Two infinite families of Archimedean maps of higher genera

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Abstract

The well known infinite families of prisms and antiprisms on the sphere were, for long time, not considered as Archimedean solids for reasons not fully understood. In this paper we describe the first two infinite families of Archimedean maps on higher genera which we call "generalized" prisms and "generalized" antiprisms.

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1 Introduction

Archimedean solids is an old topic that still fascinates by its beauty. The way it is so well spread in the Internet reflects the strong global interest that it induces on people in general. While Platonic solids are the birth of regular maps on any genus, Archimedean solids are only now moving to higher genera. It is not hard to see why; the definition of Archimedean solid, when viewed as a topological object on the sphere (not in $\mathbb{R}^3$), is somewhat evasive and clearly very restricted to the sphere. There seems to be no single and clear definition of Archimedean "solid" that can be extended to all orientable or not, or even with boundary, connected surfaces. Cayley maps seem to be close, since almost all Archimedean solids (i.e. on the sphere) are Cayley maps, with only two exceptions: the great rhombicosidodecahedron and the great rhombicuboctahedron (see Table 1).

By the classical definition, an Archimedean solid is a 3-dimensional convex polyhedron whose facets (here called faces) are regular convex polygons of two or more different types, such that any two vertices can be transformed one into the other by applying a solid's symmetry. According to this definition there

are infinitely many Archimedean solids (or convex polyhedra) arranged in 13 *classical* Archimedean solids (not counting the mirror images of the snub cube and the snub dodecahedron), plus two infinite families of *prisms* and *antiprisms*, see Table 1. These are attributed to Plato, Archimedes and others, although it was Kepler that put the classification into the organized form that is known today. Modern proofs are given by Walsh [11] (1972) and by Ball and Coxeter [2] (1987). More detailed information, as well as nice pictures, can be found in Wolfram MathWorld [13] and in the popular Wikipedia [12], to name but a few. By dropping the restriction "two or more different types" from the above definition, the list of Archimedean solids will include the five Platonic solids (tetrahedron, cube, octahedron, icosahedron and dodecahedron).

The classification of the Archimedean solids can be obtained by purely combinatorial methods, viewing convex polyhedra as graphs embedded in the sphere, in which case the graph will be *polyhedral*, that is, planar, simple and 3-connected. Within this view, Archimedean solids are particular maps having a combinatorial symmetry. Attempts to generalize Archimedean solids to higher genera have been sparse and essentially focused on the torus. This has been the case of Grünbaum and Shephard [5], by means of semi-regular tilings of the plane, Babai [1], Proulx [10], Gross and Tucker [6], by means of vertex-transitive group actions on the torus. To higher genera there is only one known attempted generalization given by Karabáš and Nedela [8].

For genus $g > 1$ there are no known infinite families of Archimedean maps. In this paper we give the first two of such examples.

2 Preliminaries

In the following, all graphs will be assumed to be connected and the term *surface* will always mean an orientable, connected, compact surface without boundary. Groups will be always finite and, as usual in group theory, we denote by $|g|$ the order of an element g of a group G, whose order (size) is $|G|$. For elements $x_1, \ldots, x_n$ of a group G, $\langle x_1, \ldots, x_n \rangle$ denotes the group generated by $\{x_1, \ldots, x_n\}$.

A (topological) *map* is an embedding $\mathcal{M} : \Gamma \to \mathcal{S}$ of a graph Γ into a surface $\mathcal{S}$, called the *supporting surface* of $\mathcal{M}$, such that the connected components of $\mathcal{S} \setminus \mathcal{M}(\Gamma)$, called *faces* of $\mathcal{M}$, are simply connected subspaces of $\mathcal{S}$.

Fixing an orientation on the supporting surface $\mathcal{S}$ of the map $\mathcal{M}$, we can describe $\mathcal{M}$ combinatorially by a triple $(D; R, L)$, where D is the set of *darts* (directed edges or arcs, pictured as half edges) of the embedded graph $\mathcal{M}(\Gamma)$, R is the permutation of D which cyclically permutes darts around vertices of $\mathcal{M}(\Gamma)$, according to the fixed orientation on $\mathcal{S}$, and L is the involutory permutation of D, possibly with fixed points, whose orbits are the edges (including loops and semi-edges) of $\mathcal{M}(\Gamma)$. Due to the connectivity of the underlying graph $\mathcal{M}(\Gamma)$, the (permutation) group $Mon(\mathcal{M}) = \langle R, L \rangle$, called the *monodromy group* of $\mathcal{M}$, acts transitively on D.

Conversely, given such a triple $\mathcal{M} = (D; R, L)$, defining vertices, edges and faces of $\mathcal{M}$ as orbits of the action of $\langle R \rangle$, $\langle L \rangle$ and $\langle RL \rangle$ on D, respectively, while incidence is given by nonempty intersecting orbits, we get an embedding of a graph Γ on a surface $\mathcal{S}$ determined by its genus, given by the well known Euler formula.

