

V Dia do Geogebra
Começando nos primeiros anos
Escola Superior de Educação de Lisboa
9 de maio

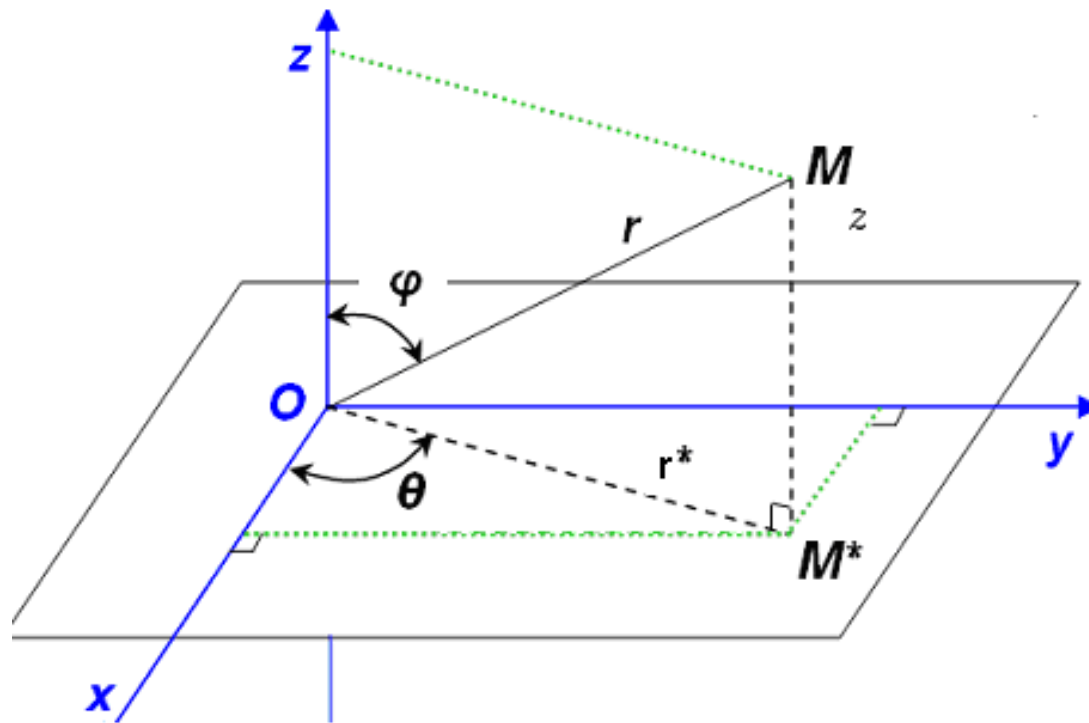


Pavimentar a esfera

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Lisboa 2015

1. Coordenadas Esféricas

(r, θ, φ)



$$M = (x, y, z)$$

$$M^* = (x, y, 0)$$

$$z = r \sin(90 - \phi) = r \cos \phi$$

$$r^* = r \cos(90 - \phi) = r \sin \phi$$

$$x = r^* \cos \theta = r \sin \phi \cos \theta$$

$$y = r^* \sin \theta = r \sin \phi \sin \theta$$

$$z = r \cos \phi$$

$$r \in [0, \infty), \theta \in [0, 2\pi], \phi \in [0, \pi]$$

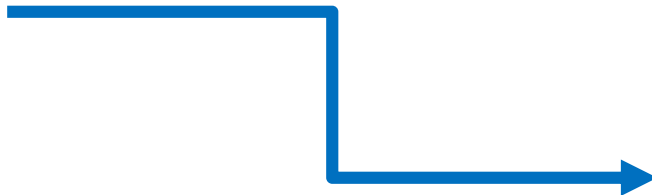
Coordenadas
Esféricas



Coordenadas
Cartesianas

$$\mathbf{M} = (r, \theta, \varphi)$$

$$\mathbf{M} = (x, y, z)$$



$$\begin{aligned}x &= r^* \cos \theta = r \sin \phi \cos \theta \\y &= r^* \sin \theta = r \sin \phi \sin \theta \\z &= r \cos \phi\end{aligned}$$

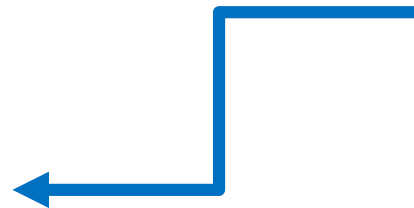
$$\mathbf{M} = (r, \theta, \varphi)$$

$$\mathbf{M} = (x, y, z)$$

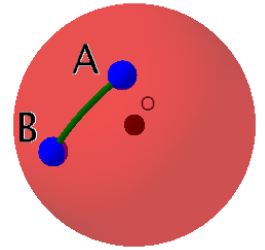
$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\phi = \arctan \frac{y}{x}$$

$$\theta = \arctan \frac{\sqrt{x^2 + y^2}}{z}$$



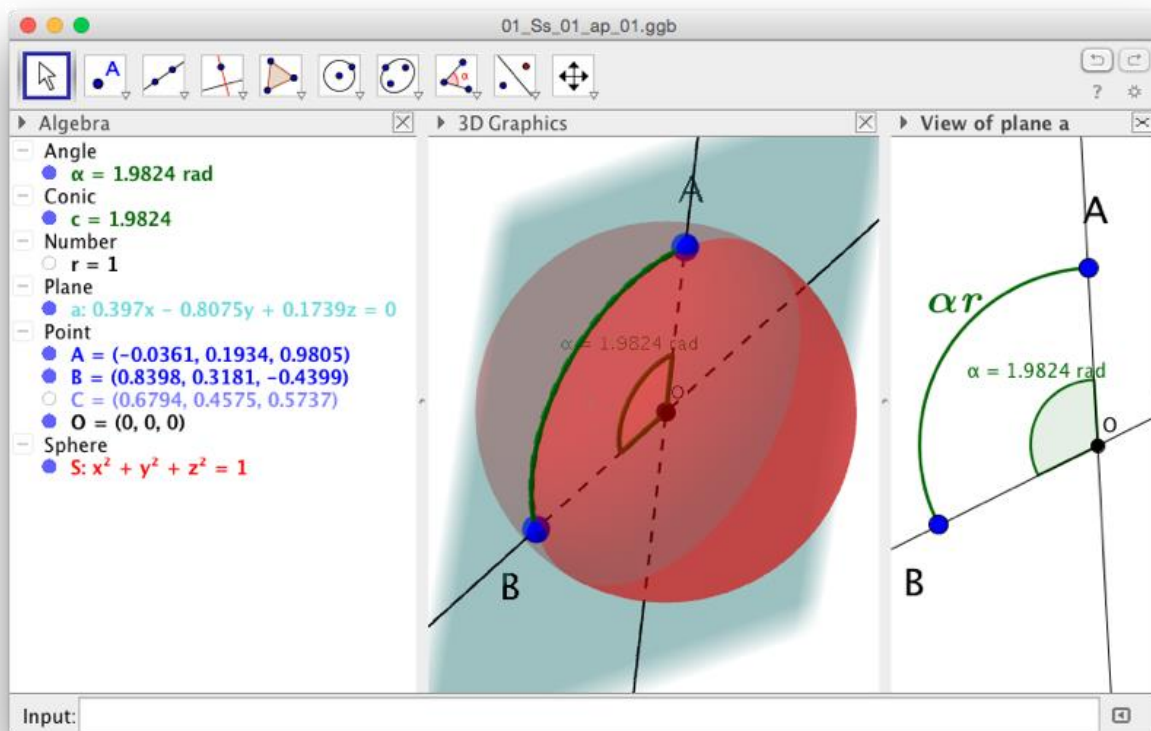
2. A ferramenta segmento esférico



01_Ss_01_t.ggb

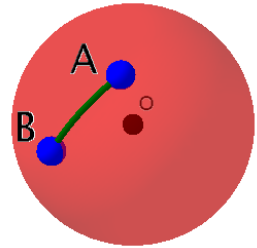
01_Ss_01_it.ggt

Questão: Dados dois pontos A e B sobre a esfera construir o arco ou segmento esférico que une A a B.



