Research Article

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Invertibility characterization of Wiener–Hopf plus Hankel operators on variable exponent Lebesgue spaces via even asymmetric factorization

Abstract: We obtain invertibility and Fredholm criteria for the Wiener–Hopf plus Hankel operators acting between variable exponent Lebesgue spaces on the real line. Such characterizations are obtained via the so-called even asymmetric factorization which is applied to the Fourier symbols of the operators under study.

Keywords: Wiener–Hopf operator, Hankel operator, variable exponent Lebesgue spaces, invertibility, Fredholm property, even asymmetric factorization

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Dedicated to the memory of Academician N. Muskhelishvili on the occasion of his 125th birthday anniversary

1 Introduction

The variable exponent Lebesgue spaces have been intensively investigated in the last years. The concept appeared for the first time in the literature in 1931 due to Orlicz [19] and the growing interest in these spaces is connected to applications in problems of fluid dynamics, elasticity theory, calculus of variations and differential equations (see, e.g., [20, 22]).

Although variable exponent Lebesgue spaces preserve many of the properties of standard Lebesgue spaces (as can be seen in [13] and [18]), some difficulties arise in the study of convolution type operators since, in general, such operators are not bounded in these spaces. In particular, some invertibility and Fredholm properties of Wiener–Hopf plus Hankel operators, already studied for standard Lebesgue spaces (see, e.g., [1, 2, 4, 5, 6, 7, 9, 10, 14, 15]) attract, nowadays, an increasing interest in the framework of variable exponent Lebesgue spaces. The same situation occurs even for the boundedness of somehow more classical operators, like the Hardy–Littlewood maximal operator and the Cauchy singular integral operator; see, e.g., [11, 12, 13, 16, 17, 21].

The main goal of the present paper is to obtain invertibility and Fredholm criteria for the Wiener–Hopf plus Hankel operators acting between variable exponent Lebesgue spaces upon appropriate factorizations of the Fourier symbols of the operators.

In order to define the operators under study, we will start with the definition of variable Lebesgue spaces. Let $p : \mathbb{R} \rightarrow [1, \infty]$ be a measurable a.e. finite function. We denote by $L^{p(\cdot)}(\mathbb{R})$ the set of all

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complex-valued functions f on $\mathbb{R}$ such that

$$I_{p(\cdot)}\left(\frac{f(x)}{\lambda}\right) := \int_{\mathbb{R}} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx < \infty$$

for some $\lambda > 0$. This set becomes a Banach space when equipped with the norm

$$\|f\|_{p(\cdot)} := \inf\{\lambda > 0 : I_{p(\cdot)}(f/\lambda) \leq 1\}.$$

The space $L^{p(\cdot)}(\mathbb{R})$ is called *variable exponent Lebesgue space*, and if $p(\cdot) = p$ is constant, then $L^{p(\cdot)}(\mathbb{R})$ is nothing but the standard Lebesgue space $L^p(\mathbb{R})$.

We will always be assuming that

$$1 < p_- := \operatorname{ess\,inf}_{x \in \mathbb{R}} p(x) \leq \operatorname{ess\,sup}_{x \in \mathbb{R}} p(x) =: p_+ < \infty. \quad (1.1)$$

Under these conditions, the space $L^{p(\cdot)}(\mathbb{R})$ is separable and reflexive, and its dual space is isomorphic to $L^{q(\cdot)}(\mathbb{R})$, where $q(\cdot)$ is the conjugate exponent function defined by

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1 \quad (x \in \mathbb{R}).$$

Additionally, with condition (1.1) we have that $\|\phi I\|_{\mathcal{L}(L^{p(\cdot)}(\mathbb{R}))} \leq \|\phi\|_{L^\infty(\mathbb{R})}$ for functions $\phi \in L^\infty(\mathbb{R})$.

Moreover, $L^{p(\cdot)}(\mathbb{R}_+)$ denotes the variable exponent Lebesgue space of complex-valued functions on the positive half-line $\mathbb{R}_+ = (0, +\infty)$. The subspace of $L^{p(\cdot)}(\mathbb{R})$ formed by all functions supported in the closure of $\mathbb{R}_+$ is denoted by $L_+^{p(\cdot)}(\mathbb{R})$ and $L_-^{p(\cdot)}(\mathbb{R})$ represents the subspace of $L^{p(\cdot)}(\mathbb{R})$ formed by all the functions supported in the closure of $\mathbb{R}_- := (-\infty, 0)$.

We are now able to present in a detailed way the main operators of this work. We will consider the Wiener–Hopf plus Hankel operators acting between the variable exponent $p(\cdot)$ Lebesgue spaces, denoted by

$$W_\phi + H_\phi : L_+^{p(\cdot)}(\mathbb{R}) \rightarrow L^{p(\cdot)}(\mathbb{R}_+),$$

with W_ϕ and H_ϕ being Wiener–Hopf and Hankel operators defined by

$$W_\phi = r_+ \mathcal{F}^{-1} \phi \mathcal{F}, \quad H_\phi = r_+ \mathcal{F}^{-1} \phi \mathcal{F} J, \quad (1.2)$$

respectively. Here r_+ represents the operator of restriction from $L^{p(\cdot)}(\mathbb{R})$ into $L^{p(\cdot)}(\mathbb{R}_+)$, $\mathcal{F}^{-1}$ denotes the inverse of the Fourier transformation $\mathcal{F}$, ϕ is the so-called Fourier symbol, and $J : L_+^{p(\cdot)}(\mathbb{R}) \rightarrow L^{p(\cdot)}(\mathbb{R})$ is the reflection operator given by the rule $J\varphi(x) = \tilde{\varphi}(x) = \varphi(-x)$ which throughout the paper will always be considered for the even functions $p(\cdot)$ only (so that J will be a bounded operator in variable exponent Lebesgue spaces).

2 Auxiliary operators and relations

The boundedness of a wide variety of operators (and in particular of Wiener–Hopf and Hankel operators) follows from the boundedness of the maximal operator on variable exponent Lebesgue spaces.

