GeoGebra, Complex Maps and Riemann Sphere

Abstract:

This paper gives a vision of the work we have been carried out over the past two years, presented at the meetings “III Dia GeoGebra Portugal”, Aveiro University, May, 2013” and “IV Dia GeoGebra Portugal”, Superior School of Education of Oporto Polytechnic Institute, May 2014, and the improvements done so far having in mind didactical purposes. In the first meeting, we have shown how we could use GeoGebra to create colouring domains enabling the representation of complex function graphics and opening a promising path for the exploration of properties of functions of two real variables. In the second meeting we have unveiled the relation between Mobius transformations and the movements of a Riemann Sphere. Now our focus of attention is in the use of this software as a tool for learn, teach and research complex analysis

Domain Colouring in GeoGebra and Representations of Functions from C to C.

The colouring domain is a procedure popularized by Frank Farris (1997), previously used by Larry Crone and Hans Lundmark, based on the use of a spectrum of colours that act as elements of replacement of the not accessible dimension and it was used to represent complex functions of complex variable. Applying this technique and using GeoGebra (Breda et al., 2013) we may obtain graphic representations of complex functions, weaving some considerations about the information through the analysis of these graphics.

Considering a complex function f from C to C, the f(z) real part, f1(x,y), and the f(z) imaginary part, f2(x,y), are functions from R^2 to R. Accordingly, these functions may be represented graphically in a 3D window, but the graph of f need to be represented in another way. We will do it by means of the domain colouring as shown in the 2D graphical view. Figure 1illustates the colouring domain of the identity function, f, which corresponds to the typical color domain of the complex plane.

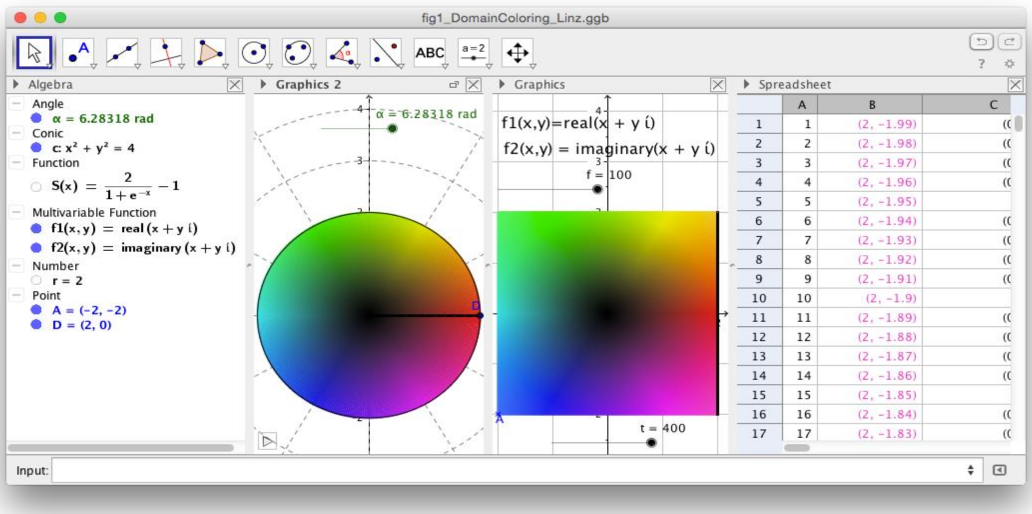


Figure 1 - Application creating the scan of a coloring two-dimensional domain in GeoGebra.

Taking, for example, the square [-2,2] × [-2,2], we may consider for initial free point A=(-2,-2) and then generate in the column B of the worksheet, a list of 400 points numbered by the index of column A (Ai = i, i = 1, ..., 400), defining them by Bi = I + (v, Ai / f) and taking for parameter variation v the interval [0,4]. Making the necessary adjustments and using an analogous procedure we can also have the colored circle.

To assign a color to each point we have made use of the points' trace properties. The mathematical idea behind it is the use of |f(z)|, arg(z) and the position of f(z) in relation to the north pole of the Riemann sphere, identifying the south pole with the zero of the complex plane. Then, for hue we use the angle defined by the complex number f(z) and the real axis, Angle[(1, 0), (0,), (f1(x(B1), y(B1)), f2(x(B1), y(B1)))] / (2π). For the saturation we use the value of the function S in |f(z)|, using S(Distance[(0, 0), (f1(x(B1), y(B1)), f2(x(B1) y(B1)))]), in the parameter s,see figure 2, . Finally, for lightness we use the angle defined by f(z) and the north pole, 2Angle[(0, 0), (0, 0, 2), (f1(x(B1), y(B1)), f2(x(B1), y(B1)))] / π.

This technique may also be used to visualize the action of the Möbius transformations in the complex plane, via stereographic projection, as explained in the followings sections.

