A Scale Variational Principle of Herglotz

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Abstract

The Herglotz problem is a generalization of the fundamental problem of the calculus of variations. In this paper, we consider a class of non-differentiable functions, where the dynamics is described by a scale derivative. Necessary conditions are derived to determine the optimal solution for the problem. Some other problems are considered, like transversality conditions, the multi-dimensional case, higher-order derivatives and for several independent variables.

Keywords: calculus of variations; scale derivative.

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1 Introduction

The calculus of variations deals with optimization of a given functional, whose algebraic expression is the integral of a given function, that depends on time, space and the velocity of the trajectory:

\[ x \mapsto \int_{a}^{b} L(t, x(t), \dot{x}(t)) \, dt. \]

The variational principle of Herglotz can be seen as an extension of such classical theories, but instead of an integral, we have the functional as a solution of a differential equation (see [9, 10]):

\[ \begin{cases} \dot{z}(t) = L(t, x(t), \dot{x}(t)), & \text{with } t \in [a, b], \\ z(a) = z_a. \end{cases} \]

Without the dependence of \( z \), we can convert this problem into a calculus of variations problem. In fact, integrating the differential equation

\[ \dot{z}(t) = L(t, x(t), \dot{x}(t)) \]

from \( a \) to \( b \), we obtain

\[ z(b) = \int_{a}^{b} \left[ L(t, x(t), \dot{x}(t)) + \frac{z_a}{b-a} \right] \, dt. \]

Recently, more advances were made namely proving Noether’s type theorems for the variational principle of Herglotz (see e.g. [5, 6, 7, 8, 9, 12]). The aim of this paper is to consider the Herglotz problem, but the trajectories \( x(\cdot) \) may be non-differentiable functions. We believe that this situation may model more efficiently certain physical problems, like fractals.

The organization of the paper is the following. In Section 2 we define what is a scale derivative, following the concept as presented in [2], and we present some of its main properties, like the algebraic rules, integration by parts formula, etc. In Section 3 we prove our new results. After presenting the Herglotz scale problem, we prove a necessary condition that every extremizer must fulfill. Some generalizations of the main result are also presented to complete the study.
2 Scale calculus

We review some definitions and the main results from [2] that we will need. For more on the subject, see references [1, 2, 3].

From now on, let \(\alpha, \beta, h\) be reals in \([0,1]\) with \(\alpha + \beta > 1\) and \(h \ll 1\), and consider \(I := [a - h, b + h]\).

**Definition 1.** Let \(f : I \rightarrow \mathbb{R}\) be a function. The delta derivative of \(f\) at \(t\) is defined by
\[
\Delta_h[f](t) := \frac{f(t + h) - f(t)}{h}, \quad \text{for} \quad t \in [a - h, b],
\]
and the nabla derivative of \(f\) at \(t\) is defined by
\[
\nabla_h[f](t) := \frac{f(t) - f(t - h)}{h}, \quad \text{for} \quad t \in [a, b + h].
\]

If \(f\) is differentiable, then
\[
\lim_{h \to 0} \Delta_h[f](t) = \lim_{h \to 0} \nabla_h[f](t) = f'(t).
\]

These two operators can be combined into a single one, where the real part is the mean value of such operators, and the complex part measures the difference between them.

**Definition 2.** The \(h\)-scale derivative of \(f\) at \(t\) is given by
\[
\frac{\Box_h f}{\Box t}(t) = \frac{1}{2} [(\Delta_h[f](t) + \nabla_h[f](t)) + i(\Delta_h[f](t) - \nabla_h[f](t))], \quad \text{for} \quad t \in [a, b].
\]

For complex valued functions \(f\), such definition is extended by
\[
\frac{\Box_h f}{\Box t}(t) = \frac{\Box_h \text{Re} f}{\Box t}(t) + i \frac{\Box_h \text{Im} f}{\Box t}(t).
\]

We now explain how to drop the dependence on the parameter \(h\) in the definition of the scale derivative. First, consider the set \(C^0_{\text{conv}}([a,b] \times [0,1], \mathbb{C})\) of the functions \(g \in C^0([a,b] \times [0,1], \mathbb{C})\) for which the limit
\[
\lim_{h \to 0} g(t, h)
\]
exists for all \(t \in [a,b]\), and let \(E\) be a complementary space of \(C^0_{\text{conv}}([a,b] \times [0,1], \mathbb{C})\) in \(C^0([a,b] \times [0,1], \mathbb{C})\).