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2. A ferramenta segmento esférico



01_Ss_01_t.ggb
01_Ss_01-it.ggt

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Algebra

- Angle
 - $\alpha = 1.9824 \text{ rad}$
- Conic
 - $c = 1.9824$
- Number
 - $r = 1$
- Plane
 - $a: 0.397x - 0.8075y + 0.1739z = 0$
- Point
 - $A = (-0.0361, 0.1934, 0.9805)$
 - $B = (0.8398, 0.3181, -0.4399)$
 - $C = (0.6794, 0.4575, 0.5737)$
 - $O = (0, 0, 0)$
- Sphere
 - $S: x^2 + y^2 + z^2 = 1$

View of plane a

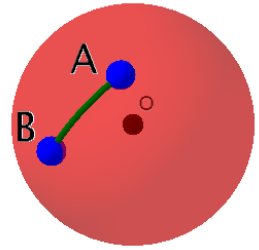
Construction Protocol

No.	Name	Definition	Value
1	Number r		$r = 1$
2	Point O		$O = (0, 0, 0)$
3	Sphere S	Sphere with center O and radius r	$S: x^2 + y^2 + z^2 = 1$
4	Point A	Point in S	$A = (-0.0361, 0.1934, \dots)$
5	Point B	Point in S	$B = (0.8398, 0.3181, \dots)$
6	Text text1		"A"
7	Text text2		"B"
8	Arc c	SphereSegment[A, B, O, r]	$c = 1.9824$
9	Plane a	Plane through O, B, A	$a: 0.397x - 0.8075y + \dots$
10	Angle α	Angle between B, O, A	$\alpha = 1.9824 \text{ rad}$
11	Line b	Line through B, O	$b: X = (0.8398, 0.318 \dots$
12	Line d	Line through A, O	$d: X = (-0.0361, 0.19 \dots$
13	Text text3		" αr "
14	Point C	Point on c	$C = (0.6794, 0.4575, \dots$

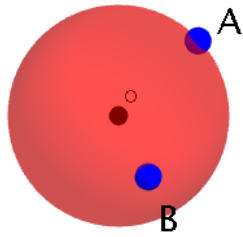
14 / 14

Input:

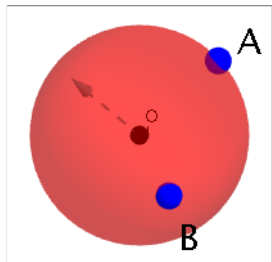
2. A ferramenta segmento esférico



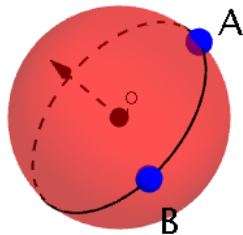
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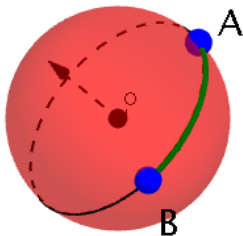
$r=1$
 $O=(0,0,0)$
 $S=\text{Sphere}[O, r]$
 $A=\text{PointIn}[S]$
 $B=\text{PointIn}[S]$



$\text{PerpendicularVector}[\text{Plane}[A, O, B]]$



$\text{Circle}[O, B, \text{PerpendicularVector}[\text{Plane}[A, O, B]]]$

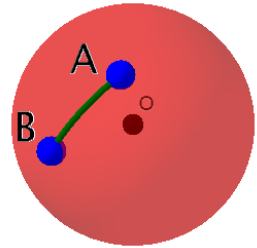


$\text{Arc}[\text{Circle}[O, B, \text{Vector}[\text{PerpendicularVector}[\text{Plane}[A, O, B]]]], B, A]$

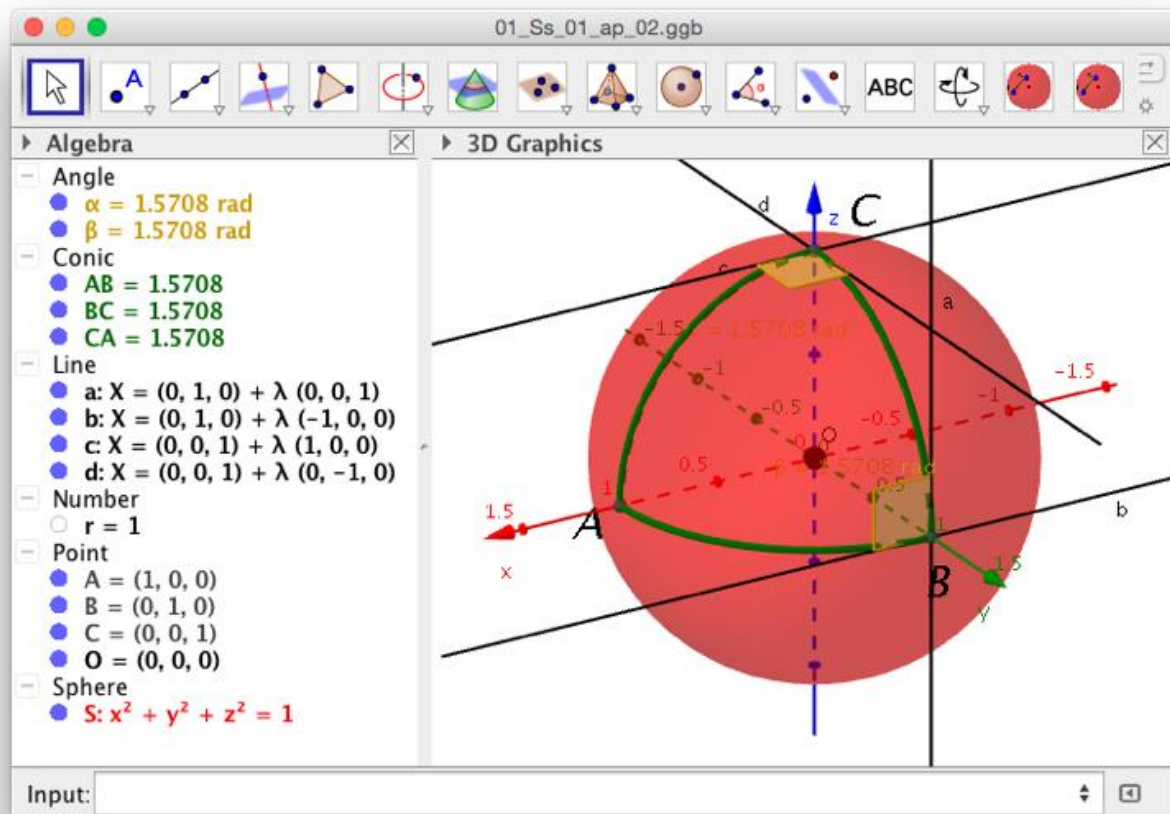
SphereSegment[A, B, O, r] ou SegmentoEsférico[A,B,O,r]

3. Triângulos esféricos, lados

Triângulo esférico ABC, onde A, B e C correspondem à intersecção, no primeiro octante, da esfera de centro em $O=(0,0,0)$ com os eixos coordenados.



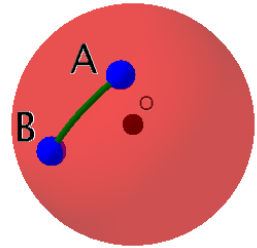
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slide_07_app.ggb

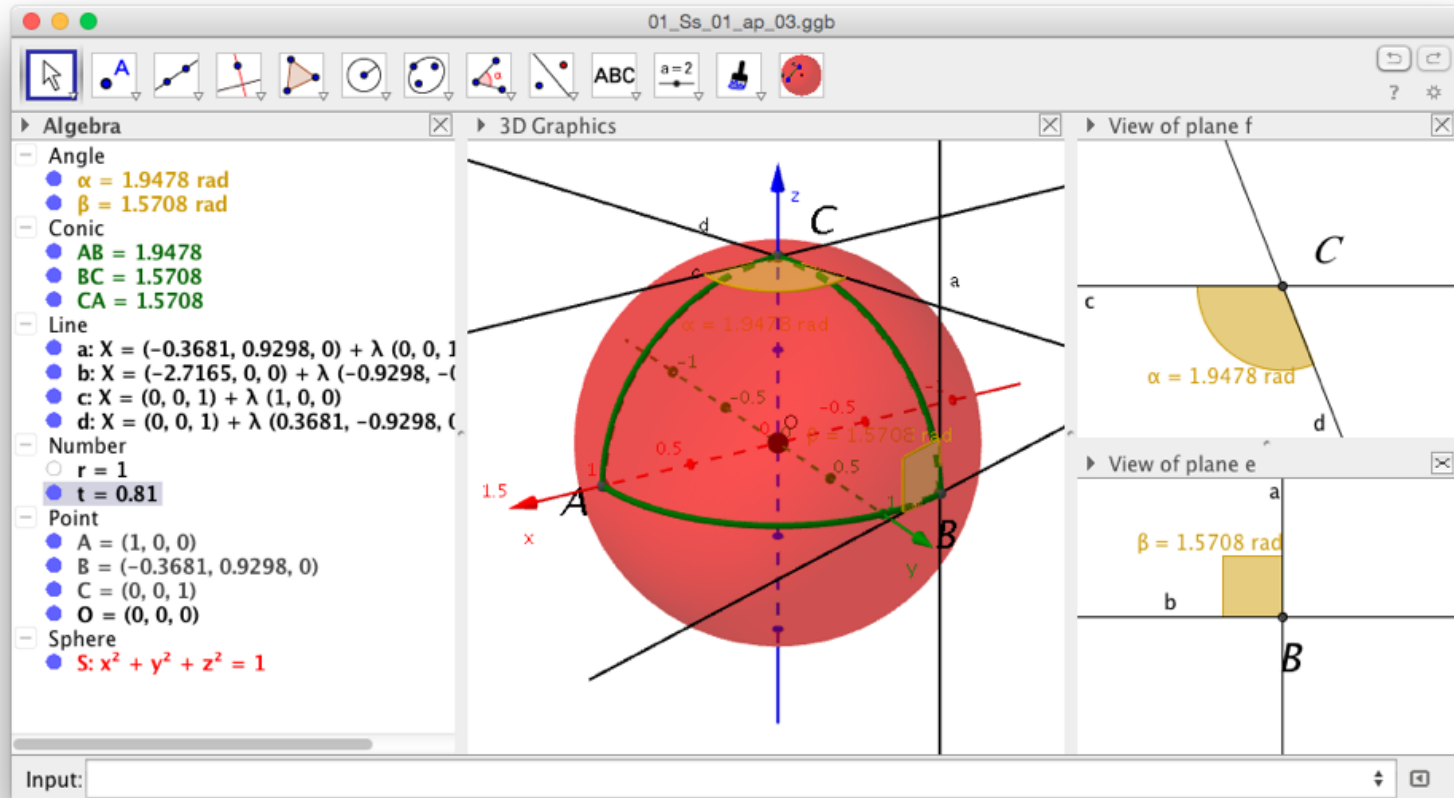
Trata-se de um **triângulo equilátero** onde todos os lados esféricos medem $\pi/2$ (1,50708; 5CD), e em que os seus **ângulos são todos retos**, isto é medem $\pi/2$ rad.