A *morphism* from a (combinatorial) map $\mathcal{M} = (D; R, L)$ to a (combinatorial) map $\mathcal{M}' = (D'; R', L')$ is a function $\phi : D \to D'$ such that $\phi R = R'\phi$ and $\phi L = L'\phi$. If there is a morphism from $\mathcal{M}$ to $\mathcal{M}'$ then, the assignment $R \mapsto R'$, $L \mapsto L'$ extends to an epimorphism from $Mon(\mathcal{M})$ to $Mon(\mathcal{M}')$. Due to the transitivity of the respective actions of $Mon(\mathcal{M}')$ and $Mon(\mathcal{M})$, any morphism from $\mathcal{M}$ to $\mathcal{M}'$ is onto and uniquely determined by the image of a dart. Therefore morphisms are also called *coverings* and injective coverings are called *isomorphisms*. Recalling that a (combinatorial) map $\mathcal{M} = (D; R, L)$ arises from a (topological) map $\mathcal{M} : \Gamma \to \mathcal{S}$ by fixing an orientation on the supporting surface $\mathcal{S}$, isomorphisms from $\mathcal{M}$ to $\mathcal{M}$ will be called *orientation preserving automorphisms* of $\mathcal{M}$. Changing the orientation on the supporting surface $\mathcal{S}$, the same (topological) map $\mathcal{M} : \Gamma \to \mathcal{S}$ will be combinatorially described by the triple $(D; R^{-1}, L)$, called the *mirror image* of $(D; R, L)$. Hence, we call isomorphisms from $\mathcal{M} = (D; R, L)$ to $(D; R^{-1}, L)$ *orientation reversing automorphisms* of $\mathcal{M}$. The (permutation) group $Aut(\mathcal{M})$ of all automorphisms of the map $\mathcal{M}$ is the (full) *automorphism group* of $\mathcal{M}$. Topologically, an automorphism of the map $\mathcal{M} : \Gamma \to \mathcal{S}$ is an automorphism of the underlying graph $\mathcal{M}(\Gamma)$ which extends to a self-homeomorphism of the supporting surface $\mathcal{S}$. The group $Aut^+(\mathcal{M})$ of orientation preserving automorphisms of $\mathcal{M} = (D; R, L)$, which is by definition the centralizer of $Mon(\mathcal{M})$ in the symmetric group on D, is a subgroup of $Aut(\mathcal{M})$ of index at most two. If $Aut^+(\mathcal{M}) = Aut(\mathcal{M})$ then $\mathcal{M}$ and its mirror image are not isomorphic. Otherwise, $Aut^+(\mathcal{M})$ has index two in $Aut(\mathcal{M})$ and there is an orientation reversing automorphism (also called a *reflection*) sending $\mathcal{M}$ to its mirror image.

Any automorphism group $G \leqslant Aut(\mathcal{M})$ of a map $\mathcal{M} = (D; R, L)$ gives rise to a covering ϕ from $\mathcal{M}$ to the *quotient map* $\mathcal{M}/G$. Each dart of the quotient map $\mathcal{M}/G$ is a set $Gx = \{g(x) : g \in G\}$, $x \in D$, and we have well-defined permutations R' and L' of $D' = \{Gx : x \in D\}$ given by $R'(Gx) = GR(x)$ and $L'(Gx) = GL(x)$, $x \in D$, describing the quotient map $\mathcal{M}/G$ as a triple $(D'; R', L')$. The function $\phi : D \to D',\ x \mapsto Gx$ is then a covering from $\mathcal{M}$ to $\mathcal{M}/G$.

A map $\mathcal{M}$ will be called *vertex-transitive*[1] if the (full) automorphism group $Aut(\mathcal{M})$ acts transitively on vertices. Any vertex-transitive map $\mathcal{M}$ gives rise to a one- or two- vertex quotient map by factoring $\mathcal{M}$ by its orientation preserving automorphism group $Aut^+(\mathcal{M})$. In [8], a vertex-transitive map $\mathcal{M}$ is called of *type I* or of *type II*, according to the quotient map $\mathcal{M}/Aut^+(\mathcal{M})$ having one or two vertices respectively.

Cayley maps are particular embeddings of Cayley graphs. Given a group G with a set of generators S such that $1 \notin S = S^{-1}$, the *Cayley graph* (G, S)

[1]For some authors $\mathcal{M}$ is vertex-transitive if $Aut^+(\mathcal{M})$ acts transitively on vertices.

(determined by G and S) is the graph with vertex set G and edges $\{g, gs\}$, $g \in G$, $s \in S$. Calling a graph *simple* if it has no loops, no semi-edges and no *multiple edges*, that is, edges incident with the same vertices, we have that any Cayley graph is a connected vertex-transitive simple graph. A *Cayley map* is an embedding of a Cayley graph (G, S) into an *oriented* surface $\mathcal{S}$ (that is, a surface with a fixed orientation) such that the action of R on the set of darts $G \times S$ is the same at each vertex $g \in G$, that is, there is a cyclic permutation ρ of S such that $R(g, s) = (g, \rho(s))$ for any $s \in S$. In other words, a Cayley map is a group G with a cyclically ordered set S of generators of G such that $1 \notin S = S^{-1}$. So we will denote a Cayley map by $(G; (s_0, ..., s_{n-1}))$. It follows directly from the definition that, for any Cayley map $\mathcal{M}$, $A = Aut^+(\mathcal{M})$ acts transitively on vertices, giving a one-vertex quotient map $\mathcal{M}/A$.

