \theta^{(m)}(0)}{x-0} = \lim_{x\to 0^-} \frac{\theta^{(m)}(x)}{x} = 0.$$

Therefore, $\theta^{(m+1)}(0) = 0$ and, consequently, $\theta \in C^{m+1}$. Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$\varphi(x) = \theta(x)\theta(1-x) = \begin{cases} e^{-\frac{1}{x}-\frac{1}{1-x}} & , \ x \in]0,1[\\ 0 & , \ x \in \mathbb{R}\backslash]0,1[. \end{cases}$$

As φ is a product of two smooth functions, φ is also smooth. Now let $\varphi_{\alpha,\beta} : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$\begin{aligned}\varphi_{\alpha,\beta}(x) =& \varphi\left(\frac{x-\alpha}{\beta-\alpha}\right) = \theta\left(\frac{x-\alpha}{\beta-\alpha}\right)\theta\left(\frac{\beta-x}{\beta-\alpha}\right) \\ =& \begin{cases} e^{\frac{\alpha-\beta}{x-\alpha}+\frac{\alpha-\beta}{\beta-x}} & , \ x \in]\alpha,\beta[\\ 0 & , \ x \in \mathbb{R}\backslash]\alpha,\beta[. \end{cases}\end{aligned}$$

Thus, there exists a smooth function $\upsilon = \varphi_{\alpha,\beta}$ such that $\upsilon(x) > 0$ for all $x \in]\alpha,\beta[$ and $\upsilon(x) = 0$ for all $x \in \mathbb{R}\backslash]\alpha,\beta[$.

□

Definition 1.2.4 (Inner Product of Functions) *The vector space of all real valued continuous functions on a closed interval* $[a,b]$ *is an inner product space, whose inner product is defined by*

$$\langle f, g\rangle = \int_a^b f(x)g(x)dx, \quad f,g \in C[a,b].$$

The following Lemma is known as the Fundamental Lemma of the Calculus of Variations.

Lemma 1.2.3 *Let*

$$H := \{h \in C^2[a,b] : h(a) = h(b) = 0\}$$

be a set. If $\langle h, g\rangle = 0$ *for all* $h \in H$ *and if* $g : [a,b] \to \mathbb{R}$ *is a continuous function, then* $g = 0$ *on the interval* $[a,b]$.

Proof: Suppose that $g(c) \neq 0$ for some $c \in [a,b]$. Without loss of generality we will assume that $g(c) > 0$. Since g is continuous on the interval $[a,b]$, exists a subinterval $]\alpha,\beta[$ of $[a,b]$ such that $g(x) > 0$ for all $x \in]\alpha,\beta[$. By Lemma 1.2.2 there is a smooth function $\upsilon = \varphi_{\alpha,\beta}$ such that $\upsilon(x) > 0$ for all $x \in]\alpha,\beta[$ and $\upsilon(x) = 0$ for all $x \in [a,b]\backslash]\alpha,\beta[$. So, $\upsilon \in H$ and

$$\langle \upsilon, g\rangle = \int_a^b \upsilon(x)g(x)dx = \int_\alpha^\beta \upsilon(x)g(x)dx > 0.$$

Therefore, there exists $\upsilon \in H$ such that $\langle \upsilon, h\rangle \neq 0$. Consequently, $g = 0$ on $[a,b]$.

□

Remark 1.2.2 As the function $\varphi_{\alpha,\beta}$ of the proof of the Lemma 1.2.2 is smooth, the above Lemma remains valid if $h \in C^n[a,b]$ for $n \in \mathbb{N}$.

With the following theorem we will derive a necessary condition for a smooth function to be a solution of (P_{CV1}).

Theorem 1.2.4 *Let* S *be the set defined by*

$$S = \{y \in C^2[a,b] : y(a) = y_a \text{ and } y(b) = y_b\}$$

and $J : S \to \mathbb{R}$ *be a functional of the form*

$$J(y) = \int_a^b f(x, y(x), y'(x))dx,$$

where y_a *and* y_b *are given real numbers and* f *has continuous partial derivatives of the second order with respect to* x, y *and* y'. *If* $y \in S$ *is an extremizer for* J, *then*

$$\frac{\partial f}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y(x), y'(x))\right) = 0 \tag{1.5}$$

for all $x \in [a,b]$.

Proof: Suppose that $y \in S$ is an extremizer for J. Let us consider the variations $y + \epsilon h \in S$, where $|\epsilon| \ll 1$ and $h \in C^2[a,b]$. All these variations can be generated by an appropriate set H of functions h. As the variations considered are in S and the endpoints are fixed, H should be defined by

$$H := \{h \in C^2[a,b] : h(a) = h(b) = 0\}.$$

Let j be the function defined by

$$j(\epsilon) = \int_a^b f(x, y(x) + \epsilon h(x), y'(x) + \epsilon h'(x))dx. \tag{1.6}$$

Note that $j(\epsilon) = J(y + \epsilon h)$ and for the function j the variable is ϵ and y and h are fixed. Consequently, y' and h' are also fixed. As y is a solution of (P_{CV1}), then $\epsilon = 0$ is an extremizer of j. Therefore, by Theorem 1.2.1 and using integration by parts, we get

$$\begin{aligned}
& j'(0) = 0 \\
\Leftrightarrow\ & \frac{d}{d\epsilon} \int_a^b f(x, y(x) + \epsilon h(x), y'(x) + \epsilon h'(x))dx\Big|_{\epsilon=0} = 0 \\
\Leftrightarrow\ & \int_a^b \frac{df}{d\epsilon}(x, y(x) + \epsilon h(x), y'(x) + \epsilon h'(x))\Big|_{\epsilon=0} dx = 0 \\
\Leftrightarrow\ & \int_a^b \left(\frac{\partial f}{\partial y}(x, y(x), y'(x))h(x) + \frac{\partial f}{\partial y'}(x, y(x), y'(x))h'(x) \right) dx = 0 \\
\Leftrightarrow\ & \int_a^b \left(\frac{\partial f}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y(x), y'(x)) \right) \right) h(x)\ dx \\
& + \left[\frac{\partial f}{\partial y'}(x, y(x), y'(x))h(x) \right]_a^b = 0 \\
\Leftrightarrow\ & \int_a^b \left(\frac{\partial f}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y(x), y'(x)) \right) \right) h(x)dx = 0.
\end{aligned}$$

As f has continuous partial derivatives of second order, by Lemma 1.2.3, we have that for all $x \in [a,b]$

$$\frac{\partial f}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y(x), y'(x)) \right) = 0.$$

This concludes the proof.

□

The second-order ordinary differential Equation (1.5) is generally nonlinear and it is called the **Euler–Lagrange equation**. We can write it, in a more concise way, as

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0.$$

Remark 1.2.3 In the Equation (1.2) as $n \longrightarrow \infty$ we have that $\Delta x \longrightarrow 0$ and it becomes the Euler–Lagrange equation (1.5).

Definition 1.2.5 (Extremal) *If y is a smooth function and satisfies the Euler–Lagrange equation with respect to J, then y is called an* ***extremal*** *for J.*

Definition 1.2.6 (First Variation) *The quantity $\delta J(h, y) = j'(0)$, where j is given by* (1.6)*, is called the* ***first variation*** *of J at y in the direction h.*

Now we revisit the problem of Example 1.2.1.

Example 1.2.2 For the problem

$$\begin{aligned} \max \quad & J(y) = \int_0^1 (y'(x))^2 - y^2(x) - 2xy(x)\, dx \\ \text{s.t.} \quad & y(0) = 1 \\ & y(1) = 2 \end{aligned}$$

the Lagrangian is

$$f(x, y, y') = (y')^2 - y^2 - 2xy$$

and the Euler–Lagrange equation (1.5) gives

$$y''(x) = -x - y(x).$$

So,

$$y(x) = c_1 \sin(x) + c_2 \cos(x) - x,$$

where c_1 and c_2 are real constants. As $y(0) = 1$ and $y(1) = 2$, we have that $c_1 = \frac{3-\cos(1)}{\sin(1)}$ and $c_2 = 1$. Therefore,

$$y(x) = \frac{3 - \cos(1)}{\sin(1)} \sin(x) + \cos(x) - x$$

and this function $y(x)$ is the extremal (a candidate for maximizer) for the given problem. We can confirm these results with the help of Maple and the following code:

```
with(VariationalCalculus);
F := (diff(y(x), x))^2-y(x)^2-2*x*y(x);
eqEL := EulerLagrange(F, x, y(x));
```

returns the Euler–Lagrange equation

```
{-2*x-2*y(x)-2*(diff(y(x), x, x))}.
```

To solve this equation we execute

```
dsolve({op(eqEL), y(0) = 1, y(1) = 2}, y(x))
```

and we obtain

```
y(x) = sin(x)*(3-cos(1))/sin(1)+cos(x)-x
```

that is the solution that we determined previously. We can also see the graphic of $y(x)$ in Figure 1.5.

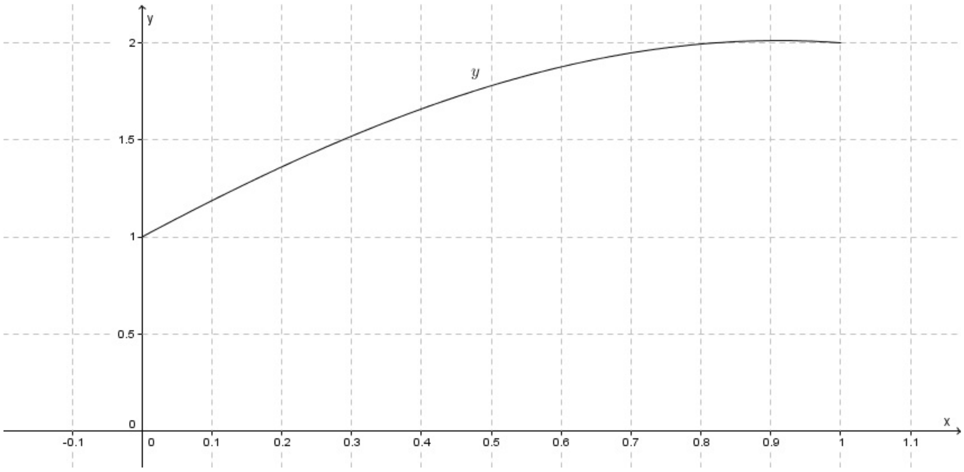

Figure 1.5: Graphic of the extremal $y(x)$ to problem of Example 1.2.2.

Example 1.2.3 For the problem

$$\begin{aligned} \max \quad & J(y) = \int_0^1 (y'(x))^2 - 2xy(x)\ dx \\ \text{s.t.} \quad & y(0) = 1 \\ & y(1) = 3 \end{aligned}$$

the Lagrangian is

$$f(x, y, y') = (y')^2 - 2xy.$$

The function $y(x) = -\frac{x^3}{6} + x + 1$ satisfies the Euler–Lagrange equation (1.5), but as $y(1) = \frac{11}{6} \neq 3$, it isn't solution of the given problem.

Example 1.2.4 For the problem

$$\begin{aligned} \max \quad & J(y) = \int_0^1 -(y'(x) - 1)^2\ dx \\ \text{s.t.} \quad & y(0) = 0 \\ & y(1) = 1 \end{aligned}$$

the Lagrangian is

$$f(x, y, y') = -(y' - 1)^2$$

and the Euler–Lagrange equation (1.5) is given by

$$\frac{d}{dx} 2(y'(x) - 1) = 0.$$

Note that $y^*(x) = x$ is a solution of the Euler–Lagrange equation. Therefore, y^* is an extremal for J. As $J(y) \leq 0$ for all y and $J(y^*) = 0$, we have that y^* is actually a (global) maximizer for the given problem.

Now we intend to derive a sufficient condition for a smooth function to be a solution of (P_{CV1}).

Definition 1.2.7 (Concave Function) *The function $f(x, y, z)$ is concave in $M \subseteq \mathbb{R}^3$ for the variables y and z if $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ exist and are continuous and the condition*

$$f(x, y + y_1, z + z_1) - f(x, y, z) \leq \frac{\partial f}{\partial y}(x, y, z) y_1 + \frac{\partial f}{\partial z}(x, y, z) z_1$$

holds for every $(x, y, z), (x, y + y_1, z + z_1) \in M$.

Theorem 1.2.5 *If the function $f(x, y, y')$ of the problem (P_{CV1}) is concave in $[a, b] \times \mathbb{R}^2$ for the variables y and y', then each solution y of the Euler–Lagrange equation (1.5) is a solution of the problem (P_{CV1}).*

Proof: Let $h \in H$ be a function, where H is as defined in the proof of Theorem 1.2.4, and ϵ such that $|\epsilon| \ll 1$. So,

$$\begin{aligned}
& J(y + \epsilon h) - J(y) \\
&= \int_a^b f(x, y(x) + \epsilon h(x), y'(x) + \epsilon h'(x)) - f(x, y(x), y'(x)) \, dx \\
&\leq \int_a^b \frac{\partial f}{\partial y}(x, y(x), y'(x)) \epsilon h(x) + \frac{\partial f}{\partial y'}(x, y(x), y'(x)) \epsilon h'(x) \, dx \\
&= \epsilon \int_a^b \left(\frac{\partial f}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx} \frac{\partial f}{\partial y'}(x, y(x), y'(x)) \right) h(x) \, dx \\
&= 0.
\end{aligned}$$

Therefore, as $J(y + \epsilon h) - J(y) \leq 0$, we have that y is a solution of (P_{CV1}).

□

Example 1.2.5 The function f of the Example 1.2.4 is concave, because

$$\begin{aligned}
& f(u, v+v_1, w+w_1) - f(u,v,w) \leq \frac{\partial f}{\partial v}(u,v,w)v_1 + \frac{\partial f}{\partial w}(u,v,w)w_1 \\
\Leftrightarrow & -(w+w_1-1)^2 + (w-1)^2 \leq -2(w-1)w_1 \\
\Leftrightarrow & -w_1^2 \leq 0
\end{aligned}$$

is true for all $(u, v+v_1, w+w_1), (u,v,w) \in \mathbb{R}^3$. Now, by the Theorem 1.2.5, we can conclude again that the extremal $y^*(x) = x$ is a maximizer for the problem of the Example 1.2.4.

Particular Cases

Now we are going to analyse three cases where the Euler–Lagrange equation can be simplified. Suppose that the functional given by

$$J(y) = \int_a^b f(x, y(x), y'(x))dx$$

satisfies the conditions of Theorem 1.2.4.

1. **First case:** y does not appear explicitly in the integrand.
   In this case the functional is of the form

   $$J(y) = \int_a^b f(x, y'(x))dx$$

   and the Euler–Lagrange equation becomes

   $$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y'(x))\right) = 0 \Rightarrow \frac{\partial f}{\partial y'}(x, y'(x)) = c,$$

   where c is a constant of integration.

2. **Second case:** The independent variable x does not appear explicitly in the integrand (so called autonomous case).
   In this case the functional is of the form

   $$J(y) = \int_a^b f(y(x), y'(x))dx.$$

   Theorem 1.2.6 *Let J be a functional such that*

   $$J(y) = \int_a^b f(y(x), y'(x))dx \tag{1.7}$$

   and define the function G by

   $$G(y, y') = y'\frac{\partial f}{\partial y'}(y, y') - f(y, y').$$

   Then, $G(y(x), y'(x))$ is constant along any extremal y of (1.7).

Proof: By definition of extremal (Definition 1.2.5),

$$\frac{\partial f}{\partial y}(y,y') - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}(y,y')\right) = 0.$$

Consequently, we have

$$\begin{aligned}
\frac{dG}{dx}(y,y') &= y''\frac{\partial f}{\partial y'}(y,y') + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(y,y')\right) - \frac{d}{dx}f(y,y') \\
&= y''\frac{\partial f}{\partial y'}(y,y') + y'\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(y,y')\right) \\
&\quad - \left(y'\frac{\partial f}{\partial y}(y,y') + y''\frac{\partial f}{\partial y'}(y,y')\right) \\
&= -y'\left(\frac{\partial f}{\partial y}(y,y') - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}(y,y')\right)\right) \\
&= -y' \times 0 \\
&= 0.
\end{aligned}$$

Thus, $G(y(x), y'(x))$ is constant along any extremal y of (1.7).

□

Remark 1.2.4 As $G(y(x), y'(x))$ is constant along any extremal, then $G(y(x), y'(x))$ is constant along any extremizer of (1.7).

3. **Third case (a degenerate case)**: the integrand is linear in y'. Suppose that J is a functional of the form

$$J(y) = \int_a^b A(x, y(x))y'(x) + B(x, y(x))dx,$$

where A and B are smooth functions of x and y. In this case, the Euler–Lagrange equation is

$$\frac{dA}{dx}(x,y) - \left(y'\frac{\partial A}{\partial y}(x,y) + \frac{\partial B}{\partial y}(x,y)\right) = 0. \tag{1.8}$$

Note that

$$\begin{aligned}
\frac{dA}{dx}(x,y) &= \frac{\partial A}{\partial x}(x,y) + y'\frac{\partial A}{\partial y}(x,y) \\
\Leftrightarrow \frac{dA}{dx}(x,y) - y'\frac{\partial A}{\partial y}(x,y) &= \frac{\partial A}{\partial x}(x,y).
\end{aligned}$$

So, we can rewrite Equation (1.8) as

$$\frac{\partial A}{\partial x}(x,y) - \frac{\partial B}{\partial y}(x,y) = 0 \Leftrightarrow \frac{\partial A}{\partial x}(x,y) = \frac{\partial B}{\partial y}(x,y) =: g(x,y). \quad (1.9)$$

Then,

$$A(x,y) = \int g(x,y)dx + f_A(y)$$

and

$$B(x,y) = \int g(x,y)dy + f_B(x),$$

where f_A and f_B are functions. Let ϕ be the function defined by

$$\phi := \int\int g(x,y)dxdy + \int f_A(y)dy + \int f_B(x)dx.$$

So, we have that

$$\frac{\partial \phi}{\partial x} = \int g(x,y)dy + f_B(x) = B(x,y)$$

and

$$\frac{\partial \phi}{\partial y} = \int g(x,y)dx + f_A(y) = A(x,y).$$

In conclusion, if Equation (1.9) is an identity for all $x \in [a,b]$ and for all $y \in S$, this implies the existence of a smooth function ϕ such that

$$\frac{\partial \phi}{\partial y}(x,y) = A(x,y), \; \frac{\partial \phi}{\partial x}(x,y) = B(x,y).$$

Thus, as f is the integrand

$$f = \frac{\partial \phi}{\partial y}y' + \frac{\partial \phi}{\partial x} = \frac{d\phi}{dx} \Leftrightarrow f dx = d\phi.$$

Consequently,

$$J(y) = \int_a^b d\phi = \phi(b, y(b)) - \phi(a, y(a)).$$

Conclusions:

(a) The value of J is independent of y, therefore the integrand is path independent.

(b) J depends only on ϕ and the points $(a, y(a))$ and $(b, y(b))$.

Therefore, we can formulate the following Theorem and prove it.

