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Discrete Hypercomplex Function Theory and its Applications

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Abstract. Recently, one can observe an increased interest in discrete function theories and their applications. Although we will give a broader overview in our talk we would like to give a closer idea on the topic and its applications. To this end we present the question of boundary values of discrete monogenic functions in this short text. We also show their applicability in the theory of discrete Riemann boundary value problems (Riemann BVP's). The grid itself was chosen in view of applications to image processing, such as discrete monogenic functions.

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PRELIMINARIES

Let us begin with some basic facts of discrete function theory. Without loss of generality we restrict ourselves to the (for practical applications) most important case $d = 3$.

Consider the grid $h\mathbb{Z}^3$, with orthonormal basis e_k, $k = 1,2,3$, with $h > 0$ being the lattice constant (mesh size) and the standard forward and backward differences $\partial_h^{\pm j}$ given by

$$\partial_h^{+j} f(mh) = h^{-1}(f(mh+e_jh) - f(mh)), \quad \partial_h^{-j} f(mh) = h^{-1}(f(mh) - f(mh-e_jh)),$$

for $hm = h(m_1e_1 + m_2e_2 + m_3e_3) \in h\mathbb{Z}^3$. For a discrete function theory we introduce a discrete Dirac operator which factorizes the star-Laplacian Δ_h by splitting each basis element $e_k, k = 1,2,3$, into two basis elements e_k^+ and $e_k^-, k = 1,2,3$, i.e., $e_k = e_k^+ + e_k^-, k = 1,2,3$, corresponding to the forward and backward directions. The new basis elements should satisfying the following multiplication relations:

$$\begin{cases} e_j^- e_k^- + e_k^- e_j^- &= \quad 0, \\ e_j^+ e_k^+ + e_k^+ e_j^+ &= \quad 0, \\ e_j^+ e_k^- + e_k^- e_j^+ &= \quad -\delta_{jk}, \end{cases} \tag{1}$$

where δ_{jk} is the Kronecker delta. These basis elements $\{e_1^\pm, e_2^\pm, e_3^\pm\}$ generate the complexified Clifford algebra $\mathbb{C}_3 = \mathbb{C} \otimes_\mathbb{R} \mathbb{R}_{0,3}$. Keep in mind that the norm is no longer preserved under multiplication. In fact, we have only the estimate (see [3])

$$|ab| \leq 2^3|a||b|, \quad a,b \in \mathbb{C}_3.$$

Furthermore, we consider functions defined on $(\emptyset \neq) G \subset h\mathbb{Z}^3$ and taking values in $\mathbb{C}_3$. Properties like l_p-summability $(1 \leq p < \infty)$ and so on, are defined for a $\mathbb{C}_3$-valued function by being ascribed to each component. The corresponding spaces are denoted, respectively, by $l_p(G, \mathbb{C}_3), (1 \leq p < +\infty)$ and so on.

The discrete Dirac operator D^{+-} and its adjoint operator D^{-+} are defined by

$$D_h^{+-} = \sum_{j=1}^{3} e_j^+ \partial_h^{+j} + e_j^- \partial_h^{-j}, \quad D_h^{-+} = \sum_{j=1}^{3} e_j^+ \partial_h^{-j} + e_j^- \partial_h^{+j}.$$

Each operator factorizes the star-Laplacian (a self-adjoint operator)

$$\Delta_h = \sum_{j=1}^{3} \partial_h^{+j} \partial_{\mathbf{h}}^{-\mathbf{j}},$$

i.e.,

$$(D_h^{+-})^2 = (D_h^{-+})^2 = -\Delta_h.$$

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This idea is different from the case of discrete complex analysis [6]. For more details we refer the reader to the literature, e.g. [1],[5],[3],[4].

DISCRETE BOUNDARY VALUES

The principal difference between discrete and continuous boundary values is that in the first ones are defined on two boundary layers while only one layer is enough in the second case. This is a direct consequence of having two difference operators (forward and backward differences) instead of a single derivative.