Define \(\pi\) the projection of \(C^0_{\text{conv}}([a,b] \times [0,1], \mathbb{C}) \oplus E\) onto \(C^0_{\text{conv}}([a,b] \times [0,1], \mathbb{C}),\)
\[
\pi : C^0_{\text{conv}}([a,b] \times [0,1], \mathbb{C}) \oplus E \rightarrow C^0_{\text{conv}}([a,b] \times [0,1], \mathbb{C})
\]
\[
g := g_{\text{conv}} + g_E \quad \mapsto \quad \pi(g) = g_{\text{conv}}.
\]

Using these definitions, we arrive at the main concept of [2].

**Definition 3.** The scale derivative of \(f \in C^0(I, \mathbb{C})\), denoted by \(\frac{\Box f}{\Box t}\), is defined by
\[
\frac{\Box f}{\Box t}(t) := \frac{\Box_h f}{\Box t}(t), \quad t \in [a, b],
\]
where
\[
\left(\frac{\Box_h f}{\Box t}\right)(t) = \lim_{h \to 0} \pi \left(\frac{\Box_h f}{\Box t}(t)\right).
\]

**Definition 4.** Given \(f : I^n = [a - nh, b + nh] \rightarrow \mathbb{C}\), define the higher-order scale derivative of \(f\) by
\[
\frac{\Box^n f}{\Box t^n}(t) = \frac{\Box f}{\Box t} \left(\frac{\Box^{n-1} f}{\Box t^{n-1}}\right)(t), \quad t \in [a, b],
\]
where \(\frac{\Box f^1}{\Box t^1} := \frac{\Box f}{\Box t}\) and \(\frac{\Box f^n}{\Box t^n} := f\).
We will adopt the notation $\square^n f(t)$ instead of $\frac{\partial^n f}{\partial t^n}(t)$ when there is no danger of confusion. Scale partial derivatives are also considered here. They are defined as in the standard case.

**Definition 5.** Let $f: \prod_{i=1}^n [a_i, b_i + h] \to \mathbb{R}$ be a function. Define, for each $i \in \{1, \ldots, n\}$,

$$\Delta_h^i[f](t_1, \ldots, t_n) := \frac{f(t_1, \ldots, t_{i-1}, t_i+h, t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n)}{h},$$

for $t_i \in [a_i - h, b_i]$ and for $t_j \in [a_j - h, b_j + h]$ if $j \neq i$, and

$$\nabla_h^i[f](t_1, \ldots, t_n) := \frac{f(t_1, \ldots, t_{i-1}, t_i, t_{i+1}, \ldots, t_n) - f(t_1, \ldots, t_{i-1}, t_i+h, t_{i+1}, \ldots, t_n)}{h},$$

for $t_i \in [a_i, b_i + h]$ and for $t_j \in [a_j - h, b_j + h]$, if $j \neq i$. The $h$-scale partial derivative of $f$ with respect to the $i$-th coordinate is given by

$$\frac{\partial_h f}{\partial t_i}(t_1, \ldots, t_n) = \frac{1}{2} \left[ (\Delta_h^i[f] + \nabla_h^i[f]) + i \left( \Delta_h^i[f] - \nabla_h^i[f] \right) \right],$$

for $t_i \in [a_i, b_i]$.

The definition of partial scale derivatives $\square f/\square t_i$ is clear. In what follows, we will denote

$$C^\alpha_\square([a, b], \mathbb{K}) := \{ f \in C^0(I^n, \mathbb{K}) \mid \frac{\partial^k f}{\partial t_k} \in C^0(I^{n-k}, \mathbb{C}), k = 1, 2, \ldots, n \}, \quad \mathbb{K} = \mathbb{R} \text{ or } \mathbb{K} = \mathbb{C}.$$

**Definition 6.** Let $f \in C^0(I, \mathbb{C})$ and $\alpha \in [0, 1]$. We say that $f$ is Hölderian of Hölder exponent $\alpha$ if there exists a constant $C > 0$ such that, for all $s, t \in I$,

$$|f(t) - f(s)| \leq C|t - s|^\alpha,$$

and we write $f \in H^\alpha(I, \mathbb{C})$, or simply $f \in H^\alpha$ when there is no danger of mislead.

We say that $f(t_1, \ldots, t_n) \in H^\alpha$ if $f(t_1, \ldots, t_{i-1}, t_i + h, t_{i+1}, \ldots, t_n) \in H^\alpha$, for all $i \in \{1, \ldots, n\}$ and for all $t_j \in [a_j, b_j]$, $j \neq i$.

