Some results in fractional Clifford analysis*

N. Vieira‡

‡ CIDMA - Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
E-mail: mferreira@ua.pt,nloureirovieira@gmail.com

Abstract

What is nowadays called (classic) Clifford analysis consists in the establishment of a function theory for functions belonging to the kernel of the Dirac operator. While such functions can very well describe problems of a particle with internal $SU(2)$-symmetries, higher order symmetries are beyond this theory. Although many modifications (such as Yang-Mills theory) were suggested over the years they could not address the principal problem, the need of a n-fold factorization of the d'Alembert operator.

In this paper we present the basic tools of a fractional function theory in higher dimensions, for the transport operator ($\alpha = \frac{1}{2}$), by means of a fractional correspondence to the Weyl relations via fractional Riemann-Liouville derivatives. A Fischer decomposition, fractional Euler and Gamma operators, monogenic projection, and basic fractional homogeneous powers are constructed.

Keywords: Fractional monogenic polynomials, Fischer decomposition, Fractional Dirac operator, Riemann-Liouville fractional derivative, Stationary transport operator.

1 Introduction

In the last decades the interest in fractional calculus increased substantially. This fact is due to on the one hand different problems can be considered in the framework of fractional derivatives like, for example, in optics and quantum mechanics, and on the other hand fractional calculus gives us a new degree of freedom which can be used for more complete characterization of an object or as an additional encoding parameter.

Over the last decades F. Sommen and his collaborators developed a method for establishing a higher dimension function theory based on the so-called Weyl relations [1, 3, 2]. In more restrictive settings it is nowadays called Howe dual pair technique (see [6]). Its focal point is the construction of an operator algebra (classically $\mathfrak{osp}(1|2)$) and the resulting Fischer decomposition.

The aim of this paper is to present a Fischer decomposition, when considering the fractional Dirac operator defined via Riemann-Liouville derivatives, where the fractional parameter is equal to $\frac{1}{2}$ (which leads to the case of the stationary transport operator). The results presented here correspond to a restriction of correspondent ones presented in [7] for the particular case of $\alpha = \frac{1}{2}$.

In the Preliminaries we recall some basic facts about Clifford analysis and fractional calculus. In Section 3, we introduce the corresponding Weyl relations for this fractional setting and the notion of a fractional homogeneous polynomial. In the same section we present the fractional correspondence to the Fischer decomposition. In the final section we construct the projection of a given fractional homogeneous polynomial into the space of fractional homogeneous monogenic polynomials. We also calculate the dimension of the space of fractional homogeneous monogenic polynomials.

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2 Preliminaries

We consider the d-dimensional vector space $\mathbb{R}^d$ endowed with an orthonormal basis $\{e_1, \cdots,$
$e_d\}$. We define the universal real Clifford algebra $\mathbb{R}_{0,d}$ as the 2^d-dimensional associative algebra which obeys the multiplication rules $e_i e_j + e_j e_i = -2\delta_{i,j}$. A vector space basis for $\mathbb{R}_{0,d}$ is generated by the elements $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1, \ldots, h_k\} \subset M = \{1, \ldots, d\}$, for $1 \leq h_1 < \cdots < h_k \leq d$. An important subspace of the real Clifford algebra $\mathbb{R}_{0,d}$ is the so-called space of paravectors $\mathbb{R}_1^d = \mathbb{R} \bigoplus \mathbb{R}^d$, being the sum of scalars and vectors. An important property of algebra $\mathbb{R}_{0,d}$ is that each non-zero vector $x \in \mathbb{R}_1^d$ has a multiplicative inverse given by $\frac{\overline{x}}{||x||^2}$. Now, we introduce the complexified Clifford algebra $\mathbb{C}_d$ as the tensor product

$$\mathbb{C} \otimes \mathbb{R}_{0,d} = \left\{ w = \sum_A w_A e_A, \ w_A \in \mathbb{C}, A \subset M \right\},$$

where the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e., $ie_j = e_j i$ for all $j = 1, \ldots, d$.

An $\mathbb{C}_d-$valued function f over $\Omega \subset \mathbb{R}_1^d$ has representation $f = \sum_A e_A f_A$, with components $f_A : \Omega \to \mathbb{C}$. Properties such as continuity are understood component-wisely. Next, we recall the Euclidean Dirac operator $D = \sum_{j=1}^d e_j \ \partial_{x_j}$, which factorizes the d-dimensional Euclidean Laplacian, i.e., $D^2 = -\Delta = -\sum_{j=1}^d \partial x_j^2$. A $\mathbb{C}_d$-valued function f is called *left-monogenic* if it satisfies $Du = 0$ on Ω (resp. *right-monogenic* if it satisfies $uD = 0$ on Ω). For more details about Clifford algebras and monogenic function we refer [2].