One-vertex maps will be denoted by $X_n(\lambda)$. Combinatorially $X_n(\lambda) = (\mathbb{Z}_n; \rho, \lambda)$ where ρ is the permutation $\rho(i) = i + 1 \pmod n$ of $\mathbb{Z}_n$. If a quotient map $\mathcal{M}/G$ is a one-vertex map, then $\mathcal{M}/G \cong X_n(\lambda)$ for some involution λ of $\mathbb{Z}_n$, where n is the number of darts of $\mathcal{M}/G$. In this case, there is a covering ϕ from $\mathcal{M} = (D; R, L)$ to $X_n(\lambda)$ inducing an epimorphism from $Mon(\mathcal{M}) = \langle R, L \rangle$ to $Mon(X_n(\lambda)) = \langle \rho, \lambda \rangle$ sending R to ρ and L to λ.

3 Archimedean maps

Given a map $\mathcal{M} : \Gamma \to \mathcal{S}$, we call closed walks on $\mathcal{M}(\Gamma)$ whose vertices and edges are all the vertices and all the edges incident with a face of $\mathcal{M}$, *$\mathcal{M}$-facial walks* (see [9]). We say that $\mathcal{M}$ is *polyhedral* if (1) Γ is a simple graph, (2) faces have valency at least 3, (3) any $\mathcal{M}$-facial walk is a cycle and (4) any two $\mathcal{M}$-facial walks are either disjoint or their intersection is just a vertex or an edge of $\mathcal{M}(\Gamma)$. We note that, as a consequence of Proposition 5.5.12 in [9], any polyhedral map has a 3-connected simple underlying graph.

As polyhedral maps are natural generalization of convex polyhedra ([9], p. 151), the following generalization of Archimedean solids seems to be natural, despite the first generalization includes a convention for spherical maps, which makes rather suspicious to call it "natural".

Definition. An *Archimedean map* is a polyhedral vertex-transitive map.

As a consequence of the fact that $Aut^+(\mathcal{M})$ has index at most 2 in $Aut(\mathcal{M})$, we have that $Aut^+(\mathcal{M})$ acts on vertices of a vertex-transitive map $\mathcal{M}$ with at most two orbits. According to the action of $Aut^+(\mathcal{M})$ on vertices or on darts of $\mathcal{M}$, we may distinguish four disjoint classes of Archimedean maps, ordered by the increasing Archimedean degree defined next.

Definition. An Archimedean map $\mathcal{M}$ will be called:

- *perfect Archimedean* if $Aut^+(\mathcal{M})$ does not act transitively on vertices;
- *accurate Archimedean* if $Aut^+(\mathcal{M})$ acts regularly on vertices;

- *fair Archimedean* if $Aut^+(\mathcal{M})$ acts transitively on vertices but not regularly neither on vertices nor on darts;
- *Platonic* if $Aut^+(\mathcal{M})$ acts regularly on darts, that is, if $\mathcal{M}$ is a regular map.

Remark. Earlier definition of Archimedean solid requires a map $\mathcal{M}$ to have "two or more different types of faces". This "regularity destruction" forces the size of $Aut^+(\mathcal{M})$ to be less than the number of darts of $\mathcal{M}$. We see the extreme case, i.e. when an Archimedean map $\mathcal{M}$ has the smallest size of $Aut^+(\mathcal{M})$ for a fixed number of darts, as the one that best fits this spirit of "regularity destruction", hence the name "perfect" for this situation. Those that have $Aut^+(\mathcal{M})$ acting regularly on vertices, are not perfect but still fits the spirit just right, hence the name "accurate". The "fair" Archimedean maps fit the definition more "loosely". The Platonic ones do not fit the earlier definition at all and so they were not considered as Archimedean in the past.

Denoting respectively by D and V the sets of darts and vertices of the Archimedean map $\mathcal{M}$, we have the inequality $\frac{|V|}{2} \leqslant |Aut^+(\mathcal{M})| \leqslant |D|$. Let the quotient

$$\delta(\mathcal{M}) = \frac{|Aut^+(\mathcal{M})|}{|V|}$$

denote de *Archimedean degree* of $\mathcal{M}$. One can easily see that $\mathcal{M}$ is perfect if and only if $\delta(\mathcal{M}) = \frac{1}{2}$, accurate if and only if $\delta(\mathcal{M}) = 1$, fair if and only if $1 < \delta(\mathcal{M}) < \nu$, where ν is the common valency (degree) of the vertices of $\mathcal{M}$, and Platonic if and only if $\delta(\mathcal{M}) = \nu$.