4. Triângulos esféricos, ângulos



01_Ss_01_t.ggb
01_Ss_01_it.ggt

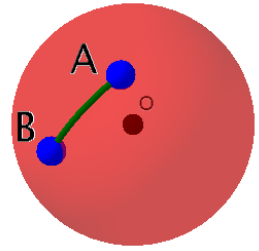
Triângulo esférico ABC, com o ponto **C** situado num dos polos relativos ao equador definido por **A** e **B**.



01_Ss_01_ap_03.ggb

Trata-se de um **triângulo isósceles** em que dois dos seus lados medem $\pi/2$ (1,5708; 5CD), e dois dos seus ângulos são retos, medem $\pi/2$ rad.

5. Um triângulo genérico na esfera



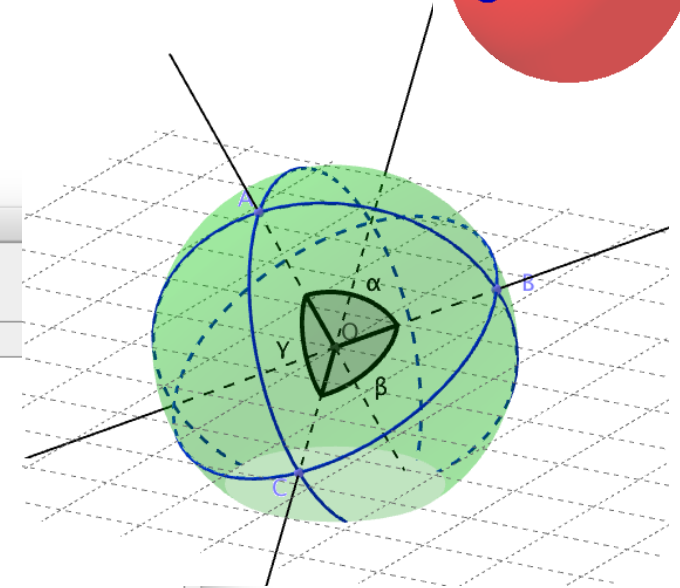
slide_09_app.ggb

Folha Algébrica | Folha Gráfica 3D

- Cónica
 - $AB = 1.56$
 - $BC = 1.83$
 - $CA = 1.74$
- Esfera
 - $S: x^2 + y^2 + z^2 = 1$
- Número
 - $r = 1$
- Ponto
 - $A = (0.32, -0.4, 0.86)$
 - $B = (0.07, -0.9, -0.43)$
 - $C = (-0.97, 0.1, 0.22)$
 - $O = (0, 0, 0)$

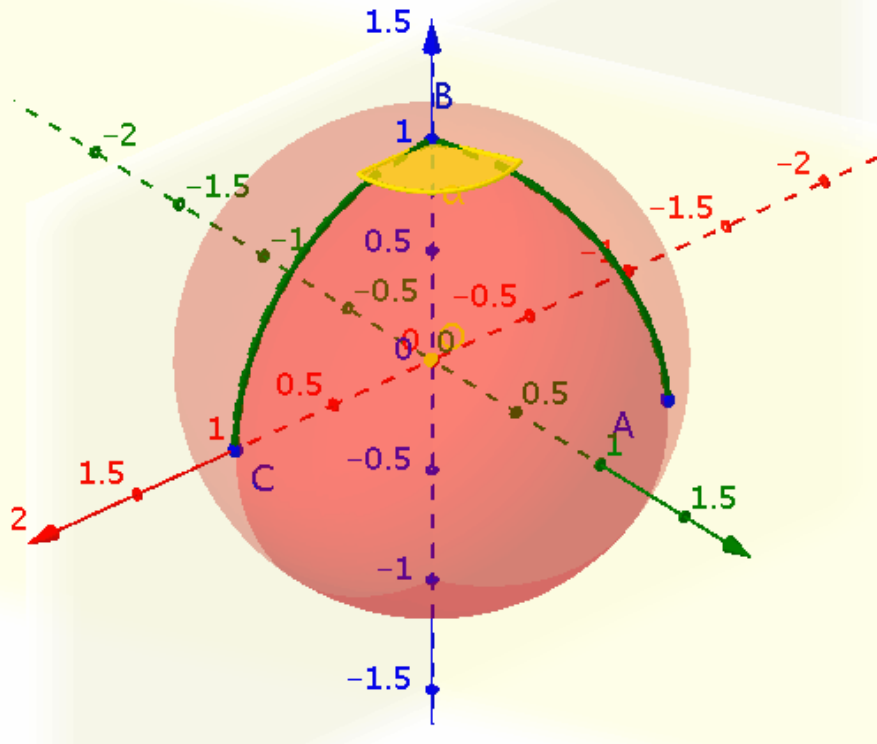
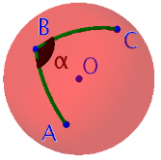
Sphere Segment AB
`Arc[Circle[O, B, Vector[PerpendicularVector[Plane[A, O, B]]], B, A]`
Sphere Segment AC
`SphereSegment[C, A, O, r]`
Sphere Segment CB

Entrada:

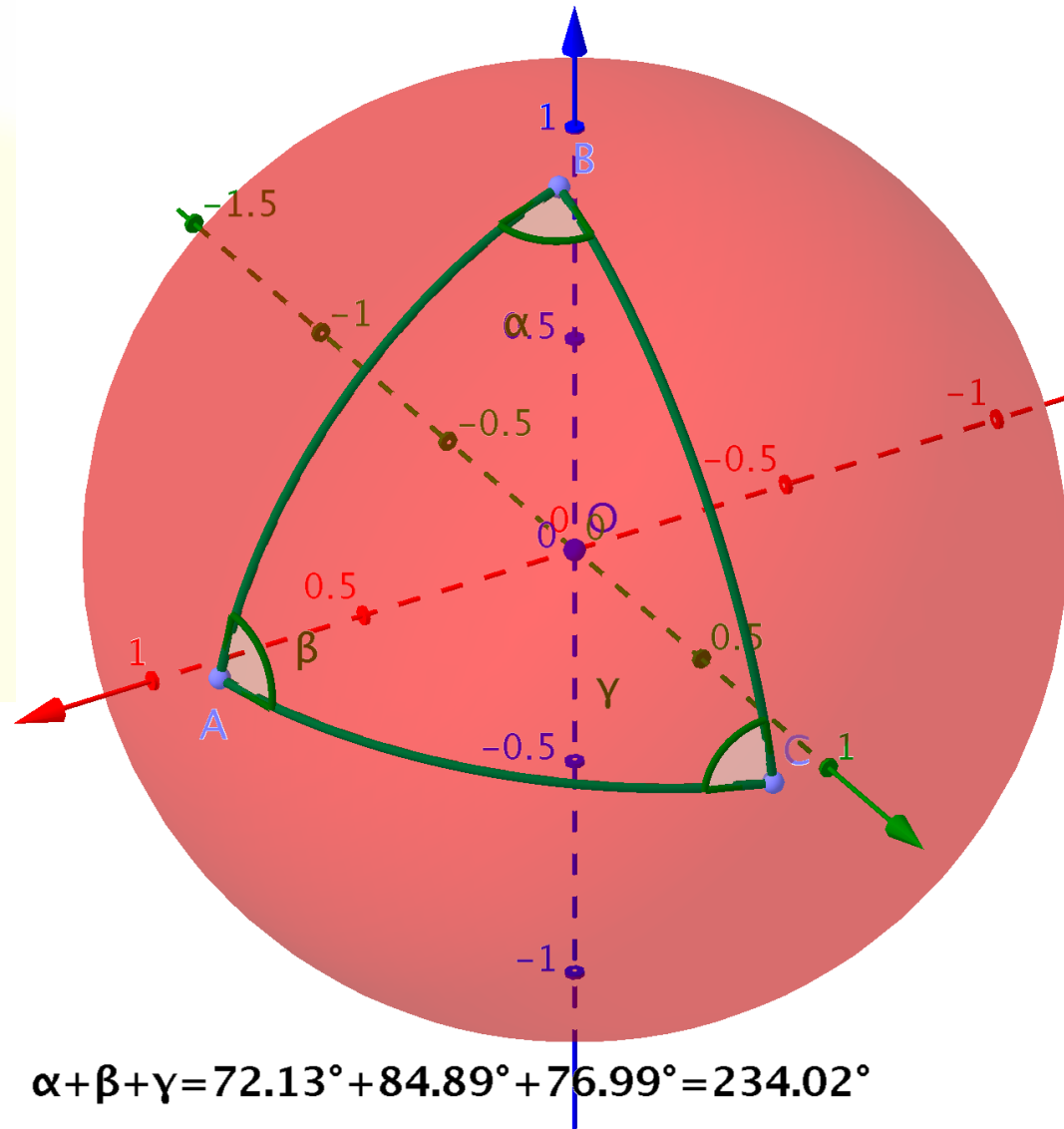


slide_09_app.ggb

4. Ângulos entre dois segmentos ...