The most widely known definition of the fractional derivative is the so-called Riemann-Liouville definition:

$$\left(D_{a^+}^\alpha f\right)(x) = \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f(t)}{(x-t)^{\alpha-n+1}} \, dt, \qquad n = [\alpha]+1, \quad a, x > 0. \tag{1}$$

where $[\alpha]$ means the integer part of α. In [4] fractional derivative (1) was successfully applied in the definition of the fractional correspondent of the Dirac operator in the context of Clifford analysis. In fact, for the particular case of $\alpha = \frac{1}{2}$, the fractional Dirac operator corresponds to $\mathbf{D} = \sum_{j=1}^d e_j \ \mathbf{D}_j = \sum_{j=1}^d e_j \ \ (D_j + Y_j)$, where $D_j = \partial_{x_j}$ and $Y_j = \frac{1}{2(\xi_j - x_j)}$ with $\xi = (\xi_1, \ldots, \xi_d)$ the observer time vector. Moreover, we have that $\mathbf{DD} = D$ with D the Euclidian Dirac operator, i.e., the stationary transport operator. A $\mathbb{C}_n$-valued function f is called *fractional left-monogenic* if it satisfies $\mathbf{D}u = 0$ on Ω (resp. *fractional right-monogenic* if it satisfies $u\mathbf{D} = 0$ on Ω). We observe that due to the definition of $\mathbf{D}$ we have that

$$\mathbf{D} \left(\prod_{i=1}^d (\xi_i - x_i)^{\frac{1}{2}} \right) = 0, \tag{2}$$

i.e., $\prod_{i=1}^d (\xi_i - x_i)^{\frac{1}{2}}$ is a fractional monogenic function. The fractional power $(\xi_i - x_i)^{\frac{1}{2}}$ corresponds to $(\xi_j - x_j)^{\frac{1}{2}}$ if $\xi_j \geq x_j$, or $(x_j - \xi_j)^{\frac{1}{2}} i$ if $\xi_j < x_j$, with $j = 0, 1, \ldots, d$. From now until the end of the paper, we consider paravectors of the form $\underline{\mathbf{x}} = \mathbf{x}_0 + \mathbf{x}$, where $\mathbf{x} = \sum_{j=1}^d e_j \ \mathbf{x}_j$ with $\mathbf{x}_j = \frac{\xi_j - x_j}{2}$.

3 Weyl relations and Fractional Fischer decomposition

The aim of this section is to provide the basic tools for a function theory for the fractional Dirac operator defined via fractional Riemann-Liouville derivatives for the particular case of $\alpha = \frac{1}{2}$.

3.1 Fractional Weyl relations

Now we introduce the fractional correspondence of the classical Euler and Gamma operators. Furthermore, we show that the two natural operators $\mathbf{D}$ and $\mathbf{x}$, considered as odd elements, generate a finite-dimensional Lie superalgebra in the algebra of endomorphisms generated by the partial fractional Riemann-Liouville derivatives, the basic *vector variables* $\mathbf{x}_j$ (seen as multiplication operators), and the basis of the Clifford algebra e_j.

In order to obtain our results, we use some standard technique in higher dimensions, namely we study the commutator and the anti-commutator between $\mathbf{x}$ and $\mathbf{D}$. We start proposing the following fractional Weyl relations

$$[\ \mathbf{D}_i, \mathbf{x}_i\ ] = \mathbf{D}_i \ \mathbf{x}_i - \mathbf{x}_i \ \mathbf{D}_i = -\frac{1}{2}, \tag{3}$$

with $i=1,\ldots,d$. This leads to the following relations for $\mathbf{x}$ and $\mathbf{D}$:

$$\{\mathbf{D},\mathbf{x}\} = \mathbf{D}\mathbf{x}+\mathbf{x}\mathbf{D} = -2\mathbb{E}+\frac{d}{2}, \qquad [\mathbf{x},\mathbf{D}] = \mathbf{x}\mathbf{D}-\mathbf{D}\mathbf{x} = -2\mathbf{\Gamma}-\frac{d}{2}, \tag{4}$$

where $\mathbb{E}$, $\mathbf{\Gamma}$ are, respectively, the fractional Euler and Gamma operators of order $\frac{1}{2}$, and have the following expressions

$$\mathbb{E} = \sum_{i=1}^{d} \mathbf{x}_i\ \mathbf{D}_j, \qquad \mathbf{\Gamma} = \sum_{i<j} e_i e_j\left(\mathbf{x}_i\ \mathbf{D}_j - \mathbf{D}_i\ \mathbf{x}_j\right). \tag{5}$$

From (5) we derive

$$[\mathbf{x},\mathbb{E}] = \frac{1}{2}\ \mathbf{x}, \qquad [\mathbf{D},\mathbb{E}] = -\frac{1}{2}\ \mathbf{D}, \tag{6}$$

which allow us to conclude that we have a finite dimensional Lie superalgebra generated by $\mathbf{x}$ and $\mathbf{D}$, isomorphic to $\mathfrak{osp}(1|2)$. Now we introduce the definition of fractional homogeneity of a polynomial by means of the fractional Euler operator.