The classification of spherical Archimedean maps (Archimedean solids) is done by their local types. A *local type* of a map $\mathcal{M}$ at a vertex v is the ordered set of valencies of the faces adjacent to v written in a cyclic order (according to a fixed orientation of the supporting surface). The local type of an Archimedean map, which does not depend on the choice of the vertex due to vertex-transitivity, will be written in a multiplicative form like (3.4.6.4) or $(3^4.5) = (3.3.3.3.5)$. These are known as Cundy and Rolett symbols. Archimedean maps whose local type has no repeating factors are necessarily perfect since then, adjacent vertices must belong to different orbits under the action of orientation-preserving automorphisms.

Cayley maps are not necessarily Archimedean; those that are Archimedean, include all accurate Archimedean, some fair Archimedean and some Platonic maps. This is a consequence of the well known fact that a map $\mathcal{M}$ is Cayley if and only if $Aut^+(\mathcal{M})$ has a subgroup G acting regularly on vertices, which is not the case for perfect Archimedean maps and not always guaranteed for fair Archimedean maps or Platonic ones. Moreover, as any Cayley map has a simple underlying graph, a Cayley map is Archimedean if and only if it is polyhedral. Thus, even regular Cayley maps need not to be Archimedean. For example, $(\mathbb{Z}_n, (1, -1))$ is a regular Cayley map (on the sphere) which is not Archimedean (since its underlying graph is not 3-connected, polyhedrality fails). The Cayley

map $(\mathbb{Z}_6, (2,3,4))$ (on the torus) is neither regular nor Archimedean (see Figure 1), since the facial walk given by the non-triangular face is not a cycle. For higher genera see section 4, where two other families of Cayley maps which are not Archimedean arise from Proposition 4.2 and Proposition 4.4.

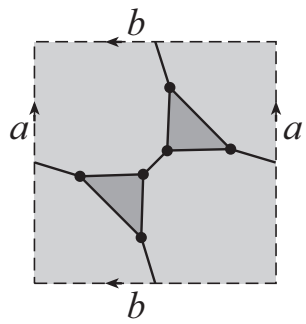


Figure 1: The Cayley map $(\mathbb{Z}_6, (2,3,4))$ (on the torus).

Remark. By definition, a map is Platonic if and only if it is a polyhedral regular map. As regular maps are classified up to genus 101 (see [3]), regular Cayley maps, as well as Platonic maps up to genus 101 can be extracted from this classification. Although Proposition 2.1 in [4] gives necessary and sufficient conditions for a regular map to be Cayley, the classification of regular Cayley maps (on any genus) is still open.

Fair Archimedean maps seem to be relatively rare, especially on low genus. There are no fair Archimedean maps on genus 0 (see Table 1). On genus 1 (torus), of the 11 local types of vertex-transitive maps only three local types support fair Archimedean maps (see Table 2). According to the census [7], there are no fair Archimedean maps on genus 2, there are five on genus 3 (of which 2 are non-Cayley) and three on genus 4 (of which 2 are non-Cayley). The first examples of fair Archimedean non-Cayley maps appear on genus 3.

Archimedean maps on the sphere

Besides the well known Platonic solids (Platonic maps), the Archimedean maps on the sphere are listed in Table 1 below. It displays a picture of the map (taken from Wikipedia [12]), its local type and its class according to the definition given in the previous section. Note that the great rhombicosidodecahedron and the great rhombicuboctahedron, being perfect Archimedean maps, are not Cayley maps, while the others Archimedean maps on the sphere, being accurate, are all Cayley maps.

Map	Picture	Type	Class
Cuboctahedron		(3.4.3.4)	accu
Great rhombicosidodecahedron		(4.6.10)	perf
Great rhombicuboctahedron		(4.6.8)	perf
Icosidodecahedron		(3.5.3.5)	accu
Small rhombicosidodecahedron		(3.4.5.4)	accu
Small rhombicuboctahedron		(3.4^3)	accu
Snub cube		$(3^4.4)$	accu
Snub dodecahedron		$(3^4.5)$	accu
Truncated cube		(3.8^2)	accu
Truncated dodecahedron		(3.10^2)	accu
Truncated icosahedron		(5.6^2)	accu
Truncated octahedron		(4.6^2)	accu
Truncated tetrahedron		(3.6^2)	accu
n-Prism ($n \neq 4$)		$(4^2.n)$	accu
n-Antiprism ($n \neq 3$)		$(3^3.n)$	accu

Table 1: The Archimedean maps on the sphere.

Archimedean maps on the torus

Toroidal Archimedean maps arise from the semi-regular planar tilings either by describing quadrilateral fundamental regions, approach followed by Gross

and Tucker [6], or by analyzing the different actions that arise from the 17 planar wallpaper groups as in Babai [1]. There are infinitely many toroidal Archimedean maps grouped in 11 possible local types (see Table 2), up to mirror image. With the exception of all the perfect Archimedean maps, which are non-Cayley maps, all the remaining Archimedean maps on the torus are Cayley.