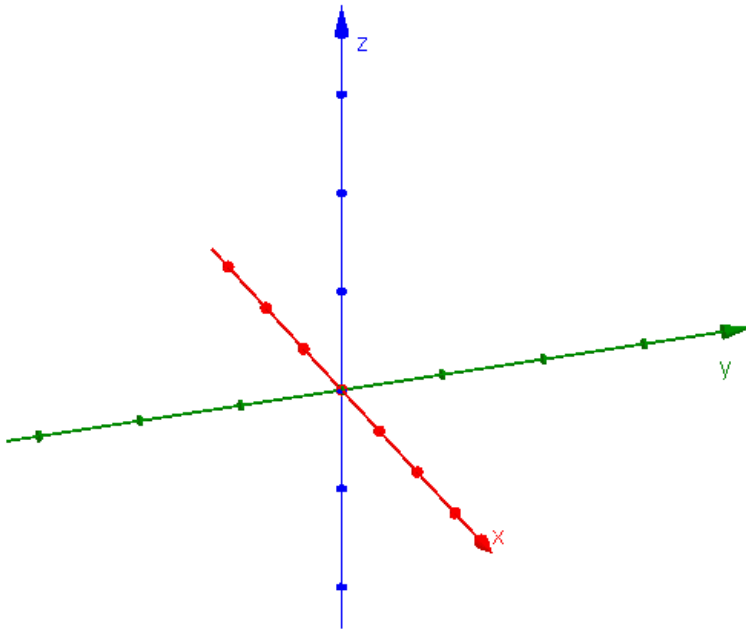
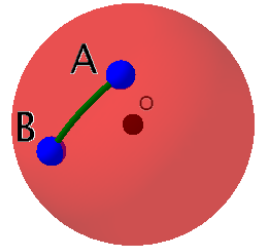
slide_11_app_01.ggb



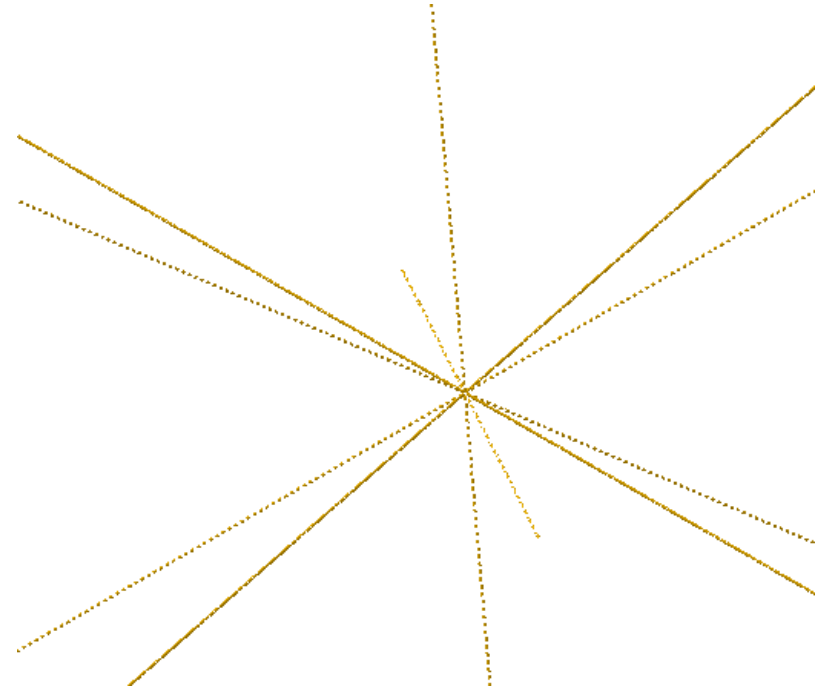
slide_11_app_02.ggb

$$\alpha + \beta + \gamma = 72.13^\circ + 84.89^\circ + 76.99^\circ = 234.02^\circ$$

5. Outros polígonos esféricos

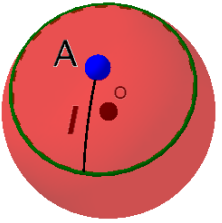


slide_10_app_01.ggb

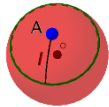


slide_10_app_02.ggb
b

7. Compasso esférico




Questão: Dados dois pontos, A e B , construir o lugar geométrico dos pontos da esfera que distam de A a medida l .

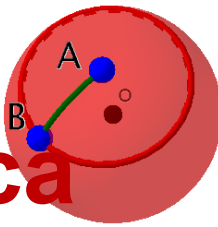


, **SphereCompass**[A , l , O , r]

This tool construct the locus of points equidistant of A at the spherical distance l .


Tool Name	Sphere Compass
Command Name	SphereCompass
Syntax	SphereCompass[A , l , O , r]
Tool Help	Given A, distance l, and centre and radius of sphere, draw the locus of points equidistant of A at the spheric distance l.
Icon	
Script	<pre>r=1 O=(0,0,0) S=Sphere[O, r] A=PointIn[S] l=1 IntersectConic[S, Sphere[A, Rotate[A, l rad, O, Plane[A, line[O,IntersectPath[x + Oz = 0, y + Oz = 0]]]]]]</pre>
File	03_ScAI_01_t.ggb

6. “Circunferências” na geometria esférica

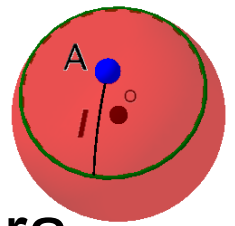


Pontos equidistantes de A e à distância esférica AB

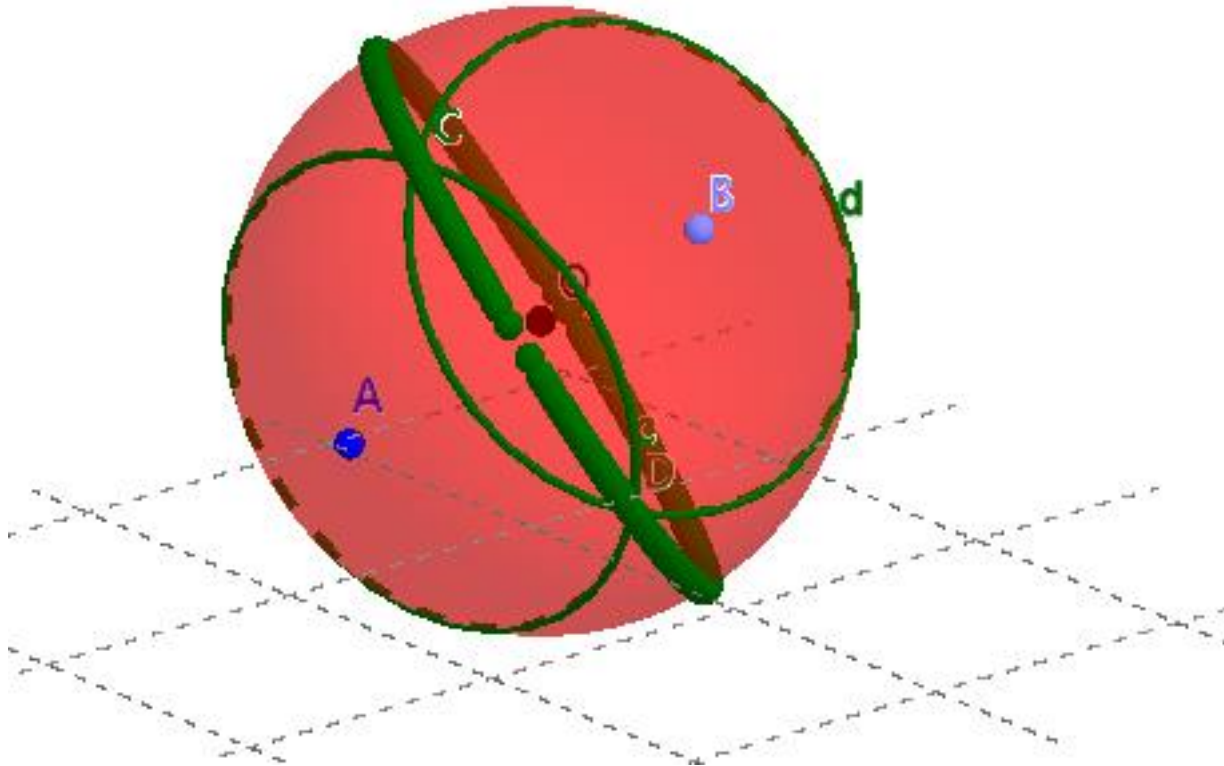
 **SphereEquidistantPoints[A, B, O, r]**
This tool construct the locus of points equidistant of A at the spherical distance AB.