Given $f \in L_{\operatorname{loc}}^1(\mathbb{R})$, the Hardy–Littlewood maximal operator M is defined by

$$(Mf)(x) := \sup_{x \in \Omega} \frac{1}{|\Omega|} \int_{\Omega} |f(y)| dy,$$

where the supremum is taken over all intervals $\Omega \subset \mathbb{R}$ containing x , and the Cauchy singular integral operator S is defined by

$$(Sf)(x) := \frac{1}{\pi i} \int_{\mathbb{R}} \frac{f(\tau)}{\tau - x} d\tau,$$

where the integral is understood in the principal value sense.

Theorem 2.1. (cf., e.g., [16, Theorem 2.1.]) *Let $p : \mathbb{R} \rightarrow [1, \infty]$ be a measurable function satisfying (1.1). If the Hardy–Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R})$, then the Cauchy singular integral operator S is bounded on $L^{p(\cdot)}(\mathbb{R})$.*

The following result states a sufficient condition on $p(\cdot)$ for M to be bounded on $L^{p(\cdot)}(\mathbb{R})$.

Theorem 2.2 (cf., e.g., [12, 13]). *Let $p : \mathbb{R} \rightarrow [1, \infty]$ satisfy (1.1). In addition, suppose that there exist constants A_0 and A_∞ such that $p(\cdot)$ satisfies*

$$|p(x) - p(y)| \leq \frac{A_0}{-\log|x - y|}, \quad |x - y| \leq \frac{1}{2},$$

and

$$|p(x) - p(y)| \leq \frac{A_\infty}{\log(e + |x|)}, \quad |x| \leq |y|.$$

Then the Hardy–Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R})$.

Let $\mathcal{P}(\mathbb{R})$ denote the class of exponents $p : \mathbb{R} \rightarrow [1, \infty]$ which are continuous on $\mathbb{R}$ and satisfy (1.1), and let $\mathcal{B}(\mathbb{R})$ denote the set of all $p(\cdot) \in \mathcal{P}(\mathbb{R})$ such that M is bounded on $L^{p(\cdot)}(\mathbb{R})$. Additionally, let $\mathcal{B}_e(\mathbb{R})$ represent the set of all even functions $p(\cdot) \in \mathcal{B}(\mathbb{R})$.

A function $\phi \in L^\infty(\mathbb{R})$ is called a *Fourier multiplier* on $L^{p(\cdot)}(\mathbb{R})$ ($p(\cdot) \in \mathcal{B}(\mathbb{R})$) if the operator $W_\phi^0 := \mathcal{F}^{-1}\phi\mathcal{F}$ acting on $L^2(\mathbb{R}) \cap L^{p(\cdot)}(\mathbb{R})$, extends uniquely to a bounded operator on $L^{p(\cdot)}(\mathbb{R})$. The set of all Fourier multipliers on $L^{p(\cdot)}(\mathbb{R})$ is denoted by $\mathcal{M}^{p(\cdot)}(\mathbb{R})$.

It follows that for $\phi \in \mathcal{M}^{p(\cdot)}(\mathbb{R})$, with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$, the Wiener–Hopf and Hankel operators defined in (1.2), with $W_\phi : L_+^{p(\cdot)}(\mathbb{R}) \rightarrow L^{p(\cdot)}(\mathbb{R}_+)$ and $H_\phi : L_+^{p(\cdot)}(\mathbb{R}) \rightarrow L^{p(\cdot)}(\mathbb{R}_+)$ are bounded. These are in fact necessary and sufficient conditions for the Wiener–Hopf plus Hankel operator to be bounded in variable exponent Lebesgue spaces.

Let $\ell_0 : L^{p(\cdot)}(\mathbb{R}_+) \rightarrow L_+^{p(\cdot)}(\mathbb{R})$ denote the zero extension operator from the space $L^{p(\cdot)}(\mathbb{R}_+)$ onto the space $L_+^{p(\cdot)}(\mathbb{R})$. Consider the projection operator

$$P := \mathcal{F}\ell_0 r_+ \mathcal{F}^{-1},$$

(or, equivalently, $P := (I + S)/2$) and its complementary projection $Q := I - P$. If $p(\cdot) \in \mathcal{B}(\mathbb{R})$, then $P, Q \in \mathcal{L}(L^{p(\cdot)}(\mathbb{R}))$ and $P^2 = P$ and $Q^2 = Q$ (see [16, Lemma 3.10]). Additionally, $S^* = S$, $P^* = P$ and $Q^* = Q$ are bounded linear operators in $L^{q(\cdot)}(\mathbb{R})$ (see [16, Lemma 3.11]). The images of P and Q on $L^{p(\cdot)}(\mathbb{R})$, denoted by $PL^{p(\cdot)}(\mathbb{R})$ and $QL^{p(\cdot)}(\mathbb{R})$, respectively, are closed subspaces of $L^{p(\cdot)}(\mathbb{R})$ and, moreover, $L^{p(\cdot)}(\mathbb{R})$ decomposes into the direct sum of these two subspaces, that is,

$$L^{p(\cdot)}(\mathbb{R}) = PL^{p(\cdot)}(\mathbb{R}) \oplus QL^{p(\cdot)}(\mathbb{R}).$$

We also deal with the Toeplitz and Hankel operators defined on the space $PL^{p(\cdot)}(\mathbb{R})$. More precisely, we consider the Toeplitz and Hankel operators defined by

$$\begin{aligned} \mathcal{T}_\phi &:= P\phi P : PL^{p(\cdot)}(\mathbb{R}) \rightarrow PL^{p(\cdot)}(\mathbb{R}), \\ \mathcal{H}_\phi &:= P\phi JP : PL^{p(\cdot)}(\mathbb{R}) \rightarrow PL^{p(\cdot)}(\mathbb{R}), \end{aligned}$$

respectively.

Analogously to [3, Proposition 2.10], we can derive the following identities between the Toeplitz/Wiener–Hopf and Hankel operators acting on the variable exponent Lebesgue spaces.