Here we present the following two theorems 1 and 3, where the boundary value of discrete monogenic functions in the upper, resp. lower, lattice is characterized in terms of their behaviour in the Fourier domain, as well as their corollaries, which provided us with closed formulae for the symbols of the discrete Hilbert transforms in the upper, resp. the lower, lattice. The proofs can be found in [1].

Theorem 1 *Let $f \in l_p(h\mathbb{Z}^2, \mathbb{C}_3)$ given by $f = f_1 + e_3^+ f_2 + e_3^- f_3 + e_3^+ e_3^- f_4$, with $f_i : h\mathbb{Z}^2 \to \mathbb{C}_2, i = 1,2,3,4$. Then f is the boundary value of a discrete monogenic function in the discrete upper half plane if and only if its discrete 2D-Fourier transform $F = \mathscr{F}_h f$, with*

$$F(\underline{\xi}) = F_1(\underline{\xi}) + e_3^+ F_2(\underline{\xi}) + e_3^- F_3(\underline{\xi}) + e_3^+ e_3^- F_4(\underline{\xi}), \quad \underline{\xi} \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^2,$$

satisfies the system

$$\begin{cases} \frac{h\underline{d} - \sqrt{4+h^2\underline{d}^2}}{2} F_1 + \frac{\widetilde{\underline{\xi}}_-}{\underline{d}} F_2 = 0, \\ \frac{h\underline{d} - \sqrt{4+h^2\underline{d}^2}}{2} F_3 + \frac{\widetilde{\underline{\xi}}_-}{\underline{d}} (F_1 - F_4) = 0. \end{cases} \tag{2}$$

Corollary 2 *Let $f \in l_p(h\mathbb{Z}^2, \mathbb{C}_3)$ be a boundary value of a discrete monogenic function in the upper half space. Then its 2D-Fourier transform $F = \mathscr{F}_h f$, satisfies the equation*

$$\frac{\widetilde{\underline{\xi}}_-}{\underline{d}} \left(e_3^+ \frac{h\underline{d} - \sqrt{4+h^2\underline{d}^2}}{2} + e_3^- \frac{2}{h\underline{d} - \sqrt{4+h^2\underline{d}^2}} \right) F = F.$$

Theorem 3 *Let $f \in l_p(h\mathbb{Z}^2, \mathbb{C}_3)$ given by $f = f_1 + e_3^+ f_2 + e_3^- f_3 + e_3^+ e_3^- f_4$, with $f_i : h\mathbb{Z}^2 \to \mathbb{C}_2, i = 1,2,3,4$. Then f is the boundary value of a discrete monogenic function in the discrete lower half plane if and only if its 2D-Fourier transform $F = \mathscr{F}_h f$, with*

$$F(\underline{\xi}) = F_1(\underline{\xi}) + e_3^+ F_2(\underline{\xi}) + e_3^- F_3(\underline{\xi}) + e_3^+ e_3^- F_4(\underline{\xi}), \quad \underline{\xi} \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^2,$$

satisfies the system

$$\begin{cases} \frac{h\underline{d} - \sqrt{4+h^2\underline{d}^2}}{2} F_2 - \frac{\widetilde{\underline{\xi}}_-}{\underline{d}} F_1 = 0, \\ \frac{h\underline{d} - \sqrt{4+h^2\underline{d}^2}}{2} (F_1 - F_4) - \frac{\widetilde{\underline{\xi}}_-}{\underline{d}} F_3 = 0. \end{cases} \tag{3}$$

Corollary 4 *Let $f \in l_p(h\mathbb{Z}^2, \mathbb{C}_3)$ be a boundary value of a discrete monogenic function in the lower half space. Then its 2D-Fourier transform $F = \mathscr{F}_h f$, satisfies the equation*

$$-\frac{\widetilde{\underline{\xi}}_-}{\underline{d}} \left(e_3^+ \frac{2}{h\underline{d} - \sqrt{4+h^2\underline{d}^2}} + e_3^- \frac{h\underline{d} - \sqrt{4+h^2\underline{d}^2}}{2} \right) F = F. \tag{4}$$

Both theorems and their corollaries allow us to introduce upper and lower discrete Hardy spaces as the space of functions which fulfill (2) and (3), respectively.

