Bodies with mirror surface invisible from two points

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Abstract

We consider a setting where a bounded set with a piecewise smooth boundary in Euclidean space is identified with a body with mirror surface, and the billiard in the complement of the set is identified with the dynamics of light rays outside the body in the framework of geometric optics. We show that in this setting it is possible to construct a body invisible from two points.

1 Introduction

The problems studied in the framework of billiard invisibility involve the mathematical design of bodies with well-defined surfaces whose scattering map preserves certain trajectories of a flow of elastic particles. The main practical application of this study is the optical shielding: by surrounding an object by a specially designed mirror surface, it is possible to create an illusion of invisibility from given points or directions.

From the practical viewpoint, this approach is a low-tech alternative to the major modern attempts at achieving invisibility, which are primarily focussed around the design of metamaterials that allow the bending of electromagnetic waves around the concealed object. The idea was first suggested in [15], and tangible results in this direction were achieved by research teams at Duke University, first in 2006 when a imperfect prototype of microwave-range cloaking device was created [9], and then in 2012, when remarkable results were achieved completely cloaking a centimetre-thin board, even though in one direction only. Another development in this direction [17] is a successful concealing of a nanoscale object under a carpet cloak made of layers of silicon oxide and silicon nitride arranged in a special way.

Another approach to invisibility was recently suggested in [18] and lead to the development of a working invisibility cloak based on calcite crystals (also see [3]). The cloak works only under one light polarization: it is essentially two-dimensional, although works at all angles.

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The first work that targets the problem of designing a body invisible in a direction in the framework of mirror invisibility appears in [1] and is motivated by the problem of constructing a nonconvex body or zero resistance. The authors demonstrated that there exists a (connected and even simply connected) body invisible in one direction: if this body is manufactured out of perfectly reflective mirrors, a laser beam sent through this construction in the direction of invisibility would leave the body along the same trajectory. Remarkably, in [8] it is shown that this body is also invisible in acoustic waves.

This research led to several intriguing mathematical problems. Some of them, proposed by Sergei Tabachnikov [16], ask whether it is possible to design a body with a mirror surface invisible in two directions or a body invisible from a point. The former problem was solved in [11]: it was demonstrated that a parabolic construction can be used to produce a body invisible in two directions in three-dimensional case. This body consists of two connected components. It is possible to construct two- and three-dimensional infinitely connected fractal bodies invisible in two and three directions respectively [12]. Note that it is impossible to construct a body invisible in all directions [11], and at least in two-dimensional case there is no piecewise smooth body invisible in a countable number of directions [13]. A somewhat related development that should be mentioned here is the phenomena best known under the term of digital sundial. It is possible to construct fractal bodies such that their projections on almost all planes can be prescribed up to a set of zero measure (see [5]). In this framework, Burdzy and Kulczycki [2] showed that there are two-dimensional bodies which consist of an infinite number of linear segments, and which are almost invisible in almost all directions.

In addition to the fascinating research problems discussed above, billiard invisibility is linked to some fundamental properties of dynamical systems. As it is pointed out in [10] (see also [7]), billiard invisibility is related to the long-standing Ivrii's conjecture. The Ivrii's conjecture states that the measure of periodic trajectories in a piecewise smooth planar billiard is zero, and, in particular, for any natural number n the measure of periodic trajectories of period n is zero. The conjecture was proved for several important special cases (the most recent breakthrough is [6], where the authors proved the nonexistence of billiards with an open set of periodic quadrilateral orbits).

Recently we have identified bodies (connected but not simply connected) invisible from a point [10, 14]. In this paper we provide a construction of a body invisible from 2 points. The body has infinitely many connected components. To our best knowledge, this construction demonstrates a new property of confocal conics. Thus, if you look with two eyes open from a fixed position, the body disappears — becomes completely invisible. The previous result (on a body invisible from a point) meant that you needed to look at the body with one eye open and the other closed. The eye(s) should be placed at the point(s) from which the body is invisible.

The main result of the paper is the following Theorem 1.

Theorem 1. *Given two different points in Euclidean space* $\mathbb{R}^d$, $d = 2, 3, \dots$, *there exists* *a body in $\mathbb{R}^d$ invisible from these points.*

We also show that mixed-type invisibility, where two points are substituted with a point and a direction, also holds true.

Theorem 2. *Given a points $A \in \mathbb{R}^d$ and a vector $v \in S^{d-1}$, there exists a body in $\mathbb{R}^d$ invisible simultaneously from the point A and in the direction v.*

The rest of the paper is devoted to the proof of these results.

2 Invisibility from two points

We begin with reminding the relevant definitions, then explain our construction and prove that it is invisible from two points.