Let us consider the complex function $f(z)=z^4$ -1. Using the coloring domain to represent its component functions f1(x,y)=real($(x+ί y)^4$ -1) and f2(x,y) = imaginary($(x+ί y)^4$ -1), we obtain the graphics illustrated in figure 2.

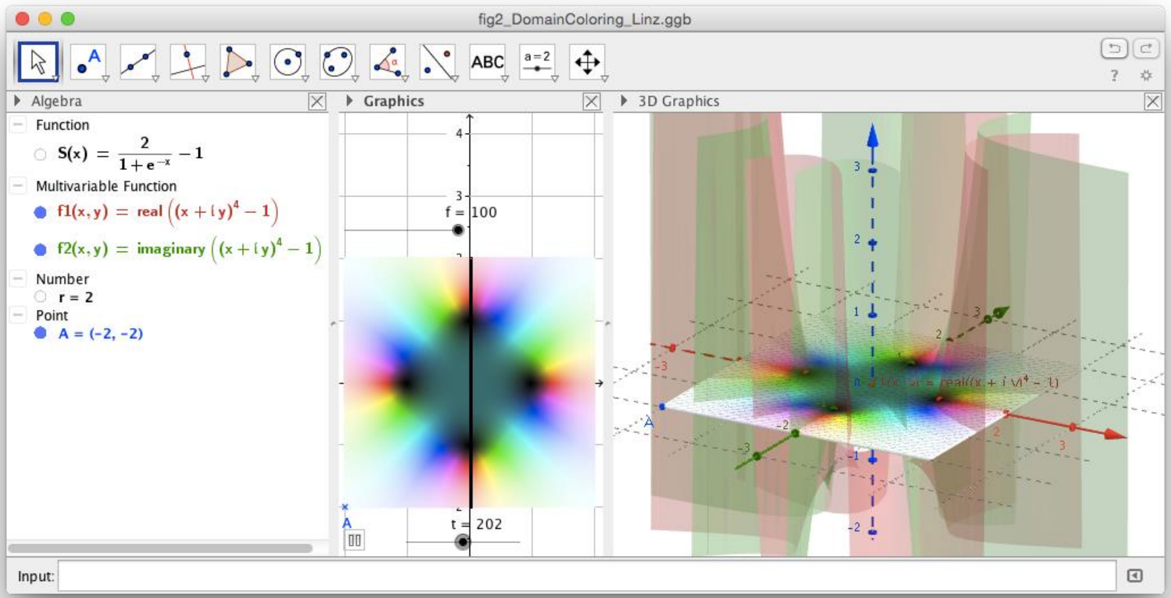


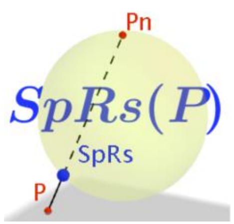
Figure 2 - The behavior of $f(z)=z^4$ -1 and its component functions.

By means of the colouring domain, defined in GeoGebra, the four roots of the polynomial z^4 -1 in C are easily visualized and we may also analyze the behavior of f around the zeros.

Domain Colouring in The Riemann Sphere

The definition of a stereographic projection of the unit sphere S^2 requires a projection point (a point P_0 on the sphere) and a projection plane α (plane perpendicular to the axis defined by the sphere center and the projection point). Geometrically its action can be described as follows: given a point P on the sphere, distinct from the projection point P_0 , its image, by the stereographic projection regarding P_0 and the projection plan α, is the point P' obtained by the intersection of the straight line PP_0 with α. This correspondence is a bijection and so we may identify a sphere without a point with a plane.

For the creation of a tool providing the stereographic projection of a point in the Riemann sphere, we have done as follows:



px=2
py=2
pz=2
Pn=(px,py,pz)
P=1+i
r=1
SpRs=Intersect[line[Pn, P], Sphere[(x(Pn),y(Pn),z(Pn)- r), r], 2]

Tool name: StereographicProjectionRiemannSphere

Description: Given: the pole, 3D point, and the radius of Riemann Sphere: and P(in the Argand Plan) obtain the Stereographic Projection of P in the Riemann Sphere.

Command name: SpRs

We may assign a colouring criteria to the points in the Argand plan and to their corresponding projections on the Riemann sphere using the domain colouring technique, see figure 3.