Proposition 2.3. *Let $\phi, \varphi \in \mathcal{M}^{p(\cdot)}(\mathbb{R})$ and $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$ be such that*

$$\mathcal{T}_\phi, \mathcal{T}_\varphi, \mathcal{H}_\phi, \mathcal{H}_\varphi \in \mathcal{L}(PL^{p(\cdot)}(\mathbb{R}), PL^{p(\cdot)}(\mathbb{R})).$$

Then

$$\begin{aligned} \mathcal{T}_{\phi\varphi} &= \mathcal{T}_\phi \mathcal{T}_\varphi + \mathcal{H}_\phi \mathcal{H}_\varphi^\sim, \\ \mathcal{H}_{\phi\varphi} &= \mathcal{T}_\phi \mathcal{H}_\varphi + \mathcal{H}_\phi \mathcal{T}_\varphi^\sim. \end{aligned}$$

In what follows, we will also make use of the identities

$$JQ = PJ, \quad JP = QJ, \quad J^2 = I, \quad JW_\phi^0 J = W_\phi^0.$$

In order to relate the operators and to transfer certain operator properties between the related operators, we will also make use of the following notions of equivalence and equivalence after extension relations between bounded linear operators. Consider two bounded linear operators $T : X_1 \rightarrow X_2$ and $S : Y_1 \rightarrow Y_2$, acting between the Banach spaces. The operators T and S are said to be *equivalent* (written as $T \sim S$) if there are two bounded invertible linear operators, $E : Y_2 \rightarrow X_2$ and $F : X_1 \rightarrow Y_1$, such that

$$T = E S F. \quad (2.1)$$

It directly follows from (2.1) that if two operators are equivalent, then they belong to the same *invertibility class*. More precisely, one of these operators is invertible, left invertible, right invertible or only generalized invertible if and only if the other operator enjoys the same property.

We say that T is *equivalent after extension* to S (written as $T \overset{*}{\sim} S$) if there are Banach spaces Z_1 and Z_2 and invertible bounded linear operators E and F such that

$$\begin{bmatrix} T & 0 \\ 0 & I_{Z_1} \end{bmatrix} = E \begin{bmatrix} S & 0 \\ 0 & I_{Z_2} \end{bmatrix} F,$$

where I_{Z_1} and I_{Z_2} represent the identity operators in Z_1 and Z_2 , respectively.

If two operators are equivalent after extension, then they belong to the same invertibility class.

Theorem 2.4. *Let $\phi \in \mathcal{M}^{p(\cdot)}(\mathbb{R})$ with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. The Wiener–Hopf plus Hankel operator*

$$W_\phi + H_\phi : L_+^{p(\cdot)}(\mathbb{R}) \rightarrow L^{p(\cdot)}(\mathbb{R}_+)$$

is equivalent to the Toeplitz plus Hankel operator

$$\mathcal{T}_\phi + \mathcal{H}_\phi = P\phi(I + J)P : PL^{p(\cdot)}(\mathbb{R}) \rightarrow PL^{p(\cdot)}(\mathbb{R}).$$

Proof. Consider the Wiener–Hopf plus Hankel operator rewritten in the form

$$W_\phi + H_\phi = r_+ \mathcal{F}^{-1} \phi(I + J) \mathcal{F}. \quad (2.2)$$

Applying the invertible operator $\mathcal{F}\ell_0 : L^{p(\cdot)}(\mathbb{R}_+) \rightarrow PL^{p(\cdot)}(\mathbb{R})$ from the left of (2.2) and the invertible operator $\ell_0 r_+ \mathcal{F}^{-1} : PL^{p(\cdot)}(\mathbb{R}) \rightarrow L_+^{p(\cdot)}(\mathbb{R})$ from the right, we obtain

$$\begin{aligned} \mathcal{F}\ell_0(W_\phi + H_\phi)\ell_0 r_+ \mathcal{F}^{-1} &= \mathcal{F}\ell_0 r_+ \mathcal{F}^{-1} \phi(I + J) \mathcal{F}\ell_0 r_+ \mathcal{F}^{-1} \\ &= P\phi(I + J)P \\ &= \mathcal{T}_\phi + \mathcal{H}_\phi, \end{aligned}$$

exhibiting therefore, in explicit form, the announced equivalence relation. $\square$

3 Even asymmetric factorization

Here we introduce the notion of even asymmetric factorization for invertible functions $\phi \in \mathcal{M}^{p(\cdot)}(\mathbb{R})$, with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$, in the space $L^{p(\cdot)}(\mathbb{R})$. This type of factorization will play an important role in the characterization of the invertibility and Fredholm property of the Wiener–Hopf plus Hankel operators under study.

Let us fix the notation (for $x \in \mathbb{R}$):

$$\lambda_\pm(x) := x \pm i, \quad \lambda(x) := \sqrt{x^2 + 1}, \quad \zeta(x) := \frac{x - i}{x + i}.$$

Moreover, let us introduce the following auxiliary operators

$$P_J = \frac{I + J}{2}, \quad Q_J = \frac{I - J}{2}$$

acting on $L^{p(\cdot)}(\mathbb{R})$. Because $J^2 = I$, these operators are complementary projections. Let us denote the image of P_J by $L_e^{p(\cdot)}(\mathbb{R})$, i.e.,

$$L_e^{p(\cdot)}(\mathbb{R}) := \{f \in L^{p(\cdot)}(\mathbb{R}) : f = \tilde{f}\}.$$

Additionally, we also consider the space $L_o^{p(\cdot)}(\mathbb{R})$ defined by

$$L_o^{p(\cdot)}(\mathbb{R}) := \{f \in L^{p(\cdot)}(\mathbb{R}) : f = -\tilde{f}\}.$$

