ЧЕБЫШЕВСКИЙ СБОРНИК
Том 12 Выпуск 4 (2011)

A SPECTRUM ASSOCIATED WITH MINKOWSKI DIAGONAL continued fraction

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Let α be real irrational number. The function $\mu_\alpha(t)$ is defined as follows.
The Legendre theorem states that if

$$\left|\alpha - \frac{A}{Q}\right| < \frac{1}{2Q^2}, \quad (A,Q) = 1 \tag{1}$$

then the fraction $\frac{A}{Q}$ is a convergent fraction for the continued fraction expansion of α. The converse statement is not true. It may happen that $\frac{A}{Q}$ is a convergent to α but (1) is not valid. One should consider the sequence of the denominators of the convergents to α for which (1) is true. Let this sequence be

$$Q_0 < Q_1 < \cdots < Q_n < Q_{n+1} < \cdots .$$

Then for $\alpha \notin \mathbb{Q}$ the function $\mu_\alpha(t)$ is defined by

$$\mu_\alpha(t) = \frac{Q_{n+1} - t}{Q_{n+1} - Q_n} \cdot ||Q_n\alpha|| + \frac{t - Q_n}{Q_{n+1} - Q_n} \cdot ||Q_{n+1}\alpha||, \quad Q_n \leqslant t \leqslant Q_{n+1}.$$

From the other hand, for every ν one of the consecutive convergent fractions $\frac{p_\nu}{q_\nu}, \frac{p_{\nu+1}}{q_{\nu+1}}$ to α satisfies (1). So either

$$(Q_n, Q_{n+1}) = (q_\nu, q_{\nu+1})$$

for some ν, or

$$(Q_n, Q_{n+1}) = (q_{\nu-1}, q_{\nu+1})$$

for some ν.

Actually the function $\mu_\alpha(t)$ was considered by Minkowski [4]. There exists an alternative geometric definition of $\mu_\alpha(t)$. Some related facts were discussed in [3, 6].

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[1]Department of Mathematics, Aveiro University, Aveiro 3810, Portugal. This work was supported by *FEDER* founds through *COMPETE*–Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese founds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT–Fundção para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690

The quantity
$$\mathfrak{m}(\alpha)=\limsup_{t\to+\infty} t\cdot\mu_\alpha(t).$$
was considered in [6]. An explicit formula for the value of $\mathfrak{m}(\alpha)$ in terms of continued fraction expansion for α was proved in [6]. It is as follows. Put

$$\mathfrak{m}_n(\alpha)=\begin{cases} G(\alpha_\nu^*,\alpha_{\nu+2}^{-1}), & \text{if } (Q_n,Q_{n+1})=(q_{\nu-1},q_{\nu+1}) \text{ with some } \nu,\\ F(\alpha_{\nu+1}^*,\alpha_{\nu+2}^{-1}), & \text{if } (Q_n,Q_{n+1})=(q_{\nu},q_{\nu+1}) \text{ with some } \nu,\end{cases} \tag{2}$$

where
$$G(x,y)=\frac{x+y+1}{4},\quad F(x,y)=\frac{(1-xy)^2}{4(1+xy)(1-x)(1-y)}$$
and α_ν,α_ν^* come from continued fraction expansion to

$$\alpha=[a_0;a_1,a_2,\dots,a_t,\dots]$$

in such a way:
$$\alpha_\nu=[a_\nu;a_{\nu+1},...],\quad \alpha_\nu^*=[0;a_\nu,a_{\nu-1},...,a_1].$$
Then
$$\mathfrak{m}(\alpha)=\limsup_{n\to+\infty}\mathfrak{m}_n(\alpha),$$
The specrtum

$$\mathbb{M}=\{m\in\mathbb{R}:\ \exists\alpha\in\mathbb{R}\setminus\mathbb{Q} \text{ such that } m=\mathfrak{m}(\alpha)\}.$$

was studied in [6]. It was proven there that $\mathbb{M}\subset\left[\frac14,\frac12\right]$ and that $\frac14,\frac12\in\mathbb{M}$. However no further structure of the spectrum $\mathbb{M}$ is known.