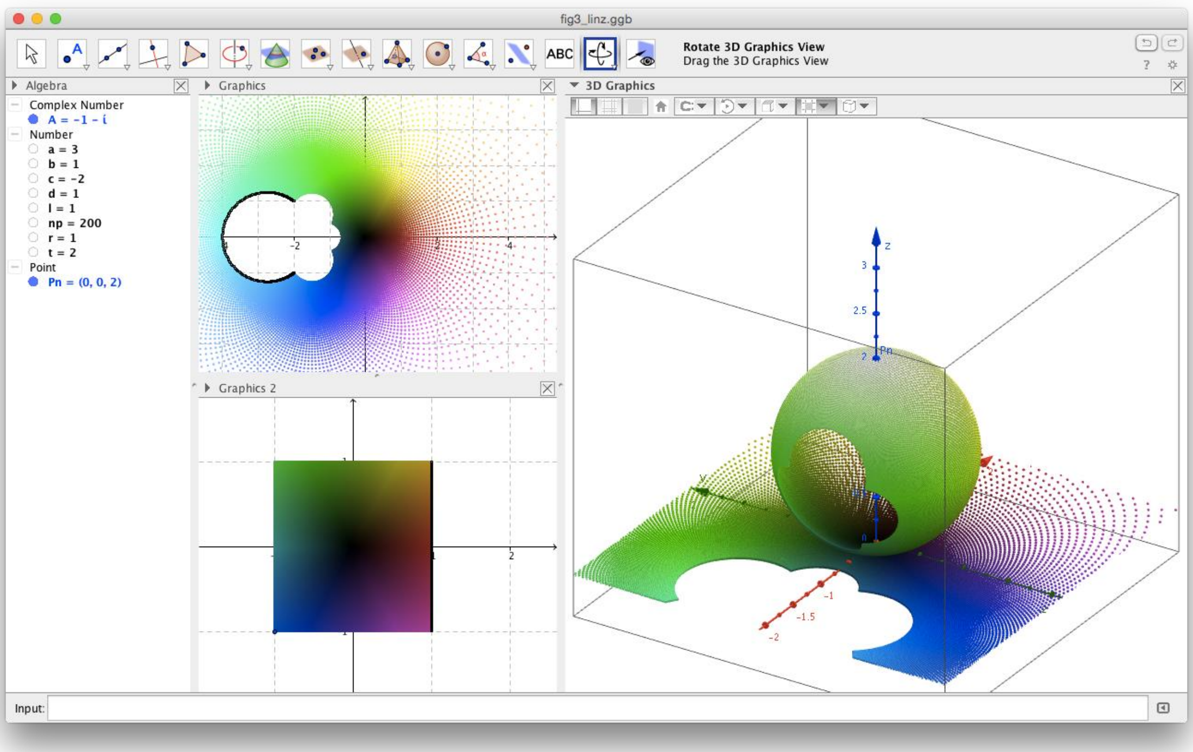


Figure 3 - Domain coloring of the identity map and the corresponding colouring of the Riemann sphere

In this procedure, one of the difficulties we are faced with is that traces of points in the three-dimensional view are not preserved not allowing the exploration of all capabilities of visualization

Möbius Transformation

Using several tools we are capable to construct an application allowing the visualization of the maim properties of Möbius transformations, namely their domain, co-domain and their projections in the Riemann sphere. We may simultaneously obtain several views of related phenomena that allow to infer and to study known properties or obtain new findings.

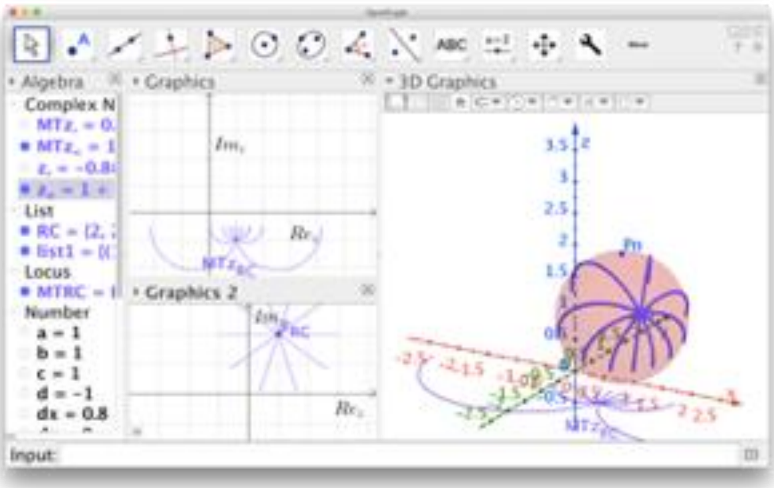


Figure 4 - Effects of a centered grid by Möbius transformation and itd projection in the Riemamn sphere.

Conclusions

In order to use all the capabilities of the coloring domain technique some improvements of GeoGebra are needed. It would be very fruitful if, in future versions, were possible to collect the color information of the locus of points in 2D or 3D; to save patterns of graphics of complex functions and patterns of the Riemann sphere surface of a new object. If so, it would be possible to see the geometric effects of actions of complex functions and in the coloring of the Riemann sphere.

Considering the dynamic characteristic of geometrical transformations, it would be also interesting to improve the way to get animated gif's or even videos. This feature would be of great help to relate Möbius transformations with movements of the sphere.

In spite of the functionality of the created tool, the performance is quite low, its effects are not quickly visualized and the procedures are too exigent for the hardware.