Tool Name	Sphere Compass
Command Name	SphereCompass
Syntax	SphereCompass[A, l, O, r]
Tool Help	Given A, distance l, and centre an radius of sphere, draw the locus of points equidistant of A at the spherical distance l.
Icon	
Script	<pre>r=1 O=(0,0,0) S=Sphere[O, r] A=PointIn[S] l=1 IntersectConic[S, Sphere[A, Rotate[A, l rad, O, Plane[A, line[O,IntersectPath[x + Oz = 0, y + Oz = 0]]]]]]]</pre>
File	03_ScAI_01_t.ggb

8. Compasso esférico e mediatrizes

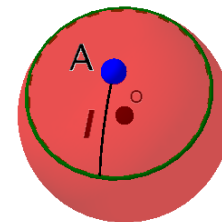


Pontos equidistantes de dois pontos fixos na esfera

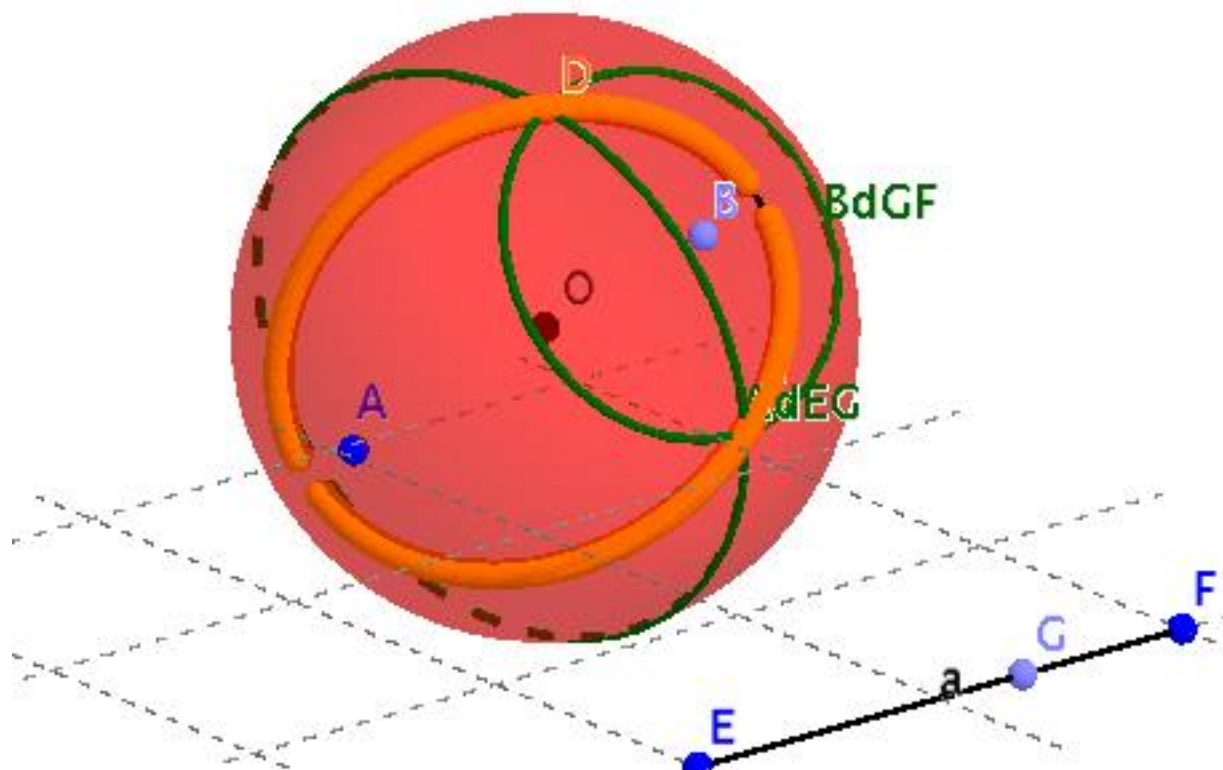


slide_14_app.ggb

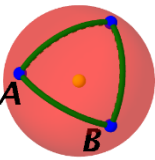
9. Compasso esférico e elipse



Uma elipse na geometria esférica



slide_15_app.ggb



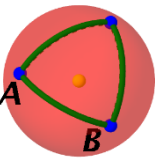
10. Triângulo equilátero esférico genérico

Command	icon
<pre>A=PointIn[S] B=PointIn[S] C=Intersect[IntersectConic[Sphere[B, A], S], IntersectConic[Sphere[A, B], S], 2] ABC={SphereSegment[A, B, O, r], SphereSegment[B, Intersect[IntersectConic[Sphere[B, A], S], IntersectConic[Sphere[A, B], S], 2], O, r], SphereSegment[Intersect[IntersectConic[Sphere[B, A], S], IntersectConic[Sphere[A, B], S], 2], A, O, r]}</pre>	
SphericEquilateralTriangle[C, B, O, r]	

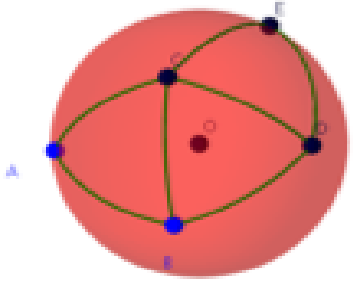
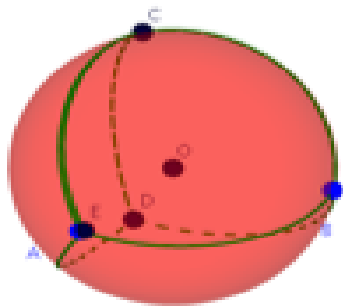
A ferramenta **SphericEquilateralTriangle[C, B, O, r]** foi concebida para a construção de triângulos esféricos equiláteros.

Aplicações:

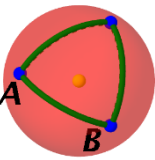
1. Explorar as pavimentações esféricas por triângulos equiláteros;
2.

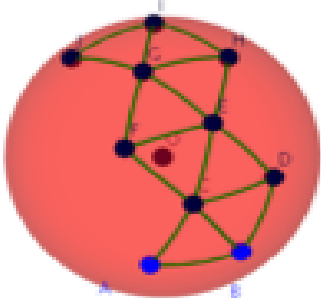
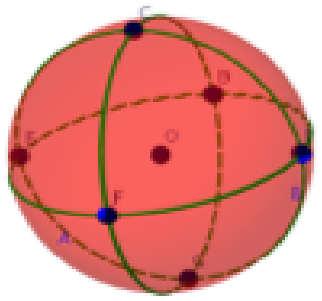


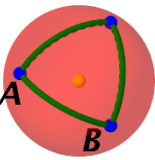
11. Pavimentar com 3 triângulos

N.º de triângulos equiláteros	... sobre um hemisfério	Medida do segmento AB	... a fechar sobre a esfera	Medida do segmento AB
3		1,05 rad		1,908 rad (v.a.) $\arccos(-1/3)$ ou $2 \arctan(\sqrt{2})$ (v.e.)

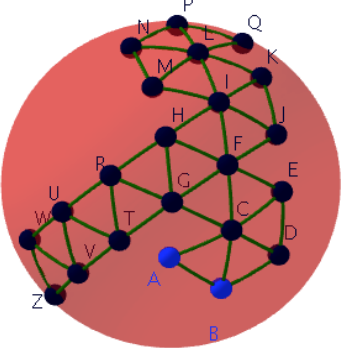
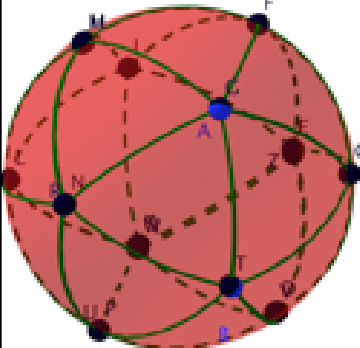
12. Pavimentar a esfera com 8 triângulos

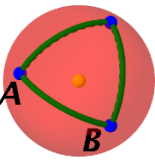


N.º de triângulos equiláteros	... sobre um hemisfério	Medida do segmento AB	... a fechar sobre a esfera	Medida do segmento AB
8		0,64 rad		1,574 rad (v.a.) $\pi/2$ (v.e.)



13. Pavimentar com 20 triângulos

N.º de triângulos equiláteros	... sobre um hemisfério	Medida do segmento AB	... a fechar sobre a esfera	Medida do segmento AB
20		0,40 rad		<p>1,107 rad (v.a.)</p> $2 \arcsin\left(\frac{\sqrt{5-\sqrt{5}}}{\sqrt{10}}\right)$ <p>2 arcsin($\sqrt{(5-\sqrt{5})}/\sqrt{10}$) (v.e.)</p>



13. Pavimentar com losangos

Pavimentação dodecaédrica não regular, por “quadriláteros” losangos não rectângulos.