Type	Class	Type	Class	Type	Class
(3.3.3.3.3.3)	Plat, fair	(3.4.6.4)	accu	(4.6.12)	perf
(3.3.3.3.6)	accu	(3.6.3.6)	fair	(4.8.8)	perf, accu
(3.3.3.4.4)	accu	(3.12.12)	accu	(6.6.6)	Plat, accu
(3.3.4.3.4)	accu	(4.4.4.4)	Plat, fair		

Table 2: The 11 local types of toroidal Archimedean maps.

Archimedean maps of higher genera

Fixing the genus g of the supporting surface, according to Euler formula (see (2) in next section), there is a finite number of Archimedean maps of genus $g > 1$. According to the census given in [7], we have the following numbers of Archimedean maps (counted up to isomorphism and mirror image):

- On genus 2 there are 17 Archimedean maps: 8 perfect (non-Cayley) and 9 accurate (Cayley).
- On genus 3 there are 103 Archimedean maps: 25 perfect (non-Cayley), 69 accurate (Cayley), 5 fair (4 Cayley and 1 non-Cayley) and 4 Platonic (2 Cayley and 2 non-Cayley).
- On genus 4 there are 111 Archimedean maps: 35 perfect (non-Cayley), 71 accurate (Cayley), 3 fair (2 Cayley and 1 non-Cayley) and 2 Platonic (1 Cayley and 1 non-Cayley).

4 Generalized prisms and antiprisms

We describe now two families of accurate Archimedean maps called *generalized prisms* and *generalized antiprisms*. They show that there are at least two (accurate) Archimedean maps on any even genus.

Any member of the two families is a Cayley map $(G; (s_0, \ldots, s_{n-1}))$, that is, a map $\mathcal{M} = (D; R, L)$ with $D = G \times S$, where $S = \{s_0, \ldots, s_{n-1}\}$ and $R, L : D \to D$ are given by

$$\begin{aligned} R(g, s_i) &= (g, s_{\rho(i)}) \quad \text{with} \quad \rho(i) = i + 1 \mod n \quad \text{and} \\ L(g, s_i) &= (gs_i, s_{\lambda(i)}) \quad \text{with} \quad \lambda(i) = j \text{ such that } s_j = {s_i}^{-1}. \end{aligned}$$

For any $a \in G$, $(g, s_i) \mapsto (ag, s_i)$ is an orientation preserving automorphism of $\mathcal{M}$. So, we can regard G as a subgroup of $Aut^+(\mathcal{M})$ that acts regularly on vertices. Factoring $\mathcal{M}$ by $G < Aut^+(\mathcal{M})$ we get a one-vertex map $X_n(\lambda)$ (see end

of section 2). Vertices, edges and faces of $\mathcal{M}$ are (in one-to-one correspondence with) the orbits of the action of (the subgroups generated by) R, L and RL on $G \times S$, respectively. Hence, the number of vertices of $\mathcal{M}$ is $|G|$ and the number of edges is $\frac{|G|n}{2}$. The set of faces of $\mathcal{M}$ is partitioned in equivalence classes, each class determined by faces covering the same face of the one-vertex map $X_n(\lambda)$. Since elements of G are automorphisms of $\mathcal{M}$, faces in the same equivalence class have the same valency, which is a multiple of the valency of the face of $X_n(\lambda)$ they cover. The multiplicity is determined by the order $|g|$ of an element g of G. To be more precise, we first remak that, recursively we have

$$(RL)^{\alpha}(g, s_i) = \left(g s_i s_{\rho\lambda(i)} s_{(\rho\lambda)^2(i)} \dots s_{(\rho\lambda)^{\alpha-1}(i)}, s_{(\rho\lambda)^{\alpha}(i)}\right) , \tag{1}$$

for any $\alpha \in \mathbb{N}$. Let α_i be the length of the orbit of $i \in \mathbb{Z}_n$ by $\rho\lambda$. Then, looking at the action of $(RL)^{\alpha_i}$ on (g, s_i), we get that the length of the orbit of (g, s_i) by RL is $\alpha_i m_i$, where $m_i = |s_i s_{\rho\lambda(i)} \dots s_{(\rho\lambda)^{\alpha_i-1}(i)}|$. This order m_i, which is independent of g, is constant on the orbit of i by $\rho\lambda$, since conjugation preserves order. Therefore $\mathcal{M}$ has local type $(\alpha_0 m_0, \dots, \alpha_{n-1} m_{n-1})$, up to cyclic permutations. Using the Euler formula and the orbit-counting theorem (Burnside's Lemma) we easily get the Euler characteristic χ of the supporting surface of $\mathcal{M}$,

$$\chi = |G| \left(1 - \frac{n}{2} + \sum_{i \in \mathbb{Z}_n} \frac{1}{\alpha_i m_i}\right) . \tag{2}$$

Generalized prisms

Consider the family of Cayley maps

$$\mathcal{P}_k = (G; (s_0, ..., s_5)) = (D_{2k}; (x, y, z, y^{-1}, u, u^{-1}))$$

defined on the dihedral group $D_{2k} = \langle y, z \mid y^{2k} = z^2 = (yz)^2 = 1 \rangle$, with $x = y^k$ and $u = y^{k-1}$, where $k \geqslant 3$. Since $s_{\lambda(0)} = {s_0}^{-1} = s_0$ (x is an involution), $s_{\lambda(1)} = {s_1}^{-1} = s_3$, $s_{\lambda(2)} = {s_2}^{-1} = s_2$ (z is an involution) and $s_{\lambda(4)} = {s_4}^{-1} = s_5$, then $\mathcal{P}_k$ covers the one-vertex map $X_6(\lambda)$ with $\lambda = (1, 3)(4, 5)$ and local type $(3^2.2^2.3.1)$.