Since we have two boundary layers one could think that the discrete boundary values are independent on both layers. This is not the case as can be seen in the next theorem where we introduce the operators A^+, A^-, which reconstruct the boundary data in the $0-$layer from the knowledge of boundary data in the $1-$layer, resp. $-1-$layer, thus giving the complete boundary data of a function in the upper, resp. lower, discrete Hardy space.

Theorem 5 *An arbitrary function $f \in l_p(\mathbb{Z}^2, \mathbb{C}_3)$ can be decomposed into a pair of functions $P_+ f$ and $Q_+ f$ where $P_+ f \in h_p^+$, i.e. it can be extended to the zero layer via its action in the Fourier domain by*

$$e_3^- F^{+,0} = \frac{\widetilde{\underline{\xi}}_-}{\underline{d}} \frac{h\underline{d} + \sqrt{4 + h^2 \underline{d}^2}}{2} \left(-e_3^- F_1^{+,1} + e_3^- e_3^+ F_3^{+,1} \right) := A^+ F^{+,1},$$

with $F^{+,1} = \mathscr{F}_h f$ and $F^{+,1}$ fulfills (2). In the same way an arbitrary function $f \in l_p(\mathbb{Z}^2, \mathbb{C}_3)$ can be decomposed into a pair of functions $P_- f$ and $Q_- f$ where $P_- f \in h_p^-$, i.e. it can be extended to the zero layer via its action in the Fourier domain by

$$\begin{aligned} e_3^+ F^{-,0} &= \frac{2\sqrt{4 + h^2 \underline{d}^2}}{h\underline{d} + \sqrt{4 + h^2 \underline{d}^2}} \left(-e_3^+ F_1^{-,-1} - e_3^+ e_3^- F_3^{-,-1} \right) \\ &+ \frac{\widetilde{\underline{\xi}}_-}{\underline{d}} \left(\frac{h^2 \underline{d}^2 - 1 - h\underline{d}\sqrt{4 + h^2 \underline{d}^2}}{h\underline{d} - \sqrt{4 + h^2 \underline{d}^2}} \right) \left(e_3^+ F_2^{-,-1} - e_3^+ e_3^- \left(F_1^{-,-1} - F_4^{-,-1} \right) \right) := A^- F^{-,-1} \end{aligned}$$

with $F^{-,-1} = \mathscr{F}_h f$ and $F^{-,-1}$ fulfills (3).

DISCRETE RIEMANN BVP

The above considerations allow us to study discrete Riemann boundary value problems. Two examples are given below. The proofs can be found in [2] although they are essentially an adaptation of the classic proofs using the results from the previous section.

Theorem 6 *The problem*

$$\begin{aligned} D_h^{-+} f(n) &= 0, n \in \mathbb{Z}^3 \backslash \{n_3 = 0\}, \\ f(\underline{n}, 1) &= g(\underline{n}), \underline{n} \in \mathbb{Z}^2. \end{aligned}$$

with $g \in h_p^+(\mathbb{Z}^2)$ has a unique solution given by the respective Hardy/Plemelj-projections.

Theorem 7 *The problem*

$$\begin{aligned} D_h^{-+} f(n) &= 0, n \in \mathbb{Z}^3 \backslash \{n_3 = 0\}, \\ A^+ \left(f(\underline{n}, 1) \right) &= A^- \left(f(\underline{n}, -1) \right) + g(\underline{n}), \underline{n} \in \mathbb{Z}^2. \end{aligned}$$

with $g \in l_p(\mathbb{Z}^2)$ has a unique solution given by the respective Hardy/Plemelj-projections.

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