**Theorem 1.** For all $f \in H^\alpha$ and $g \in H^2$, we have

$$\frac{\partial(f \cdot g)}{\partial t}(t) = \frac{\partial f}{\partial t}(t)g(t) + f(t) \frac{\partial g}{\partial t}(t), \quad t \in [a, b].$$

**Theorem 2.** Let $f \in C^1_\square([a, b], \mathbb{R})$ be such that

$$\lim_{h \to 0} \int_a^b \frac{\partial_h f}{\partial t} (t) dt = 0,$$

where $\frac{\partial_h f}{\partial t} := \left( \frac{\partial_h f}{\partial t} \right)_{\text{conv}} + \left( \frac{\partial_h f}{\partial t} \right)_E$. Then,

$$\int_a^b \frac{\partial_h f}{\partial t}(t) dt = f(b) - f(a).$$

As a consequence, we have the following integration by parts formula. If

$$\lim_{h \to 0} \int_a^b \left( \frac{\partial_h (f \cdot g)}{\partial t} \right)_E (t) dt = 0,$$

where $f \in H^\alpha$ and $g \in H^2$, then

$$\int_a^b \frac{\partial f}{\partial t}(t) \cdot g(t) dt = [f(t)g(t)]_a^b - \int_a^b f(t) \cdot \frac{\partial g}{\partial t}(t) dt.$$
3 The scale variational principle of Herglotz

The (classical) variational principle of Herglotz is described in the following way. Consider the differential equation

\[
\begin{align*}
\dot{z}(t) &= L(t, x(t), \dot{x}(t), z(t)), \quad \text{with } t \in [a, b] \\
z(a) &= z_a \\
x(a) &= x_a, x(b) = x_b,
\end{align*}
\]

where \( x, z \) and \( L \) are smooth functions. We wish to find \( x \) (and the correspondent solution \( z \) of the system) such that \( z(b) \) attains an extremum. The necessary condition is a second-order differential equation:

\[
\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} + \frac{\partial L}{\partial z} \frac{\partial z}{\partial x},
\]

for all \( t \in [a, b] \). This can be seen as an extension of the basic problem of calculus of variations. If \( L \) does not depend on \( z \), then integrating the differential equation along the interval \([a, b] \), we get

\[
\left\{ \begin{array}{l}
\int_a^b \left[ L(t, x(t), \dot{x}(t)) + \frac{z_a}{b-a} \right] dt \to \text{extremize} \\
x(a) = x_a, x(b) = x_b.
\end{array} \right.
\]

As is well known, many physical phenomena are characterized by non-differentiable functions (e.g. generic trajectories of quantum mechanics [4], scale-relativity without the hypothesis of space-time differentiability [11]). The usual procedure is to replace the classical derivative by a scale derivative, and consider the space of continuous (and non-differentiable) functions. The scale calculus of variations approach was studied in [1, 2, 3] for a certain concept of scale derivative \( \Box x(t) \):

\[
\left\{ \begin{array}{l}
\int_a^b L(t, x(t), \Box x(t)) \to \text{extremize} \\
x(a) = x_a, x(b) = x_b.
\end{array} \right.
\]

Motivated by this problem, we define the fundamental scale variational principle of Herglotz. First we need to define what extremum is.

**Definition 7.** We say that \( z \in C^1([a, b], \mathbb{C}) \) attains an extremum at \( t = b \) if \( z'(b) = 0 \).

The problem is then stated in the following way. Consider the system

\[
\begin{align*}
\dot{z}(t) &= L(t, x(t), \Box x(t), z(t)), \quad \text{with } t \in [a, b] \\
z(a) &= z_a \\
x(a) &= x_a, x(b) = x_b.
\end{align*}
\]

(4)

For simplicity, define

\[
[x, z](t) := (t, x(t), \Box x(t), z(t)).
\]

We assume that

1. the trajectories \( x \) are in \( H^\alpha \cap C^1_\Box([a, b], \mathbb{R}) \), \( \Box x \in H^\alpha \) and the functional \( z \) in \( C^2([a, b], \mathbb{C}) \),
2. for each \( x \), there exists a unique solution \( z \) of the system (4)
3. \( z_a, x_a, x_b \) are fixed numbers,
4. the Lagrangian \( L : [a, b] \times \mathbb{R} \times C^2 \to \mathbb{C} \) is of class \( C^2 \).