Theorem 1.2.7 *Suppose that the functional J satisfies the conditions of Theorem 1.2.4 and the Euler–Lagrange equation (1.5) reduces to an identity. Then, the integrand must be linear in y' and the value of the functional is independent of y.*

Proof: If the Euler–Lagrange equation reduces to an identity, then

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \ \forall x \in [a,b] \text{ and } \forall y \in S$$

$$\Leftrightarrow \frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial y'^2} y'' = 0, \ \forall x \in [a,b] \text{ and } \forall y \in S. \tag{1.10}$$

As y'' appears only in the last term, this can not be cancelled with any other term of the above equation, and as the Equation (1.10) must hold for all $y \in S$ we can conclude that

$$\frac{\partial^2 f}{\partial y'^2} = 0 \Rightarrow \frac{\partial f}{\partial y'} = A(x,y) \Rightarrow f = A(x,y)y' + B(x,y)$$

for some functions A and B. So, the Euler–Lagrange equation is

$$\frac{\partial A}{\partial x}(x,y) = \frac{\partial B}{\partial y}(x,y)$$

for all $x \in [a,b]$ and for all $y \in S$. So, we have that

$$J(y) = \int_a^b d\phi = \phi(b, y(b)) - \phi(a, y(a)) = \phi(b, y_b) - \phi(a, y_a)$$

and therefore, the value of the functional is independent of y (a, b, y_a and y_b are given).

□

1.2.3 Some Generalizations for the VPFE

VPFE for Functionals Containing Second-Order Derivatives

A procedure similar to the one of Section 1.2.2 can be done if the functional J also contains second-order derivatives.

Theorem 1.2.8 *Let S be the set defined by*

$$S = \{y \in C^4[a,b] : y^{(m)}(a) = y_a^{(m)} \text{ and } y^{(m)}(b) = y_b^{(m)} \text{ for } m = 0, 1\}$$

and $J : S \to \mathbb{R}$ be a functional of the form

$$J(y) = \int_a^b f(x, y(x), y'(x), y''(x))dx, \tag{1.11}$$

where $y_a^{(m)}$ *and* $y_b^{(m)}$ *for* $m = 0, 1$ *are given real numbers and* f *has continuous partial derivatives of the third order with respect to* x*,* y*,* y' *and* y''*. If* $y \in S$ *is an extremizer for* J*, then*

$$\frac{\partial}{\partial y} f(x, y(x), y'(x), y''(x)) - \frac{d}{dx}\left(\frac{\partial}{\partial y'} f(x, y(x), y'(x), y''(x))\right)$$
$$+\frac{d^2}{dx^2}\left(\frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x))\right) = 0 \tag{1.12}$$

for all $x \in [a, b]$*.*

Proof: Suppose that $y \in S$ is an extremizer for J. Again, let us consider the variations $y + \epsilon h \in S$, where $|\epsilon| \ll 1$ and $h \in C^4[a, b]$. Now the set H should be defined by

$$H := \{h \in C^4[a, b] : h(a) = h'(a) = h(b) = h'(b) = 0\}.$$

Let j be the function defined by

$$j(\epsilon) = \int_a^b f(x, y(x) + \epsilon h(x), y'(x) + \epsilon h'(x), y''(x) + \epsilon h''(x))dx.$$

It is known that if $y \in S$ is an extremizer for J, then, by Theorem 1.2.1,

$$\begin{aligned}
& j'(0) = 0 \\
\Leftrightarrow\ & \frac{d}{d\epsilon}\int_a^b f(x, y(x) + \epsilon h(x), y'(x) + \epsilon h'(x), y''(x) + \epsilon h''(x))\Big|_{\epsilon=0} dx = 0 \\
\Leftrightarrow\ & \int_a^b \frac{d}{d\epsilon} f(x, y(x) + \epsilon h(x), y'(x) + \epsilon h'(x), y''(x) + \epsilon h''(x))\Big|_{\epsilon=0} dx = 0 \\
\Leftrightarrow\ & \int_a^b \frac{\partial}{\partial y} f(x, y(x), y'(x), y''(x))h(x)dx \\
& + \int_a^b \frac{\partial}{\partial y'} f(x, y(x), y'(x), y''(x))h'(x)dx \\
& + \int_a^b \frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x))h''(x)dx = 0.
\end{aligned}$$

Now we will eliminate the terms $h'(x)$ and $h''(x)$ in the previous equation using integration by parts.

Elimination of the term $h'(x)$:

$$\begin{aligned}
&\int_a^b \frac{\partial}{\partial y'} f(x, y(x), y'(x), y''(x)) h'(x) dx \\
&= \left[\frac{\partial}{\partial y'} f(x, y(x), y'(x), y''(x)) h(x) \right]_a^b \\
&\quad - \int_a^b \frac{d}{dx} \left(\frac{\partial}{\partial y'} f(x, y(x), y'(x), y''(x)) \right) h(x) dx \\
&= - \int_a^b \frac{d}{dx} \left(\frac{\partial}{\partial y'} f(x, y(x), y'(x), y''(x)) \right) h(x) dx.
\end{aligned}$$

Elimination of the term $h''(x)$:

$$\begin{aligned}
&\int_a^b \frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x)) h''(x) dx \\
&= \left[\frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x)) h'(x) \right]_a^b \\
&\quad - \int_a^b \frac{d}{dx} \left(\frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x)) \right) h'(x) dx \\
&= - \int_a^b \frac{d}{dx} \left(\frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x)) \right) h'(x) dx \\
&= - \left[\frac{d}{dx} \left(\frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x)) \right) h(x) \right]_a^b \\
&\quad + \int_a^b \frac{d^2}{dx^2} \left(\frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x)) \right) h(x) dx \\
&= \int_a^b \frac{d^2}{dx^2} \left(\frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x)) \right) h(x) dx.
\end{aligned}$$

As f has continuous partial derivatives of the third order, by Lemma 1.2.3, we have that for all $x \in [a, b]$

$$\begin{aligned}
&\frac{\partial}{\partial y} f(x, y(x), y'(x), y''(x)) - \frac{d}{dx} \left(\frac{\partial}{\partial y'} f(x, y(x), y'(x), y''(x)) \right) \\
&+ \frac{d^2}{dx^2} \left(\frac{\partial}{\partial y''} f(x, y(x), y'(x), y''(x)) \right) = 0.
\end{aligned}$$

This concludes the proof.

□

We can write the Euler–Lagrange equation (1.12) in a more concise way, as

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) + \frac{d^2}{dx^2} \left(\frac{\partial f}{\partial y''} \right) = 0.$$

Definition 1.2.8 *The solutions* y *of* (1.12) *are called extremals for the functional defined in* (1.11).

Particular Cases for Functionals Containing Second-Order Derivatives

Now we will also analyse three cases where the Euler–Lagrange equation (1.12) is simplified. We suppose that the functional given by

$$J(y) = \int_a^b f(x, y(x), y'(x), y''(x))dx$$

satisfies the conditions of Theorem 1.2.8.

1. **First case:** y does not appear explicitly in the integrand.
   In this case the functional is of the form

   $$J(y) = \int_a^b f(x, y'(x), y''(x))dx$$

   and the Euler–Lagrange equation (1.12) is

   $$\begin{aligned} & -\frac{d}{dx}\left(\frac{\partial f}{\partial y'}(x, y'(x), y''(x))\right) + \frac{d^2}{dx^2}\left(\frac{\partial f}{\partial y''}(x, y'(x), y''(x))\right) = 0 \\ \Rightarrow & -\frac{\partial f}{\partial y'}(x, y'(x), y''(x)) + \frac{d}{dx}\left(\frac{\partial f}{\partial y''}(x, y'(x), y''(x))\right) = c, \end{aligned}$$

   where c is a constant of integration.

2. **Second case:** The independent variable x does not appear explicitly in the integrand (autonomous case).
   In this case the functional is of the form

   $$J(y) = \int_a^b f(y(x), y'(x), y''(x))dx.$$

   Theorem 1.2.9 *Let* J *be a functional such that*

   $$J(y) = \int_a^b f(y(x), y'(x), y''(x))dx \tag{1.13}$$

   and define G *by*

   $$G(y, y', y'') = y''\frac{\partial f}{\partial y''} - y'\left(\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}\right) - f.$$

   Then, $G(y(x), y'(x), y''(x))$ *is constant along any extremal* y *of* (1.13).

Proof: Suppose that y is an extremal for J. So, by Definition 1.2.8

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0$$

for all $x \in [a, b]$. Thus, we get

$$\begin{aligned}
\frac{dG}{dx}(y, y', y'') &= \frac{d}{dx}\left(y''\frac{\partial f}{\partial y''} - y'\left(\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}\right) - f\right) \\
&= y'''\frac{\partial f}{\partial y''} + y''\frac{d}{dx}\frac{\partial f}{\partial y''} - y''\left(\frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}\right) \\
&\quad - y'\left(\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) - \frac{d}{dx}f \\
&= y'''\frac{\partial f}{\partial y''} + y''\frac{\partial f}{\partial y'} - y'\frac{d^2}{dx^2}\frac{\partial f}{\partial y''} + y'\frac{d}{dx}\frac{\partial f}{\partial y'} \\
&\quad - y'\frac{\partial f}{\partial y} - y''\frac{\partial f}{\partial y'} - y'''\frac{\partial f}{\partial y''} \\
&= - y'\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''}\right) \\
&= - y' \times 0 \\
&= 0.
\end{aligned}$$

Therefore, $G(y(x), y'(x), y''(x))$ is constant along any extremal y of (1.13).

3. **Third case (a degenerate case)**: the integrand is linear in y''. Suppose that J is a functional of the form

$$J(y) = \int_a^b A(x, y(x), y'(x))y''(x) + B(x, y(x), y'(x))dx, \qquad (1.14)$$

where A and B are smooth functions of x, y and y'. The Euler–Lagrange equation (1.12) associated to (1.14) is

$$\begin{aligned}
&\frac{d^2}{dx^2}A(x, y, y') - \frac{d}{dx}\left(y''\frac{\partial A}{\partial y'} + \frac{\partial B}{\partial y'}\right) + y''\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y} = 0 \\
\Rightarrow\ &\frac{d}{dx}\left(\frac{\partial A}{\partial x} + y'\frac{\partial A}{\partial y} + y''\frac{\partial A}{\partial y'}\right) - \left(y'''\frac{\partial A}{\partial y'} + y''\frac{d}{dx}\frac{\partial A}{\partial y'} + \frac{d}{dx}\frac{\partial B}{\partial y'}\right) \\
&+ y''\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y} = 0 \\
\Rightarrow\ &\frac{d}{dx}\frac{\partial A}{\partial x} + y''\frac{\partial A}{\partial y} + y'\frac{d}{dx}\frac{\partial A}{\partial y} + y'''\frac{\partial A}{\partial y'} + y''\frac{d}{dx}\frac{\partial A}{\partial y'} - y'''\frac{\partial A}{\partial y'} \\
&- y''\frac{d}{dx}\frac{\partial A}{\partial y'} - \frac{d}{dx}\frac{\partial B}{\partial y'} + y''\frac{\partial A}{\partial y} + \frac{\partial B}{\partial y} = 0
\end{aligned}$$

$$
\begin{aligned}
&\Rightarrow \frac{d}{dx}\frac{\partial A}{\partial x} + 2y''\frac{\partial A}{\partial y} + y'\frac{d}{dx}\frac{\partial A}{\partial y} - \frac{d}{dx}\frac{\partial B}{\partial y'} + \frac{\partial B}{\partial y} = 0 \\
&\Rightarrow \frac{\partial^2 A}{\partial x^2} + y'\frac{\partial^2 A}{\partial y \partial x} + y''\frac{\partial^2 A}{\partial y' \partial x} + 2y''\frac{\partial A}{\partial y} + y'\left(\frac{\partial^2 A}{\partial x \partial y} + y'\frac{\partial^2 A}{\partial y^2} + y''\frac{\partial^2 A}{\partial y' \partial y}\right) \\
&\quad - \frac{\partial^2 B}{\partial x \partial y'} - y'\frac{\partial^2 B}{\partial y \partial y'} - y''\frac{\partial^2 B}{\partial y'^2} + \frac{\partial B}{\partial y} = 0 \\
&\Rightarrow y''\left(\frac{\partial^2 A}{\partial y' \partial x} + 2\frac{\partial A}{\partial y} + y'\frac{\partial^2 A}{\partial y' \partial y} - \frac{\partial^2 B}{\partial y'^2}\right) \\
&\quad + y'\left(\frac{\partial^2 A}{\partial y \partial x} + \frac{\partial^2 A}{\partial x \partial y} + y'\frac{\partial^2 A}{\partial y^2} - \frac{\partial^2 B}{\partial y \partial y'}\right) \\
&\quad + \left(\frac{\partial^2 A}{\partial x^2} - \frac{\partial^2 B}{\partial x \partial y'} + \frac{\partial B}{\partial y}\right) = 0.
\end{aligned}
$$

As the functions A and B depend only on x, y and y', the coefficients of y'' and y' and the other terms of the previous equation depend only on x, y and y'. So, in this case the Euler–Lagrange equation is a differential equation of at most second-order.

Remark 1.2.5 A differential equation of second order usually has two arbitrary constants of integration. The problem defined in Theorem 1.2.8 has four boundary conditions. This means that the necessary condition of optimality usually leads to a impossible problem.

VPFE for Functionals Containing Derivatives of Order n

Analogously, we can obtain similar results when the functional J contains derivatives of order $n \in \mathbb{N}$. By mathematical induction, we can prove the following formula of integration by parts:

$$
\begin{aligned}
&\int_a^b t(x)h^{(n)}(x)dx \\
&= \left[\sum_{i=0}^{n-1}(-1)^i t^{(i)}(x)h^{(n-i-1)}(x)\right]_a^b + \int_a^b (-1)^n t^{(n)}(x)h(x)dx, \qquad (1.15)
\end{aligned}
$$

where $t, h : C^n[a,b] \longrightarrow \mathbb{R}$ are two functions. With Equation (1.15), we prove easily the following theorem.

Theorem 1.2.10 *Let S be the set defined by*

$$S = \{y \in C^{2n}[a,b] : y^{(m)}(a) = y_a^{(m)} \text{ and } y^{(m)}(b) = y_b^{(m)} \text{ for } 0 \le m \le n-1\}$$

and $J : S \to \mathbb{R}$ be a functional of the form

$$J(y) = \int_a^b f(x, y(x), y'(x), \ldots, y^{(n)}(x))dx,$$

where $y_a^{(m)}$ *and* $y_b^{(m)}$ *for* $m = 0, \ldots, n-1$ *are given real numbers and* f *has continuous partial derivatives of the* $(n+1)$*th order with respect to* x*,* y*,* $y', \ldots, y^{(n)}$*. If* $y \in S$ *is an extremizer for* J*, then*

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \ldots + (-1)^n \frac{d^n}{dx^n}\frac{\partial f}{\partial y^{(n)}} = 0$$

for all $x \in [a, b]$*.*

VPFE for Functionals Containing Several Dependent Variables

In this section we will derive the Euler–Lagrange equations for the fixed variational problem where the functional depends on several dependent variables and one independent variable. Consider that $\boldsymbol{y} = (y_1, \ldots, y_k)$ and $\boldsymbol{y}' = (y_1', \ldots, y_k')$, where $k \in \mathbb{N}$. Let $\boldsymbol{C}_k^2[a, b]$ be the set defined by

$$\boldsymbol{C}_k^2[a, b] = \{(y_1, \ldots, y_k) : y_1, \ldots, y_k \in C^2[a, b]\}.$$

Theorem 1.2.11 *Let* S *be the set defined by*

$$S = \{\boldsymbol{y} \in \boldsymbol{C}_k^2[a, b] : \boldsymbol{y}(a) = \boldsymbol{y}_a \text{ and } \boldsymbol{y}(b) = \boldsymbol{y}_b\}$$

and $J : S \longrightarrow \mathbb{R}$ *be a functional of the form*

$$J(\boldsymbol{y}) = \int_a^b f(x, \boldsymbol{y}(x), \boldsymbol{y}'(x))dx, \tag{1.16}$$

where $\boldsymbol{y}_a$ *and* $\boldsymbol{y}_b$ *are given vectors,* f *is a function that has continuous partial derivatives of the second order with respect to* x*,* y_i *and* y_i' *for* $i = 1, \ldots, k$*. If* $\boldsymbol{y} \in S$ *is an extremizer for* J*, then*

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \ \forall i = 1, \ldots, k. \tag{1.17}$$

Proof: By definition of S the set of variations H is defined by

$$H = \{\boldsymbol{h} \in \boldsymbol{C}_k^2[a, b] : \boldsymbol{h}(a) = \boldsymbol{h}(b) = \boldsymbol{0}\}.$$

Note that $\boldsymbol{h} = (h_1, \ldots, h_k)$, where $h_i \in C^2[a, b]$ for $i = 1, \ldots, k$. Suppose that $\boldsymbol{y}$ is an extremizer for J. Also here, we can consider the variations $\boldsymbol{y} + \epsilon\boldsymbol{h}$, where $|\epsilon| \ll 1$ and $\boldsymbol{h} \in H$.

Let j be the function defined by

$$j(\epsilon) = J(\boldsymbol{y} + \epsilon\boldsymbol{h}) = \int_a^b f(x, \boldsymbol{y}(x) + \epsilon\boldsymbol{h}(x), \boldsymbol{y}'(x) + \epsilon\boldsymbol{h}'(x))dx.$$

As $\boldsymbol{y} \in S$ is an extremizer of J, then $\epsilon = 0$ is an extremizer for j. Consequently, $j'(0) = 0$. Computing, we obtain

$$\begin{aligned}
& j'(0) = 0 \\
\Leftrightarrow\ & \frac{d}{d\epsilon} \int_a^b f(x, \boldsymbol{y} + \epsilon \boldsymbol{h}, \boldsymbol{y}' + \epsilon \boldsymbol{y}')\bigg|_{\epsilon=0} dx = 0 \\
\Leftrightarrow\ & \int_a^b \frac{d}{d\epsilon} f(x, \boldsymbol{y} + \epsilon \boldsymbol{h}, \boldsymbol{y}' + \epsilon \boldsymbol{h}')\bigg|_{\epsilon=0} dx = 0 \\
\Leftrightarrow\ & \int_a^b \sum_{i=1}^k \left(\frac{\partial f}{\partial y_i} h_i + \frac{\partial f}{\partial y_i'} h_i' \right) dx = 0, \ \forall\ \boldsymbol{h} \in H. \qquad (1.18)
\end{aligned}$$

The above equation is more complicated than those previously studied, but good choices of functions $\boldsymbol{h} \in H$ can simplify it as we will see. For $i = 1, \ldots, k$ let H_i be the set of functions in H defined by

$$H_i = \{\boldsymbol{h} \in H : h_j = 0 \text{ if } j \neq i\}.$$

If the above equation is checked for all $\boldsymbol{h} \in H$, then it is also satisfied for all $\boldsymbol{h} \in H_i$, with $i = 1, \ldots, k$. Thus, by Equation (1.18) we have that

$$\int_a^b \left(\frac{\partial f}{\partial y_i} h_i + \frac{\partial f}{\partial y_i'} h_i' \right) dx = 0, \ \forall i = 1, \ldots, k.$$

If the above equation is checked for all $\boldsymbol{h} \in H_i$, then it is also satisfied for all $\boldsymbol{h} \in H_i$ such that $h_i(a) = h_i(b) = 0$. So, we have that

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y_i'} = 0, \ \forall i = 1, \ldots, k.$$

This concludes the proof.