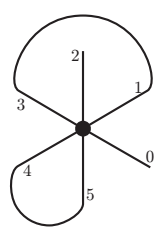


Figure 2: The one-vertex map $X_6(\lambda)$ with $\lambda = (1, 3)(4, 5)$.

For convenience, in the following we set $d(k) = \frac{2k}{\gcd(2k, k-1)}$, which is k if k is odd and $2k$ if k is even.

Proposition 4.1 *The Cayley map $\mathcal{P}_k$ has $4k$ vertices, $12k$ edges, $4k$ triangular faces and $2k$ rectangular faces. If k is odd, then $\mathcal{P}_k$ has four k-gonal faces, local type $(3^2.4^2.3.k)$ and genus $k-1$. If k is even, then $\mathcal{P}_k$ has two $2k$-gonal faces, local type $(3^2.4^2.3.2k)$ and genus k.*

Proof. The number of vertices of $\mathcal{P}_k$ is $|D_{2k}| = 4k$ and the number of edges is $\frac{|D_{2k}|6}{2} = 12k$. Using the above notation, as the orbits of $\rho\lambda$ are $\{0,1,4\}$, $\{2,3\}$ and $\{5\}$, we have $\alpha_0 = \alpha_1 = \alpha_4 = 3$, $\alpha_2 = \alpha_3 = 2$, $\alpha_5 = 1$ and $m_0 = m_1 = m_4$, $m_2 = m_3$ with:

$$m_0 = |s_0\, s_1\, s_4| = |xyu| = 1, \text{ since } xyu = y^{2k} = 1,$$

$$m_2 = |s_2\, s_3| = |zy^{-1}| = |yz| = 2,$$

$$m_5 = |s_5| = |u^{-1}| = |u| = d(k).$$

Hence $\mathcal{P}_k$ has local type $(3^2.4^2.3.d(k))$. Therefore $\mathcal{P}_k$ has $|D_{2k}|\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right) = 4k$ triangular faces, $|D_{2k}|\left(\frac{1}{4}+\frac{1}{4}\right) = 2k$ rectangular faces and $\frac{|D_{2k}|}{d(k)}$ of the faces are $d(k)$-gons. Euler formula (2) conclude the proof. $\square$

Proposition 4.2 *$\mathcal{P}_k$ is Archimedean if and only if k is odd.*
The family $\{\mathcal{P}_{2m+1}\}_{m\in\mathbb{N}}$ is composed of accurate Archimedean maps.

Proof. Since $\mathcal{P}_k$ is a Cayley map, it has a simple underlying graph and $Aut(\mathcal{P}_k)$ acts transitively on vertices of $\mathcal{P}_k$. Therefore, we have only to check polyhedrality. From the previous proposition, we know that $\mathcal{P}_k$ has local type $(3^2.4^2.3.d(k)) = (\alpha_i m_i)_{i\in\mathbb{Z}_6}$. Then the valencies of the six (not necessarily distinct) faces $f_0, ..., f_5$ around (by some fixed orientation) the vertex $v = 1$ (or any other vertex), have valencies 3, 3, 4, 4, 3 and $d(k)$. Regarding faces as sets of vertices, and following the action of RL on darts given by (1) (according to the order given by the local type) we have:

$$f_0 = \{1, s_0, s_0 s_{\rho\lambda(0)}\} = \{1, s_0, s_0 s_1\} = \{1, y^k, y^{k+1}\};$$

$$f_1 = \{1, s_1, s_1 s_{\rho\lambda(1)}\} = \{1, s_1, s_1 s_4\} = \{1, y, y^k\};$$

$$f_2 = \{1, s_2, s_2 s_{\rho\lambda(2)}, s_2 s_{\rho\lambda(2)} s_{(\rho\lambda)^2(2)}\} = \{1, s_2, s_2 s_3, s_2 s_3 s_2\} = \{1, z, zy^{2k-1}, y\};$$

$$f_3 = \{1, s_3, s_3 s_{\rho\lambda(3)}, s_3 s_{\rho\lambda(3)} s_{(\rho\lambda)^2(3)}\} = \{1, s_3, s_3 s_2, s_3 s_2 s_3\} = \{1, y^{2k-1}, zy, z\};$$

$$f_4 = \{1, s_4, s_4 s_{\rho\lambda(3)}\} = \{1, s_4, s_4 s_0\} = \{1, y^{k-1}, y^{2k-1}\};$$