Observe that the solution \( z(t) \) actually is a function on three variables, to know \( z = z(t, x(t), \Box x(t)) \). When there is no danger of mislead, we will simply write \( z(t) \). We are interested in finding a trajectory \( x \) for which the corresponding solution \( z \) is such that \( z(b) \) attains an extremum. In particular, what necessary conditions such solutions must fulfill. These equations are called Euler-Lagrange.
equation types. Again, problem (4) can be reduced to the scale variational problem in case $L$ is independent of $z$:

$$
\int_a^b L \left[ (t, x(t), \Box x(t)) + \frac{z_a}{b-a} \right] dt \rightarrow \text{extremize.}
$$

**Theorem 3.** If the pair $(x, z)$ is a solution of problem (4), and $\frac{\partial L}{\partial x} [x, z] \in H^\alpha (I, \mathbb{C})$ ($\alpha \in [0, 1]$), then $(x, z)$ is a solution of the equation

$$
\Box \left( \frac{\partial L}{\partial x} [x, z](t) \right) = \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial z} [x, z](t) \frac{\partial L}{\partial \Box x} [x, z](t),
$$

for all $t \in [a, b]$.

**Proof.** Let $\epsilon$ be an arbitrary real, and consider variation functions of $x$ of type $x(t) + \epsilon \eta(t)$, with $\eta \in H^\beta (I, \mathbb{R}) \cap C^1([a, b], \mathbb{R})$ ($\beta \in [0, 1]$), $\eta(a) = \eta(b) = \Box \eta(a) = 0$, and

$$
\lim_{h \to 0} \int_a^b \left( \Box h \left( \lambda(t) \frac{\partial L}{\partial x} [x, z](t) \eta(t) \right) \right) dt = 0.
$$

The corresponding rate of change of $z$, caused by the change of $x$ in the direction of $\eta$, is given by

$$
\theta(t) = \frac{d}{d\epsilon} \left. \left[ z(t, x(t) + \epsilon \eta(t), \Box x(t) + \epsilon \Box \eta(t)) \right] \right|_{\epsilon=0}.
$$

Then

$$
\dot{\theta}(t) = \frac{d}{d\epsilon} \left. \left[ z(t, x(t) + \epsilon \eta(t), \Box x(t) + \epsilon \Box \eta(t)) \right] \right|_{\epsilon=0} = \left. \frac{\partial L}{\partial x} [x, z](t) \eta(t) + \frac{\partial L}{\partial z} [x, z](t) \Box \eta(t) + \frac{\partial L}{\partial \Box x} [x, z](t) \theta(t) \right|_{\epsilon=0}.
$$

We obtain a first order linear differential equation on $\theta$, whose solution is

$$
\lambda(b)\theta(b) - \theta(a) = \int_a^b \lambda(t) \left[ \frac{\partial L}{\partial x} [x, z](t) \eta(t) + \frac{\partial L}{\partial z} [x, z](t) \Box \eta(t) \right] dt,
$$

where

$$
\lambda(t) = \exp \left( -\int_a^t \frac{\partial L}{\partial z} [x, z](\tau) d\tau \right).
$$

Using the fact that $\theta(a) = \theta(b) = 0$, we get

$$
\int_a^b \lambda(t) \left[ \frac{\partial L}{\partial x} [x, z](t) \eta(t) + \frac{\partial L}{\partial z} [x, z](t) \Box \eta(t) \right] dt = 0.
$$

Integrating by parts the second term, we obtain

$$
\int_a^b \left[ \lambda(t) \frac{\partial L}{\partial x} [x, z](t) - \frac{\partial}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial \Box x} [x, z](t) \right) \right] \eta(t) dt + \left[ \eta(t) \lambda(t) \frac{\partial L}{\partial \Box x} [x, z](t) \right]_a^b = 0.
$$

Since $\eta(a) = \eta(b) = 0$, and $\eta$ is an arbitrary function elsewhere,

$$
\lambda(t) \frac{\partial L}{\partial x} [x, z](t) - \frac{\partial}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial \Box x} [x, z](t) \right) = 0,
$$

for all $t \in [a, b]$. Since the function $t \mapsto \lambda(t)$ is differentiable, and the function $t \mapsto \frac{\partial L}{\partial z} [x, z](t)$ is in $H^\alpha$, it follows that

$$
\lambda(t) \left( \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial z} [x, z](t) \frac{\partial L}{\partial \Box x} [x, z](t) - \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \Box x} [x, z](t) \right) \right) = 0.
$$
Finally, since \( \lambda(t) > 0 \), for all \( t \), we get

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial x_i} [x, z](t) \right) = \frac{\partial L}{\partial x_i} [x, z](t) + \frac{\partial L}{\partial x} [x, z](t) \frac{\partial L}{\partial x} [x, z](t),
\]

for all \( t \in [a, b] \).