In this paper we consider the spectrum

$$\mathbb{I}=\{m\in\mathbb{R}:\ \exists\alpha\in\mathbb{R}\setminus\mathbb{Q} \text{ such that } \mathfrak{i}(\alpha)=m\},$$

where
$$\mathfrak{i}(\alpha)=\liminf_{n\to\infty}\mathfrak{m}_n(\alpha)$$
(however, compared to $\mathfrak{m}(\alpha)$, this quantity has no clear Diophantine sense).

It is clear that
$$\min\mathbb{I}=\frac14,\quad \max\mathbb{I}=\frac12.$$

Theorem. *There exists positive ω_0 such that*

$$\left[\frac14,\omega_0\right]\subset\mathbb{I}.$$

The proof is based ol M. Hall's ideas (see [2]). It uses technique from [5].

Remark. *An explicit formula for ω_0 may be obtained from the proof below. It is interesting to get optimal estimates for the value of ω_0.*

We need some well known results.

Recall the definition of a τ-set $\mathcal{F} \subset \mathbb{R}$. The set $\mathcal{F}$ must be of the form

$$\mathcal{F} = \mathcal{S} \setminus \left(\bigcup_{\nu+1}^{\infty} \Delta_\nu \right),$$

where $\mathcal{S} \subset \mathbb{R}$ is a segment, and $\Delta_\nu \subset \mathcal{S}$, $\nu = 1, 2, 3, ...$ is an ordered sequence of disjoint intervals. Moreover for every t if

$$\mathcal{S} \setminus \left(\bigcup_{\nu+1}^{t-1} \Delta_\nu \right) = \bigcup_{j=1}^{r} \mathcal{M}_j$$

is a union of segments $\mathcal{M}_j$ and $\Delta_t \subset \mathcal{M}_{j^*}$ then

$$\mathcal{M}_{j^*} = \mathcal{N}^1 \sqcup \Delta_t \sqcup \mathcal{N}^2,$$

and

$$\min(|\mathcal{N}^1|, |\mathcal{N}^2|) \geqslant \tau |\Delta_t|.$$

Consider the set

$$\mathcal{F}_5 = \{\alpha = [0; b_1, b_2, b_3, ...] : \ b_\nu \leqslant 5 \ \ \forall \ \nu\}$$

consisting of all irrational real numbers from the unit interval $(0, 1)$ with partial quotients bounded by 5. One can easily see that

$$\begin{cases} A = \min \mathcal{F}_5 = [0; \overline{5, 1}] = \frac{\sqrt{45}-5}{10} = 0.1708^+, \\ B = \max \mathcal{F}_5 = [0; \overline{1, 5}] = \frac{\sqrt{45}-5}{2} = 0.85410^+. \end{cases} \tag{3}$$

Put

$$\mathcal{S}_5 = [A, B] \subset [0, 1].$$

The following lemma comes from the results of the papers [1] or [7].

Lemma 1. *The set $\mathcal{F}_5$ is a τ-set with $\tau = \tau_5 = 1.788^+$.*

Let $H(x, y) : \ \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ be a function in two variables of the class $G \in C^1(\mathcal{S} \times \mathcal{S})$. Consider the set

$$\mathcal{J} = \{z \in \mathbb{R} : \ \exists\, x, y \in \mathcal{S} \ \ z = H(x, y)\}.$$

By continuousity argument $\mathcal{J}$ is a segment.