We will now present some deduction, starting from the assumption of invertibility of our main operator, which is helpful for understanding the motivation for the definition of the factorization proposed below. Suppose that $W_\phi + H_\phi$ is invertible on $L^{p(\cdot)}(\mathbb{R})$. Then, from Theorem 2.4, it follows that $\mathcal{T}_\phi + \mathcal{H}_\phi$ is also invertible. Thus there exists a function $h \in PL^{p(\cdot)}(\mathbb{R})$ such that

$$P\phi(I + J)h = \zeta$$

or, equivalently,

$$P\phi(h + \tilde{h}) - \zeta = 0.$$

Note that $\zeta \in PL^{p(\cdot)}(\mathbb{R})$. Thus it follows that

$$\phi(h + \tilde{h}) - \zeta = h_-$$

for some “minus factor” h_- , which is equivalent to

$$\phi\zeta^{-1}(h + \tilde{h}) = \zeta^{-1}h_- + 1.$$

Letting $g_- := \zeta^{-1}h_- + 1$, we have $g_- \in QL^{p(\cdot)}(\mathbb{R}) \oplus \mathbb{C}$ and

$$\phi\zeta^{-1}(h + \tilde{h}) = g_-. \quad (3.1)$$

Multiplying both sides of equation (3.1) by the factor λ_-^2 , we obtain

$$\phi(x)\lambda_-^2(x)(h(x) + \tilde{h}(x)) = \lambda_-^2(x)g_-(x).$$

Introducing in the last identity the functions

$$f_e(x) := \lambda_-^2(x)(h(x) + \tilde{h}(x)), \quad f_-(x) := \lambda_-^2(x)g_-(x),$$

we can rewrite it in the form $\phi f_e = f_-$, and therefore $\phi = f_- f_e^{-1}$. Moreover, $f_e = \tilde{f}_e$. From the definition of f_- and f_e , it follows that

$$\lambda_-^{-2}f_- \in QL^{p(\cdot)}(\mathbb{R}) \oplus \mathbb{C}, \quad \lambda_-^{-2}f_e \in L_e^{p(\cdot)}(\mathbb{R}).$$

The above analysis shows the importance of the factors λ_-^{-2} and λ^{-2} in this process and, in particular, leads to the following definition of factorization.

Definition 3.1. A function $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ (with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$) is said to admit an *even asymmetric factorization* on $L^{p(\cdot)}(\mathbb{R})$ if it can be represented in the form

$$\phi(x) = \phi_-(x)\zeta^k(x)\phi_e(x), \quad x \in \mathbb{R}, \quad k \in \mathbb{Z},$$

where the following conditions hold:

- (i) $\lambda_-^{-2}\phi_- \in QL^{p(\cdot)}(\mathbb{R}) \oplus \mathbb{C}$ and $\lambda_-^{-2}x\phi_-^{-1} \in QL^{q(\cdot)}(\mathbb{R}) \oplus \mathbb{C}$;
- (ii) $\lambda^{-2}|x|\phi_e \in L_e^{q(\cdot)}(\mathbb{R})$ and $\lambda^{-2}\phi_e^{-1} \in L_e^{p(\cdot)}(\mathbb{R})$;

(iii) the linear operator $V := \phi_e^{-1}(I + J)P\phi_-^{-1}P : PL^{p(\cdot)}(\mathbb{R}) \rightarrow L_e^{p(\cdot)}(\mathbb{R})$ is bounded.

The uniqueness of an even asymmetric factorization (up to a constant) is stated in the following proposition.

Proposition 3.2. *Assume that $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ admits the following two even asymmetric factorizations in $L^{p(\cdot)}(\mathbb{R})$, with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$:*

$$\phi(x) = \phi_-^{(1)}(x)\zeta^{k_1}(x)\phi_e^{(1)}(x) = \phi_-^{(2)}(x)\zeta^{k_2}(x)\phi_e^{(2)}(x), \quad x \in \mathbb{R}. \quad (3.2)$$

Then $k_1 = k_2$, $\phi_-^{(1)} = c\phi_-^{(2)}$ and $\phi_e = c^{-1}\phi_e^{(2)}$ for some constant $c \in \mathbb{C} \setminus \{0\}$.

Proof. Let ϕ admit two even asymmetric factorizations in $L^{p(\cdot)}(\mathbb{R})$ as in (3.2), where $\phi_-^{(1)}$, $\phi_-^{(2)}$ and $\phi_e^{(1)}$, $\phi_e^{(2)}$ have the corresponding properties of Definition 3.1. From (3.2) we immediately have that

$$\phi_-^{(1)}(x)(\phi_-^{(2)}(x))^{-1}\zeta^{k_1-k_2}(x) = \phi_e^{(2)}(x)(\phi_e^{(1)}(x))^{-1}, \quad x \in \mathbb{R}. \quad (3.3)$$

Assume, without loss of generality, that $k := k_1 - k_2 \leq 0$ and consider the following auxiliary function:

$$\psi(x) := \frac{x}{\lambda_-^4(x)}\phi_-^{(1)}(x)(\phi_-^{(2)}(x))^{-1} \in H_-^1(\mathbb{R}) \quad (3.4)$$

which by the use of the reflection operator J also leads to

$$\widetilde{\psi}(x) = \frac{-x}{\lambda_+^4(x)}\widetilde{\phi_-^{(1)}}(x)(\widetilde{\phi_-^{(2)}}(x))^{-1} \in H_+^1(\mathbb{R}) \quad (3.5)$$

(where $H_{\pm}^1(\mathbb{R})$ denote the corresponding “plus” and “minus” Hardy spaces; see, e.g., [3, §2.5]).

The right-hand side of (3.3) is an even function (since it is the product of two even functions). Hence, from (3.3), we immediately obtain that

$$\phi_-^{(1)}(x) \left(\phi_-^{(2)}(x) \right)^{-1} \zeta^{2k}(x) = \widetilde{\phi_-^{(1)}}(x) \left(\widetilde{\phi_-^{(2)}}(x) \right)^{-1}.$$

This identity together with (3.4) and (3.5) lead to the conclusion that

$$\psi(x)\zeta^{2k+4}(x) = -\widetilde{\psi}(x). \quad (3.6)$$

Due to the inclusions in (3.4) and (3.5), if $2k + 4 \leq 0$ then from (3.6) we immediately obtain that ψ is identically zero and hence we have a contradiction. This means that only two possibilities remain: either $k = -1$ or $k = 0$. Let us analyze the case where $k = -1$. In this case, we have that (3.6) turns out to be equivalent to

$$\lambda_-^2(x)\psi(x) = -\lambda_+^2(x)\widetilde{\psi}(x).$$

From the last equation and (3.4) and (3.5), it follows that ψ must be a constant. Thus $c\zeta^2 = -c$ (with $c \in \mathbb{C}$), which implies that $c = 0$ and, consequently, $\psi = 0$. Hence, for $k = -1$ we would have a contradiction.