□

Observations:

1. In general, the above condition is a system of k second-order differential equations for the k unknown functions $y_1, \ldots, y_k$.

2. If $\boldsymbol{y}$ satisfies the above system, then Equation (1.18) is verified for all $\boldsymbol{h} \in H$.

Definition 1.2.9 *The solutions* $\boldsymbol{y}$ *of* (1.17) *are called extremals for the functional defined in* (1.16).

Particular Cases for Several Dependent Variables

Here, we will also analyse three cases where the Euler–Lagrange equation (1.17) is simplified. We suppose that the functional given by

$$J(\boldsymbol{y}) = \int_a^b f(x, \boldsymbol{y}(x), \boldsymbol{y}'(x))dx$$

satisfies the conditions of Theorem 1.2.11.

1. **First case:** $\boldsymbol{y}$ does not appear explicitly in the integrand.
   In this case the functional is of the form

   $$J(\boldsymbol{y}) = \int_a^b f(x, \boldsymbol{y}'(x))dx.$$

   So, writing the Euler–Lagrange equation (1.17) we have:

   $$\begin{aligned} & \frac{d}{dx}\frac{\partial f}{\partial y_i'}(x, \boldsymbol{y}'(x)) = 0, \ \forall i = 1, \ldots, k \\ \Leftrightarrow \ & \frac{\partial f}{\partial y_i'}(x, \boldsymbol{y}'(x)) = c_i, \ \forall i = 1, \ldots, k, \end{aligned}$$

   where c_i is a constant of integration for all $i = 1, \ldots, k$.

2. **Second case:** The independent variable x does not appear explicitly in the integrand (autonomous case).
   In this case the functional is of the form

   $$J(\boldsymbol{y}) = \int_a^b f(\boldsymbol{y}(x), \boldsymbol{y}'(x))dx.$$

   Theorem 1.2.12 *Let J be a functional such that*

   $$J(\boldsymbol{y}) = \int_a^b f(\boldsymbol{y}(x), \boldsymbol{y}'(x))dx \tag{1.19}$$

   and define the function G by

   $$G(\boldsymbol{y}, \boldsymbol{y}') = \sum_{i=1}^{k} y_i' \frac{\partial f}{\partial y_i'} - f.$$

   Then, $G(\boldsymbol{y}(x), \boldsymbol{y}'(x))$ is constant along any extremal $\boldsymbol{y}$ of (1.19).

   Proof: Suppose that $\boldsymbol{y}$ is an extremal for J. So, by Definition 1.2.9 we know that

   $$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \ \forall i = 1, \ldots, k.$$

Thus,

$$\begin{aligned}
\frac{dG}{dx}(\boldsymbol{y}, \boldsymbol{y}') &= \frac{d}{dx}\left(\sum_{i=1}^{k} y_i' \frac{\partial f}{\partial y_i'} - f\right) = \sum_{i=1}^{k}\left(y_i'' \frac{\partial f}{\partial y_i'} + y_i' \frac{d}{dx}\frac{\partial f}{\partial y_i'}\right) - \frac{df}{dx} \\
&= \sum_{i=1}^{k}\left(y_i'' \frac{\partial f}{\partial y_i'} + y_i' \frac{d}{dx}\frac{\partial f}{\partial y_i'} - y_i' \frac{\partial f}{\partial y_i} - y_i'' \frac{\partial f}{\partial y_i'}\right) \\
&= \sum_{i=1}^{k} y_i' \left(\frac{d}{dx}\frac{\partial f}{\partial y_i'} - \frac{\partial f}{\partial y_i}\right) = \sum_{i=1}^{k}(y_i' \times 0) = 0.
\end{aligned}$$

So, $G(\boldsymbol{y}(x), \boldsymbol{y}'(x))$ is constant along any extremal $\boldsymbol{y}$ of (1.19).

□

3. **Third case (a degenerate case)**: Let $F = F(x, \boldsymbol{y})$ be any smooth function and let M be a function defined by

$$M(x, \boldsymbol{y}, \boldsymbol{y}') = \sum_{i=1}^{k} A_i(x, \boldsymbol{y}) y_i' + B(x, \boldsymbol{y}),$$

where $A_i(x, \boldsymbol{y}) = \frac{\partial F}{\partial y_i}$ and $B(x, \boldsymbol{y}) = \frac{\partial F}{\partial x}$. We will verify that the Euler–Lagrange equations (1.17) for the functional

$$J(\boldsymbol{y}) = \int_a^b M(x, \boldsymbol{y}(x), \boldsymbol{y}'(x)) dx$$

are satisfied for any smooth function $\boldsymbol{y}$. For $j = 1, \ldots, k$

$$\begin{aligned}
&\frac{\partial M}{\partial y_j} - \frac{d}{dx}\frac{\partial M}{\partial y_j'} \\
&= \left(\sum_{i=1}^{k} y_i' \frac{\partial^2 F}{\partial y_j \partial y_i} + \frac{\partial^2 F}{\partial y_j \partial x}\right) - \frac{d}{dx}\frac{\partial F}{\partial y_j} \\
&= \sum_{i=1}^{k} y_i' \frac{\partial^2 F}{\partial y_j \partial y_i} + \frac{\partial^2 F}{\partial y_j \partial x} - \frac{\partial^2 F}{\partial x \partial y_j} - \sum_{i=1}^{k} y_i' \frac{\partial^2 F}{\partial y_i \partial y_j} \\
&= 0,
\end{aligned}$$

because F is a smooth function. Therefore, the Euler–Lagrange equations (1.17) for the functional $J(\boldsymbol{y})$ are satisfied for any smooth function $\boldsymbol{y}$. We have just proved the following result.

Theorem 1.2.13 *Consider the problem that consists in finding* $\boldsymbol{y} \in S$ *(see Theorem 1.2.11) that extremizes*

$$J(\boldsymbol{y}) = \int_a^b \left(\sum_{i=1}^{k} A_i(x, \boldsymbol{y}(x)) y_i'(x) + B(x, \boldsymbol{y}(x))\right) dx. \tag{1.20}$$

Let $F(x, \boldsymbol{y}) = \int B(x, \boldsymbol{y})dx$. If $\frac{\partial F}{\partial y_i} = A_i(x, \boldsymbol{y})$, then any $\boldsymbol{y} \in S$ is an extremal of (1.20).

1.3 The Isoperimetric Problem

In this section we will study the isoperimetric problem (IP) given by

$$\begin{aligned}
(IP) \quad \max \quad & J(y) = \int_a^b f(x, y(x), y'(x))dx \\
\text{s.t.} \quad & y(a) = y_a \\
& y(b) = y_b \\
& I(y) = \int_a^b g(x, y(x), y'(x))dx = L, \qquad (1.21)
\end{aligned}$$

where $J, I : C^2[a, b] \longrightarrow \mathbb{R}$ are functionals, f and g are two smooth functions of x, y and y', y_a and y_b are fixed reals and L is a specified constant. Conditions like (1.21) are called **Isoperimetric Constraints**. We intend to derive a necessary condition for a smooth function to be a solution of (IP). Recall the following theorem [41, p. 77].

Theorem 1.3.1 (Lagrange Multiplier Rule) *Let $\Omega \subset \mathbb{R}^n$ be a region and let $f, g : \Omega \longrightarrow \mathbb{R}$ be two smooth functions. If f has a local extremum at $\overline{\boldsymbol{x}} \in \Omega$ subject to the condition $g(\boldsymbol{x}) = 0$ for $\boldsymbol{x} \in \Omega$ and if $\nabla g(\overline{\boldsymbol{x}}) \neq \boldsymbol{0}$, then there is a number λ such that*

$$\nabla(f(\overline{\boldsymbol{x}}) - \lambda g(\overline{\boldsymbol{x}})) = \boldsymbol{0}.$$

Theorem 1.3.2 *Suppose that $y \in C^2[a, b]$ is a solution of the problem (IP). Then, there exists $(\lambda_0, \lambda_1) \neq (0, 0)$ such that*

$$\frac{\partial K}{\partial y} - \frac{d}{dx}\frac{\partial K}{\partial y'} = 0, \qquad (1.22)$$

where $K = \lambda_0 f - \lambda_1 g$.

1. *If y is not an extremal for I, then we can take $\lambda_0 = 1$.*

2. *If y is an extremal for I, then we can take $\lambda_0 = 0$ and $\lambda_1 = 1$.*

Proof: Let $y \in C^2[a, b]$ be a solution of the problem (IP). Consider the variations $y + \epsilon_1 h_1 + \epsilon_2 h_2$, where $|\epsilon_1| \ll 1$, $|\epsilon_2| \ll 1$, $h_1, h_2 \in C^2[a, b]$ and $h_m(a) = h_m(b) = 0$ for $m = 1, 2$. For a fixed choice of h_1 and h_2 we can regard $J(y+\epsilon_1 h_1+\epsilon_2 h_2)$ and $I(y+\epsilon_1 h_1+\epsilon_2 h_2)$ as functions of ϵ_1 and ϵ_2. So we consider

$$j(\epsilon_1,\epsilon_2)=J(y+\epsilon_1 h_1+\epsilon_2 h_2)$$

and

$$i(\epsilon_1,\epsilon_2)=I(y+\epsilon_1 h_1+\epsilon_2 h_2).$$

Thus, we can convert the problem (IP) to a finite-dimensional constrained optimization problem (IP') given by

$$\begin{aligned}(IP')\quad \max\quad & j(\epsilon_1,\epsilon_2)=\int_a^b f(x,y+\epsilon_1 h_1+\epsilon_2 h_2,y'+\epsilon_1 h_1'+\epsilon_2 h_2')dx\\ \text{s.t.}\quad & i(\epsilon_1,\epsilon_2)-L=0.\end{aligned}$$

As y is a solution of (IP), so $(\epsilon_1,\epsilon_2)=(0,0)$ is a solution of (IP'). Therefore, $i(0,0)=L$.

Suppose that y is not an extremal for I. Since

$$\frac{\partial i}{\partial \epsilon_m}(0,0)=\int_a^b\left(\frac{\partial g}{\partial y}-\frac{d}{dx}\frac{\partial g}{\partial y'}\right)h_m dx,\quad \text{for } m=1,2,$$

and y is not an extremal for I, then, without loss of generality, there exists h_2 such that $\frac{\partial i}{\partial \epsilon_2}(0,0)\neq 0$. So $\nabla i(0,0)\neq(0,0)$. Consider the function $\widehat{g}:\Lambda\to\mathbb{R}$ defined in the neighbourhood $\Lambda\subseteq\mathbb{R}^2$ of the point $(0,0)$ by

$$\widehat{g}(\epsilon_1,\epsilon_2)=i(\epsilon_1,\epsilon_2)-L.$$

We know that $\widehat{g}(0,0)=i(0,0)-L=0$, $\widehat{g}$ is differentiable with respect to ϵ_2 and $\frac{\partial\widehat{g}}{\partial\epsilon_2}$ is continuous in Λ, because g is smooth. As $\frac{\partial\widehat{g}}{\partial\epsilon_2}(0,0)\neq 0$, we have that, by the Implicit Function Theorem [41, p. 266–267], there are the neighbourhoods I_{ϵ_1} of $\epsilon_1=0$ and I_{ϵ_2} of $\epsilon_2=0$ and the function $\phi:I_{\epsilon_1}\to\mathbb{R}$ such that

1. $\phi(0)=0$,
2. For all $\epsilon_1\in I_{\epsilon_1}$ we have that $(\epsilon_1,\phi(\epsilon_1))\in\Lambda$ and $\widehat{g}(\epsilon_1,\phi(\epsilon_1))=0$.

Therefore, we can write ϵ_2 as a function of ϵ_1, that is, $\epsilon_2=\phi(\epsilon_1)$ and we can assert that there is a subfamily of variations that satisfies the isoperimetric constraint. Concluding, the function h_1 can be regarded as arbitrary, but the term $\epsilon_2 h_2$ can be viewed as a "correction term", that is, the term $\epsilon_2 h_2$ ensures that $y+\epsilon_1 h_1+\epsilon_2 h_2$ satisfies the isoperimetric condition (1.21). Therefore, h_2 is not arbitrary.

By Theorem 1.3.1, as j and i are smooth functions, $(\epsilon_1,\epsilon_2)=(0,0)$ is a solution of (IP') and $\nabla i(0,0)\neq(0,0)$, we know that there is a constant λ_1 such that

$$\begin{aligned}&\nabla(j(0,0)-\lambda_1 i(0,0))=(0,0) \qquad (1.23)\\ \Rightarrow\ &\frac{\partial}{\partial\epsilon_m}\left(j(\epsilon_1,\epsilon_2)-\lambda_1 i(\epsilon_1,\epsilon_2)\right)\Big|_{(\epsilon_1,\epsilon_2)=(0,0)}=0 \text{ for } m=1,2.\end{aligned}$$

Note that

$$\frac{\partial j}{\partial \epsilon_m}(\epsilon_1, \epsilon_2)\bigg|_{(\epsilon_1,\epsilon_2)=(0,0)} = \int_a^b \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) h_m \; dx \text{ for } m = 1, 2$$

and

$$\frac{\partial i}{\partial \epsilon_m}(\epsilon_1, \epsilon_2)\bigg|_{(\epsilon_1,\epsilon_2)=(0,0)} = \int_a^b \left(\frac{\partial g}{\partial y} - \frac{d}{dx}\frac{\partial g}{\partial y'}\right) h_m \; dx \text{ for } m = 1, 2.$$

So, by Equation (1.23) we have that

$$\int_a^b \left(\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'}\right) h_1 dx = 0,$$

where $F = f - \lambda_1 g$ ($\lambda_0 = 1$). As h_1 is arbitrary, we know by Lemma 1.2.3 that

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\frac{\partial F}{\partial y'} = 0$$

for any extremizer y of (IP).

Now suppose that y is an extremal for I. This case is obvious, because for $\lambda_0 = 0$ and $\lambda_1 = 1$ we obtain $K = -g$. As y is an extremal for I, we have that Equation (1.22) is satisfied.

□

Example 1.3.1 Let us verify that there is a function that is an extremal of the isoperimetric problem given by

$$\begin{aligned} \max \quad & J(y) = \int_0^1 y^2(x)(y')^2(x) \; dx \\ \text{s.t.} \quad & y(0) = 1 \\ & y(1) = 2 \\ & I(y) = \int_0^1 y^2(x) \; dx = \frac{7}{3}. \end{aligned}$$

If y isn't an extremal for I, then $K = y^2(x)(y')^2(x) - \lambda_1 y^2(x)$ and the Euler–Lagrange equation (1.22) is given by

$$y^2(x)y''(x) = -y(x)(y')^2(x) - \lambda_1 y(x).$$

Using the software Maple, the solution of this equation is

$$y(x) = 0 \vee y(x) = -\sqrt{-\lambda_1 x^2 - 2c_1 x + 2c_2} \vee y(x) = \sqrt{-\lambda_1 x^2 - 2c_1 x + 2c_2}.$$

The functions $y(x) = 0$ and $y(x) = -\sqrt{-\lambda_1 x^2 - 2c_1 x + 2c_2}$ are not admissible, since $y(0) > 0$. For $y(x) = \sqrt{-\lambda_1 x^2 - 2c_1 x + 2c_2}$, we have that

$$\begin{cases} y(0) = 1 \\ y(1) = 2 \\ I(y) = \frac{7}{3} \end{cases} \Rightarrow \begin{cases} c_1 = -1 \\ c_2 = \frac{1}{2} \\ \lambda_1 = -1. \end{cases}$$

This system has a real solution. So, the function $y(x) = \sqrt{x^2 + 2x + 1}$ is an extremal for the isoperimetric problem.

Now consider that y is an extremal for I. Then $y(x) = 0$ and we had already seen that this function is not a solution for the problem.

Concluding, the unique extremal of the isoperimetric problem is

$$y(x) = \sqrt{x^2 + 2x + 1} = \sqrt{(x+1)^2} = x + 1$$

for $x \in [0, 1]$.

1.4 Variational Problem with a Variable Endpoint

Throughout the text to refer a variational problem with a variable endpoint we will just write VPVE.

1.4.1 Natural Boundary Conditions

In the previous section we studied variational problems with fixed endpoints, that is, our goal was to determine the extremizers for a functional $J : C^{2n}[a, b] \longrightarrow \mathbb{R}$ given by

$$J(y) = \int_a^b f(x, y(x), y'(x), \ldots, y^{(n)}(x))dx$$

subject to given boundary conditions, for $n \in \mathbb{N}$. These conditions take the form $y^{(m)}(a) = y_a^{(m)}$ and $y^{(m)}(b) = y_b^{(m)}$ for $m = 0, \ldots, n-1$, where $y_a^{(m)}$ and $y_b^{(m)}$ are known real numbers. However, there are variational problems for which we don't know all of these boundary conditions, in other words, sometimes $y_a^{(m)}$, or $y_b^{(m)}$, is unknown for some $m = 0, \ldots, n-1$. When this happens our objective is to find the extremizers for a given functional and also to determine the unknown boundary conditions that extremizers satisfy. We will see that the methods of calculus of variations always supply exactly the right number of boundary conditions, even if no boundary conditions are imposed. There are essentially two types of boundary conditions:

1. The boundary conditions that are imposed on the problem.

2. The boundary conditions that arise from the variational process (natural boundary conditions).

Natural Boundary Conditions for $n = 1$

Consider the functional $J : C^2[a, b] \longrightarrow \mathbb{R}$ given by

$$J(y) = \int_a^b f(x, y(x), y'(x))dx,$$

where f is a smooth function. Now, no boundary conditions are imposed on y and we want to determine the extremizers $y \in C^2[a, b]$ for J. Then, we derive a necessary condition for J to have an extremum at y. Suppose that y is an extremizer for J and consider again the variations $y + \epsilon h$, where $|\epsilon| \ll 1$ and $h \in C^2[a, b]$. As no boundary conditions are imposed we don't require that $h(a) = 0$ and $h(b) = 0$. Therefore, for all $h \in C^2[a, b]$

$$\delta J(h, y) = \int_a^b \left(\frac{\partial f}{\partial y} h + \frac{\partial f}{\partial y'} h' \right) dx = 0$$
$$\Leftrightarrow \int_a^b \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) h \; dx + \left[\frac{\partial f}{\partial y'} h \right]_a^b = 0. \tag{1.24}$$

For the variational problem with fixed endpoints the term $\left[\frac{\partial f}{\partial y'} h\right]_a^b$ vanished, because $h(a) = h(b) = 0$. Nevertheless, in the present variational problem this term doesn't vanish for all $h \in C^2[a, b]$. So, Equation (1.24) is valid for all $h \in C^2[a, b]$. In particular, it is valid for all $h \in C^2[a, b]$ such that $h(a) = h(b) = 0$. Thus, by previously study of VPFE for $n = 1$ we know that

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} = 0 \tag{1.25}$$

for any y at which J has an extremum. Combining the conditions (1.24) and (1.25), we obtain

$$\left[\frac{\partial f}{\partial y'} h \right]_a^b = 0, \; \forall h \in C^2[a, b]. \tag{1.26}$$

Thus, we can always find functions $h \in C^2[a, b]$ such that

1. $h(a) = 0$ and $h(b) \neq 0$. So,

$$\left[\frac{\partial f}{\partial y'} \right]_b = 0. \tag{1.27}$$

2. $h(a) \neq 0$ and $h(b) = 0$. So,

$$\left[\frac{\partial f}{\partial y'} \right]_a = 0. \tag{1.28}$$

Concluding, if J has an extremizer $y \in C^2[a,b]$ and no boundary conditions are imposed, then y must satisfy the Euler–Lagrange equation (1.25) along with Equations (1.27) and (1.28), because the Equation (1.26) is satisfied for all $h \in C^2[a,b]$. The Equations (1.27) and (1.28) are evaluated at b and a, respectively. Therefore, they are boundary conditions. These conditions are not imposed, they arise in the variational process. They are the natural boundary conditions.