Definition 1. A *body* is a bounded set with finitely or countably many connected components, where each component is either a domain with a piecewise smooth boundary, or a piece of smooth hypersurface, and at least one component is a domain.

Definition 2. A body $B \subset \mathbb{R}^d$ is said to be *invisible from a point $O \in \mathbb{R}^d \setminus B$*, if for almost all $v \in S^{d-1}$ the billiard particle in $\mathbb{R}^d \setminus B$ emanating from O with the initial velocity v, after a finite number of reflections from ∂B will eventually move freely with the same velocity v along a straight line containing O.

If the point O is infinitely distant, we get the notion of a body invisible in a direction. Namely, we have the following definition.

Definition 3. A body B is *invisible in a direction $v \in S^{d-1}$*, if for almost all straight lines in $\mathbb{R}^d$ with the director vector v, the billiard particle in $\mathbb{R}^d \setminus B$ that initially moves along this line with velocity v, after a finite number of reflections from ∂B will finally move freely along the same line with the same velocity v.

2.1 Construction of a body invisible from two points

In this section we construct a 2D body invisible from two different points A_1 and A_2. An invisible body in three or higher dimensions is obtained by rotating the 2D body around the axis A_1A_2.

Let s be the symmetry axis of the system $\{A_1, A_2\}$, i.e. s is a straight line perpendicular to the segment A_1A_2. Choose two different arbitrary points on s such that they both lie on the same side of the segment A_1A_2 (see Fig. 1 (a)). Denote the point nearest to A_1A_2 by L and the other one by K. Let C_1 be the intersection of A_1K with the line through A_2 and L, and let C_2 be the symmetric intersection of A_2K with the line through A_1 and L (see Fig. 1 (a)). Choose a point O on LK. Let D_2 be the intersection of A_2K with the

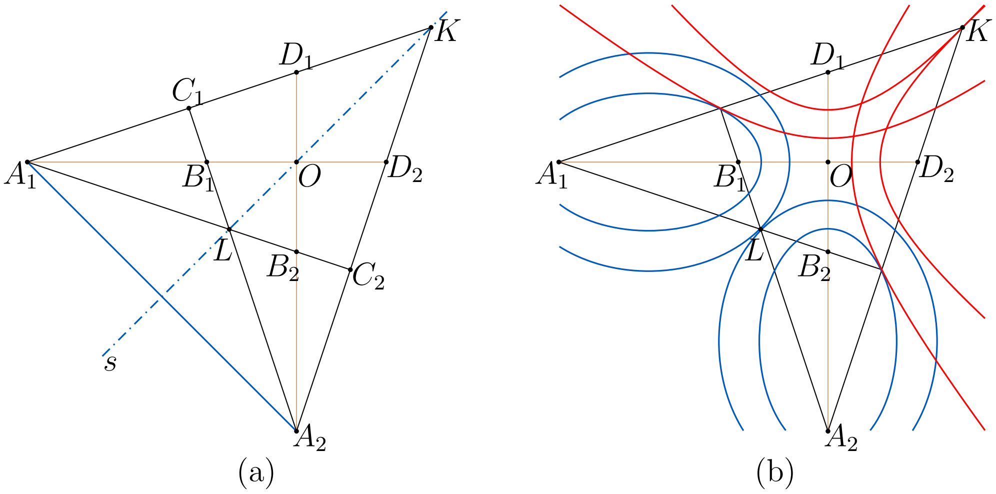


Figure 1: Construction of the eight basic curves

line through $A_1$and O and let B_1 be the intersection of the segments A_1D_2 and C_1L. The points D_1 and B_2 are constructed symmetrically (see Fig. 1 (b)).

Draw two confocal ellipses with foci A_1 and B_1 through the points C_1 and L and two confocal hyperbolas with foci A_2 and D_1 through the points C_1 and K. Repeat this symmetrically on the other side of s (see Fig. 1 (b)). Thus, we have 8 curves of second order through the points C_1, K, C_2, L; each curve is either an ellipse or a hyperbola; there are 2 curves through each point. In the sequel by hyperbola we mean the branch of hyperbola that passes through the corresponding point. Each curve (an ellipse or a branch of hyperbola) bounds a convex set; by exterior of an ellipse or a hyperbola we mean the exterior of the corresponding set.

Observe that the curves through each of the points C_1, K, C_2, L are tangent to each other at these points, and hence intersect only at these points. This is due to the property shared by ellipses and hyperbolas: the tangent line at a point on the curve always makes equal angles with the lines drawn through this point and foci. Hence it is not difficult to see that all of the 8 curves bisect the relevant angles.