Definition 3.1 *A polynomial P_l is called fractional homogeneous of degree $l\in\mathbb{N}_0$, if and only if $\mathbb{E}P_l = -\frac{l}{2}\ P_l$.*

We observe that from the previous definition the basic fractional homogeneous powers are given by $\prod_{j=1}^{d}\mathbf{x}_j^{\beta_j}$, with $l=|\beta|=\beta_1+\ldots+\beta_d$. In combination with the first relation in (6) this definition also implies that the multiplication of a fractional homogeneous polynomial of degree l by $\mathbf{x}$, results in a fractional homogeneous polynomial of degree $l+1$, and thus may be seen as a raising operator. Moreover, we can also ensure that for a fractional homogeneous polynomial P_l of degree l, $\mathbf{D}P_l$ is a fractional homogeneous polynomial of degree $l-1$. Furthermore, Weyl's relations (3) enable us to construct fractional homogeneous polynomials, recursively.

3.2 Fractional Fischer decomposition

A fractional Fischer inner product of two fractional homogeneous polynomials P and Q would have the following form

$$\langle P(\mathbf{x}), Q(\mathbf{x})\rangle = \mathrm{Sc}\left[\overline{P(\partial_{\mathbf{x}})}\ Q(\mathbf{x})\right], \tag{7}$$

where $\partial_{\mathbf{x}}$ represents $\mathbf{D}_j$, and $P(\partial_{\mathbf{x}})$ is a differential operator obtained by replacing in the polynomial P each variable $\mathbf{x}_j$ by the corresponding fractional derivative, i.e. D_j+Y_j. From (7) we have that for any polynomial P_{l-1} of homogeneity $l-1$ and any polynomial Q_l of homogeneity l the relation $\langle \mathbf{x}\ P_{l-1},\ Q_l\rangle = \langle P_{l-1}, \mathbf{D}Q_l\rangle$. This fact allows us to obtain the following result:

Theorem 3.2 *For each $l\in\mathbb{N}_0$ we have $\Pi_l = \mathcal{M}_l + \mathbf{x}\ \Pi_{l-1}$, where Π_l denotes the space of fractional homogeneous polynomials of degree l and $\mathcal{M}_l$ denotes the space os fractional monogenic homogeneous polynomials of degree l. Moreover, the subspaces $\mathcal{M}_k$ and $\mathbf{x}\ \Pi_{l-1}$ are orthogonal with respect to the Fischer inner product (7).*

The proof is analogous to the proof of Theorem 3.5 in [7] but considering $\alpha=\frac{1}{2}$, and therefore we omit it from the paper. As a result of the previous theorem we obtain the fractional Fischer decomposition with respect to the fractional Dirac operator $\mathbf{D}$.

Theorem 3.3 *Let P_l be a fractional homogeneous polynomial of degree l. Then*

$$P_l = M_l + \mathbf{x}\ M_{l-1} + \mathbf{x}^2\ M_{l-2} + \ldots + \mathbf{x}^l\ M_0, \tag{8}$$

where each M_j denotes the fractional homogeneous monogenic polynomial of degree j. More specifically, $M_0=P_0$ and $M_l=\{u\in P_l:\quad \mathbf{D}u=0\}$.

The spaces represented in (8) are orthogonal to each other with respect to the Fischer inner product (7). This a consequence of the construction of the fractional Euler operator $\mathbb{E}$ (see (5)), and in particular of (4).

3.3 Explicit formulae

Here we obtain an explicit formula for the projection $\pi_{\mathcal{M}}(P_l)$ of a given fractional homogeneous polynomial P_l into the space of fractional homogeneous monogenic polynomials. We start with the following auxiliary result:

Theorem 3.4 *For any fractional homogeneous polynomial P_l and any positive integer s, we have* $\mathbf{D}\mathbf{x}^s P_l = g_{s,l}\mathbf{x}^{s-1}P_l + (-1)^s\mathbf{x}^s\mathbf{D}P_l$, *where* $g_{2k,l} = k$ *and* $g_{2k+1,l} = k + l + \frac{d}{2}$.