The process, previously studied, is completely methodical, because:

1. If boundary conditions are imposed at a and at b, then the variational formulation requires $h(a) = h(b) = 0$. Therefore, there are none natural boundary condition.

2. If only one boundary condition is imposed at a, then the variational formulation requires $h(a) = 0$, but $h(b)$ is free. So, the problem is supplemented by the natural boundary condition (1.27).

3. If only one boundary condition is imposed at b, then the variational formulation requires $h(b) = 0$, but $h(a)$ is free. So, the problem is supplemented by the natural boundary condition (1.28).

4. If no boundary conditions are imposed, then we have both natural boundary conditions (1.27) and (1.28).

Natural Boundary Conditions for $n = 2$

Consider the functional $J : C^4[a,b] \longrightarrow \mathbb{R}$ given by

$$J(y) = \int_a^b f(x, y(x), y'(x), y''(x))dx,$$

where f is a smooth function. Our goal is the same as the case $n = 1$. What differs is that here the integrand also depends on y''. By Section 1.2.3, if y is an extremizer for J, we have that for all $h \in C^4[a,b]$

$$\delta J(h,y) = \int_a^b \left(\frac{\partial f}{\partial y} h + \frac{\partial f}{\partial y'} h' + \frac{\partial f}{\partial y''} h'' \right) dx = 0$$

$$\Leftrightarrow \int_a^b \left(\frac{\partial f}{\partial y} h - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) h - \frac{d}{dx}\left(\frac{\partial f}{\partial y''} \right) h' \right) dx + \left[\frac{\partial f}{\partial y''} h' \right]_a^b + \left[\frac{\partial f}{\partial y'} h \right]_a^b = 0$$

$$\Leftrightarrow \int_a^b \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} \right) h \; dx + \left[\frac{\partial f}{\partial y''} h' \right]_a^b$$

$$+ \left[\left(\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''} \right) h \right]_a^b = 0. \tag{1.29}$$

If Equation (1.29) is satisfied for all $h \in C^4[a,b]$, then it is also satisfied for all $h \in C^4[a,b]$ such that $h(a) = h(b) = h'(a) = h'(b) = 0$. So, we have that

$$\int_a^b \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} \right) h \; dx = 0$$

$$\Leftrightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0 \tag{1.30}$$

for any y at which J has an extremum. Combining the conditions (1.29) and (1.30), we obtain

$$\left[\frac{\partial f}{\partial y''} h' + \left(\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''} \right) h \right]_a^b = 0, \;\; \forall h \in C^4[a,b].$$

Thus, we can always find functions $h \in C^4[a,b]$ such that

1. $h(a) \neq 0$, but $h(b) = h'(a) = h'(b) = 0$. So,
$$\left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''} \right]_a = 0.$$

2. $h(b) \neq 0$, but $h(a) = h'(a) = h'(b) = 0$. So,
$$\left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''} \right]_b = 0.$$

3. $h'(a) \neq 0$, but $h(a) = h(b) = h'(b) = 0$. So,
$$\left[\frac{\partial f}{\partial y''} \right]_a = 0.$$

4. $h'(b) \neq 0$, but $h(a) = h(b) = h'(a) = 0$. So,
$$\left[\frac{\partial f}{\partial y''} \right]_b = 0.$$

If $y(a)$, $y(b)$, $y'(a)$ and $y'(b)$ are fixed, we do not obtain any natural boundary condition. If one of them is free, then we obtain a correspondent supplementary condition. For example, if $y(a)$ is free, we obtain

$$\left[\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''} \right]_a = 0.$$

Natural Boundary Conditions for Several Dependent Variables

Consider the functional $J : \boldsymbol{C}_k^2[a,b] \longrightarrow \mathbb{R}$ given by

$$J(\boldsymbol{y}) = \int_a^b f(x, \boldsymbol{y}(x), \boldsymbol{y}'(x))dx,$$

where f is a smooth function. If no boundary conditions are imposed on $\boldsymbol{y}$ at a and at b, then it is not required that $\boldsymbol{h}(a) = (0,\ldots,0)$ and that $\boldsymbol{h}(b) = (0,\ldots,0)$. So, for all $\boldsymbol{h} \in \boldsymbol{C}_k^2[a,b]$ we have that

$$\delta J(\boldsymbol{h}, \boldsymbol{y}) = \int_a^b \sum_{i=1}^k \left(\frac{\partial f}{\partial y_i} h_i + \frac{\partial f}{\partial y_i'} h_i' \right) dx = 0.$$

In particular, the above equation is satisfied for all $\boldsymbol{h} \in H_i$, where H_i is defined by

$$H_i = \{\boldsymbol{h} \in \boldsymbol{C}_k^2[a,b] : h_j = 0 \text{ if } j \neq i\}.$$

Therefore, for all $i = 1,\ldots,k$ and for all $\boldsymbol{h} \in H_i$ we have that

$$\int_a^b \left(\frac{\partial f}{\partial y_i} h_i + \frac{\partial f}{\partial y_i'} h_i' \right) dx = 0.$$

Integrating by parts, we obtain

$$\int_a^b \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} \right) h_i \; dx + \left[\frac{\partial f}{\partial y_i'} h_i \right]_a^b = 0. \tag{1.31}$$

If the above equation is satisfied for all $\boldsymbol{h} \in H_i$, then it is also satisfied for all $\boldsymbol{h} \in H_i$ such that $h_i(a) = h_i(b) = 0$. Therefore,

$$\int_a^b \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} \right) h_i \; dx = 0, \;\; \forall i = 1,\ldots,k.$$

Thus,

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx}\frac{\partial f}{\partial y_i'} = 0, \;\; \forall i = 1,\ldots,k \tag{1.32}$$

for any $\boldsymbol{y}$ at which J has an extremum. Combining the conditions (1.31) and (1.32), we obtain

$$\left[\frac{\partial f}{\partial y_i'} h_i \right]_a^b = 0, \;\; \forall i = 1,\ldots,k \text{ and } \forall \boldsymbol{h} \in H_i.$$

So, we can always find functions $\boldsymbol{h}$ such that

1. $h_i(a) = 0$ and $h_i(b) \neq 0$. Then,

$$\left[\frac{\partial f}{\partial y_i'} \right]_b = 0, \;\; \forall i = 1,\ldots,k.$$

2. $h_i(a) \neq 0$ and $h_i(b) = 0$. Then,

$$\left[\frac{\partial f}{\partial y_i'}\right]_a = 0, \ \forall i = 1, \ldots, k.$$

The conclusions that can be drawn are similar to those we have discussed previously.

1.4.2 The General Case

Previously, we assumed that the endpoint b of the integral was fixed. Now we consider a more general case where the endpoint of the integral is a variable of the problem. We intend to solve the following problem:

$$\begin{aligned}(P_{GC}) \quad \max \quad & J(\overline{x}, y) = \int_a^{\overline{x}} f(x, y(x), y'(x))dx \\ \text{s.t.} \quad & y(a) = y_a,\end{aligned}$$

where $(\overline{x}, y) \in \,]a, b] \times C^2[a, b]$. We use the approach of Chiang in [12]. As $\overline{x}$ and $y(\overline{x})$ are free, their achievement will be obtained through the variational process, in other words, now we intend to determine y, $y(\overline{x})$ and $\overline{x}$ that maximize the above problem (P_{GC}). We will only study this case, but the procedure is easily extended to the case where the initial point of the integral $\overline{x}_a$ is a variable of problem and $\overline{x}_a \in [a, b[$.

Theorem 1.4.1 *Let $(\overline{x}, y)$ be a solution of the problem (P_{GC}). Then, for all $x \in [a, \overline{x}]$, the solution $(\overline{x}, y)$ satisfies*

1. $[f]_{\overline{x}} = 0$,
2. $\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y, y') = 0$ *(Euler–Lagrange equation),*
3. $\left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0$ *(natural boundary condition).*

Proof: Let us consider the variations $\overline{x} + \epsilon\Delta x$ and $y + \epsilon h$, where $|\epsilon| \ll 1$, $\Delta x \in \mathbb{R}$ and $h \in C^2[a, b]$. Note that $h(a) = 0$, because y_a is given. Let j be the function defined by

$$j(\epsilon) = \int_a^{\overline{x}+\epsilon\Delta x} f(x, y + \epsilon h, y' + \epsilon h')dx.$$

As $(\overline{x}, y)$ is a solution of (P_{GC}), we know that $j'(0) = 0$. Recall that, if $\widehat{f}$ is continuous and g is differentiable, then the function

$$\widehat{j}(\epsilon) = \int_a^{g(\epsilon)} \widehat{f}(x, \epsilon)dx$$

is differentiable and

$$\widehat{j}'(\epsilon) = g'(\epsilon)\widehat{f}(g(\epsilon), \epsilon) + \int_a^{g(\epsilon)} \frac{\partial \widehat{f}}{\partial \epsilon}(x, \epsilon)dx.$$

Therefore,

$$\begin{aligned} j'(\epsilon) &= \Delta x f(\overline{x} + \epsilon\Delta x, y + \epsilon h, y' + \epsilon h') \\ &\quad + \int_a^{\overline{x}+\epsilon\Delta x} \frac{\partial f}{\partial \epsilon}(x, y + \epsilon h, y' + \epsilon h')dx \\ \\ &= \Delta x f(\overline{x} + \epsilon\Delta x, y + \epsilon h, y' + \epsilon h') \\ &\quad + \int_a^{\overline{x}+\epsilon\Delta x} \frac{\partial f}{\partial y}(x, y + \epsilon h, y' + \epsilon h')h(x)dx \\ &\quad + \int_a^{\overline{x}+\epsilon\Delta x} \frac{\partial f}{\partial y'}(x, y + \epsilon h, y' + \epsilon h')h'(x)dx. \end{aligned}$$

When $\epsilon = 0$ we have that

$$\begin{aligned} j'(0) &= \Delta x f(\overline{x}, y, y') + \int_a^{\overline{x}} \frac{\partial f}{\partial y}(x, y, y')h(x) + \frac{\partial f}{\partial y'}(x, y, y')h'(x)dx \\ &= \Delta x[f]_{\overline{x}} + \int_a^{\overline{x}} \left(\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y, y')\right) h(x)dx + \left[\frac{\partial f}{\partial y'}h\right]_{\overline{x}} \\ &= 0. \end{aligned}$$

Thus,

$$\Delta x[f]_{\overline{x}} + \int_a^{\overline{x}} \left(\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y, y')\right) h(x)dx \ + \left[\frac{\partial f}{\partial y'}h\right]_{\overline{x}} = 0. \tag{1.33}$$

As Δx and h are arbitrary, we can conclude that for all $x \in [a, \overline{x}]$

$$\Delta x[f]_{\overline{x}} = 0,$$

$$\int_a^{\overline{x}} \left(\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y, y')\right) h(x)dx = 0$$

and

$$\left[\frac{\partial f}{\partial y'}h\right]_{\overline{x}} = 0.$$

This implies that

$$[f]_{\overline{x}} = 0,$$

$$\frac{\partial f}{\partial y}(x, y, y') - \frac{d}{dx}\frac{\partial f}{\partial y'}(x, y, y') = 0 \tag{1.34}$$

and

$$\left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0.$$

This concludes the proof.

□

Note that the Euler–Lagrange equation is also a necessary condition when the endpoint of the integral is a variable of the problem.

With the Figure 1.6 we can understand better the total variation in y.

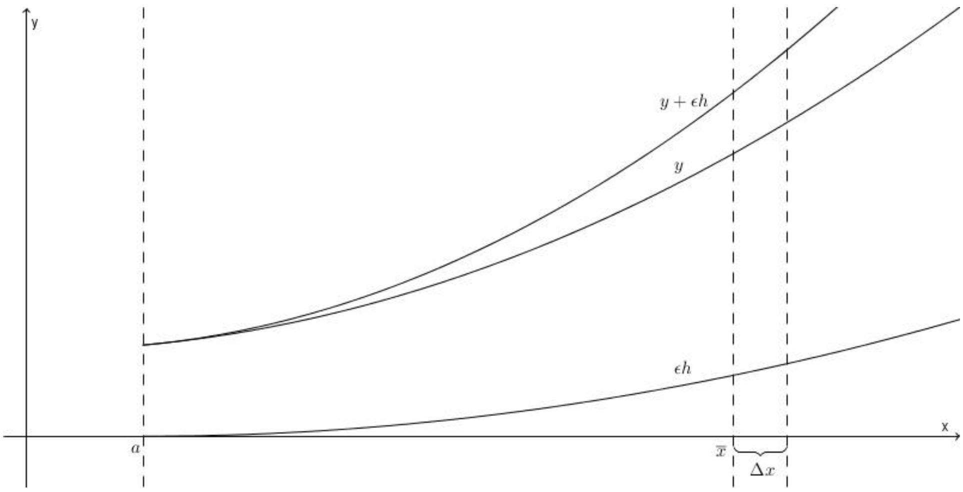

Figure 1.6: Free end-time problem

From now on we will assume that $\epsilon = 1$. As we can observe on Figure 1.6, the total variation in y is given by

$$\Delta y = (y + h)(\overline{x} + \Delta x) - y(\overline{x}). \tag{1.35}$$

By Taylor's Theorem [41, p. 262–264], we know that

$$(y + h)(\overline{x} + \Delta x) - (y + h)(\overline{x}) = (y' + h')(\overline{x})\Delta x + O(\Delta x^2).$$

As h is arbitrary we choose $h'(\overline{x}) = 0$. So,

$$\begin{gathered}(y + h)(\overline{x} + \Delta x) - (y + h)(\overline{x}) = y'(\overline{x})\Delta x + O(\Delta x^2)\\ \Leftrightarrow h(\overline{x}) = (y + h)(\overline{x} + \Delta x) - y(\overline{x}) - y'(\overline{x})\Delta x + O(\Delta x^2).\end{gathered}$$

By Equation (1.35) we have that

$$h(\overline{x}) = \Delta y - y'(\overline{x})\Delta x + O(\Delta x^2).$$

Therefore, replacing $h(\overline{x})$ by $h(\overline{x}) = \Delta y - y'(\overline{x})\Delta x + O(\Delta x^2)$ on the Equation (1.33) and using Equation (1.34) we obtain

$$\begin{aligned}
&\Delta x[f]_{\overline{x}} + \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} \left(\Delta y - y'(\overline{x})\Delta x + O(\Delta x^2)\right) = 0 \\
\Leftrightarrow\ &\Delta x \left[f - \frac{\partial f}{\partial y'} y'\right]_{\overline{x}} + \Delta y \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} + O(\Delta x^2) = 0 \\
\Rightarrow\ &\Delta x \left[f - \frac{\partial f}{\partial y'} y'\right]_{\overline{x}} + \Delta y \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0. \qquad (1.36)
\end{aligned}$$

So, we can conclude that

$$\left[f - \frac{\partial f}{\partial y'} y'\right]_{\overline{x}} = 0$$

and

$$\left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0.$$

Unlike the Euler–Lagrange equation, the above equations are relevant only on the point $x = \overline{x}$. These equations bridge the gap caused by the missing boundary condition (in this case for the terminal point). The Equation (1.36) is the **general transversality condition** and, depending on the specific conditions of each problem, it can be written in various forms. Let's see the following cases.

Specialized Transversality Conditions

1. Vertical Terminal Line:

   Suppose that $\overline{x}$ is fixed and $y(\overline{x})$ is arbitrary. So, there are no changes in $\overline{x}$ which implies that $\Delta x = 0$. Therefore, the general transversality condition (1.36) is simplified to

   $$\Delta y \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0. \qquad (1.37)$$

   As Δy is arbitrary, we must have

   $$\left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0$$

   to satisfy Equation (1.37). Note that the above equation is a natural boundary condition, as we already studied (cf. (1.27)).

2. Horizontal Terminal Line:

   Supposed that $\overline{x}$ is free and that $y(\overline{x})$ is fixed. Consequently, we have that Δx is arbitrary and that $\Delta y = 0$. So, the general transversality condition (1.36) is simplified and it is given by

$$\Delta x \left[f - \frac{\partial f}{\partial y'} y' \right]_{\overline{x}} = 0. \tag{1.38}$$

   Similarly, as Δx is arbitrary, we must have

$$\left[f - \frac{\partial f}{\partial y'} y' \right]_{\overline{x}} = 0 \tag{1.39}$$

   to check the Equation (1.38).

3. Terminal Curve:

   Now let $\Delta x \neq 0$ and $\Delta y \neq 0$, simultaneously. We only know that $y(\overline{x}) = \phi(\overline{x})$, where ϕ is a given curve. As $y(\overline{x}) = \phi(\overline{x})$ and as Δx is a small arbitrary number, we conclude that

$$\Delta y = \phi'(\overline{x})\Delta x + O(\Delta x^2).$$

   Thus,

$$\begin{aligned} \Delta x \left[f - \frac{\partial f}{\partial y'} y' \right]_{\overline{x}} + \phi'(\overline{x})\Delta x \left[\frac{\partial f}{\partial y'} \right]_{\overline{x}} = 0 \\ \Leftrightarrow \Delta x \left[f + \frac{\partial f}{\partial y'} \left(\phi' - y' \right) \right]_{\overline{x}} = 0. \end{aligned}$$

   Because Δx is arbitrary, we must have

$$\left[f + \frac{\partial f}{\partial y'} \left(\phi' - y' \right) \right]_{\overline{x}} = 0.$$

   The above equation is another transversality condition. In the two previous cases, or we didn't know $y(\overline{x})$, or we didn't know $\overline{x}$. Here we don't know the two, simultaneously, and so we have to determine both. Thus, we need to know two conditions that are:

$$\left[f + \frac{\partial f}{\partial y'} \left(\phi' - y' \right) \right]_{\overline{x}} = 0 \tag{1.40}$$

   and

$$y(\overline{x}) = \phi(\overline{x}). \tag{1.41}$$

4. Truncated Vertical Terminal Line:

Now we consider that $\overline{x}$ is fixed and $y(\overline{x}) \geq y_{min}$, where y_{min} is a minimum permissible level for the vertical axis. So, we can analyse two possibilities: $y(\overline{x}) > y_{min}$, or $y(\overline{x}) = y_{min}$.