For the case $k = 0$, which means that $k_1 = k_2$, from (3.3) we have that

$$\phi_-^{(1)}(x)(\phi_-^{(2)}(x))^{-1} = \widetilde{\phi_-^{(1)}}(x)(\widetilde{\phi_-^{(2)}}(x))^{-1}.$$

Consequently, $\phi_-^{(1)}(x)(\phi_-^{(2)}(x))^{-1} = c$ for a constant $c \in \mathbb{C} \setminus \{0\}$. Thus $\phi_-^{(1)} = c\phi_-^{(2)}$ and $\phi_e^{(1)} = c^{-1}\phi_e^{(2)}$. $\square$

4 Equivalent operators

We will relate the Wiener–Hopf plus Hankel operators with the following operators:

$$\begin{aligned} \mathcal{U}_\phi &:= P\phi P_J : L_e^{p(\cdot)}(\mathbb{R}) \rightarrow PL^{p(\cdot)}(\mathbb{R}), \\ \mathcal{V}_\phi &:= Q_J\phi^{-1}Q : QL^{p(\cdot)}(\mathbb{R}) \rightarrow L_o^{p(\cdot)}(\mathbb{R}). \end{aligned}$$

Proposition 4.1. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ and $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. The operator $\mathcal{U}_\phi \in \mathcal{L}(L_e^{p(\cdot)}(\mathbb{R}), PL^{p(\cdot)}(\mathbb{R}))$ is equivalent to the Wiener–Hopf plus Hankel operator $W_\phi + H_\phi \in \mathcal{L}(L_+^{p(\cdot)}(\mathbb{R}), L^{p(\cdot)}(\mathbb{R}_+))$.*

Proof. First, recall that from Theorem 2.4 we already know that

$$W_\phi + H_\phi \sim P\phi(I + J)P.$$

Then let us consider the operators

$$\begin{aligned} R_1 &:= \frac{1}{\sqrt{2}}(I + J)P : PL^{p(\cdot)}(\mathbb{R}) \rightarrow L_e^{p(\cdot)}(\mathbb{R}) \\ R_2 &:= \frac{1}{\sqrt{2}}P(I + J) : L_e^{p(\cdot)}(\mathbb{R}) \rightarrow PL^{p(\cdot)}(\mathbb{R}). \end{aligned}$$

It is easily seen that $R_1R_2 = P_J$ and $R_2R_1 = P$ (the identity operators on $L_e^{p(\cdot)}(\mathbb{R})$ and $PL^{p(\cdot)}(\mathbb{R})$, respectively). Thus R_1 and R_2 are the inverses to each other and a direct computation yields that

$$\mathcal{U}_\phi = \frac{1}{\sqrt{2}}P\phi(I + J)PR_2,$$

which clearly shows the equivalence relation between $\mathcal{U}_\phi$ and $\mathcal{T}_\phi + \mathcal{H}_\phi$ and, consequently, between $\mathcal{U}_\phi$ and $W_\phi + H_\phi$. $\square$

Lemma 4.2. *Let X and Y be linear normed spaces, $A : X \rightarrow Y$ a linear invertible operator, $P_1 : X \rightarrow P_1X$ and $P_2 : Y \rightarrow P_2Y$ be linear projections, and let $Q_1 := I - P_1$ and $Q_2 := I - P_2$. Then $P_2AP_1 : P_1X \rightarrow P_2Y$ is equivalent after extension to $Q_1A^{-1}Q_2 : Q_2Y \rightarrow Q_1X$.*

Proof. Let $A : X \rightarrow Y$ be an invertible operator with inverse $A^{-1} : Y \rightarrow X$. We can rewrite A^{-1} in the following matrix form upon the use of the subspaces defined by the projections P_1, Q_1, P_2 and Q_2 :

$$A^{-1} = \begin{bmatrix} P_1A^{-1}Q_2 & P_1A^{-1}P_2 \\ Q_1A^{-1}Q_2 & Q_1A^{-1}P_2 \end{bmatrix} : Q_2Y \times P_2Y \rightarrow P_1X \times Q_1X.$$

A direct computation yields

$$\begin{bmatrix} Q_1A^{-1}Q_2 & 0 \\ 0 & P_2 \end{bmatrix} = B \begin{bmatrix} P_2AP_1 & 0 \\ 0 & Q_1 \end{bmatrix} A^{-1}, \quad (4.1)$$

where B is the invertible matrix operator defined by

$$B := \begin{bmatrix} 1 & -Q_1A^{-1}P_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & Q_1 \\ P_2 & P_2AQ_1 \end{bmatrix}.$$

Thus (4.1) shows the equivalence after extension between $Q_1A^{-1}Q_2$ and P_2AP_1 . $\square$

Proposition 4.3. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. The operator $\mathcal{U}_\phi : L_e^{p(\cdot)}(\mathbb{R}) \rightarrow PL^{p(\cdot)}(\mathbb{R})$ is invertible if and only if $\mathcal{V}_\phi : QL^{p(\cdot)}(\mathbb{R}) \rightarrow L_o^{p(\cdot)}(\mathbb{R})$ is invertible.*

Proof. This result follows directly from Lemma 4.2. Choosing $P_1 := P_J$, $P_2 := P$, $Q_1 := Q_J$ and $Q_2 := Q$, we derive that $P\phi P_J$ is equivalent after extension to the operator $Q_J\phi^{-1}Q$. Consequently, $\mathcal{U}_\phi$ is invertible if and only if $\mathcal{V}_\phi$ is invertible. $\square$

The following corollary appears now as a direct and natural consequence of the above results.