Lemma 2. *Suppose that the derivatives $\partial H/\partial x, \partial H/\partial y$ do not take zero values on the box $\mathcal{S} \times \mathcal{S}$. Suppose that $\mathcal{F}$ is τ-set and $\mathcal{S} = [\min \mathcal{F}, \max \mathcal{F}]$. Suppose that*

$$\tau \geqslant \max_{x,y \in \mathcal{S}} \ \max \left(\left| \frac{\partial H/\partial x}{\partial H/\partial y} \right|, \left| \frac{\partial H/\partial y}{\partial H/\partial x} \right| \right). \tag{4}$$

Then

$$\{z:\ \exists x, y \in \mathcal{F} \quad \text{such that} \quad z = H(x,y)\} = \mathcal{J}.$$

Lemma 2 is a staightforward generalization of a result from [5]. We do not give its proof here as the proof follows the argument from [5] word-by-word.

Now we are able to conclude the proof of Theorem.

We consider pairs of integers (R_1, R_2) of the form

$$(R_1, R_2) \;=\; (R, R) \;\text{ or }\; (R, R+1) \tag{5}$$

with $R \geqslant 6$. Consider a function

$$H_{R_1,R_2}(x,y) = F\left(\frac{1}{R_1+x}, \frac{1}{R_2+y}\right).$$

For R_1, R_2 under consideration the function $H_{R_1,R_2}(x,y)$ decreases both in x and in y.

For $0 < x, y < 1$ put

$$\varphi(x,y) = (1 - 3x + 3xy - x^2y)(1-x).$$

For any $y \in (0,1)$ the function $\varphi(x,y)$ decreases in x. For any $x \in (0,1)$ the function $\varphi(x,y)$ increases in y. Now

$$\frac{\partial F/\partial y}{\partial F/\partial x} = \frac{\varphi(x,y)}{\varphi(y,x)},$$

and

$$\left|\frac{\partial H_{R_1,R_2}/\partial y}{\partial H_{R_1,R_2}/\partial x}\right| = \frac{\varphi\left(\frac{1}{R_1+x}, \frac{1}{R_2+y}\right)}{\varphi\left(\frac{1}{R_2+y}, \frac{1}{R_1+x}\right)} \left(\frac{R_1+x}{R_2+y}\right)^2.$$

Easy calculation shows that for $R_1, R_2 \geqslant 6$ one has

$$\max_{x,y\in\mathcal{S}_5} \max\left(\left|\frac{\partial H_{R_1,R_2}/\partial x}{\partial H_{R_1,R_2}/\partial y}\right|, \left|\frac{\partial H_{R_1,R_2}/\partial y}{\partial H_{R_1,R_2}/\partial x}\right|\right) =$$

$$= \frac{\varphi\left(\frac{1}{R_1+B}, \frac{1}{R_2+A}\right)}{\varphi\left(\frac{1}{R_2+A}, \frac{1}{R_1+B}\right)} \left(\frac{R_1+B}{R_2+A}\right)^2 \leqslant \frac{\varphi\left(\frac{1}{R_1+B}, \frac{1}{R_1+A}\right)}{\varphi\left(\frac{1}{R_1+A}, \frac{1}{R_1+B}\right)} \left(\frac{R_1+B}{R_1+A}\right)^2 \leqslant$$

$$\leqslant \frac{\varphi\left(\frac{1}{6+B}, \frac{1}{6+A}\right)}{\varphi\left(\frac{1}{6+A}, \frac{1}{6+B}\right)} \left(\frac{6+B}{6+A}\right)^2 = 1.363^{+} < \tau_5.$$

Here A and B are defined in (3) and in the last inequalities we use the bounds $6 \leqslant R_1 \leqslant R_2$ which follows from (5).