(a) If we suppose that $y(\overline{x}) > y_{min}$, then Δy is arbitrary. So, we have that

$$\left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0.$$

(b) If we suppose that $y(\overline{x}) = y_{min}$, then $\Delta y \geq 0$ and, consequently, Δy is not completely arbitrary. Assuming that $h(\overline{x}) > 0$, we have

$$\begin{cases} h(\overline{x}) > 0 \\ (y + \epsilon h)(\overline{x}) \geq y(\overline{x}) \end{cases} \quad \Rightarrow \quad \epsilon \geq 0.$$

Note that

$$j'(0) = \lim_{\epsilon \longrightarrow 0} \frac{j(\epsilon) - j(0)}{\epsilon}.$$

For a maximization problem, $j(\epsilon) - j(0) \leq 0$ and we know that $\epsilon \geq 0$. So, we can conclude that $j'(0) \leq 0$. Thus, $\Delta y \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} \leq 0$. As $\Delta y \geq 0$, we have that

$$\left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} \leq 0.$$

So, the transversality condition for a maximization problem can be obtained as follows:

$$\begin{cases} \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0 \text{ for } y(\overline{x}) > y_{min} \\ \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} \leq 0 \text{ for } y(\overline{x}) = y_{min} \end{cases} \quad \Rightarrow \begin{cases} \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} \leq 0 \\ y(\overline{x}) \geq y_{min} \\ (y(\overline{x}) - y_{min}) \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0. \end{cases}$$

Similarly, if the problem is to minimize J, then we have the following transversality condition:

$$\begin{cases} \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} \geq 0 \\ y(\overline{x}) \geq y_{min} \\ (y(\overline{x}) - y_{min}) \left[\frac{\partial f}{\partial y'}\right]_{\overline{x}} = 0. \end{cases}$$

5. Truncated Horizontal Terminal Line:

   Now we consider that $y(\overline{x})$ is fixed and $\overline{x} \leq x_{max} (\leq b)$, where x_{max} is a maximum permissible level for the horizontal axis. So, we can also analyse two possibilities: $\overline{x} < x_{max}$ and $\overline{x} = x_{max}$. Analogously, the transversality condition for a maximization problem is given by

$$\begin{cases} \left[f - \frac{\partial f}{\partial y'} y'\right]_{\overline{x}} \geq 0 \\ \overline{x} \leq x_{max} \\ (\overline{x} - x_{max}) \left[f - \frac{\partial f}{\partial y'} y'\right]_{\overline{x}} = 0 \end{cases}$$

   and for a minimization problem is given by

$$\begin{cases} \left[f - \frac{\partial f}{\partial y'} y'\right]_{\overline{x}} \leq 0 \\ \overline{x} \leq x_{max} \\ (\overline{x} - x_{max}) \left[f - \frac{\partial f}{\partial y'} y'\right]_{\overline{x}} = 0. \end{cases}$$

Example 1.4.1 (Horizontal Terminal Line) [12, p. 67–69] For the problem

$$\begin{aligned} \max \quad & J(y) = \int_0^{\overline{x}} xy'(x) + (y'(x))^2 \, dx \\ \text{s.t.} \quad & y(0) = 1 \\ & y(\overline{x}) = 10 \end{aligned}$$

the Lagrangian is

$$f(x, y, y') = xy' + (y')^2$$

and the Euler–Lagrange equation (1.5) gives

$$y''(x) = -\frac{1}{2}.$$

So,

$$y(x) = -\frac{x^2}{4} + c_1 x + c_2,$$

where c_1 and c_2 are real constants. As $y(0) = 1$, we have that $c_2 = 1$. By the transversality condition (1.39), we have that

$$[xy' + (y')^2 - (x + 2y')y']_{\overline{x}} = 0 \Leftrightarrow [y']_{\overline{x}} = 0 \Leftrightarrow c_1 = \frac{\overline{x}}{2}.$$

Thus, as $c_1 = \frac{\overline{x}}{2}$, $c_2 = 1$ and $y(\overline{x}) = 10$, we have that

$$y(\overline{x}) = -\frac{\overline{x}^2}{4} + \frac{\overline{x}^2}{2} + 1 = 10 \Leftrightarrow \overline{x} = 6.$$

So, the function $y(x) = -\frac{x^2}{4} + 3x + 1$ is the extremal (a candidate for maximizer) for the given problem.

Example 1.4.2 (Terminal Curve) [12, p. 66–67] For the problem

$$\begin{aligned} \max \quad & J(y) = \int_0^{\overline{x}} \sqrt{1 + (y'(x))^2}\, dx \\ \text{s.t.} \quad & y(0) = 1 \\ & y(\overline{x}) = 2 - \overline{x} \end{aligned}$$

the Lagrangian is

$$f(x, y, y') = \sqrt{1 + (y')^2},$$

the curve ϕ is given by $\phi(x) = 2 - x$ and the Euler–Lagrange equation (1.5) gives

$$-(y'(x))^2 y''(x)(1 + (y'(x))^2)^{-\frac{3}{2}} + y''(x)(1 + (y'(x))^2)^{-\frac{1}{2}} = 0.$$

So,

$$y(x) = c_1 x + c_2,$$

where c_1 and c_2 are real constants. As $y(0) = 1$, we have that $c_2 = 1$. By the condition (1.40), we have that

$$\begin{aligned} & [\sqrt{1 + (y')^2} + (1 + (y')^2)^{-\frac{1}{2}} y'(-1 - y')]_{\overline{x}} = 0 \\ \Leftrightarrow\ & [1 + (y')^2 + y'(-1 - y')]_{\overline{x}} = 0 \\ \Leftrightarrow\ & c_1 = 1. \end{aligned}$$

Thus, the function $y(x) = x + 1$ is the extremal (a candidate for maximizer) for the given problem. Furthermore, by the condition (1.41), we have that

$$y(\overline{x}) = \overline{x} + 1 = 2 - \overline{x} \Leftrightarrow \overline{x} = \frac{1}{2}.$$

Consequently, we obtain that

$$y(\overline{x}) = \frac{3}{2}.$$

Chapter 2

The Optimal Control

2.1 Introduction

The Optimal Control theory is an extension of the Calculus of Variations as we will see later. This branch of mathematics is recent. In the beginning of the Cold War (1945–1991) the USA and the USSR gave great importance to mathematicians and their theories to develop defence techniques, because this area had been recognized as advantageous during Second World War (1939–1945). Therefore, several mathematicians developed solution methods for problems which nowadays are considered as problems of Optimal Control. An example of this, are the minimum time interception problems for fighter aircraft.

So, the conventional wisdom asserts that the Optimal Control was born about 60 years ago due to the *Pontryagin Maximum Principle* carried out by Lev Semenovich Pontryagin (1908–1988), a Russian mathematician, and his group.

In this chapter we are going to study a basic problem of Optimal Control with free and bounded control and to establish the connection between the Calculus of Variations and the Optimal Control. We can see several approaches to Optimal Control, for example, in [6, 7, 24, 38].

2.2 The Basic Problem of Optimal Control

Consider the following definitions of piecewise continuous function and of piecewise differentiable function.

Definition 2.2.1 (Piecewise Continuous Function) *Let $I \subseteq \mathbb{R}$ be an interval (finite, or infinite). We say that $y : I \to \mathbb{R}$ is a piecewise continuous function if y is continuous at each $x \in I$, with the possible exception of a finite number of points $\widehat{x}$ of I, and if*

$$y(\widehat{x}) = \lim_{x \longrightarrow \widehat{x}^+} y(x),$$

or

$$y(\widehat{x}) = \lim_{x \longrightarrow \widehat{x}^-} y(x).$$

We write $y \in PC(I, \mathbb{R})$.

Definition 2.2.2 (Piecewise Differentiable Function) *Let* $y : I \to \mathbb{R}$ *be a continuous function in* I *and differentiable at each* $x \in I$, *with the possible exception of a finite number of points of* I. *Furthermore, suppose that* y' *is continuous whenever it is defined. Then, we say that* y *is a piecewise differentiable function. We write* $y \in PC^1(I, \mathbb{R})$.

Remark 2.2.1 Consider the functions $y_i : I \to \mathbb{R}$ for $i = 1, \ldots, k$, where $k \in \mathbb{N}$. Note that $\boldsymbol{y} = (y_1, \ldots, y_k)$. When

- $y_i \in PC(I, \mathbb{R})$ for all $i = 1, \ldots, k$, we write $\boldsymbol{y} \in PC(I, \mathbb{R}^k)$.
- $y_i \in PC^1(I, \mathbb{R})$ for all $i = 1, \ldots, k$, we write $\boldsymbol{y} \in PC^1(I, \mathbb{R}^k)$.

Problem Statement: The basic problem of Optimal Control consists of finding a pair $(\boldsymbol{y}, \boldsymbol{u})$ that solves the following problem (P_{OC})

$$\begin{aligned}(P_{OC}) \quad \max \quad & J(\boldsymbol{y}, \boldsymbol{u}) = \int_a^b f(x, \boldsymbol{y}(x), \boldsymbol{u}(x))dx \\ \text{s.t.} \quad & \boldsymbol{y}'(x) = g(x, \boldsymbol{y}(x), \boldsymbol{u}(x)), \ \forall\ x \in]a, b[\\ & \boldsymbol{y}(a) = \boldsymbol{y}_a,\end{aligned}$$

where a, $b \in \mathbb{R}$ such that $a < b$, $f \in C^1([a, b] \times \mathbb{R}^{k+m}, \mathbb{R})$, $g \in C^1([a, b] \times \mathbb{R}^{k+m}, \mathbb{R}^k)$, $\boldsymbol{y} \in PC^1([a, b], \mathbb{R}^k)$ and $\boldsymbol{u} \in PC([a, b], \mathbb{R}^m)$ with $k, m \in \mathbb{N}$. The vector $\boldsymbol{u}(x) = (u_1(x), \ldots, u_m(x)) \in \mathbb{R}^m$ is called the control (or controller) and $\boldsymbol{y}(x) = (y_1(x), \ldots, y_k(x)) \in \mathbb{R}^k$ is the state.

As in the Calculus of Variations, we will derive necessary conditions for the pair $(\boldsymbol{y}, \boldsymbol{u})$ to be a solution of the problem (P_{OC}).

Theorem 2.2.1 (The Pontryagin Maximum Principle for (P_{OC})) *If* $(\boldsymbol{y}, \boldsymbol{u})$ *is an optimal pair for the problem* (P_{OC}), *then there exists* $\boldsymbol{\lambda} \in PC^1([a, b], \mathbb{R}^k)$ *such that*

1. $\boldsymbol{\lambda}(b) = \boldsymbol{0}$ *(Transversality Condition),*
2. $\boldsymbol{y}'(x) = \frac{\partial H}{\partial \boldsymbol{\lambda}}(x, \boldsymbol{y}(x), \boldsymbol{u}(x), \boldsymbol{\lambda}(x))$ *(Control System),*
3. $\boldsymbol{\lambda}'(x) = -\frac{\partial H}{\partial \boldsymbol{y}}(x, \boldsymbol{y}(x), \boldsymbol{u}(x), \boldsymbol{\lambda}(x))$ *(Adjoint Equation),*
4. $\frac{\partial H}{\partial \boldsymbol{u}}(x, \boldsymbol{y}(x), \boldsymbol{u}(x), \boldsymbol{\lambda}(x)) = \boldsymbol{0}$ *(Optimality Condition),*

where $H(x, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\lambda}) = f(x, \boldsymbol{y}, \boldsymbol{u}) + \boldsymbol{\lambda} \cdot g(x, \boldsymbol{y}, \boldsymbol{u})$ *is the Hamiltonian.*

Remark 2.2.2 The Adjoint System is formed by the second and the third items of Theorem 2.2.1, i.e., by the Control System and by the Adjoint Equation.

Proof: Suppose that $(\boldsymbol{y}, \boldsymbol{u})$ is an optimal pair for the problem (P_{OC}). Let us consider the variations $\boldsymbol{u}^\epsilon = \boldsymbol{u} + \epsilon \boldsymbol{h}$, where $\boldsymbol{h} = (h_1, \ldots, h_m) \in PC([a,b], \mathbb{R}^m)$ and $|\epsilon| \ll 1$. So, $\boldsymbol{u}^\epsilon \in PC([a,b], \mathbb{R}^m)$. Note that

1. $\lim\limits_{\epsilon \longrightarrow 0} \boldsymbol{u}^\epsilon(x) = \boldsymbol{u}(x), \ \forall x \in [a,b]$.
2. $\frac{\partial \boldsymbol{u}^\epsilon}{\partial \epsilon}(x)\Big|_{\epsilon=0} = \boldsymbol{h}(x), \ \forall x \in [a,b]$.

Let $\boldsymbol{y}^\epsilon(x)$ be the state variable corresponding to the control $\boldsymbol{u}^\epsilon(x)$. By formulation of the problem (P_{OC}), we know that

1. $\frac{d\boldsymbol{y}^\epsilon}{dx}(x) = \left(\frac{dy_1^\epsilon}{dx}(x), \ldots, \frac{dy_k^\epsilon}{dx}(x)\right) = g(x, \boldsymbol{y}^\epsilon(x), \boldsymbol{u}^\epsilon(x))$.
2. $\boldsymbol{y}^\epsilon(a) = \boldsymbol{y}_a$.

We have that $\lim\limits_{\epsilon \longrightarrow 0} \boldsymbol{y}^\epsilon(x) = \boldsymbol{y}(x)$ and that there is the derivative

$$\frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon}(x) = \left(\frac{\partial y_1^\epsilon}{\partial \epsilon}(x), \ldots, \frac{\partial y_k^\epsilon}{\partial \epsilon}(x)\right)$$

for all $x \in [a,b]$.

Consider a function $\boldsymbol{\lambda} \in PC^1([a,b], \mathbb{R}^k)$. By the Fundamental Theorem of Integral Calculus we have that

$$\int_a^b \frac{d}{dx}(\boldsymbol{\lambda}(x) \cdot \boldsymbol{y}^\epsilon(x))dx = [\boldsymbol{\lambda}(x) \cdot \boldsymbol{y}^\epsilon(x)]_a^b.$$

So,

$$\int_a^b \frac{d}{dx}(\boldsymbol{\lambda}(x) \cdot \boldsymbol{y}^\epsilon(x))dx + [\boldsymbol{\lambda}(x) \cdot \boldsymbol{y}^\epsilon(x)]_b^a = 0.$$

Thus, we have that

$$\begin{aligned}
J(\boldsymbol{y}^\epsilon, \boldsymbol{u}^\epsilon) &= \int_a^b f(x, \boldsymbol{y}^\epsilon(x), \boldsymbol{u}^\epsilon(x))dx \\
&= \int_a^b \left(f(x, \boldsymbol{y}^\epsilon(x), \boldsymbol{u}^\epsilon(x)) + \frac{d}{dx}(\boldsymbol{\lambda}(x) \cdot \boldsymbol{y}^\epsilon(x))\right) dx \\
&\quad + [\boldsymbol{\lambda}(x) \cdot \boldsymbol{y}^\epsilon(x)]_b^a = 0 \\
&= \int_a^b f(x, \boldsymbol{y}^\epsilon(x), \boldsymbol{u}^\epsilon(x))dx \\
&\quad + \int_a^b (\boldsymbol{\lambda}'(x) \cdot \boldsymbol{y}^\epsilon(x) + \boldsymbol{\lambda}(x) \cdot g(x, \boldsymbol{y}^\epsilon(x), \boldsymbol{u}^\epsilon(x)))dx \\
&\quad + [\boldsymbol{\lambda}(x) \cdot \boldsymbol{y}^\epsilon(x)]_b^a = 0.
\end{aligned}$$

To simplify notation, let A be a matrix $k \times m$, $v = (v_1, \ldots, v_m)$ and $w = (w_1, \ldots, w_k)$. By Av (wA) we mean the vector obtained as product of the matrices Av $(w^T A)$. As $(\boldsymbol{y}, \boldsymbol{u})$ is a solution for the problem (P_{OC}), we know that

$$
\begin{aligned}
0 &= \frac{d}{d\epsilon} J(\boldsymbol{y}^\epsilon, \boldsymbol{u}^\epsilon)\bigg|_{\epsilon=0} \\
&= \int_a^b \left(\frac{\partial f}{\partial \boldsymbol{y}} \cdot \frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon} + \frac{\partial f}{\partial \boldsymbol{u}} \cdot \frac{\partial \boldsymbol{u}^\epsilon}{\partial \epsilon} \right)\bigg|_{\epsilon=0} dx \\
&+ \int_a^b \left(\boldsymbol{\lambda}' \cdot \frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon} + \boldsymbol{\lambda} \left[\frac{\partial g}{\partial \boldsymbol{y}} \frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon} + \frac{\partial g}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{u}^\epsilon}{\partial \epsilon} \right] \right)\bigg|_{\epsilon=0} dx \\
&- \boldsymbol{\lambda}(b) \cdot \frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon}(b)\bigg|_{\epsilon=0} \\
&= \int_a^b \left(\frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon} \cdot \left[\frac{\partial f}{\partial \boldsymbol{y}} + \boldsymbol{\lambda}' \right] + \frac{\partial f}{\partial \boldsymbol{u}} \cdot \frac{\partial \boldsymbol{u}^\epsilon}{\partial \epsilon} \right)\bigg|_{\epsilon=0} dx \\
&+ \int_a^b \boldsymbol{\lambda} \left(\frac{\partial g}{\partial \boldsymbol{y}} \frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon} + \frac{\partial g}{\partial \boldsymbol{u}} \frac{\partial \boldsymbol{u}^\epsilon}{\partial \epsilon} \right)\bigg|_{\epsilon=0} dx - \boldsymbol{\lambda}(b) \frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon}(b)\bigg|_{\epsilon=0} \\
&= \int_a^b \left(\frac{\partial f}{\partial \boldsymbol{y}} + \boldsymbol{\lambda}' + \boldsymbol{\lambda} \frac{\partial g}{\partial \boldsymbol{y}} \right) \cdot \frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon}\bigg|_{\epsilon=0} dx \\
&+ \int_a^b \left(\frac{\partial f}{\partial \boldsymbol{u}} + \boldsymbol{\lambda} \frac{\partial g}{\partial \boldsymbol{u}} \right) \cdot \boldsymbol{h}\bigg|_{\epsilon=0} dx - \boldsymbol{\lambda}(b) \frac{\partial \boldsymbol{y}^\epsilon}{\partial \epsilon}(b)\bigg|_{\epsilon=0}, \qquad (2.1)
\end{aligned}
$$

because

$$\frac{\partial \boldsymbol{u}^\epsilon}{\partial \epsilon} = \boldsymbol{h}.$$

For each $i \in \{1, \ldots, k\}$ we are going to choose $\lambda_i(x)$ so that the coefficients of $\frac{\partial y_i^\epsilon}{\partial \epsilon}\Big|_{\epsilon=0}$ are equal zero. So, for all $x \in [a, b]$, $\boldsymbol{\lambda}(x)$ should satisfy the following conditions:

- $\boldsymbol{\lambda}'(x) = -\left(\frac{\partial f}{\partial \boldsymbol{y}} + \boldsymbol{\lambda}(x) \frac{\partial g}{\partial \boldsymbol{y}} \right) = -\frac{\partial H}{\partial \boldsymbol{y}}(x, \boldsymbol{y}(x), \boldsymbol{u}(x), \boldsymbol{\lambda}(x))$.
- $\boldsymbol{\lambda}(b) = \mathbf{0}$.