`Intersect[xAxis, S]`

`Intersect[yAxis, S]`

`Intersect[zAxis, S]`

`Intersect[Line[(0, 0, 0), (1, 1, 1)],`

`S]`
`Intersect[Line[(0, 0, 0), (-1, -1, 1)],`

`S]`
`Intersect[Line[(0, 0, 0), (1, -1, 1)],`

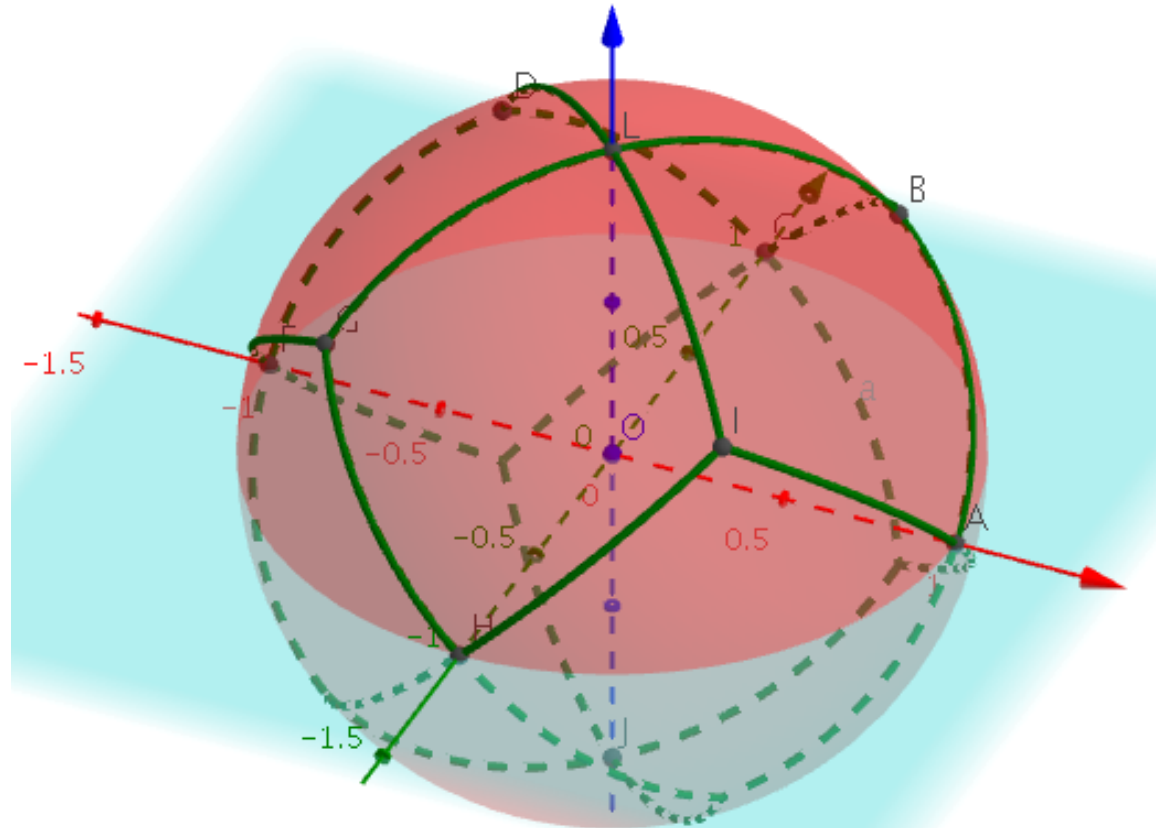
`S]`



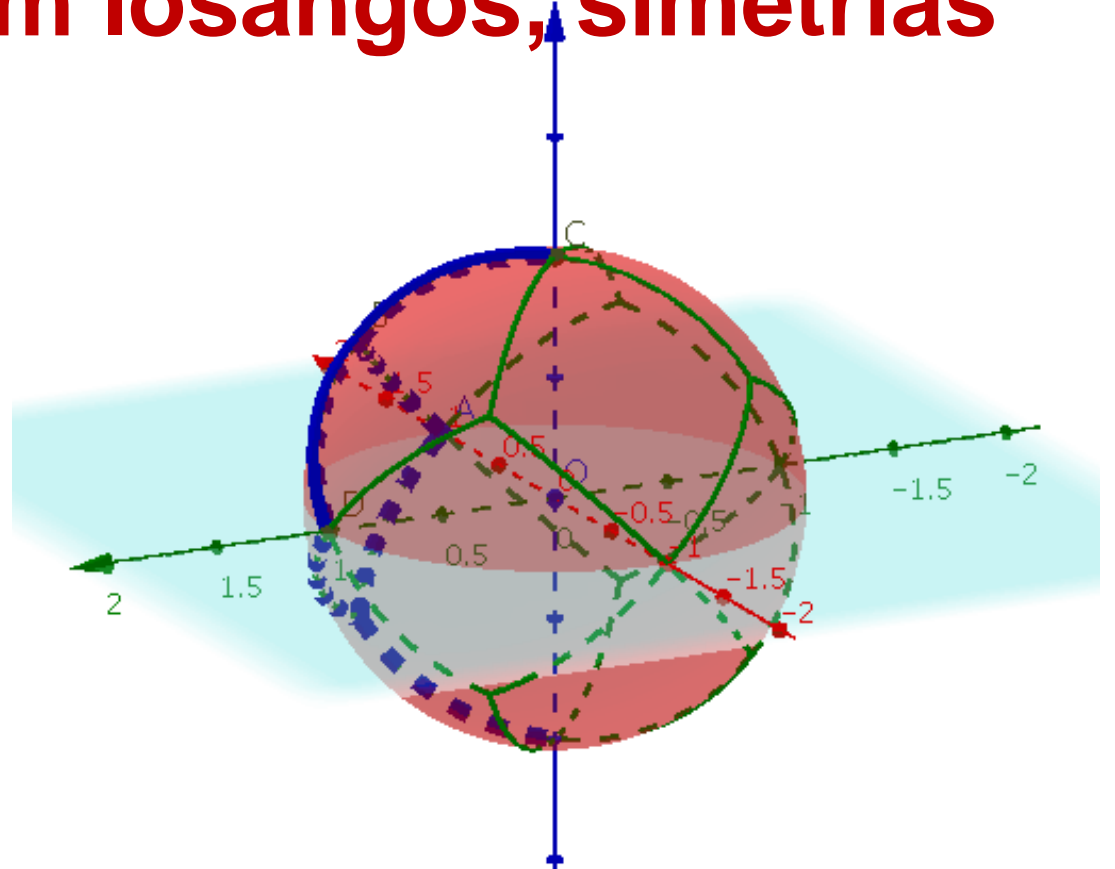
`AB=SphereSegment[A, B, O, r]`

`z = 0`

`Reflect[AB, a]`



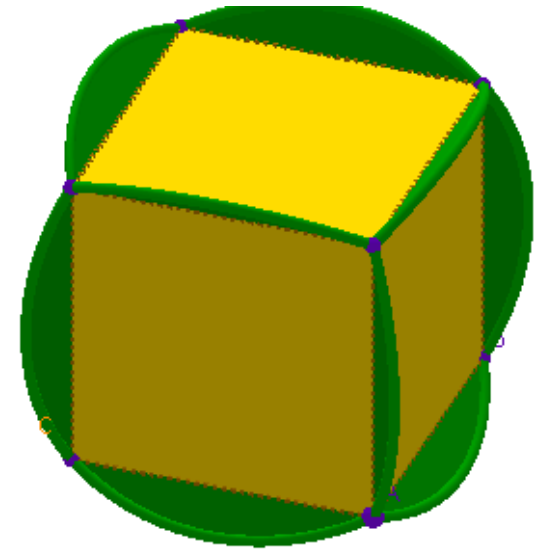
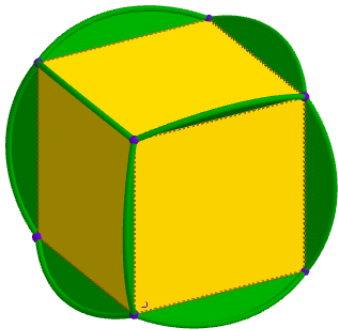
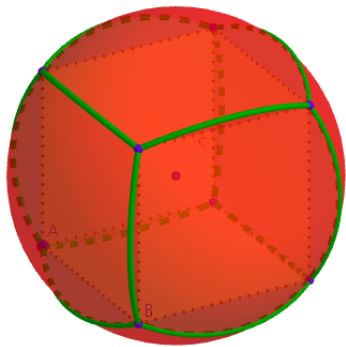
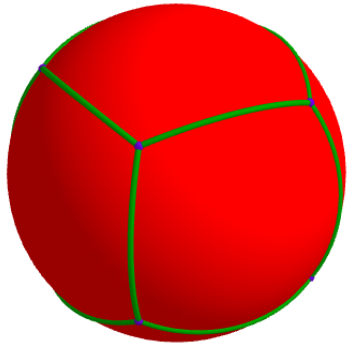
13. Pavimentar com losangos, simetrias



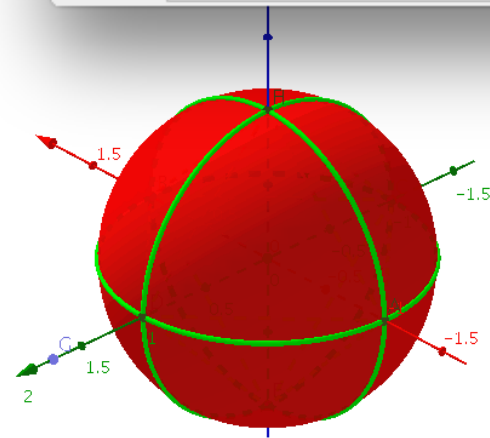
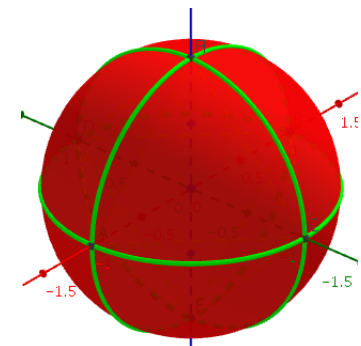
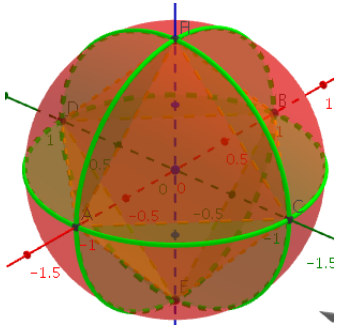
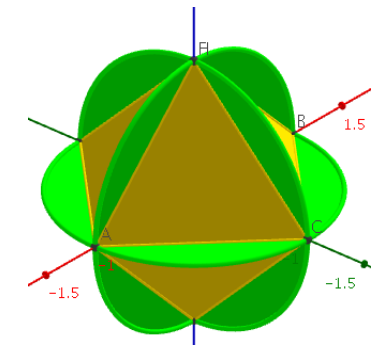
```
A=(r,0,0)
B=Intersect[Ray[(0, 0, 0), (1, 1, 1)], S,2]
C=(r,0,0)
D=(0, r, 0)

mod={SphereSegment[A, B, O, r], SphereSegment[D, B, O, r], SphereSegment[C, B, O, r]}
Rotate[mod, pi / 2, zAxis]
Rotate[mod, pi, zAxis]
Rotate[mod, 3pi / 2, zAxis]
Reflect[half, z=0]
half=Sequence[Rotate[{{SphereSegment[A, B, O, r], SphereSegment[D, B, O, r], SphereSegment[C, B, O, r]}, i* pi / 2, zAxis], i, 0, 3 ]
```