Corollary 4.4. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. The following assertions are equivalent:*

- (i) $W_\phi + H_\phi \in \mathcal{L}(L_+^{p(\cdot)}(\mathbb{R}), L^{p(\cdot)}(\mathbb{R}_+))$ is invertible;
- (ii) $\mathcal{U}_\phi \in \mathcal{L}(L_e^{p(\cdot)}(\mathbb{R}), PL^{p(\cdot)}(\mathbb{R}))$ is invertible;
- (iii) $\mathcal{V}_\phi \in \mathcal{L}(QL^{p(\cdot)}(\mathbb{R}), L_o^{p(\cdot)}(\mathbb{R}))$ is invertible.

5 Invertibility via even asymmetric factorization

In this section, we obtain the necessary and sufficient conditions for the invertibility of Wiener–Hopf plus Hankel operators on variable exponent Lebesgue spaces in terms of the even asymmetric factorization.

Lemma 5.1. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. If $\mathcal{U}_\phi$ is invertible, then there exist functions $f_- \neq 0$ and f_e such that*

$$f_- = \phi f_e \quad (5.1)$$

and

$$\lambda_-^{-2} f_- \in QL^{p(\cdot)}(\mathbb{R}) \oplus \mathbb{C}, \quad \lambda^{-2} f_e \in L_e^{p(\cdot)}(\mathbb{R}). \quad (5.2)$$

Proof. If $\mathcal{U}_\phi$ is invertible, then $\text{im } \mathcal{U}_\phi = PL^{p(\cdot)}(\mathbb{R})$. Thus there exists a function $h_e \in L_e^{p(\cdot)}(\mathbb{R}) \setminus \{0\}$ such that $\mathcal{U}_\phi h_e = \zeta$, where $\zeta \in PL^{p(\cdot)}(\mathbb{R})$. It follows that

$$P\phi h_e - \zeta = 0.$$

Thus

$$\phi h_e - \zeta = g_-, \quad g_- \in QL^{p(\cdot)}(\mathbb{R})$$

and, consequently,

$$\zeta^{-1} \phi h_e = \zeta^{-1} g_- + 1.$$

Let $h_- := \zeta^{-1} g_- + 1$. It follows that $h_- \in QL^{p(\cdot)}(\mathbb{R}) \oplus \mathbb{C}$ (with $h_- \neq 0$) and $h_- = \zeta^{-1} \phi h_e$.

Having defined $f_-(x) := \lambda_-^2(x) h_-(x)$, we obtain that

$$\lambda_-^{-2} f_- = \zeta^{-1} \phi h_e.$$

Thus f_- satisfies the required condition in (5.2).

On the other hand, having defined $f_e(x) := \lambda^2(x) h_e(x)$, we have that f_e is an even function and also that $\lambda^{-2} f_e \in L_e^{p(\cdot)}(\mathbb{R})$. Additionally, factorization (5.1) holds true. $\square$

Lemma 5.2. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$, and suppose that $\mathcal{V}_\phi$ is invertible. Then there exist functions $g_- \neq 0$ and g_e such that*

$$g_- = g_e \phi^{-1}$$

and

$$\lambda_-^{-2} x g_- \in QL^{q(\cdot)}(\mathbb{R}) \oplus \mathbb{C}, \quad \lambda_-^{-2} |x| g_e \in L_e^{q(\cdot)}(\mathbb{R}).$$

Proof. If $\mathcal{V}_\phi \in \mathcal{L}(QL^{p(\cdot)}(\mathbb{R}), L_o^{p(\cdot)}(\mathbb{R}))$ is invertible, then $\mathcal{V}_\phi^* = Q\overline{\phi^{-1}}Q_J \in \mathcal{L}(L_o^{q(\cdot)}(\mathbb{R}), QL^{q(\cdot)}(\mathbb{R}))$ is also invertible.

Let $C : \varphi \mapsto \overline{\varphi}$ be the complex conjugate operator on $L^{q(\cdot)}(\mathbb{R})$. Due to the circumstance that

$$C\overline{\phi}C = \phi, \quad SC + CS = 0, \quad CQC = P,$$

it follows that

$$P\phi^{-1}Q_J = C(Q\overline{\phi^{-1}}Q_J)C$$

is also invertible and thus $\text{im}(P\phi^{-1}Q_J) = PL^{q(\cdot)}(\mathbb{R})$.

Let $h_o \in L_o^{q(\cdot)}(\mathbb{R})$ be such that $P\phi^{-1}Q_J h_o = \zeta$ or, equivalently, $P\phi^{-1}h_o - \zeta = 0$. Thus

$$\phi^{-1}h_o - \zeta = f_-, \quad f_- \in QL^{q(\cdot)}(\mathbb{R})$$

and, consequently,

$$\zeta^{-1} f_- + 1 = \zeta^{-1} \phi^{-1} h_o.$$

Let us take $h_- := \zeta^{-1} f_- + 1 \in QL^{q(\cdot)}(\mathbb{R}) \oplus \mathbb{C}$ and define $g_-(x) := \frac{\lambda_-^2(x)}{x} h_-(x)$ ($x \neq 0$). Therefore we obtain

$$g_-(x) = \frac{\lambda_-^2(x)}{x} \phi^{-1}(x) h_o(x)$$

and

$$x\lambda_-^{-2}(x)g_-(x) = \zeta^{-1}(x)\phi^{-1}(x)h_o(x) \in QL^{q(\cdot)}(\mathbb{R}) \oplus \mathbb{C}.$$

Choosing now $g_e(x) := \frac{\lambda^2(x)}{x}h_o(x)$ ($x \neq 0$), we have that g_e is an even function, $|x|\lambda^{-2}g_e \in L_e^{q(\cdot)}(\mathbb{R})$ and the desired factorization holds true. $\square$

Theorem 5.3. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ and $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. The Wiener–Hopf plus Hankel operator*

$$W_\phi + H_\phi \in \mathcal{L}(L_+^{p(\cdot)}(\mathbb{R}), L^{p(\cdot)}(\mathbb{R}_+))$$

is invertible if and only if ϕ admits an even asymmetric factorization in $L^{p(\cdot)}(\mathbb{R})$ with index $k = 0$.

Proof. From Corollary 4.4, we know already that $W_\phi + H_\phi$ is invertible if and only if $\mathcal{U}_\phi$ is invertible.