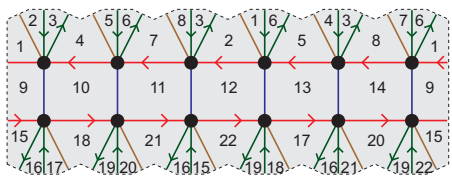
$$\begin{aligned} f_5 = \left\{1, s_5, s_5 s_{\rho\lambda(5)}, \ldots, s_5 s_{\rho\lambda(5)} \ldots s_{(\rho\lambda)^{d(k)-2}(5)}\right\} &= \left\{{s_5}^i : i = 0, \ldots, d(k)-1\right\} \\ &= \left\{y^{(k+1)i} : i = 0, \ldots, d(k)-1\right\}. \end{aligned}$$

Notice that $\rho\lambda = (0,1,4)(2,3)$ and

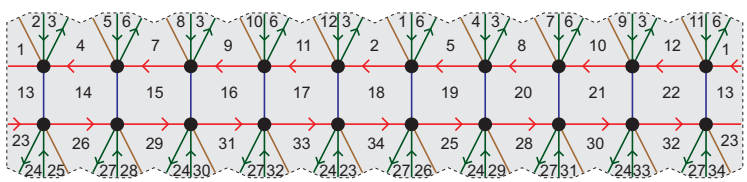
$$(s_0, ..., s_5) = (x, y, z, y^{-1}, u, u^{-1})) = (y^k, y, z, y^{2k-1}, y^{k-1}, y^{k+1}).$$

This shows that $f_0, \ldots, f_5$ are distinct faces, and thus around any vertex there are six distinct faces, proving that any $\mathcal{P}_k$-facial walk is a cycle. If k is even, then $f_4 = \{1, y^{k-1}, y^{2k-1}\} \subseteq f_5 = \{y^i : i = 0, ..., 2k-1\}$ and therefore $\mathcal{P}_k$ is not polyhedral. If k is odd, then $f_5 = \left\{y^{2i} : i = 0, ..., k-1\right\}$ and any two faces containing vertex 1 either have two adjacent vertices in common or 1 is the unique common vertex. As this will happen for faces around any vertex, $\mathcal{P}_k$ is polyhedral, i.e. $\mathcal{P}_k$ is Archimedean, in this case. $\square$

By Proposition 4.2, $\{\mathcal{P}_{2m+1}\}_{m\in\mathbb{N}}$ is a family of Archimedean maps, whose members will be called *generalized prisms*. All members of this family are accurate since there is no non-trivial automorphism sending a face around a vertex to another face around the same vertex (because their local type has no non-trivial rotational symmetry), so vertex-stabilizers are trivial.



$\mathcal{P}_3 = \big(D_6; (x \quad y \quad z \quad y^{-1} \quad u \quad u^{-1})\big)$ with 22 faces and genus 2.



$\mathcal{P}_5 = \big(D_{10}; (x \quad y \quad z \quad y^{-1} \quad u \quad u^{-1})\big)$ with 34 faces and genus 4.

Figure 3: The first two maps of the family of generalized prisms (numbers label faces).

In $\mathcal{P}_5$, the faces numbered 3, 6, 24 and 27 are pentagons.

Generalized antiprisms

Consider now the family of Cayley maps

$$\mathcal{A}_k = (G; (s_0, \ldots, s_6)) = (D_{2k}; (x, y, z, w, y^{-1}, u, u^{-1}))$$

defined on the dihedral group $D_{2k} = \langle y, z \mid y^{2k} = z^2 = (yz)^2 = 1\rangle$ with $x = y^k$, $u = y^{k-1}$ and $w = y^{-1}z$, where $k \geqslant 3$. As $s_{\lambda(0)} = {s_0}^{-1} = s_0$ (x is an involution), $s_{\lambda(1)} = {s_1}^{-1} = s_4$, $s_{\lambda(2)} = {s_2}^{-1} = s_2$ (z is an involution), $s_{\lambda(2)} = {s_3}^{-1} = s_3$ (w is an involution) and $s_{\lambda(5)} = {s_5}^{-1} = s_6$, then $\mathcal{A}_k$ covers the one-vertex map $X_7(\lambda)$ with $\lambda = (1,4)(5,6)$ and local type $(3^6.1)$.

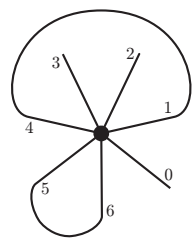


Figure 4: The one-vertex map $X_7(\lambda)$ with $\lambda = (1,4)(5,6)$.