14. Pavimentação da esfera e o cubo



15. Pavimentação da esfera e octaedros



slide_23_app_b.ggb

Algebra

- Number
 - $t = 0$
- Surface
 - $a(a,b) = (\sin(a) \cos(b), \cos(a) \cos(b), \sin(b))$
 - $b(a,b) = (\sin(a) \cos(b), \cos(a) \cos(b), \sin(b))$
 - $c(a,b) = (\sin(a) \cos(b), \cos(a) \cos(b), \sin(b))$
 - $d(a,b) = (\sin(a) \cos(b), \cos(a) \cos(b), \sin(b))$
 - $e(a,b) = (\sin(a) \cos(b), \cos(a) \cos(b), -\sin(b))$
 - $f(a,b) = (\sin(a) \cos(b), \cos(a) \cos(b), -\sin(b))$
 - $g(a,b) = (\sin(a) \cos(b), \cos(a) \cos(b), -\sin(b))$
 - $h(a,b) = (\sin(a) \cos(b), \cos(a) \cos(b), -\sin(b))$

3D Graphics

Input:

slide_23_app_a.ggb

slide_23_app_b.ggb

16. Pavimentação da esfera e tetraedro

The screenshot displays a geometry software interface with the following components:

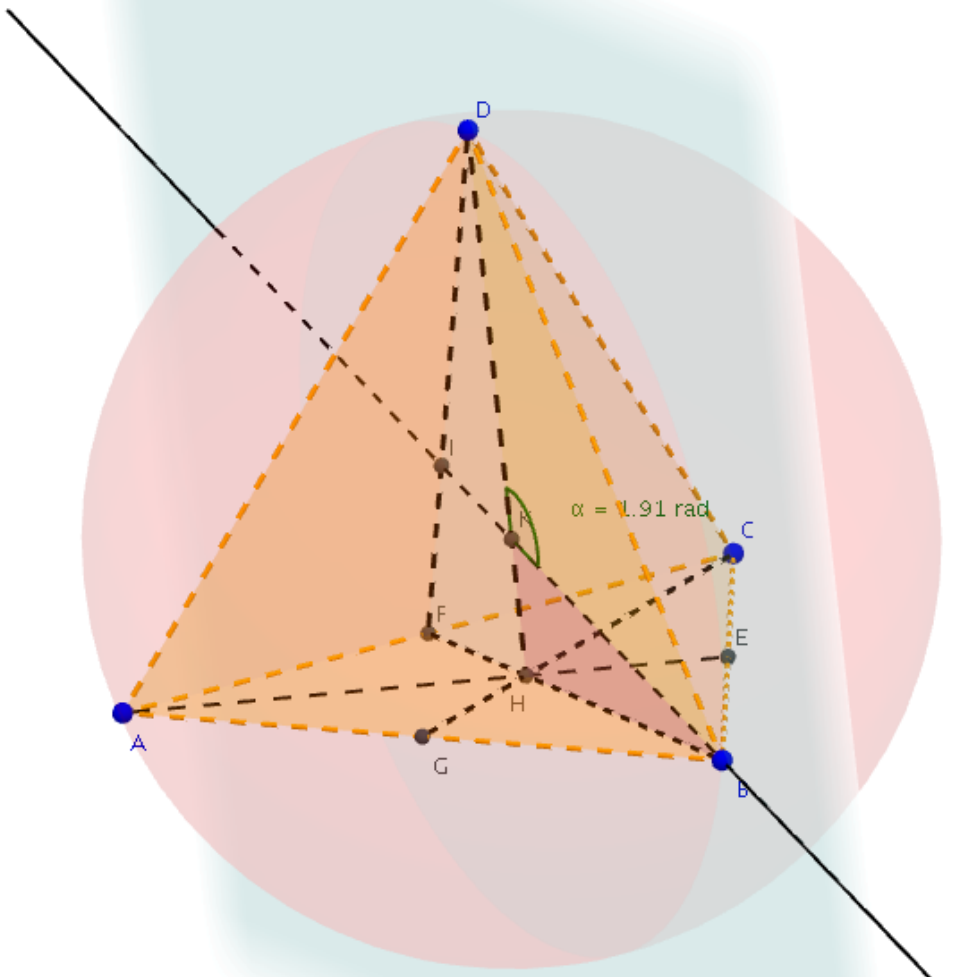
- Algebra Panel:**
 - Angle: $\beta = 1.9106332362 \text{ rad}$
 - Conic: $d = 1.9106332362$, $e = 1.9106332362$, $f = 1.9106332362$, $k = 1.9106332362$, $p = 1.9106332362$, $q = 1.9106332362$, $s = 1.9106332362$
 - Line: $i: X = (0, 0, 0) + \lambda (-0.29801026, \dots)$
 - Number: $a = 2.0943951024$, $b = 1.9106332362$, $c = 0.8235987756$, $l = 1$, $r = 1$, $t = 1$, $\alpha = 2.0853981634$
 - Sphere: $S: x^2 + y^2 + z^2 = 1$
 - Tetrahedron: $j = 0.5132002393$
- 3D Graphics Panel:** Shows a 3D view of a red sphere with a green curve on its surface. The coordinate system has a vertical z-axis and a horizontal y-axis.
- CAS Panel:**
 - 1: $2 \arctan(\sqrt{2})$
 $\rightarrow 2 \operatorname{atan}(\sqrt{2})$
 - 2: $2 \operatorname{atan}(\sqrt{2})$
 ≈ 1.9106332362
- View of plane g:** Shows a 2D view of a green curve in a plane. The angle $\beta = 1.9106332362 \text{ rad}$ is indicated between a line and the curve.

16. Pavimentação da esfera e tetraedro

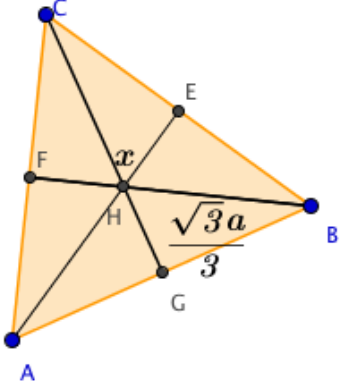
prova1.ggb

3D Graphics

View of plane created from face ABC

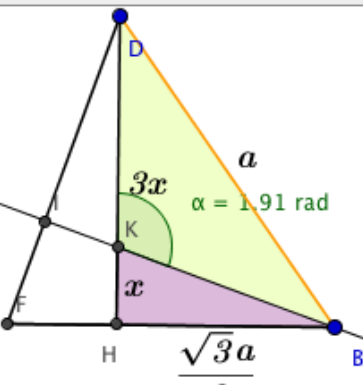


$\frac{CH}{HG} = \frac{3.62}{1.81} = 2$
 $\frac{DK}{KH} = \frac{3.84}{1.28} = 3$
 $\frac{CH}{AC} = \frac{3.62}{6.28} = 0.58, \frac{\sqrt{3}}{3} = 0.58$
 $x = \frac{\sqrt{6}a}{12}$
 $r = \frac{\sqrt{6}a}{3}$



$\pi - \arctan\left(\frac{\frac{\sqrt{3}a}{3}}{\frac{\sqrt{6}a}{3}}\right) = \pi - \arctan(2\sqrt{2}) = 1.91$