**Remark 1.** Assume that the set of state functions \( x \) is \( C^1([a, b], \mathbb{R}) \). Then equation (5) becomes

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial x} [x, z](t) \right) = \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial x} [x, z](t) \frac{\partial L}{\partial x} [x, z](t),
\]

which is the generalized variational principle of Herglotz as in [10].

**Theorem 4.** Let the pair \( (x, z) \) be a solution of the problem (4), but now \( x(b) \) is free. Then \( (x, z) \) is a solution of the equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial x} [x, z](t) \right) = \frac{\partial L}{\partial x} [x, z](t) + \frac{\partial L}{\partial x} [x, z](t) \frac{\partial L}{\partial x} [x, z](t),
\]

for all \( t \in [a, b] \), and verifies the transversality condition

\[
\frac{\partial L}{\partial x} [x, z](b) = 0.
\]

**Proof.** Following the proof of Theorem 3, the Euler-Lagrange equation is deduced. Then

\[
\left[ \eta(t) \lambda(t) \frac{\partial L}{\partial x} [x, z](t) \right]^b_a = 0.
\]

Since \( \eta(a) = 0 \) and \( \eta(b) \) is arbitrary, we obtain the transversality condition. \( \square \)

**Multi-dimensional case**

For simplicity, we considered so far one state function \( x \) only, but the multi-dimensional case \((x_1, \ldots, x_n)\) is easily studied.

**Theorem 5.** Let \( \alpha \in \]0, 1[ \) and let the vector \((x_1, \ldots, x_n, z)\) be a solution of the problem: find \((x_1, \ldots, x_n)\) that extremizes \( z(b) \), with

\[
\begin{align*}
\dot{z}(t) &= L(t) x_1(t), \ldots, x_n(t), \Box x_1(t), \ldots, \Box x_n(t), z(t), \quad \text{with} \quad t \in [a, b] \\
z(a) &= z_0 \\
x_i(a) &= x_{i_0}, \quad x_i(b) = x_{i_0}
\end{align*}
\]

where, for all \( i \in \{1, \ldots, n\} \),

1. the trajectories \( x_i \) are in \( H^\alpha \cap C^1([a, b], \mathbb{R}) \) and the functional \( z \) in \( C^2([a, b], \mathbb{C}) \),
2. \( z_0, x_{i_0}, x_{i_0} \) are fixed numbers,
3. \( \frac{\partial L}{\partial x_i} [x_1, \ldots, x_n, z] \in H^\alpha(I, \mathbb{C}) \)
4. the Lagrangian \( L : [a, b] \times \mathbb{R}^n \times \mathbb{C}^{n+1} \rightarrow \mathbb{C} \) is of class \( C^2 \).

Then, for all \( i \in \{1, \ldots, n\} \), \((x_1, \ldots, x_n, z)\) is a solution of the equation

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial x_i} [x_1, \ldots, x_n, z](t) \right) = \frac{\partial L}{\partial x_i} [x_1, \ldots, x_n, z](t) + \frac{\partial L}{\partial z} [x_1, \ldots, x_n, z](t) \frac{\partial L}{\partial x_i} [x_1, \ldots, x_n, z](t),
\]

for all \( t \in [a, b] \).
**Theorem 6.** Let the vector \((x_1, \ldots, x_n, z)\) be a solution of the problem as stated in Theorem 5, but now \(x_i(b)\) is free, for all \(i \in \{1, \ldots, n\}\). Then, for all \(i \in \{1, \ldots, n\}\), \((x_1, \ldots, x_n, z)\) is a solution of the equation

\[
\frac{\partial}{\partial t} \left( \frac{\partial L}{\partial \dot{x}_i} [x_1, \ldots, x_n, z](t) \right) = \frac{\partial L}{\partial x_i} [x_1, \ldots, x_n, z](t) + \frac{\partial L}{\partial z} [x_1, \ldots, x_n, z](t) \frac{\partial L}{\partial \dot{x}_i} [x_1, \ldots, x_n, z](t),
\]

for all \(t \in [a, b]\), and verifies the transversality condition

\[
\frac{\partial L}{\partial \dot{x}_i} [x_1, \ldots, x_n, z](b) = 0.
\]