We see that for any R_1, R_2 under consideration and for τ_5-set $\mathcal{F}_5$ the condition (4) is satisfied. We apply Lemma 2 to see that the image of the set $\mathcal{F}_5 \times \mathcal{F}_5$ under the mapping $H_{R_1,R_2}(x,y)$ is just the segment

$$\mathcal{J}_{R_1,R_2} = [H_{R_1,R_2}(B,B), H_{R_1,R_2}(A,A)].$$

But

$$H_{R,R}(B,B) < H_{R,R+1}(A,A)$$

and

$$H_{R,R+1}(B,B) < H_{R+1,R+1}(A,A).$$

That is why if we put

$$\omega_0 = H_{R_0,R_0}(A,A).$$

with $R_0 \geqslant 6$ we get

$$\bigcup_{R\geqslant R_0} \mathcal{J}_{R,R} \cup \bigcup_{R\geqslant R_0} \mathcal{J}_{R,R+1} = (1/4, \omega_0].$$

Take $m \in (0, \omega_0]$. Then there exists R_1, R_2 such that

$$m \in \mathcal{J}_{R_1,R_2}$$

and there exist

$$\beta = [0; b_1, b_2, ..., b_\nu, ...], \;\; \gamma = [0; c_1, c_2, ..., c_\nu, ...], \;\; \beta, \gamma \in \mathcal{F}_5,$$

such that

$$F\left(\frac{1}{R_1+\alpha}, \frac{1}{R_2+\beta}\right) = m.$$

Now we take

$$\alpha = [0; \underbrace{a_1, R_1, R_2, b_1}_{1}, \underbrace{a_2, a_1, R_1, R_2, b_1, b_2}_{2}, ..., \underbrace{a_\nu, a_{\nu-1}, ..., a_2, a_1, R_1, R_2, b_1, b_2, ..., b_{\nu-1}, b_\nu}_{\nu}, ...].$$

Standard argument shows that for n_ν defined from

$$\frac{p_{n_\nu}}{q_{n_\nu}} = [0; a_1, R_1, R_2, b_1, a_2, a_1, R_1, R_2, b_1, b_2, ..., a_\nu, a_{\nu-1}, ..., a_2, a_1, R_1]$$

one has

$$\lim_{\nu\to+\infty} F(\alpha^*_{n_\nu}, \alpha^{-1}_{n_\nu+1}) = m.$$

At the same time for $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ we have

$$\inf_{n\neq n_\nu \, \forall \nu} F(\alpha^*_n, \alpha_{n+1}) > \omega_0$$

and

$$\inf_{n\in\mathbb{Z}_+} G(\alpha^*_n, \alpha^{-1}_{n+2}) > \omega_0,$$

for large R_0. So $\mathfrak{i}(\alpha) = m$ and everything is proved. $\square$

СПИСОК ЦИТИРОВАННОЙ ЛИТЕРАТУРЫ

[1] S. Astels, Cantor sets and numbers with restricted partial quotients, Trans. Amer. Math. Soc., 352:1 (1999), 133 - 170.

[2] M. Hall (Jr.), On the sum and product of continued fractions, Annales of Mathematics, 48, No. 4 (1947), 966 - 993.

[3] I.D. Kan, N.G. Moshchevitin, J. Chaika, On Minkowski diagonal functions for two real numbers, in 'The Proceedings Diophantine Analysis and Related Fields 2011', M. Amou and M. Katsurada (Eds.), AIP Conf. Proc. No. 1385, pp. 42 - 48 (2011), American Institute of Physics, New York.

[4] H. Minkowski, Über die Annäherung an eine reelle Grösse durch ratiolale Zahlen, Math. Ann., 54 (1901), p. 91 - 124.

[5] N.G. Moshchevitin, On a theorem of M. Hall, Russian Mathematical Surveys, 1997, 52:6, 1312 - 1313.

[6] N.G. Moshchevitin, On Minkowski diagonal continued fraction, Anal. Probab. Methods Nubmer Theory, Proceedings of the conference in Palanga, Sept. 2011, E. Manstavičius et al. (Eds), to appear; preptint is available at arXiv:1202.4622v2 (2012).

[7] P.A. Pisarev, On the set of numbers representable as continued fractions with bounded partial quotients, Russian Mathematical Surveys, 2000, 55:5, 998 - 999

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Поступило 21.12.2011