Suppose that ϕ admits an even asymmetric factorization $\phi = \phi_- \phi_e$ (with $k = 0$) and let us prove that $\mathcal{U}_\phi$ is invertible.

First, we have that $\ker \mathcal{U}_\phi = \{0\}$. Indeed, if $\mathcal{U}_\phi g_e = 0$ for $g_e \in L_e^{p(\cdot)}(\mathbb{R})$, then $P\phi_- \phi_e g_e = 0$ and, consequently,

$$\phi_- \phi_e g_e = g_- \in QL^{p(\cdot)}(\mathbb{R}).$$

It follows that

$$\phi_e g_e = \phi_-^{-1} g_-,$$

where $\phi_e g_e$ is an even factor. Thus we have

$$\phi_-^{-1} g_- = \widetilde{\phi_-^{-1} g_-},$$

and, due to the properties of the spaces to which these functions belong, we conclude that $\phi_-^{-1} g_- = 0$. Consequently, $\phi_e g_e = 0$. Because $\phi_e \neq 0$ a.e., it follows that $g_e = 0$, that is, $\ker \mathcal{U}_\phi = \{0\}$.

Second, we now show that $\mathcal{U}_\phi$ is surjective. By the condition (iii) of Definition 3.1, we have that

$$V := \phi_e^{-1}(I + J)P\phi_-^{-1}P : PL^{p(\cdot)}(\mathbb{R}) \rightarrow L_e^{p(\cdot)}(\mathbb{R})$$

is a bounded linear operator. Let $f \in PL^{p(\cdot)}(\mathbb{R})$. Then we have

$$\begin{aligned} \mathcal{U}_\phi V f &= (P\phi_- \phi_e P J \phi_e^{-1}(I + J)P\phi_-^{-1}P) f \\ &= (P\phi_- \phi_e \phi_e^{-1}(I + J)P\phi_-^{-1}P) f \\ &= (P\phi_- (I + J)P\phi_-^{-1}P) f \\ &= (\mathcal{T}_{\phi_-} \mathcal{T}_{\phi_-^{-1}} + \mathcal{H}_{\phi_-} \mathcal{T}_{\phi_-^{-1}}) f \\ &= \left(\mathcal{T}_{\phi_- \phi_-^{-1}} - \mathcal{H}_{\phi_-} \widetilde{\mathcal{H}_{\phi_-^{-1}}} + \mathcal{H}_{\phi_-} \mathcal{T}_{\phi_-^{-1}} \right) f \\ &= f \end{aligned}$$

by Proposition 2.3 and using the fact that $\mathcal{H}_{\phi_-} = 0$. Since both $\mathcal{U}_\phi$ and V are bounded, it results that $\mathcal{U}_\phi V = P$. This proves that $\mathcal{U}_\phi$ is surjective.

For the reverse implication, let us suppose that $W_\phi + H_\phi$ is invertible (and, consequently, both $\mathcal{U}_\phi$ and $\mathcal{V}_\phi$ are also invertible operators). Applying Lemmas 5.1 and 5.2, it follows that $f_- = \phi f_e$ and $g_- = g_e \phi^{-1}$ (with the appropriate properties presented in those respective lemmas). Multiplying the corresponding elements in the last two identities, we obtain that $g_- f_- = g_e f_e$ and it follows that

$$g_- f_- = g_e f_e =: C$$

is a nonzero constant. Now, put $\phi_- := f_- = C g_-^{-1}$ and $\phi_e := f_e^{-1} = g_e C^{-1}$. Then

$$\phi = \phi_- \phi_e,$$

where (due to Lemmas 5.1 and 5.2) we have that these factors, ϕ_- and ϕ_e , satisfy the properties (i) and (ii) of Definition 3.1.

The operator $V : PL^{p(\cdot)}(\mathbb{R}) \rightarrow L_e^{p(\cdot)}(\mathbb{R})$, being the right inverse of $\mathcal{U}_\phi$, is a bounded linear operator. It can be verified in the same way as above.

Thus, we have just concluded that under the hypothesis of the invertibility of $W_\phi + H_\phi$ it follows that ϕ admits an even asymmetric factorization in $L^{p(\cdot)}(\mathbb{R})$ with index 0. $\square$

6 Fredholm theory via even asymmetric factorization

In this section, we establish the Fredholm theory of our Wiener–Hopf plus Hankel operators with the help of the even asymmetric factorization. For this purpose, we start by recalling some auxiliary results on the Wiener–Hopf and Hankel operators.

Let $C(\mathbb{R})$ be the set of bounded continuous functions on $\mathbb{R}$ for which the two limits at $\pm\infty$ exist and coincide. Analogously to the case of Lebesgue spaces with constant exponents, we can formulate the following proposition for Hankel operators on the variable exponent Lebesgue spaces (see, e.g., [3, Proposition 2.11]).

Proposition 6.1. *Let $\varphi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ and $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. If $\varphi \in C(\mathbb{R})$, then $H_\varphi : L_+^{p(\cdot)}(\mathbb{R}) \rightarrow L^{p(\cdot)}(\mathbb{R}_+)$ is a compact operator.*

The following theorem is the analogue of the *Coburn–Simonenko Theorem*, for Wiener–Hopf operators on variable exponent Lebesgue spaces, which states that a nonzero bounded Wiener–Hopf operator has a trivial kernel or a dense range.

Theorem 6.2 (see, e.g., [8, Theorem 4]). *If $\phi \in \mathcal{M}^{p(\cdot)}(\mathbb{R})$, with $p(\cdot) \in \mathcal{B}(\mathbb{R})$, does not vanish identically, then the kernel of W_ϕ in $L_+^{p(\cdot)}(\mathbb{R})$ is trivial or the image of W_ϕ is dense in $L^{p(\cdot)}(\mathbb{R}_+)$.*

As a direct consequence of this last theorem, we can establish the following corollary.