**Higher-order derivatives case**

**Theorem 7.** Let \(\alpha \in [0, 1]\) and let the pair \((x, z)\) be a solution of the problem: find \(x\) that extremizes \(z(b)\), with

\[
\begin{cases}
\dot{z}(t) = L(t, x, \Box x(t), \ldots, \Box^n x(t), z(t)), & \text{with } t \in [a, b] \\
\dot{x}(a) = z_a \\
\Box^i x(a) = x_{ia}, \Box^i x(b) = x_{ib}, & \text{for all } i \in \{0, \ldots, n-1\},
\end{cases}
\]

where

1. the trajectories \(x\) are in \(H^\alpha \cap C_C^n([a, b], \mathbb{R})\), \(\Box x \in H^\alpha\) and the functional \(z\) in \(C^2([a, b], \mathbb{C})\),
2. \(z_a, x_{ia}, x_{ib}\) are fixed numbers, for all \(i \in \{0, \ldots, n-1\}\),
3. \(\frac{\partial L}{\partial x^i} [x, z] \in H^\alpha(\mathbb{R}^n, \mathbb{C})\), for all \(i \in \{1, \ldots, n\}\),
4. \([x, z](t) = (t, x, \Box x(t), \ldots, \Box^n x(t), z(t))\) and \([x](t) = (t, x, \Box x(t), \ldots, \Box^n x(t))\),
5. the Lagrangian \(L : [a, b] \times \mathbb{R}^{n+1} \to \mathbb{R}\) is of class \(C^2\).

Then, \((x, z)\) is a solution of the equation

\[
\lambda(t) \frac{\partial L}{\partial x} [x, z](t) + \sum_{i=1}^n (-1)^i \frac{\partial^i}{\partial t^i} \left( \lambda(t) \frac{\partial L}{\partial \Box^i x} [x, z](t) \right) = 0,
\]

for all \(t \in [a, b]\).

**Proof.** Let \(x(t) + \epsilon \varphi(t)\) be a variation function of \(x\), with \(\epsilon \in \mathbb{R}\) and \(\eta \in H^\beta \cap C_C^n([a, b], \mathbb{R})\) (\(\beta \in [0, 1]\)). Also, assume that the variations fulfill the conditions:

1. for all \(i = 0, \ldots, n-1\), \(\Box^i \eta(a) = \Box^i \eta(b) = 0\), and \(\Box^n \eta(a) = 0\),
2. for all \(i = 1, 2, \ldots, n\) and \(k = 0, 1, \ldots, i-1\),
\[
\lim_{h \to 0} \left. \int_a^b \left( \frac{\partial^k}{\partial t^k} \left( \lambda(t) \frac{\partial L}{\partial \Box^i x} [x, z](t) \right) \right) dt \right|_{t=0} = 0.
\]

Define

\[
\theta(t) = \frac{d}{d\epsilon} z(t, x(t) + \epsilon \varphi(t), \Box x(t) + \epsilon \Box \eta(t), \ldots, \Box^n x(t) + \epsilon \Box^n \eta(t))|_{\epsilon=0}.
\]

Then

\[
\dot{\theta}(t) = \frac{\partial L}{\partial x} [x, z](t) \theta(t) + \sum_{i=1}^n \frac{\partial L}{\partial \Box^i x} [x, z](t) \Box^i \eta(t) + \frac{\partial L}{\partial z} [x, z](t) \theta(t).
\]

Solving this linear ODE, we arrive at

\[
\int_a^b \lambda(t) \left[ \frac{\partial L}{\partial x} [x, z](t) \theta(t) + \sum_{i=1}^n \frac{\partial L}{\partial \Box^i x} [x, z](t) \Box^i \eta(t) \right] dt = 0,
\]
where

\[ \lambda(t) = \exp \left( - \int_a^t \frac{\partial L}{\partial z} [x,z](\tau) d\tau \right). \]