Proposition 4.3 *The Cayley map $\mathcal{A}_k$ has $4k$ vertices, $14k$ edges and $8k$ triangular faces. If k is odd then $\mathcal{A}_k$ has four k-gonal faces, local type $(3^6.k)$ and genus $k-1$. If k is even then $\mathcal{A}_k$ has two $2k$-gonal faces, local type $(3^6.2k)$ and genus k.*

Proof. The number of vertices of $\mathcal{A}_k$ is $|D_{2k}| = 4k$ and the number of edges is $|D_{2k}|\frac{7}{2} = 14k$. As $\rho\lambda = (0,1,5)(2,3,4)$, we have that $\alpha_0 = \cdots = \alpha_5 = 3$, $\alpha_6 = 1$ and $m_0 = m_1 = m_5 = |s_0\, s_1\, s_5| = |xyu| = |y^{2k}| = 1$, $m_2 = m_3 = m_4 = |s_2\, s_3\, s_4| = |zwy^{-1}| = |z^2| = 1$, $m_6 = |s_6| = |u| = d(k)$. Therefore $\mathcal{A}_k$ has local type $(3^6.d(k))$. Hence there are $|D_{2k}|\frac{6}{3} = 8k$ triangular faces and $2gcd(2k, k-1)$ of the faces are $d(k)$-gons. Euler formula (2) complete the proof. $\square$

Proposition 4.4 *$\mathcal{A}_k$ is Archimedean if and only if k is odd.*
The family $\{\mathcal{A}_{2m+1}\}_{m\in\mathbb{N}}$ is composed of accurate Archimedean maps.

Proof. Taking into account that the faces (regarded as sets of vertices) containing vertex 1 are

$$f_0 = \{1, y^k, y^{k+1}\};\ f_1 = \{1, y, y^k\};\ f_2 = \{1, z, y\};\ f_3 = \{1, zy, z\};$$

$$f_4 = \{1, y^{2k-1}, zy\};\ f_5 = \{1, y^{k-1}, y^{2k-1}\} \qquad \text{and}$$

$$f_6 = \begin{cases} \{y^i : i = 0, \ldots, 2k-1\} & \text{if } k \text{ is even} \\ \{y^{2i} : i = 0, \ldots, k-1\} & \text{if } k \text{ is odd} \end{cases},$$

in analogy with the proof of Proposition 4.2 we have that, $\mathcal{A}_k$ is polyhedral and therefore Archimedean, if and only if k is odd.

For $m > 1$, $\mathcal{A}_{2m+1}$ is clearly accurate since its local type has no non-trivial rotational symmetry. For $m = 1$, that is, for local type $(3^6.3)$, another argument must be given. The word relation $xy^3 = 1$ says that there is a closed walk containing the vertices $1, x, xy, xy^2$, which gives rise to the monodromy element $W = R^3LR^4LR^4LRL$ in the stabilizer of $(1, x)$:

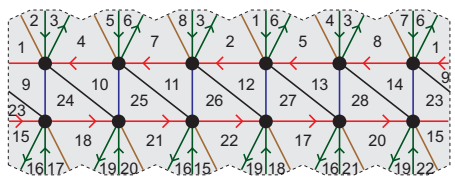
$$\begin{aligned}(1,x) \quad &\xrightarrow{L} (x,x) \xrightarrow{R} (x,y) \xrightarrow{L} (xy, y^{-1}) \xrightarrow{R^4} (xy, y) \xrightarrow{L} (xy^2, y^{-1}) \\ &\xrightarrow{R^4} (xy^2, y) \xrightarrow{L} (xy^3, y^{-1}) = (1, y^{-1}) \xrightarrow{R^3} (1, x)\,.\end{aligned}$$

Now, if there is a non-trivial automorphism φ that fixes vertex 1, we may assume that φ is a 1-step rotation around vertex 1 (since vertex-valency is prime), say $\varphi(1,x) = (1,y)$. Then

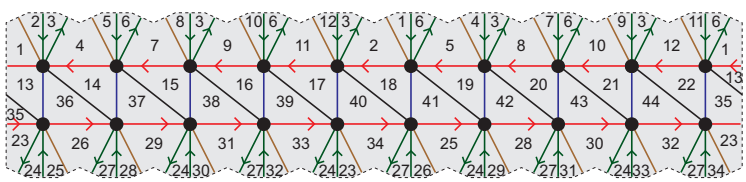
$$\varphi W(1,x) = W\varphi(1,x) \Leftrightarrow \varphi(1,x) = W(1,y) \Leftrightarrow (1,y) = W(1,y)\,,$$

that is, W fixes the dart $(1,y)$. But one can easily check that $W(1,y) = (yuwx, w) \neq (1,y)$. Hence the vertex-stabilizer must be trivial and so $\mathcal{A}_3$ is also accurate. $\square$

Members of the family of accurate Archimedean maps $\{\mathcal{A}_{2m+1}\}_{m\in\mathbb{N}}$ will be called *generalized antiprisms*.



$\mathcal{A}_3 = \big(D_6; (\,x \;\; y \;\; z \;\; w \;\; y^{-1} \;\; u \;\; u^{-1})\big)$ with 28 faces and genus 2.



$\mathcal{A}_5 = \big(D_{10}; (\,x \;\; y \;\; z \;\; w \;\; y^{-1} \;\; u \;\; u^{-1})\big)$ with 44 faces and genus 4.

Figure 5: The first two maps of the family of generalized antiprisms (numbers label faces).

In $\mathcal{A}_5$, the faces numbered 3, 6, 24 and 27 are pentagons.

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