Now we only need to prove the Optimality Condition. From (2.1) we get

$$\int_a^b \left(\frac{\partial f}{\partial \boldsymbol{u}} + \boldsymbol{\lambda} \frac{\partial g}{\partial \boldsymbol{u}} \right) \cdot \boldsymbol{h}\bigg|_{\epsilon=0} dx = 0.$$

As the above equation is valid for all $\boldsymbol{h} \in PC([a,b],\mathbb{R}^m)$, it holds in particular for $\boldsymbol{h}$ such that

$$\boldsymbol{h} = \frac{\partial f}{\partial \boldsymbol{u}} + \boldsymbol{\lambda}\frac{\partial g}{\partial \boldsymbol{u}}.$$

Thus, we have that

$$\begin{aligned}
&\int_a^b \left(\frac{\partial f}{\partial \boldsymbol{u}} + \boldsymbol{\lambda}\frac{\partial g}{\partial \boldsymbol{u}}\right)^2\Bigg|_{\epsilon=0} dx = 0 \\
\Leftrightarrow\ & \left(\frac{\partial f}{\partial \boldsymbol{u}} + \boldsymbol{\lambda}\frac{\partial g}{\partial \boldsymbol{u}}\right)^2\Bigg|_{\epsilon=0} = 0 \\
\Leftrightarrow\ & \left(\frac{\partial f}{\partial \boldsymbol{u}} + \boldsymbol{\lambda}\frac{\partial g}{\partial \boldsymbol{u}}\right)\Bigg|_{\epsilon=0} = 0,\ \forall x \in [a,b] \\
\Leftrightarrow\ & \frac{\partial H}{\partial \boldsymbol{u}}(x, \boldsymbol{y}(x), \boldsymbol{u}(x), \boldsymbol{\lambda}(x)) = \boldsymbol{0},\ \forall x \in [a,b].
\end{aligned}$$

So, we obtain the Optimality Condition.

□

Theorem 2.2.2 *If* $\boldsymbol{y}, \boldsymbol{\lambda} \in C^1([a,b],\mathbb{R}^k)$*,* $\boldsymbol{u} \in C^1([a,b],\mathbb{R}^m)$ *and* $(\boldsymbol{y}, \boldsymbol{u}, \boldsymbol{\lambda})$ *satisfies the Theorem 2.2.1, then*

$$\frac{d}{dx}H(x, \boldsymbol{y}(x), \boldsymbol{u}(x), \boldsymbol{\lambda}(x)) = \frac{\partial H}{\partial x}(x, \boldsymbol{y}(x), \boldsymbol{u}(x), \boldsymbol{\lambda}(x)).$$

Proof: Immediate computations lead to

$$\begin{aligned}
\frac{d}{dx}H =& \frac{\partial H}{\partial x} + \sum_{i=1}^{k}\frac{\partial H}{\partial y_i}y_i' + \sum_{i=1}^{m}\frac{\partial H}{\partial u_i}u_i' + \sum_{i=1}^{k}\frac{\partial H}{\partial \lambda_i}\lambda_i' \\
=& \frac{\partial H}{\partial x} + \sum_{i=1}^{k}\frac{\partial H}{\partial y_i}\frac{\partial H}{\partial \lambda_i} - \sum_{i=1}^{k}\frac{\partial H}{\partial \lambda_i}\frac{\partial H}{\partial y_i} \\
=& \frac{\partial H}{\partial x}.
\end{aligned}$$

This concludes the proof.

□

Example 2.2.1 The Variational Problem (P_1) given by

$$\begin{aligned}
(P_1) \quad \max \quad & J(y) = \int_a^b f(x, y(x), y'(x))dx \\
\text{s.t.} \quad & y(a) = y_a
\end{aligned}$$

can be transformed into a problem (P_2) of the Optimal Control given by

$$\begin{aligned}(P_2) \quad \max \quad & J(y) = \int_a^b f(x, y(x), u(x))dx \\ \text{s.t.} \quad & y'(x) = u(x) \\ & y(a) = y_a.\end{aligned}$$

We intend to find the Euler–Lagrange equation (1.5) and the natural boundary condition (1.27) by applying Theorem 2.2.1 to (P_2). By Theorem 2.2.1 we have that $H(x, y, u, \lambda) = f(x, y, u) + \lambda u$ and

$$\begin{cases} \lambda(b) = 0 \\ \frac{\partial H}{\partial \lambda} = y'(x) \\ \frac{\partial H}{\partial y} = -\lambda'(x) \\ \frac{\partial H}{\partial u} = 0 \end{cases} \Leftrightarrow \begin{cases} - \\ u(x) = y'(x) \\ \lambda'(x) = -\frac{\partial f}{\partial y} \\ \lambda(x) = -\frac{\partial f}{\partial u} \end{cases} \Leftrightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0.$$

Note that

$$\lambda(b) = 0 \Leftrightarrow \left[\frac{\partial f}{\partial u}\right]_b = \left[\frac{\partial f}{\partial y'}\right]_b = 0,$$

which is nothing else than the natural boundary condition (1.27).

Example 2.2.2 The Variational Problem $(\widehat{P}_1)$ given by

$$\begin{aligned}(\widehat{P}_1) \quad \max \quad & J(y) = \int_a^b f(x, y(x), y'(x), y''(x), y'''(x))dx \\ \text{s.t.} \quad & y(a) = y_a \\ & y'(a) = y_a^{(1)} \\ & y''(a) = y_a^{(2)}\end{aligned}$$

can be transformed into a problem of the Optimal Control. Now we consider $\boldsymbol{y}(x) = (y^0(x), y^1(x), y^2(x))$ such that $y^m(x) = y^{(m)}(x)$ for $m = 0, 1, 2$. Thus, the equivalent problem of Optimal Control is given by

$$\begin{aligned}(\widehat{P}_2) \quad \max \quad & J(y) = \int_a^b f(x, \boldsymbol{y}(x), u(x))dx \\ \text{s.t.} \quad & \boldsymbol{y}'(x) = \begin{bmatrix} y^1(x) \\ y^2(x) \\ u(x) \end{bmatrix} \\ & y^0(a) = y_a \\ & y^1(a) = y_a^{(1)} \\ & y^2(a) = y_a^{(2)}.\end{aligned}$$

Again, by Theorem 2.2.1, we have that

$$H(x, \boldsymbol{y}, u, \boldsymbol{\lambda}) = f(x, \boldsymbol{y}, u) + \boldsymbol{\lambda} \cdot \boldsymbol{y}' = f(x, \boldsymbol{y}, u) + \sum_{i=1}^{2} \lambda_i y^i + \lambda_3 u$$

and

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial \boldsymbol{\lambda}} = \boldsymbol{y}'(x) \\ \frac{\partial H}{\partial \boldsymbol{y}} = -\boldsymbol{\lambda}'(x) \\ \frac{\partial H}{\partial u} = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} \boldsymbol{y}'(x) = \boldsymbol{y}'(x) \\ \boldsymbol{\lambda}'(x) = \begin{bmatrix} -\frac{\partial f}{\partial y^0} \\ -\frac{\partial f}{\partial y^1} - \lambda_1(x) \\ -\frac{\partial f}{\partial y^2} - \lambda_2(x) \end{bmatrix} \\ \frac{\partial f}{\partial u} + \lambda_3(x) = 0. \end{array} \right.$$

So, we have that

$$\frac{d}{dx}\frac{\partial f}{\partial u} - \frac{\partial f}{\partial y^2} - \lambda_2(x) = 0 \Leftrightarrow \lambda_2(x) = \frac{d}{dx}\frac{\partial f}{\partial u} - \frac{\partial f}{\partial y^2}.$$

As $\lambda_2'(x) = -\frac{\partial f}{\partial y^1} - \lambda_1(x)$,

$$\frac{d^2}{dx^2}\frac{\partial f}{\partial u} - \frac{d}{dx}\frac{\partial f}{\partial y^2} = -\frac{\partial f}{\partial y^1} - \lambda_1(x) \Leftrightarrow \lambda_1(x) = -\frac{d^2}{dx^2}\frac{\partial f}{\partial u} + \frac{d}{dx}\frac{\partial f}{\partial y^2} - \frac{\partial f}{\partial y^1}.$$

As $\lambda_1'(x) = -\frac{\partial f}{\partial y^0}$,

$$\frac{d^3}{dx^3}\frac{\partial f}{\partial u} - \frac{d^2}{dx^2}\frac{\partial f}{\partial y^2} + \frac{d}{dx}\frac{\partial f}{\partial y^1} - \frac{\partial f}{\partial y^0} = 0.$$

As $u(x) = (y^2)'(x) = y'''(x)$, we obtain that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} - \frac{d^3}{dx^3}\frac{\partial f}{\partial y'''} = 0. \tag{2.2}$$

Note that Equation (2.2) is the Euler–Lagrange equation for $n = 3$. By Theorem 2.2.1 we also know that

$$\boldsymbol{\lambda}(b) = \boldsymbol{0} \Leftrightarrow \begin{bmatrix} \lambda_1(b) \\ \lambda_2(b) \\ \lambda_3(b) \end{bmatrix} = \begin{bmatrix} \left[-\frac{d^2}{dx^2}\frac{\partial f}{\partial y'''} + \frac{d}{dx}\frac{\partial f}{\partial y''} - \frac{\partial f}{\partial y'}\right]_b \\ \left[\frac{d}{dx}\frac{\partial f}{\partial y'''} - \frac{\partial f}{\partial y''}\right]_b \\ \left[\frac{\partial f}{\partial y'''}\right]_b \end{bmatrix} = \boldsymbol{0}$$

(Transversality Condition).

Example 2.2.3 Consider the following problem:

$$\begin{aligned} \max \quad & \int_1^5 u(x)y(x) - u^2(x) - y^2(x)\; dx \\ \text{s.t.} \quad & y'(x) = y(x) + u(x) \\ & y(1) = 2. \end{aligned} \tag{2.3}$$

Resolution by Optimal Control: By Theorem 2.2.1, the Hamiltonian is given by

$$H(x, y, u, \lambda) = uy - u^2 - y^2 + \lambda(y + u).$$

The Optimality Condition asserts that

$$\begin{aligned} & \frac{\partial H}{\partial u}(x, y(x), u(x), \lambda(x)) = 0 \\ \Leftrightarrow\ & y(x) - 2u(x) + \lambda(x) = 0 \\ \Leftrightarrow\ & \lambda(x) = -y(x) + 2u(x). \end{aligned}$$

Using the Adjoint System,

$$\begin{cases} \frac{\partial H}{\partial \lambda}(x, y(x), u(x), \lambda(x)) = y'(x) \\ \frac{\partial H}{\partial y}(x, y(x), u(x), \lambda(x)) = -\lambda'(x) \end{cases} \Leftrightarrow \begin{cases} y(x) + u(x) = y'(x) \\ \lambda'(x) = -u(x) + 2y(x) - \lambda(x). \end{cases}$$

With the Optimality Condition and the Adjoint System we have that

$$\begin{aligned} & \lambda(x) = y'(x) - 2u'(x) - u(x) + 2y(x) \\ \Leftrightarrow & -y(x) + 2u(x) - y'(x) + 2u'(x) + u(x) - 2y(x) = 0 \\ \Leftrightarrow & -3y(x) - y'(x) + 3u(x) + 2u'(x) = 0 \\ \Leftrightarrow & -3y(x) - y'(x) + 3(y'(x) - y(x)) + 2(y''(x) - y'(x)) = 0 \\ \Leftrightarrow\ & y''(x) = 3y(x) \\ \Leftrightarrow\ & y(x) = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}, \end{aligned}$$

where c_1 and c_2 are real constants. By the Transversality Condition,

$$\begin{aligned} & \lambda(5) = 0 \Leftrightarrow -y(5) + 2u(5) = 0 \Leftrightarrow -y(5) + 2(y'(5) - y(5)) = 0 \\ \Leftrightarrow\ & y'(5) = \frac{3}{2} y(5). \end{aligned}$$

Thus, we have to solve the following system to find c_1 and c_2:

$$\begin{cases} y(1) = 2 \\ y'(5) = \frac{3}{2} y(5) \end{cases} \Leftrightarrow \begin{cases} c_1 e^{\sqrt{3}} + c_2 e^{-\sqrt{3}} = 2 \\ \sqrt{3} c_1 e^{5\sqrt{3}} - \sqrt{3} c_2 e^{-5\sqrt{3}} = \frac{3}{2}(c_1 e^{5\sqrt{3}} + c_2 e^{-5\sqrt{3}}) \end{cases}$$

$$\Leftrightarrow \begin{cases} c_1 = \frac{(4\sqrt{3}+6)e^{-5\sqrt{3}}}{e^{4\sqrt{3}}(2\sqrt{3}-3) + e^{-4\sqrt{3}}(2\sqrt{3}+3)} \\ c_2 = \frac{(4\sqrt{3}-6)e^{5\sqrt{3}}}{e^{4\sqrt{3}}(2\sqrt{3}-3) + e^{-4\sqrt{3}}(2\sqrt{3}+3)}. \end{cases}$$

Concluding,

$$y(x) = \frac{(4\sqrt{3}+6)e^{\sqrt{3}(x-5)} + (4\sqrt{3}-6)e^{-\sqrt{3}(x-5)}}{e^{4\sqrt{3}}(2\sqrt{3}-3) + e^{-4\sqrt{3}}(2\sqrt{3}+3)}. \tag{2.4}$$

Resolution by the Calculus of Variations: As $y'(x) = y(x) + u(x)$ we can write the integrand f as a function of x, y and y':

$$f(x, y, y') = (y' - y)y - (y' - y)^2 - y^2.$$

Simplifying,

$$f(x, y, y') = -3y^2 + 3yy' - (y')^2.$$

The initial optimal control problem can be transformed into the following equivalent variational problem:

$$\begin{array}{ll} \max & \displaystyle\int_1^5 -3y^2(x) + 3y(x)y'(x) - (y')^2(x)\; dx \\ \text{s.t.} & y(1) = 2. \end{array}$$

By the Euler–Lagrange equation (1.5), we obtain

$$-6y(x) + 3y'(x) - \frac{d}{dx}\big(3y(x) - 2y'(x)\big) = 0 \Leftrightarrow y''(x) = 3y(x).$$

Again,

$$y(x) = c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x},$$

where c_1 and c_2 are real constants. As $y(1) = 2$ and using the natural boundary condition (1.27)

$$\begin{cases} y(1) = 2 \\ \left[\frac{\partial f}{\partial y'}\right]_5 = 0 \end{cases} \Leftrightarrow \begin{cases} y(1) = 2 \\ y'(5) = \frac{3}{2}y(5). \end{cases}$$

So,

$$y(x) = \frac{(4\sqrt{3}+6)e^{\sqrt{3}(x-5)} + (4\sqrt{3}-6)e^{-\sqrt{3}(x-5)}}{e^{4\sqrt{3}}(2\sqrt{3}-3) + e^{-4\sqrt{3}}(2\sqrt{3}+3)}.$$

Therefore, this function $y(x)$, which is the same as (2.4), is a candidate for maximizer for the given problem (2.3).

2.3 The Optimal Control Problem with Bounded Control

Sometimes, we can find problems of Optimal Control that have a bounded control, that is, $c_i \leq u_i(x) \leq d_i$, where $c_i, d_i \in \mathbb{R}$ for all $i = 1, \ldots, m$. We are going to study these problems for $m = 1$.

Problem Statement: The Optimal Control problem with bounded control, for $m = 1$, consists of finding a pair $(\boldsymbol{y}, u)$ that solves the following problem (P_{OC_b})

$$\begin{array}{lll}(P_{OC_b}) & \max & J(\boldsymbol{y}, u) = \displaystyle\int_a^b f(x, \boldsymbol{y}(x), u(x))dx \\ & \text{s.t.} & \boldsymbol{y}'(x) = g(x, \boldsymbol{y}(x), u(x)),\ \forall\, x \in\,]a, b[\\ & & \boldsymbol{y}(a) = \boldsymbol{y}_a \\ & & u(x) \in U,\end{array}$$

where $U = [c, d] \subseteq \mathbb{R}$ and $c < d$.

In this case the Optimality Condition is changed into the Maximality Condition, as we can observe in the next theorem (see [24, p. 185–187]).

Theorem 2.3.1 (The Pontryagin Maximum Principle for (P_{OC_b})) *If $(\boldsymbol{y}, u)$ is an optimal pair for the problem (P_{OC_b}), then there exists $\boldsymbol{\lambda} \in PC^1([a, b], \mathbb{R}^k)$ such that*

1. $\boldsymbol{\lambda}(b) = \mathbf{0}$ *(Transversality Condition),*

2. $\boldsymbol{y}'(x) = \frac{\partial H}{\partial \boldsymbol{\lambda}}(x, \boldsymbol{y}(x), u(x), \boldsymbol{\lambda}(x))$ *(Control System),*

3. $\boldsymbol{\lambda}'(x) = -\frac{\partial H}{\partial \boldsymbol{y}}(x, \boldsymbol{y}(x), u(x), \boldsymbol{\lambda}(x))$ *(Adjoint Equation),*

4. $u(x)$, $x \in [a, b]$, *is the solution of the problem*

$$(P_v) \quad \max_{v \in U} \quad f(x, \boldsymbol{y}(x), v) + \boldsymbol{\lambda}(x) \cdot g(x, \boldsymbol{y}(x), v)$$

   (Maximality Condition).

Remark 2.3.1 Again,

$$H(x, \boldsymbol{y}, u, \boldsymbol{\lambda}) = f(x, \boldsymbol{y}, u) + \boldsymbol{\lambda} \cdot g(x, \boldsymbol{y}, u)$$

and the Adjoint System is formed by the second and the third items of Theorem 2.3.1, i.e., by the Control System and by the Adjoint Equation.

The proof of Theorem 2.3.1 that we present here is based on the one found in [24].

Proof: The first, the second and the third points can be obtained following the same pattern as was done in Theorem 2.2.1. Then, we only have to prove the last point.

By the Transversality Condition and by the Adjoint System, we have that

$$\frac{d}{d\epsilon}J(\boldsymbol{y}^\epsilon, u^\epsilon)\bigg|_{\epsilon=0} = \int_a^b \left(\frac{\partial f}{\partial u} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial u}\right) h\bigg|_{\epsilon=0} dx,$$

as we had already seen in the proof of Theorem 2.2.1. By Taylor's Theorem [41, p. 262–264], we know that

$$J(\boldsymbol{y}^\epsilon, u^\epsilon) - J(\boldsymbol{y}, u) = \frac{d}{d\epsilon}J(\boldsymbol{y}^\epsilon, u^\epsilon)\bigg|_{\epsilon=0} \epsilon + O(\epsilon^2).$$

As $(\boldsymbol{y}, u)$ is an optimal pair for the problem (P_{OC_b}), we have that

$$J(\boldsymbol{y}^\epsilon, u^\epsilon) - J(\boldsymbol{y}, u) \leq 0.$$

This implies that

$$\frac{d}{d\epsilon}J(\boldsymbol{y}^\epsilon, u^\epsilon)\bigg|_{\epsilon=0} \epsilon + O(\epsilon^2) = \left(\int_a^b \left(\frac{\partial f}{\partial u} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial u}\right) h\bigg|_{\epsilon=0} dx\right) \epsilon + O(\epsilon^2) \leq 0.$$

Then, we have that

$$\int_a^b \left(\frac{\partial f}{\partial u}(x, \boldsymbol{y}(x), u(x)) + \boldsymbol{\lambda}(x) \cdot \frac{\partial g}{\partial u}(x, \boldsymbol{y}(x), u(x))\right) \epsilon h(x) dx \leq 0. \tag{2.5}$$

As $(\boldsymbol{y}, u)$ is an optimal pair of (P_{OC_b}), for each $x \in [a, b]$ we have that $u(x) \in \mathbb{R}$ is a point of the optimal solution u. Observe that

a) If $u(x) = c$, then $\epsilon h(x) \geq 0$. Thus, in order to verify the condition (2.5) we must have that $\frac{\partial f}{\partial u} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial u} \leq 0$ at x.

b) If $u(x) = d$, then $\epsilon h(x) \leq 0$. Thus, in order to verify the condition (2.5) we must have that $\frac{\partial f}{\partial u} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial u} \geq 0$ at x.

c) If $c < u(x) < d$, then $\epsilon h(x) \leq 0$, or $\epsilon h(x) \geq 0$. Thus, in order to verify the condition (2.5) we must have that $\frac{\partial f}{\partial u} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial u} = 0$ at x.