Integrating by parts \( n \) times, we obtain the following:

\[
\int_a^b \left[ \lambda(t) \frac{\partial L}{\partial x} [x,z](t) + \sum_{i=1}^n (-1)^i \frac{\partial^i}{\partial t^i} \left( \lambda(t) \frac{\partial L}{\partial \partial^i x} [x,z](t) \right) \right] \eta(t) dt
\]

\[ + \left[ \sum_{i=1}^n \sum_{k=0}^{i-1} (-1)^k \frac{\partial^k}{\partial t^k} \left( \lambda(t) \frac{\partial L}{\partial \partial^k x} [x,z](t) \right) \right] \eta(t) \bigg|_a^b = 0, \]

and rearranging the terms, we get

\[
\int_a^b \left[ \lambda(t) \frac{\partial L}{\partial x} [x,z](t) + \sum_{i=1}^n (-1)^i \frac{\partial^i}{\partial t^i} \left( \lambda(t) \frac{\partial L}{\partial \partial^i x} [x,z](t) \right) \right] \eta(t) dt
\]

\[ + \left[ \sum_{i=1}^n \sum_{k=i}^{i-1} (-1)^{k-i} \frac{\partial^k}{\partial t^k} \left( \lambda(t) \frac{\partial L}{\partial \partial^k x} [x,z](t) \right) \right] \eta(t) \bigg|_a^b = 0. \]

Since \( \square^i \eta(a) = \square^i \eta(b) = 0 \), for all \( i \in \{0, \ldots, n-1\} \) and \( \eta \) is arbitrary elsewhere, we get

\[ \lambda(t) \frac{\partial L}{\partial x} [x,z](t) + \sum_{i=1}^n (-1)^i \frac{\partial^i}{\partial t^i} \left( \lambda(t) \frac{\partial L}{\partial \partial^i x} [x,z](t) \right) = 0, \]

for all \( t \in [a,b] \).

**Theorem 8.** Let the pair \((x,z)\) be a solution of the problem as stated in Theorem 7, but now \( \square^i x(b) \) is free, for all \( i \in \{0, \ldots, n-1\} \). Then, \((x,z)\) is a solution of the equation

\[ \lambda(t) \frac{\partial L}{\partial x} [x,z](t) + \sum_{i=1}^n (-1)^i \frac{\partial^i}{\partial t^i} \left( \lambda(t) \frac{\partial L}{\partial \partial^i x} [x,z](t) \right) = 0, \]

for all \( t \in [a,b] \), and verifies the transversality condition

\[ \sum_{k=i}^n (-1)^{k-i} \frac{\partial^k}{\partial t^k} \left( \lambda(t) \frac{\partial L}{\partial \partial^k x} [x,z](t) \right) = 0 \quad \text{at} \quad t = b, \]

for all \( i \in \{1, \ldots, n\} \).

**Several independent variables case**

We generalize Theorem 3 for several independent variables. First we fix some notations. The variable time is \( t \in [a,b] \), \( x = (x_1, \ldots, x_n) \in \Omega := \prod_{i=1}^n [a_i, b_i] \) are the space coordinates and the state function is \( u := u(t,x) \).

**Theorem 9.** Let \( \alpha \in [0,1] \) and let the pair \((u,z)\) be a solution of the problem: find \( u \) that extremizes \( z(b) \), with

\[
\begin{align*}
\dot{z}(t) &= \int_a^b L \left( t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, z(t) \right) d^n x, \quad \text{with} \quad t \in [a,b] \\
z(a) &= z_0 \\
\forall t \in [a,b] \forall x \in \partial \Omega \\
&u(t,x) \quad \text{takes fixed values}, \\
&u(t,x) \quad \text{takes fixed values}, \quad \forall t \in [a,b] \forall x \in \Omega,
\end{align*}
\]

where, for all \( i \in \{1, \ldots, n\} \),
1. the trajectories \( u \) are in \( H^\alpha(I \times \Omega, \mathbb{R}) \cap C^1([a, b] \times \Omega, \mathbb{R}) \), \( \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_i} \in H^\alpha([a, b] \times \Omega, \mathbb{C}) \) and the functional \( z \) in \( C^2([a, b], \mathbb{C}) \).

2. \( z_a \) is a fixed number,

3. \( d^ax = dx_1 \ldots dx_n \),

4. \( \frac{\partial L}{\partial x}[u, z], \frac{\partial L}{\partial t}[u, z] \in H^\alpha(I \times \Omega, \mathbb{C}) \), where \( \frac{\partial L}{\partial t}[u, z] \) denotes the partial derivative of \( L \) with respect to the variable \( \frac{\partial u}{\partial t} \) and \( \frac{\partial L}{\partial x}[u, z] \) denotes the partial derivative of \( L \) with respect to the variable \( \frac{\partial u}{\partial x_i} \), and \( [u, z](t) = (t, x, u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, z(t)) \).