Corollary 6.3. *Except for the zero operator, a normally solvable Wiener–Hopf operator is automatically semi-Fredholm. If $\phi \in \mathcal{M}^{p(\cdot)}(\mathbb{R})$ (with $p(\cdot) \in \mathcal{B}(\mathbb{R})$) and W_ϕ is a semi-Fredholm operator, then $\dim \ker(W_\phi) = 0$ or $\dim \operatorname{coker}(W_\phi) = 0$.*

The Wiener–Hopf operators with Fourier symbols directly dependent on ζ , play an important role in the study of the Fredholm property of our Wiener–Hopf plus Hankel operators. In particular, a very useful result is presented in the next theorem (whose proof can be derived as in the case of classical Lebesgue spaces, see, e.g., [3]).

Theorem 6.4. *Let $p(\cdot) \in \mathcal{B}(\mathbb{R})$. The Wiener–Hopf operator W_ζ is Fredholm on $\mathcal{L}(L_+^{p(\cdot)}(\mathbb{R}), L^{p(\cdot)}(\mathbb{R}_+))$ and has the Fredholm index*

$$\operatorname{Ind} W_\zeta = -1.$$

Remark 6.5. Since $\operatorname{Ind} W_{\zeta^n} = n \operatorname{Ind} W_\zeta$, it follows that $\operatorname{Ind} W_{\zeta^n} = -n$.

The following two results can be found in [1] for the Toeplitz plus Hankel operators on Lebesgue spaces with a constant exponent and remain valid for the Wiener–Hopf plus Hankel operators on variable exponent Lebesgue spaces; see [8, Theorem 5].

Theorem 6.6. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$, with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. Then $\ker(W_\phi + H_\phi) = \{0\}$ or $\operatorname{coker}(W_\phi + H_\phi) = \{0\}$.*

Corollary 6.7. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$, with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. Then $W_\phi + H_\phi$ is invertible in $\mathcal{L}(L_+^{p(\cdot)}(\mathbb{R}), L^{p(\cdot)}(\mathbb{R}_+))$ if and only if $W_\phi + H_\phi$ is Fredholm with Fredholm index 0.*

We are now in the position to establish a Fredholm criterion based on the even asymmetric factorization.

Theorem 6.8. *Let $\phi \in \mathcal{GM}^{p(\cdot)}(\mathbb{R})$ with $p(\cdot) \in \mathcal{B}_e(\mathbb{R})$. The operator*

$$W_\phi + H_\phi : L_+^{p(\cdot)}(\mathbb{R}) \rightarrow L^{p(\cdot)}(\mathbb{R}_+)$$

is a Fredholm operator with Fredholm index $-k$ if and only if ϕ admits an even asymmetric factorization

$$\phi = \phi_- \zeta^k \phi_e$$

in $L^{p(\cdot)}(\mathbb{R})$ with index $k \in \mathbb{Z}$. Moreover, under the Fredholm property, the defect numbers of $W_\phi + H_\phi$ are given by

$$\dim \ker(W_\phi + H_\phi) = \max\{0, -k\}, \quad \dim \operatorname{coker}(W_\phi + H_\phi) = \max\{0, k\}.$$

Proof. Suppose that ϕ admits an even asymmetric factorization in $L^{p(\cdot)}(\mathbb{R})$ with index $k \in \mathbb{Z}$, i.e., $\phi = \phi_- \zeta^k \phi_e$ with all the properties stated in Definition 3.1. It follows that

$$\begin{aligned} W_\phi + H_\phi &= (W_{\phi_-} + H_{\phi_-}) \ell_0(W_{\zeta^k} + H_{\zeta^k}) \ell_0(W_{\phi_e} + H_{\phi_e}) \\ &= W_{\phi_-} \ell_0(W_{\zeta^k} + H_{\zeta^k}) \ell_0(W_{\phi_e} + H_{\phi_e}). \end{aligned}$$

The latter operator is equivalent to $W_{\zeta^k} + H_{\zeta^k}$ since $W_{\phi_-} \ell_0$ and $\ell_0(W_{\phi_e} + H_{\phi_e})$ are invertible operators. Additionally, since the Hankel operator H_{ζ^k} is compact (see Proposition 6.1), it follows that $W_\phi + H_\phi$ is Fredholm if and only if W_{ζ^k} is Fredholm (and, under the Fredholm property, they have the same Fredholm index). Thus from Theorem 6.4 we conclude that $W_\phi + H_\phi$ is a Fredholm operator with index $-k$.

Let us now prove the reverse implication. Assume that $W_\phi + H_\phi$ is a Fredholm operator with Fredholm index $-k$. Let us prove that ϕ admits an even asymmetric factorization with index k . To this purpose, consider the auxiliary function

$$\psi(x) := \zeta^{-k}(x) \phi(x). \quad (6.1)$$

From Proposition 2.3, we have that

$$W_\psi + H_\psi = W_{\zeta^{-k}} \ell_0(W_\phi + H_\phi) + H_{\zeta^{-k}} \ell_0(W_\phi + H_\phi).$$

Since the Hankel operator $H_{\zeta^{-k}}$ is compact (see Proposition 6.1) and the product of a compact operator with a bounded one is also compact, it follows that

$$W_\psi + H_\psi = W_{\zeta^{-k}} \ell_0(W_\phi + H_\phi) + K_1, \quad (6.2)$$

where K_1 is a compact operator. Moreover, from Theorem 6.4 (and Remark 6.5), we conclude that $W_{\zeta^{-k}}$ is a Fredholm operator with Fredholm index k . Consequently, from (6.2), we deduce that $W_\psi + H_\psi$ is a Fredholm operator with Fredholm index 0 and, from Corollary 6.7, it follows that $W_\psi + H_\psi$ is invertible. Theorem 5.3 implies that ψ admits an even asymmetric factorization in $L^{p(\cdot)}(\mathbb{R})$ with index 0. Hence, keeping in mind (6.1), we conclude that ϕ admits an even asymmetric factorization in $L^{p(\cdot)}(\mathbb{R})$ with index k .

As about the defect numbers, we know that $W_\phi + H_\phi$ has the Fredholm index $-k$. From Corollary 6.6, the kernel or the cokernel of $W_\phi + H_\phi$ is trivial. This implies the formulas for the defect numbers. $\square$

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