Note that these conditions are obtained using similar arguments as the ones used to prove Lemma 1.2.3. The previous three items can be obtained by resolution of the problem (P_v), as we will verify. To solve the problem (P_v), by Fritz–John's Theorem [8, p. 182–184], we have that

$$\begin{cases} w_0\left(\frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}\right)-w_1\frac{\partial}{\partial v}(v-d)-w_2\frac{\partial}{\partial v}(c-v)=0\\ w_1(v-d)=0\\ w_2(c-v)=0\\ w_0,w_1,w_2\geq 0\\ (w_0,w_1,w_2)\neq(0,0,0)\end{cases}$$

$$\Leftrightarrow\begin{cases} w_0\left(\frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}\right)-w_1+w_2=0\\ w_1(v-d)=0\\ w_2(c-v)=0\\ w_0,w_1,w_2\geq 0\\ (w_0,w_1,w_2)\neq(0,0,0).\end{cases}\tag{2.6}$$

If $w_0=0$, we have that

$$\begin{cases} w_1=w_2\\ w_1(v-d)=0\\ w_2(c-v)=0\\ w_1,w_2\geq 0\\ (w_1,w_2)\neq(0,0)\end{cases}\Leftrightarrow\begin{cases} w_1=w_2\\ v=d\\ v=c\\ w_1,w_2>0\end{cases}$$

and this is a contradiction, because we must have $c<d$.

Therefore, $w_0\neq 0$. Consider that $w_1'=\frac{w_1}{w_0}$ and $w_2'=\frac{w_2}{w_0}$. With these considerations the system (2.6) is equivalent to

$$\begin{cases} \frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}-w_1'+w_2'=0\\ w_1'(v-d)=0\\ w_2'(c-v)=0\\ w_1',w_2'\geq 0.\end{cases}$$

When $v=c$, we have that

$$\begin{cases} \frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}=-w_2'\leq 0\\ w_1'=0\\ w_2'(c-v)=0\\ w_2'\geq 0\end{cases}$$

and therefore we obtain a). Furthermore, when $v = d$, we have that

$$\begin{cases} \frac{\partial f}{\partial v} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial v} = w_1' \geq 0 \\ w_1'(v - d) = 0 \\ w_2' = 0 \\ w_1' \geq 0 \end{cases}$$

and therefore we obtain b). Finally, when $c < v < d$, we have that

$$\begin{cases} \frac{\partial f}{\partial v} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial v} = 0 \\ w_1' = 0 \\ w_2' = 0 \end{cases}$$

and therefore we obtain c).

Now we are going to prove that if the items a), b) and c) are true it is possible to find w_0, w_1 and w_2 that satisfy the system (2.6).

Consider the item a), in other words, suppose that $u(x) = c$. Therefore, we have that

$$\begin{cases} w_0 \left(\frac{\partial f}{\partial v} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial v} \right) - w_1 + w_2 = 0 \\ w_1(v - d) = 0 \\ w_2(c - v) = 0 \\ w_0, w_1, w_2 \geq 0 \\ (w_0, w_1, w_2) \neq (0, 0, 0) \\ v = c \\ \frac{\partial f}{\partial v} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial v} \leq 0 \end{cases} \Leftrightarrow \begin{cases} w_0 \left(\frac{\partial f}{\partial v} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial v} \right) = -w_2 \\ w_1 = 0 \\ w_0, w_2 \geq 0 \\ (w_0, w_2) \neq (0, 0) \\ \frac{\partial f}{\partial v} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial v} \leq 0. \end{cases} \tag{2.7}$$

If $\frac{\partial f}{\partial v} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial v} = 0$, we have the following system

$$\begin{cases} w_2 = 0 \\ w_1 = 0 \\ w_0 \geq 0 \\ w_0 \neq 0 \end{cases} \Leftrightarrow \begin{cases} w_1 = w_2 = 0 \\ w_0 > 0. \end{cases}$$

On the other hand, if $\frac{\partial f}{\partial v} + \boldsymbol{\lambda} \cdot \frac{\partial g}{\partial v} < 0$, we choose w_0 and w_2 such that $w_0, w_2 > 0$. This choice satisfy the first equation of system (2.7).

Consider the item b), in other words, suppose that $u(x) = d$. Therefore, we have that

$$\begin{cases} w_0\left(\frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}\right)-w_1+w_2=0 \\ w_1(v-d)=0 \\ w_2(c-v)=0 \\ w_0,w_1,w_2\geq 0 \\ (w_0,w_1,w_2)\neq(0,0,0) \\ v=d \\ \frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}\geq 0 \end{cases} \Leftrightarrow \begin{cases} w_0\left(\frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}\right)=w_1 \\ w_2=0 \\ w_0,w_1\geq 0 \\ (w_0,w_1)\neq(0,0) \\ \frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}\geq 0. \end{cases} \tag{2.8}$$

If $\frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}=0$, we have the following system

$$\begin{cases} w_1=0 \\ w_2=0 \\ w_0\geq 0 \\ w_0\neq 0 \end{cases} \Leftrightarrow \begin{cases} w_1=w_2=0 \\ w_0>0. \end{cases}$$

On the other hand, if $\frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}>0$, we choose w_0 and w_1 such that $w_0,w_1>0$. This choice satisfy the first equation of system (2.8).

Consider the item c), in other words, suppose that $c<u(x)<d$. Therefore, we have that

$$\begin{cases} w_0\left(\frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}\right)-w_1+w_2=0 \\ w_1(v-d)=0 \\ w_2(c-v)=0 \\ w_0,w_1,w_2\geq 0 \\ (w_0,w_1,w_2)\neq(0,0,0) \\ c<v<d \\ \frac{\partial f}{\partial v}+\boldsymbol{\lambda}\cdot\frac{\partial g}{\partial v}=0 \end{cases} \Leftrightarrow \begin{cases} w_1=w_2=0 \\ w_0\geq 0 \\ w_0\neq 0. \end{cases}$$

So, we must have $w_1=w_2=0$ and $w_0>0$. Concluding, the conditions of the items a), b) and c) are equivalent to the system (2.6). Thus, the fourth item of Theorem 2.3.1 is proven.

□

An example of application of Theorem 2.3.1 is given in Chapter 3.

Chapter 3

An Application of Optimal Control

3.1 Introduction

In this chapter we are going to study an optimal control problem of Diabetes *Mellitus* that was proposed by Swan in [40]. He found an exact solution using the nonlinear algebraic Ricatti equation (see [7, p. 771], [40, p. 799–802] and [43]). We are going to find the numerical solution of this problem by two different methods and then compare with the exact solution [40]. One method uses the necessary conditions of Theorem 2.3.1 and the other discretizes the problem. However, before we are going to do a brief explanation about this disease in order to understand better the problem in study (see, e.g., [13, 16, 17, 35, 40, 42]).

3.2 Diabetes *Mellitus*

Glucose is the sugar present in the blood that comes from food. It is very important for life, because our body needs sugar to produce energy necessary for normal functioning of the organs and of the tissues. For this purpose, glucose has to be transported from the blood into the cells. This transport is usually done through a hormone called insulin. Then, this hormone is responsible for the regularization of glycemia (amount of glucose in the blood).

Insulin is produzed in the pancreas, because it has specialized cells for this. About 1–2% of the tissue of the pancreas is formed by islets of Langerhans cells. We can divide them in three distinct types of endocrine cells: alpha, beta and delta cells. The alpha cells produce the hormone glucagon, when the glycemia reaches an undesirable low level. Therefore, the function of glucagon is to cause an elevation of the glucose in the blood. On the other hand, when the level of glycemia is too high, the beta cells release their insulin in order to reduce this level. Finally, delta cells produce the hormone somatostatin that inhibits the release of glucagon, or insulin, depending on the organism needs.

Diabetes *Mellitus* is a metabolic disease characterized by an abnormal and uncontrolled increase of the glycemia. It arises when the body doesn't produce enough insulin, or when there is a resistance to the insulin produced. There are several types of diabetes *mellitus*, but the three main are diabetes *mellitus* type 1, diabetes *mellitus* type 2 and gestational diabetes.

Diabetes type 1 usually affects people under 20 years, but it can arise at any age. Although this type is less common, it is the more serious. In diabetes type 1 the increase of glycemia is caused by inability of pancreas to produce insulin. This problem appears, because the beta cells are destroyed by the immune system itself. Hence, it is an autoimmune disease. It isn't known why the immune system reacts this way, but it is believed that this behaviour is related to genetic characteristics, or to some possible infections. In this case, the patient must receive daily insulin injections to control the level of this hormone. He must also control the feed and practice exercise.

Diabetes type 2 usually strikes people over 30 years that are overweight and that have cases in the family, but there are people without these characteristics that are also affected. This type is the more common form. In this case, the main causes for the increase of the glycemia are the progressive loss of efficacy of insulin (also known by "resistance to insulin") and the decrease of the insulin production by the pancreas. Nowadays it is known that due to lifestyle and inherited genes, the insulin loses efficacy and then the organism is more resistant to insulin. In this situation the pancreas reacts and produces more insulin in order to keep balanced glycemia levels. In some people the pancreas slowly begins to fail and, consequently, it isn't able to produce enough insulin to control the glycemia levels and this increases, resulting in the diabetes type 2. We know that the overweight, the excess fat in the body and the physical inactivity can worsen diabetes type 2. So, a patient with this type of diabetes should opt for healthy eating, lose weight, practice exercise, reduce the blood pressure, improve cholesterol levels and take the medication correctly. This medication usually consists in taking pills that increase the sensitivity of the tissues to insulin. A patient with diabetes type 2 only receives insulin injections when the situation is serious.

Gestational diabetes only arises during pregnancy, but it is very similar to diabetes type 2. If it is diagnosed and treated in the beginning of the pregnancy, then there are no problems or to the mother, or to the baby. It generally disappears with the birth of baby. Nevertheless, women that have this type of diabetes are more likely to have diabetes type 2 later. So, they must be careful with their health throughout life. Then, a pregnant with gestational diabetes should opt for healthy eating, practice exercise, control the blood pressure, take the medication correctly and make a careful monitoring of the baby.

3.3 An Optimal Control Problem of Diabetes *Mellitus*

Let $G(t)$ and $H(t)$ be the level of glucose and of net hormone in the blood at time t (in minutes), respectively. Note that at the time t the person isn't fasting. Consider the variables $y_1(t)$ and $y_2(t)$ defined by

$$y_1(t) = G(t) - G_0$$

and

$$y_2(t) = H(t) - H_0,$$

where G_0 and H_0 are the constant fasting values of glucose and of net hormone, respectively. The value $H(t)$ includes the weighted average of all endocrine secretions, which tend to change the glycemia. Note that if the person has high levels of glucose (hyperglycemia), then insulin is the hormone that has the most contribution to $H(t)$. On the other hand, if the person has low levels of glucose (hypoglycemia), then glucagon is the hormone that has the most contribution to $H(t)$. The control variable $u(t)$ is responsible for the rate of infusion of exogenous insulin at time t. It is obvious that $u(t) \geq 0$. When we consider that G and H are not too different from G_0 and H_0, respectively, the mathematical model of glucose and insulin interaction proposed by Ackerman in [1] is given by

$$\begin{aligned} &y_1'(t) = -m_1 y_1(t) - m_2 y_2(t) \\ &y_2'(t) = -m_3 y_2(t) + m_4 y_1(t) + u(t) \\ &y_1(0) = y_{10} \\ &y_2(0) = y_{20} \\ &m_1, m_2, m_3 > 0 \text{ and } m_4 \geq 0. \end{aligned} \tag{3.1}$$

Consider an individual with diabetes of type 1 that is with high levels of glucose. He isn't able to produce enough endogenous insulin. Therefore, the organism detects the excess of the glucose in the blood, but this situation doesn't cause an increase of the production of endogenous insulin. So, we consider $m_4 = 0$ in the Equation (3.1). Thus, he needs to administrate exogenous insulin translated by the nonnegative term $u(t)$ in the Equation (3.1). With this administration, it is expected that the levels of the glucose in the blood will decrease. In theses cases, we usually have that $G(t) > G_0$ and that $H(t) > H_0$. So, $y_1(t) > 0$ and $y_2(t) > 0$. As $m_1, m_2 > 0$, we have that $y_1'(t) < 0$ and this results in a decrease of glucose, as expected. We know that $u(t) \geq 0$ and $-m_3 y_2(t) < 0$, because $m_3 > 0$. The concentration of insulin in the blood should increase until it reaches its maximum, due to exogenous insulin administration. So, during this time it is required that $u(t) > m_3 y_2(t)$ in order to $y_2'(t) > 0$. This is obvious, because the function of infusion of exogenous insulin is to solve the low levels of endogenous insulin caused by the destruction of the beta cells in the patient with diabetes type 1. It is expected that some time after insulin administration $u(t) < m_3 y_2(t)$, that is, $y_2'(t) < 0$, because the insulin will begin to be absorbed by tissues.

In [40], Swan also uses this mathematical model and he proposes the following optimal control problem:

$$
\begin{array}{lll}
(P_{GI}) & \min & J(y_1, u) = \displaystyle\int_0^{t_f} (y_1(t) - y_d)^2 + \rho u^2(t) dt \\
& \text{s.t.} & y_1'(t) = -m_1 y_1(t) - m_2 y_2(t) \\
& & y_2'(t) = -m_3 y_2(t) + m_4 y_1(t) + u(t) \\
& & u(t) \geq 0 \\
& & y_1(0) = y_{10} \\
& & y_2(0) = y_{20},
\end{array}
$$

where y_d is a predetermined constant glucose level in a diabetic individual, ρ is a scalar weighting factor (with dimensions of $(\text{time})^2$) such that $\rho > 0$ and m_1, m_2, m_3 and m_4 are as we defined previously.

As y_d is a predetermined constant glucose level in a diabetic individual, the goal is to minimize the difference between $y_1(t)$ and y_d and the rate of infusion of exogenous insulin. Therefore, the objective function is

$$f(y_1, u) = (y_1 - y_d)^2 + \rho u^2.$$

Note that, as we consider $\rho > 0$, large controls imply large values of J.

3.4 The Necessary Conditions

Now we are going to write the necessary conditions of Theorem 2.3.1 for the problem (P_{GI}). As we are going to study a situation of hyperglycemia, we consider $m_4 = 0$.

The Hamiltonian $H = H(y_1, y_2, u, \lambda_1, \lambda_2)$ is given by

$$H = -(y_1 - y_d)^2 - \rho u^2 + \lambda_1(-m_1 y_1 - m_2 y_2) + \lambda_2(-m_3 y_2 + u).$$

The Transversality Condition is given by

$$\lambda_1(t_f) = \lambda_2(t_f) = 0.$$

The Adjoint System is given by

$$
\begin{cases}
y_1'(t) = \frac{\partial H}{\partial \lambda_1} \\
y_2'(t) = \frac{\partial H}{\partial \lambda_2} \\
\lambda_1'(t) = -\frac{\partial H}{\partial y_1} \\
\lambda_2'(t) = -\frac{\partial H}{\partial y_2}
\end{cases}
\Leftrightarrow
\begin{cases}
y_1'(t) = -m_1 y_1(t) - m_2 y_2(t) \\
y_2'(t) = -m_3 y_2(t) + u(t) \\
\lambda_1'(t) = 2(y_1(t) - y_d) + m_1 \lambda_1(t) \\
\lambda_2'(t) = m_2 \lambda_1(t) + m_3 \lambda_2(t).
\end{cases}
$$

By the Maximality Condition, we have that $u(t)$, $t \in [0, t_f]$, is the solution of

$$(P_v) \quad \max_{v \geq 0} \; H(y_1(t), y_2(t), v, \lambda_1(t), \lambda_2(t)).$$

By Fritz–John's Theorem [8, p. 182–184], we have that the solution of (P_v) is obtained solving the following system:

$$\begin{cases} w_0 \frac{\partial H}{\partial v} - w_1 \frac{\partial}{\partial v}(-v) = 0 \\ w_1(-v) = 0 \\ w_0, w_1 \geq 0 \\ (w_0, w_1) \neq (0,0) \end{cases} \Rightarrow \begin{cases} w_0(-2\rho v + \lambda_2) + w_1 = 0 \\ w_1 v = 0 \\ w_0, w_1 \geq 0 \\ (w_0, w_1) \neq (0,0). \end{cases}$$

If $w_0 = 0$ we obtained the following system

$$\begin{cases} w_1 = 0 \\ w_1 v = 0 \\ w_0, w_1 \geq 0 \\ (w_0, w_1) \neq (0,0) \end{cases}$$

that is impossible, because $(w_0, w_1) = (0,0)$. Therefore, we can consider that $w_0 = 1$. For the condition $w_1 v = 0$ we have two possibilities: $w_1 = 0$, or $v = 0$. The second possibility doesn't make sense, because the patient needs to receive exogenous insulin. So, we consider that $w_1 = 0$ and $v > 0$ in order to verify the condition $w_1 v = 0$. Thus, we have that

$$\begin{cases} v = \frac{\lambda_2}{2\rho} \\ w_1 v = 0 \\ w_0, w_1 \geq 0 \\ (w_0, w_1) \neq (0,0). \end{cases}$$

Then, we can find the solution of (P_{GI}) solving the following system:

$$\begin{cases} y_1'(t) = -m_1 y_1(t) - m_2 y_2(t) \\ y_2'(t) = -m_3 y_2(t) + \frac{\lambda_2(t)}{2\rho} \\ \lambda_1'(t) = 2(y_1(t) - y_d) + m_1 \lambda_1(t) \\ \lambda_2'(t) = m_2 \lambda_1(t) + m_3 \lambda_2(t) \\ y_1(0) = y_{10} \\ y_2(0) = y_{20} \\ \lambda_1(t_f) = 0 \\ \lambda_2(t_f) = 0. \end{cases} \tag{3.2}$$

3.5 The Exact Solution

The exact solutions $y_1(t)$ and $u(t)$ of the problem (P_{GI}) determined by Swan in [40] are

$$u(t) = -K_1 y_1(t) - K_2 y_2(t) + K$$

and

$$y_1(t) = e^{-\alpha t}\left((y_{10}-\zeta)\cos(\beta t) + \frac{1}{\beta}[\alpha(y_{10}-\zeta) - m_1 y_{10}]\sin(\beta t)\right) + \zeta,$$

where

$$\begin{aligned}
\xi &= -\sqrt{m_1^2 m_3^2 + \frac{m_2^2}{\rho}},\\
K &= \frac{m_2 y_d}{\rho \xi},\\
K_2 &= \sqrt{m_1^2 + m_3^2 + 2\sqrt{m_1^2 m_3^2 + \frac{m_2^2}{\rho}}} - (m_1 + m_3),\\
K_1 &= -\frac{K_2^2 + 2m_3 K_2}{2m_2},\\
\alpha &= \frac{K_2 + m_1 + m_3}{2},\\
\beta &= \frac{\sqrt{|m_1^2 + m_3^2 + 2\xi|}}{2},\\
\zeta &= \frac{y_d}{1 + \rho\left(\frac{m_1 m_3}{m_2}\right)^2}.
\end{aligned}$$

As $y_1'(t) = -m_1 y_1(t) - m_2 y_2(t)$ we have that

$$y_2(t) = -\frac{y_1'(t) + m_1 y_1(t)}{m_2}.$$

So, the exact solution $y_2(t)$ is

$$\begin{aligned}
y_2(t) =& \frac{e^{-\alpha t}(y_{10}-\zeta)}{m_2}(\alpha\cos(\beta t) + \beta\sin(\beta t) - m_1\cos(\beta t))\\
&+ \frac{e^{-\alpha t}[\alpha(y_{10}-\zeta) - m_1 y_{10}]}{m_2}\left(\frac{\alpha}{\beta}\sin(\beta t) - \cos(\beta t) - \frac{m_1}{\beta}\sin(\beta t)\right)\\
&- \frac{m_1 \zeta}{m_2}.
\end{aligned}$$

3.6 The Numerical Solution

Swan considered in [40] the situation in which $y_{10} = 300$ mg/dl, $y_d = 100$ mg/dl, the concentration y_{20} is null and the value of ρ is 10. By Yipintsoi in [44, p. 73, 75] the values of m_1, m_2 and m_3 for a woman of 59 years old diabetic 20 years ago, with 1.64m of height and with, approximately, 65kg of weight, were given by $m_1 = 0.0009$, $m_2 = 0.0031$ and $m_3 = 0.0415$. By Yipintsoi in [44, p. 73], we also know that she was insulin dependent. Then, we can conclude that she was diabetic type 1.