Proof: The proof follows, by induction and straightforward calculations, from the commutation between $\mathbf{D}$ and $\mathbf{x}^s$ using the relations $\mathbf{D}\mathbf{x} = -2\mathbb{E} + \frac{d}{2} - \mathbf{x}\,\mathbf{D}$ and $\mathbb{E}\mathbf{x} = \mathbf{x}\,\mathbb{E} - \frac{1}{2}\mathbf{x}$.

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Let us now compute an explicit form of the projection $\pi_{\mathcal{M}}(P_l)$.

Theorem 3.5 *Consider the constants $c_{j,l}$ defined by $c_{0,l} = 1$, and* $c_{j,l} = \frac{(-1)^j\ \Gamma\left(\frac{d}{2}+l-1-\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{d}{2}+1\right)\ \Gamma\left(1+2\left[\frac{j}{2}\right]\right)}$, *where $j = 1, \ldots, l$ and $[\cdot]$ represents the integer part. Then the map $\pi_{\mathcal{M}}$ given by*

$$\pi_{\mathcal{M}}(P_l) := P_l + c_{1,l}\ \mathbf{x}\ \mathbf{D}\ P_l + c_{2,l}\ \mathbf{x}^2\mathbf{D}^2\ P_l + \ldots + c_{l,l}\ \mathbf{x}^l\mathbf{D}^l\ P_l$$

is the projection of the fractional homogeneous polynomial P_l into the space of fractional homogeneous monogenic polynomials.

Proof: Let us consider the linear combination

$$r = a_0\ P_l + a_1\ \mathbf{x}\ \mathbf{D}\ P_l + a_2\ \mathbf{x}^2\ \mathbf{D}^2\ P_l + \ldots + a_k\ \mathbf{x}^l\ \mathbf{D}^l\ P_l,$$

with $a_0 = 1$. If there are constants a_j, $j = 1, \ldots, l$, such that $r \in \mathcal{M}_l$, then r is equal to $\pi_{\mathcal{M}}(P_l)$. Indeed, we know that $P_l = \mathcal{M}_l \oplus \mathbf{x}P_{l-1}$ and $r = P_l + Q_{l-1}$, with $Q_{l-1} = \sum_{i=1}^{l} a_i\ \mathbf{x}^i\ \mathbf{D}^i\ P_l$. Applying Theorem 3.4, we get

$$\begin{aligned}
0 &= \mathbf{D}(\pi_{\mathcal{M}}(P_l)) \\
&= \mathbf{D}\ P_l + a_1\ \mathbf{D}\ \mathbf{x}\ \mathbf{D}\ P_l + a_2\ \mathbf{D}\ \mathbf{x}^2\ \mathbf{D}^2\ P_l + \ldots + a_l\ \mathbf{D}\ \mathbf{x}^l\mathbf{D}^l\ P_l \\
&= (1 + a_1\ g_{1,l-1})\ \mathbf{D}\ P_l + (-a_1 + a_2\ g_{2,l-2})\mathbf{x}\ \mathbf{D}^2\ P_l + (a_2 + a_3\ g_{3,l-3})\ \mathbf{x}^2\ \mathbf{D}^3\ P_l \\
&\quad + \ldots + ((-1)^{l-1}a_{l-1} + a_l\ g_{l,0})\ \mathbf{x}^{l-1}\ \mathbf{D}^l\ P_l.
\end{aligned}$$

Hence if the relation $(-1)^{j-1}a_{j-1} + a_j g_{j,l-j} = 0$ holds for each $j = 1, \ldots, l$, then the function r is fractional monogenic. By induction we get $a_j = \frac{(-1)^j\ \Gamma\left(\frac{d}{2}+l-1-\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{d}{2}+1\right)\ \Gamma\left(1+2\left[\frac{j}{2}\right]\right)}$.

■

Theorem 3.6 *Each polynomial P_l can be written in a unique way as* $P_l = \sum_{j=0}^{l} \mathbf{x}^j\ M_{l-j}(P_l)$, *where* $M_{l-j}(P_l) = c'_j \sum_{n=0}^{j} c_{j,l-n}\ \mathbf{x}^n\ \mathbf{D}^n\mathbf{D}^{l-j}\ P_l$ *with $j = 0, \ldots, l$, and the coefficients c'_j are defined by* $c'_j = \frac{(-1)^j\ \Gamma\left(\frac{d}{2}+l-1-\left[\frac{j}{2}\right]\right)}{\Gamma\left(\frac{d}{2}+1\right)\ \Gamma\left(1+2\left[\frac{j}{2}\right]\right)}$.

The proof of this result is analogous to the proof of Theorem 3.9, for $\alpha = \frac{1}{2}$, in [7], and therefore we omit it from the paper.

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