5. \( L : [a, b] \times \Omega \times \mathbb{C}^{n+2} \to \mathbb{C} \) is of class \( C^2 \).

Then, \((u, z)\) is a solution of the equation
\[
\frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial t}[u, z](t) \int_\Omega \frac{\partial L}{\partial z}[u, z](t) \, d^nx - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \frac{\partial L}{\partial x_i}[u, z](t) \right) - \eta(t) = 0,
\]
for all \( t \in [a, b] \) and for all \( x \in \Omega \).

**Proof.** Let \( u(t, x) + \eta(t, x) \) be a variation function of \( u \), with \( \epsilon \in \mathbb{R} \) and \( \eta \in H^\beta(I \times \Omega, \mathbb{R}) \cap C^\beta([a, b] \times \Omega, \mathbb{R}) \) \((\beta \in [0, 1])\). Also, assume that the variations fulfill the conditions:

1. \( \eta(t, x) = 0, \quad \forall t \in [a, b] \forall x \in \partial \Omega \),
2. \( \eta(t, x) = 0, \quad \forall t \in [a, b] \forall x \in \Omega \),
3. \( \frac{\partial \eta}{\partial t}(a, x) = \frac{\partial \eta}{\partial t}(b, x) = 0, \quad \forall x \in \Omega \),
4. for all \( i = 1, 2, \ldots, n \),
\[
\lim_{h \to 0} \int_a^b \left( \frac{\partial}{\partial t} \left( \frac{\partial L}{\partial x_i}[u, z](t) \eta(t) \right) \right) \, dt = 0.
\]

and
\[
\lim_{h \to 0} \int_a^b \left( \frac{\partial}{\partial x_i} \left( \frac{\partial L}{\partial t}[u, z](t) \eta(t) \right) \right) \, dt = 0,
\]

where
\[
\lambda(t) = \exp \left( - \int_a^t \int_\Omega \frac{\partial L}{\partial z}[u, z](\tau) \, d^nx \, d\tau \right).
\]

Let
\[
\theta(t) = \frac{d}{d\epsilon} \left| \left. z(t, x, u + \epsilon \eta(t), \frac{\partial u}{\partial t}, \frac{\partial \eta}{\partial t}, \frac{\partial u}{\partial x_1}, \frac{\partial \eta}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial \eta}{\partial x_n} \right| \right|_{\epsilon=0}.
\]

Proceeding with some calculations, we arrive at the ODE
\[
\dot{\theta}(t) - \int_\Omega \frac{\partial L}{\partial \theta}[u, z](t) \, d^nx - \theta(t) = 0.
\]

Solving the ODE, and taking into consideration that \( \theta(a) = \theta(b) = 0 \), we get
\[
\int_a^b \int_\Omega \lambda(t) \left( \frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial z}[u, z](t) \frac{\partial \eta}{\partial t} + \sum_{i=1}^n \frac{\partial L}{\partial x_i}[u, z](t) \frac{\partial \eta}{\partial x_i} \right) \, d^nx \, dt = 0.
\]

Integrating by parts, and considering the boundary conditions over \( \eta \), we get
\[
\int_a^b \int_\Omega \left[ \lambda(t) \frac{\partial L}{\partial u}[u, z](t) - \frac{\partial}{\partial t} \left( \lambda(t) \frac{\partial L}{\partial z}[u, z](t) \right) - \sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \lambda(t) \frac{\partial L}{\partial x_i}[u, z](t) \right) \right] \eta \, d^nx \, dt = 0.
\]
By the arbitrariness of $\eta$, it follows that for all $t \in [a,b]$ and for all $x \in \Omega$,

$$\lambda(t) \frac{\partial L}{\partial u}[u, z](t) - \square \frac{\partial L}{\partial \square t}[u, z](t) - \sum_{i=1}^{n} \square \frac{\partial L}{\partial x_i}[u, z](t) = 0.$$ 

Since $\lambda(t) > 0$, this condition implies that

$$\frac{\partial L}{\partial u}[u, z](t) + \frac{\partial L}{\partial \square t}[u, z](t) \int_{\Omega} \frac{\partial L}{\partial \square z}[u, z](t) d^n x - \square \frac{\partial L}{\partial \square t}[u, z](t) - \sum_{i=1}^{n} \square \frac{\partial L}{\partial x_i}[u, z](t) = 0,$$

for all $t \in [a,b]$ and for all $x \in \Omega$, and the theorem is proved. \hfill $\square$

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### References