With these considerations, we are going to find a numerical solution of the problem (P_{GI}) for this woman by two different methods. First, we are going to solve the system (3.2) (so called indirect method), using the software Maple, and then we are going to propose a second procedure that solves directly the problem by discretizing it (so called direct method), using the software IPOPT (Interior Point OPTimizer). For more on the subject we refer the reader to [36]. To test the efficiency of the results, we are going to draw the exact solution given in Section 3.5 and the two numerical solutions for $t \in [0, 145]$. In Appendix B we provide the codes for the indirect (Appendix B.1) and direct (Appendix B.2) methods.

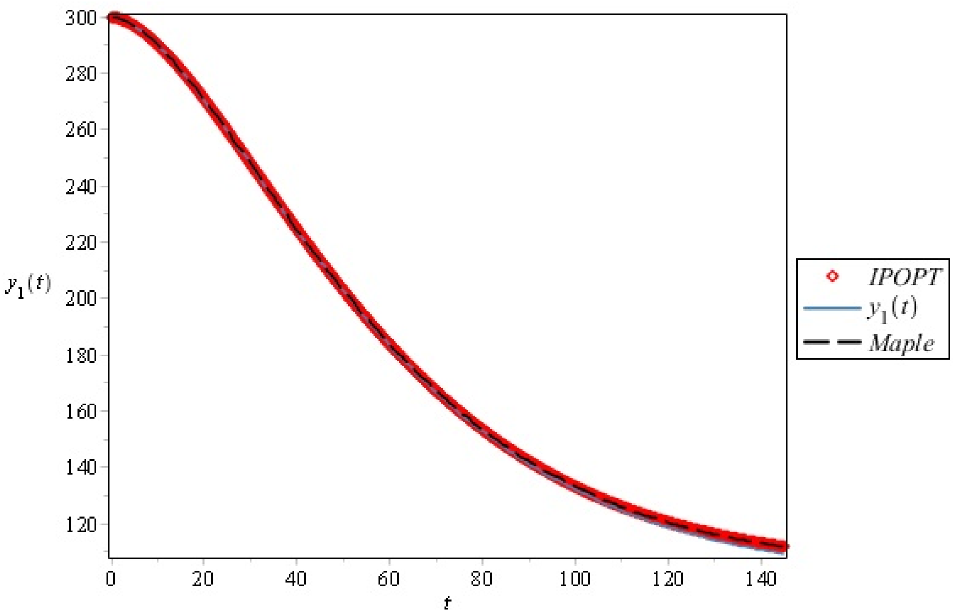

Figure 3.1: Solution $y_1(t)$ (exact versus approximations obtained by direct and indirect methods).

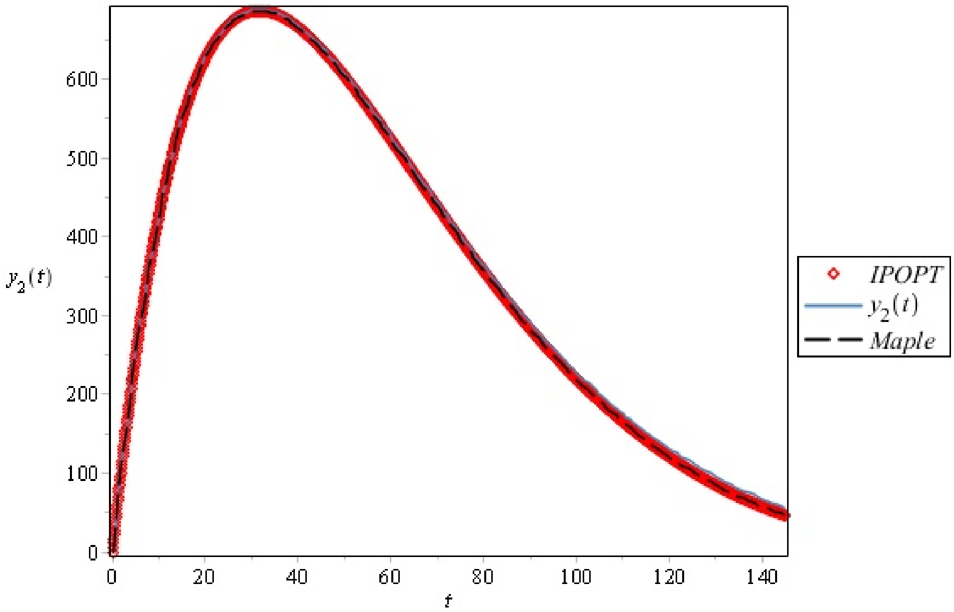

Figure 3.2: Solution $y_2(t)$ (exact versus approximations obtained by direct and indirect methods).

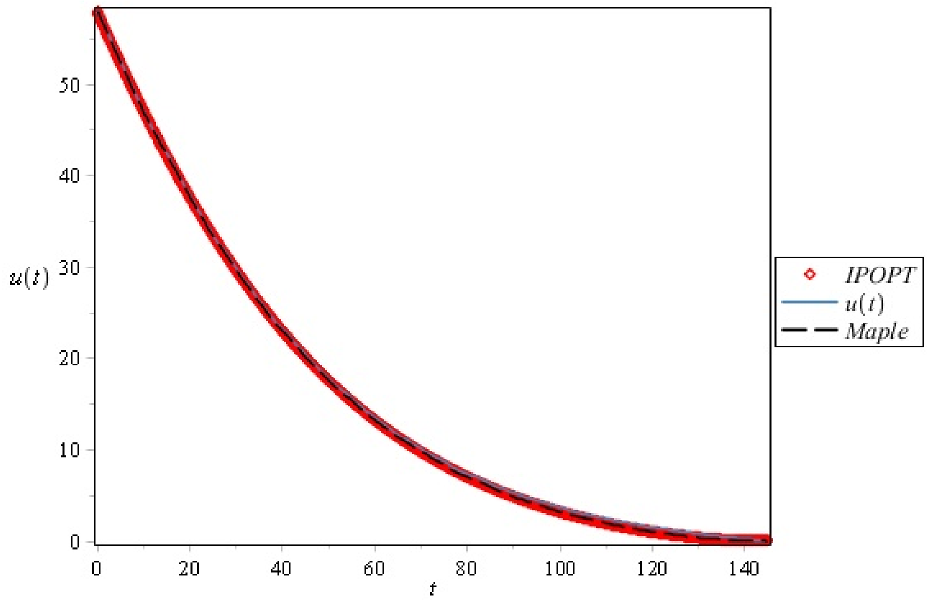

Figure 3.3: Solution $u(t)$ (exact versus approximations obtained by direct and indirect methods).

3.7 Discussion

The numerical approximations obtained by Maple and by IPOPT are very similar between them and very close to the exact solution, because as we can observe in the previous figures the dashed black line is superimposed on the line composed by the red points.

For a diabetic person, the levels of glucose in the blood are higher than those of an individual that isn't diabetic. This is caused by inability to produce enough endogenous insulin to maintain the optimal levels of glycemia. In [35] the levels of glucose in the blood (mg/dl) are:

1. $G_0 \in [70, 100]$ and $G(t) \in [70, 140]$ for a normal person, after a meal (approximately 2 hours);

2. $G_0 \in [100, 126]$ and $G(t) \in [140, 200]$ for a person in the situation of pre diabetes, after a meal (approximately 2 hours);

3. $G_0 > 126$ and $G(t) > 200$ for a person with diabetes, after a meal (approximately 2 hours).

Diabetics may not be able to maintain the levels mentioned in the items 1. and 2. So, the values proposed by Swan for y_{10}, y_{20} and y_d are coherent, because as the woman had diabetes type 1 it is possible that she was with high levels of glucose and with null levels of insulin. Therefore, this is the hormone that has the most contribution to $H(t)$ in this case.

The insulin is administrated in the patient at time $t = 0$. Consider that the time at which the concentration of insulin in the blood reaches the maximum value is t_m. As we know the exact solution of $y_2(t)$, we conclude that $t_m \simeq 31.77$ minutes. From $t = 0$ the glycemia decreases and this means that there is absorption of the insulin by tissues since the time of its administration. However, there is more insulin to enter into blood than into tissues for $t \in [0, t_m]$, because $y_2(t)$ is strictly increasing in this interval of time. For $t > t_m$ the concentration of insulin in the blood is strictly decreasing and this means that from $t = t_m$ there is more insulin to enter into tissues than into blood.

Consider that $I_i = [x_i, x_{i+1}]$ and $\Delta_i = |y_1(x_{i+1}) - y_1(x_i)|$.

i	I_i	Δ_i
1	$[0, 30]$	52.35
2	$[30, 60]$	64.46
3	$[60, 90]$	42.00
4	$[90, 120]$	22.46
5	$[120, 145]$	9.52

Table 3.1: The absolute value of the decrease of glycemia for different intervals of time.

In Table 3.1 we can observe that the absolute value of the decrease of glycemia was, approximately, 52.35 for $t \in [0, 30]$ and 64.46 for $t \in [30, 60]$.

This is supposed to happen, because for $t > t_m$ insulin has more effect. Note that for $i = 3, 4, 5$

$$\Delta_1 > \Delta_i,$$

because after 60 minutes the level of glycemia ($y_1(60) = 183.19$ mg/dl) is not so worrying as in the beginning ($y_1(0) = 300$ mg/dl).

After 145 minutes the insulin is absorbed almost entirely by tissues, because their levels in the blood are very low, as we can observe in Figure 3.2. Consequently, the level of glucose in the blood decreases from 300 mg/dl to 109.22 mg/dl. Therefore, the patient reaches a good level of glucose with the administration of insulin.

In Figure 3.3 we can observe the decreasing rate of infusion of exogenous insulin over time.

The difference between the numerical solutions and the exact solution is not significant. Concluding, we obtained good numerical approximations to the solution.

Appendix A

Euler's Method in MATLAB

```
function[x,Solution_y,deltax]=met_euler(a,b,ya,yb,n,f,color)

% This function takes as input the extremes 'a' and 'b' of the
% interval, the values of ya, yb and n, the function and the
% color of the graphic. This routine returns the array x compo-
% sed by the values x_0,...,x_{n+1}, the array y composed by
% the values y_1,...,y_{n+1} and deltax.

% Startup and given information:

  x=zeros(1,n+2);
  x(1)=a;
  x(n+2)=b;

% Calculation of delta x

  deltax=(b-a)/(n+1);

% Calculation of the values of x(i)

  for i = 2:n+1
     x(i)=x(1)+(i-1)*deltax;
  end

% Creation of the variables to determine y1,...,y(n+1)

  y=sym('y',[1 n+2]);
  y(1)=ya;
  y(n+2)=yb;
```

```
% Definition of the function phi

  phi= 0;
  for i = 1:n+1
     phi=phi+feval(f,x(i),y(i),(y(i+1)-y(i))/deltax)*deltax;
  end

  pretty(simplify(phi))
  dphi=sym('dphi',[1,n]);

  for i = 1:n
     dphi(i)=diff(phi,y(i+1));
  end

  S=solve(dphi);

  Solution_y=zeros(1,n+2);
  Solution_y(1)=ya;
  Solution_y(n+2)=yb;

  if n==1
     Solution_y(2)=S;

  else
     SNames = fieldnames(S);
     for i = 2:n+1
        Solution_y(i) = S.(SNames{i-1});
     end
  end

  plot(x,Solution_y,color,'LineWidth',2)
  grid on

end
```

The code used to solve the Example 1.2.1 is

```
g=inline('(z^2)-(y^2)-(2*x*y)');

t=linspace(0,1,500);
g2=((3-cos(1))/sin(1))*sin(t)+cos(t)-t;

[x,Solution_y,deltax]=met_euler(0,1,1,2,1,g,'--ok')
hold on
```

```
plot(t,g2,'-k','LineWidth',2)
legend('n=1','Extremal');
xlabel('x');
ylabel('y');
title('Euler''s Method of Finite Differences for n=1 versus the
extremal')
axis([0 1 0 3])

figure
[x,Solution_y,deltax]=met_euler(0,1,1,2,2,g,'--ok')
hold on
plot(t,g2,'-k','LineWidth',2)
legend('n=2','Extremal');
xlabel('x');
ylabel('y');
title('Euler''s Method of Finite Differences for n=2 versus the
extremal')
axis([0 1 0 3])

figure
[x,Solution_y,deltax]=met_euler(0,1,1,2,3,g,'--ok')
hold on
plot(t,g2,'-k','LineWidth',2)
legend('n=3','Extremal');
xlabel('x');
ylabel('y');
title('Euler''s Method of Finite Differences for n=3 versus the
extremal')
axis([0 1 0 3])

figure
[x,Solution_y,deltax]=met_euler(0,1,1,2,4,g,'--ok')
hold on
plot(t,g2,'-k','LineWidth',2)
legend('n=4','Extremal');
xlabel('x');
ylabel('y');
title('Euler''s Method of Finite Differences for n=4 versus the
extremal')
axis([0 1 0 3])
```

Appendix B

The Numerical Solution of (P_{GI})

B.1 Maple (indirect method)

```
m1 := 0.0009;
m2 := 0.0031;
m3 := 0.0415;
ro := 10;
yd := 100;
E1 := -sqrt(m1^2*m3^2+m2^2/ro);
K := m2*yd/(ro*E1);
K2 := sqrt(m1^2+m3^2+2*sqrt(m1^2*m3^2+m2^2/ro))-m1-m3;
K1 := -(K2^2+2*K2*m3)/(2*m2);
alfa := (K2+m1+m3)*(1/2);
bet := (1/2)*sqrt(abs(m1^2+m3^2+2*E1));
E2 := yd/(1+ro*(m1*m3/m2)^2);

system_ode := diff(y1(t), t) = -m1*y1(t)-m2*y2(t),
diff(y2(t), t) = -m3*y2(t)+(1/(2*ro))*lamb2(t),
diff(lamb1(t), t) = 2*(y1(t)-yd)+m1*lamb1(t),
diff(lamb2(t), t) = m2*lamb1(t)+m3*lamb2(t);

boundaryCond := y1(0) = 300, y2(0) = 0, lamb1(145) = 0,
lamb2(145) = 0;

solution := dsolve({boundaryCond, system_ode}, numeric,
range = 0 .. 145, output = listprocedure);

with(plots);
```

```
y_1 := solution[4];
y_1 := rhs(y_1);
plot(y_1(t), t = 0 .. 145, color = black, linestyle = dash,
axes = boxed, labels = ['t', 'y[1](t)'], legend = 'Maple',
legendstyle = [location = right], size = [.5, .65]);

y_2 := solution[5];
y_2 := rhs(y_2);
plot(y_2(t), t = 0 .. 145, color = black, linestyle = dash,
axes = boxed, labels = ['t', 'y[2](t)'], legend = 'Maple',
legendstyle = [location = right], size = [.5, .65]);

u := (1/20)*solution[3];
u := rhs(u);
plot(u(t), t = 0 .. 145, color = black, linestyle = dash,
axes = boxed, labels = ['t', 'u(t)'], legend = 'Maple',
legendstyle = [location = right], size = [.5, .65]);
```

B.2 AMPL for IPOPT (direct method)

```
param ti := 0;
param tf := 145;
param n := 1500;
param h := (tf-ti)/n;

param yd := 100;
param m1 := 0.0009;
param m2 := 0.0031;
param m3 := 0.0415;
param ro := 10;

###  State variables:

var y1 {i in 0..n};
var y2 {i in 0..n};

###  Initial values

s.t. ivy1   : y1[0]=300 ;
s.t. ivy2   : y2[0]=0 ;

### Control variable
```

```
var u {i in 0..n},>=0;

### Auxiliary functions for improved Euler

var fy1 {i in 0..n} = (-m1*y1[i])-(m2*y2[i]);
var fy2 {i in 0..n}   =  (-m3*y2[i])+u[i] ;

### Minimize cost

minimize cost: sum{i in 0..n}((y1[i]-yd)^2+(ro*(u[i])^2));

### Euler Method

s.t. lx {i in 0..n-1} : y1[i+1]=y1[i] + h*fy1[i]  ;
s.t. ly {i in 0..n-1} : y2[i+1]=y2[i] + h*fy2[i]  ;

##############################################################

option solver ipopt;
option ipopt_options "max_iter=9999999 acceptable_tol=1e-8";
solve;

##############################################################

display cost;
printf "------Values of t------\n";
printf {i in 0..n} "%18.10f\n",  ti+i*h;
printf "------Values of y1------\n";
printf {i in 0..n} "%18.10f\n", y1[i];
printf "------Values of y2------\n";
printf {i in 0..n} "%18.10f\n", y2[i];
printf "------Values of u------\n";
printf {i in 0..n} "%18.10f\n",  u[i];
```

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