View of plane h

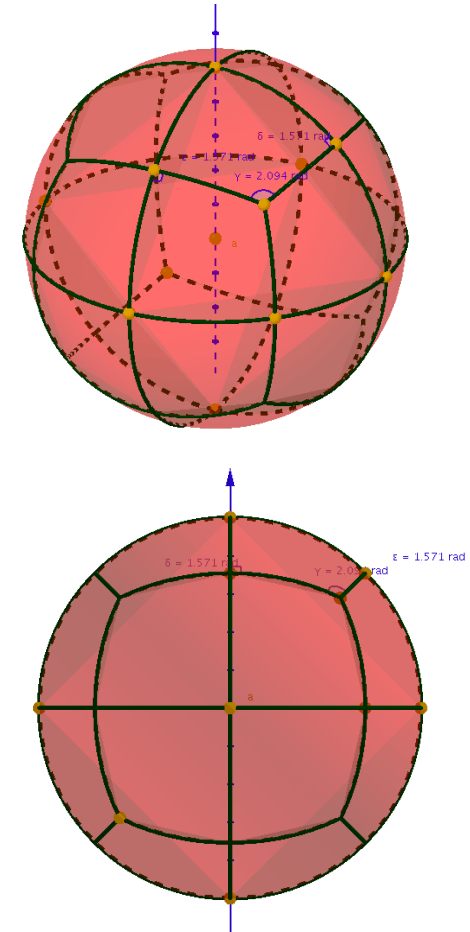
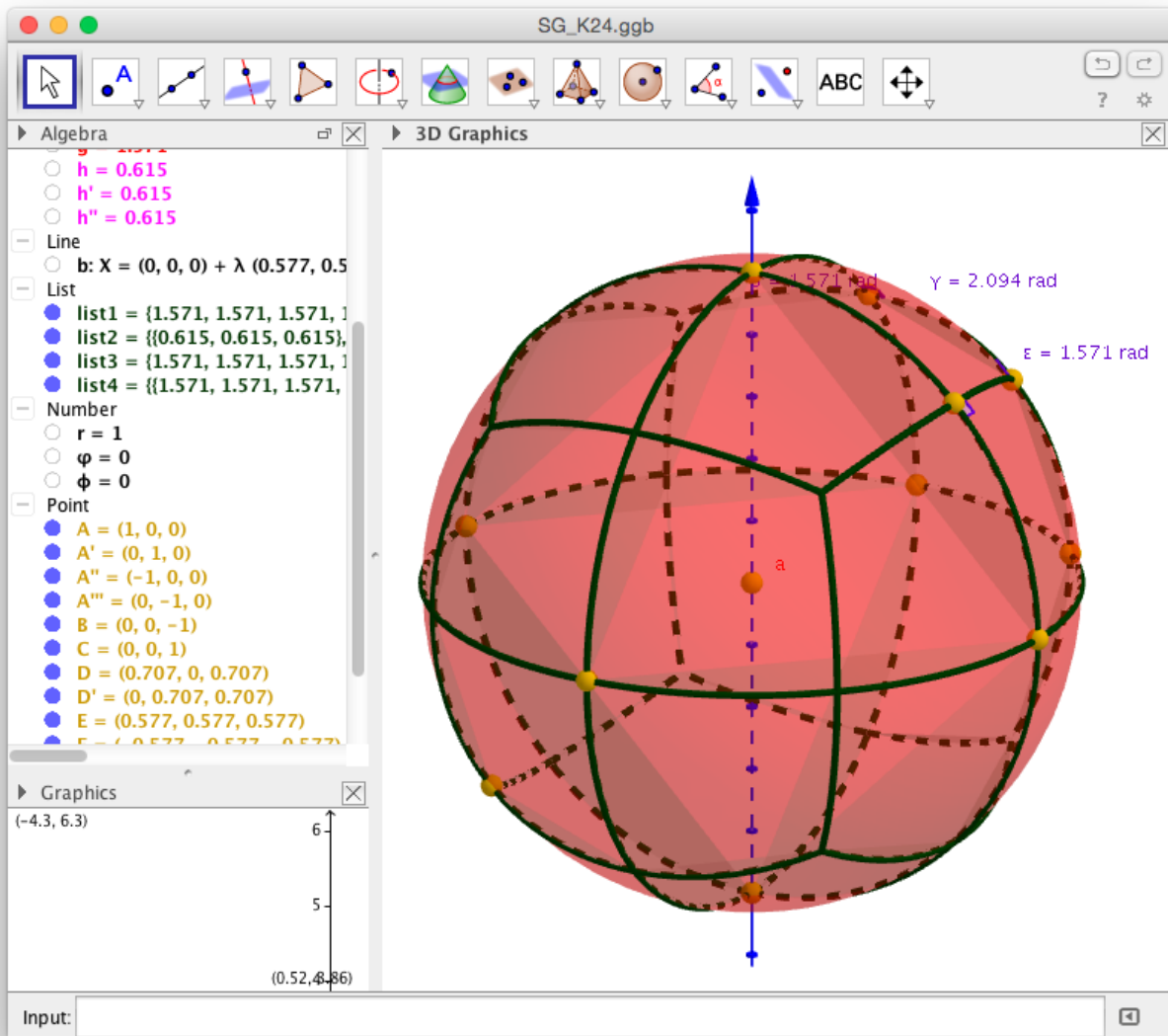


$\frac{CH}{HG} = \frac{3.62}{1.81} = 2$
 $\frac{DK}{KH} = \frac{3.84}{1.28} = 3$
 $\frac{CH}{AC} = \frac{3.62}{6.28} = 0.58, \frac{\sqrt{3}}{3} = 0.58$
 $x = \frac{\sqrt{6}a}{12}$
 $r = \frac{\sqrt{6}a}{3}$

$\pi - \arctan\left(\frac{\frac{\sqrt{3}a}{3}}{\frac{\sqrt{6}a}{3}}\right) = \pi - \arctan(2\sqrt{2}) = 1.91$

slide_25_app.ggb

17. Pavimentações Monoédricas da esfera

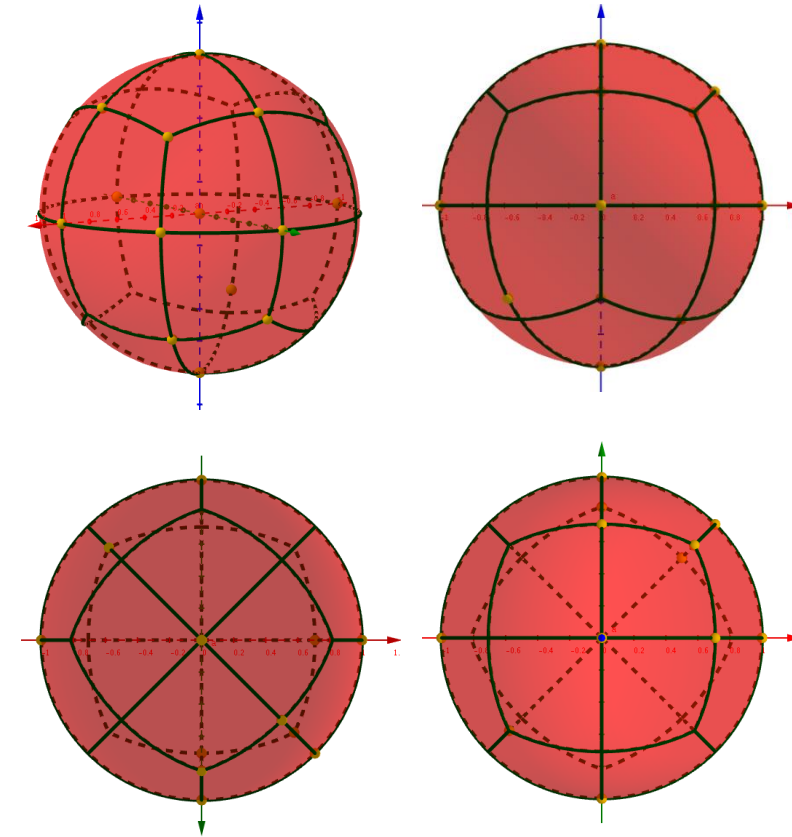
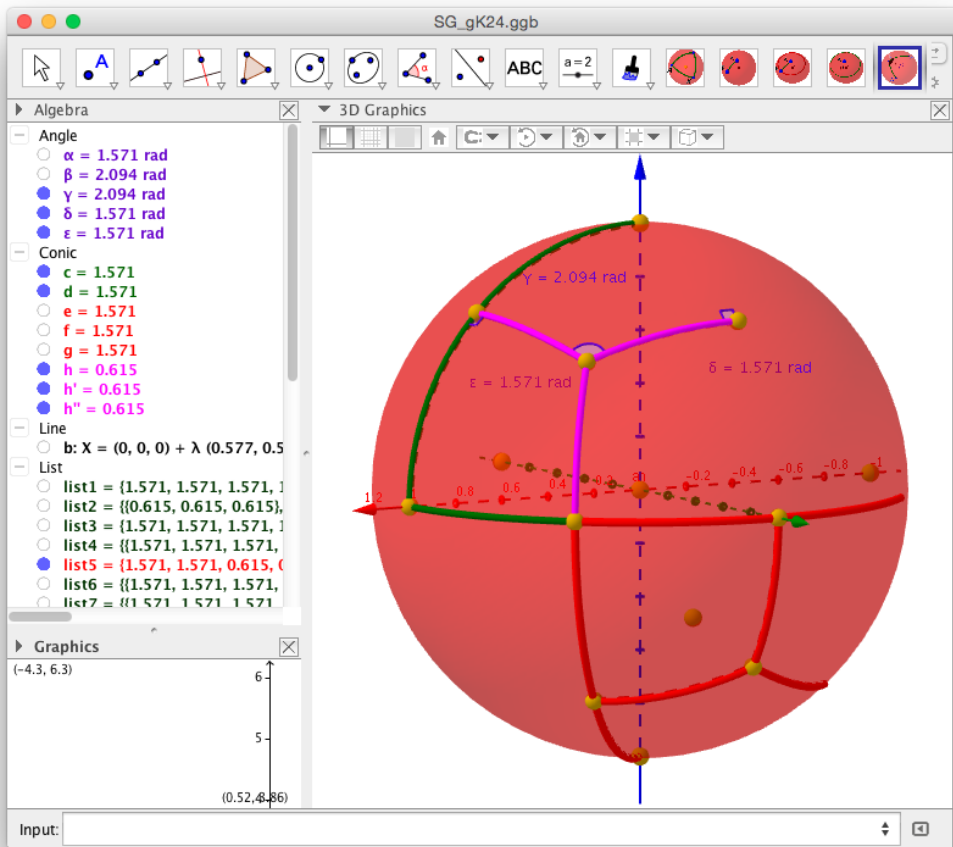


K24

SG_K24.ggb

17. Pavimentações Monoédricas da esfera

SG_gK24.ggb



gK24

$\text{Reflect}[\text{Rotate}\{c, d, h, h', h''\}, 0.7854, \text{Line}[B, C]], \text{Plane}[z = 0]]$