Take H_1 on the bisector of the angle $B_1C_1D_1$ inside the quadrangle $B_1C_1D_1O$ and denote by H_2 the point symmetric to H_1 with respect to KL. Further, let N be the point of intersection of A_1H_2 with A_2H_1, and let M be the point of intersection of A_1H_1 with A_2H_2. Obviously, N belongs to the quadrangle OB_2LB_1 and M belongs to the quadrangle OD_1KD_2, and both points belong to the line KL, which is the bisector of the angles B_1LB_2 and D_1KD_2. Since the chosen points lie on the bisectors, they belong to the exterior of the corresponding pairs of curves.

If the symmetric configuration of points is slightly changed, one can also choose the points H_1, M, H_2, N possessing the mentioned properties.

Now, let l^1 be the arc of the ellipse with foci A_1 and B_1 with the endpoints at L

and at the point of intersection of the ellipse with the line A_1N, and l^2 be the arc of the ellipse with foci A_2 and B_2 with the endpoints at L and at the point of intersection of the ellipse with the line A_2N. It follows from the construction that these points of intersection lie on the segments A_1N and A_2N, respectively, and the arcs l^1 and l^2 belong to the quadrangle bounded by the straight lines $A_1N,\ A_2N,\ C_1L,\ C_2L$. Let Q_L be the curvilinear quadrangle bounded by $l_1,\ l_2$, and by segments of the lines $A_1N,\ A_2N$. In a similar way define the curvilinear quadrangles $Q_{C_1},\ Q_{C_2}$, and Q_K; each of them is bounded by two segments of lines $A_1M,\ A_2M,\ A_1N,\ A_2N$ and by two arcs. There are 8 arcs in total (four of them: $l^1,\ l^2,\ c^1_A,\ c^1_K$, are shown in Fig. 2); each of them generates

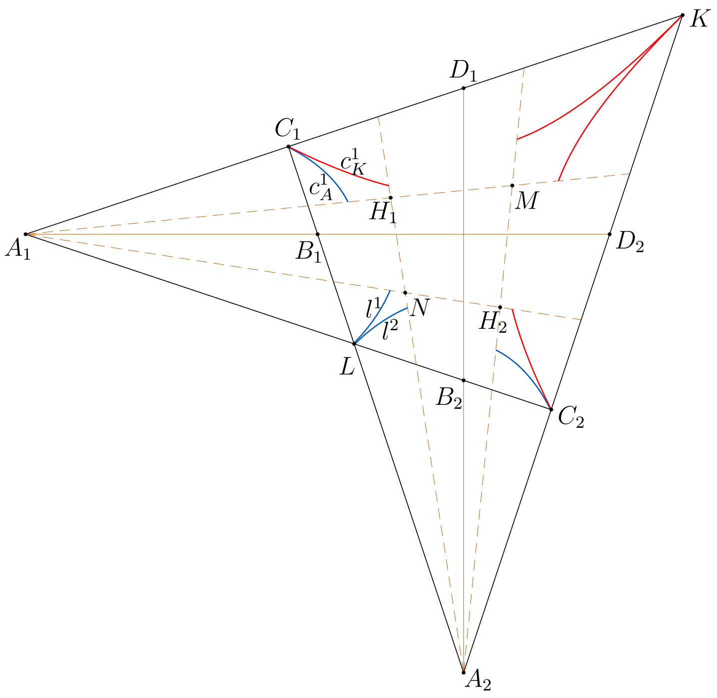


Figure 2: Construction of the initial curvilinear segments

an infinite sequence of arcs to be defined below, 8 sequences in total. Each sequence is totally contained in one of 8 quadrangles that are shown in figure and determined by their diagonals: $NB_1,\ NB_2,\ B_1H_1,\ B_2H_2,\ H_1D_1,\ H_2D_2,\ MD_1,\ MD_2$ (see Fig. 3). Let us describe the sequences $l_0,\, l_1,\, l_2, \dots$ and $c_0,\, c_1,\, c_2, \dots$ generated by the arcs l^2 and c^1_K, respectively; the other sequences are defined analogously.

The sequence $l_0,\, l_1,\, l_2, \dots$ of arcs of ellipses with foci at A_2 and B_2 is uniquely defined by the following conditions. (1) l_0 coincides with l^2. (2) The endpoints of the arc l_i, $i = 0,\, 1,\, 2, \dots$ are denoted by λ_i and ν_i; λ_i lies on the segment A_1B_2 and ν_i lies on the segment ν_0B_2. (3) λ_{i+1} lies on the segment ν_iA_2 (see Fig. 5).

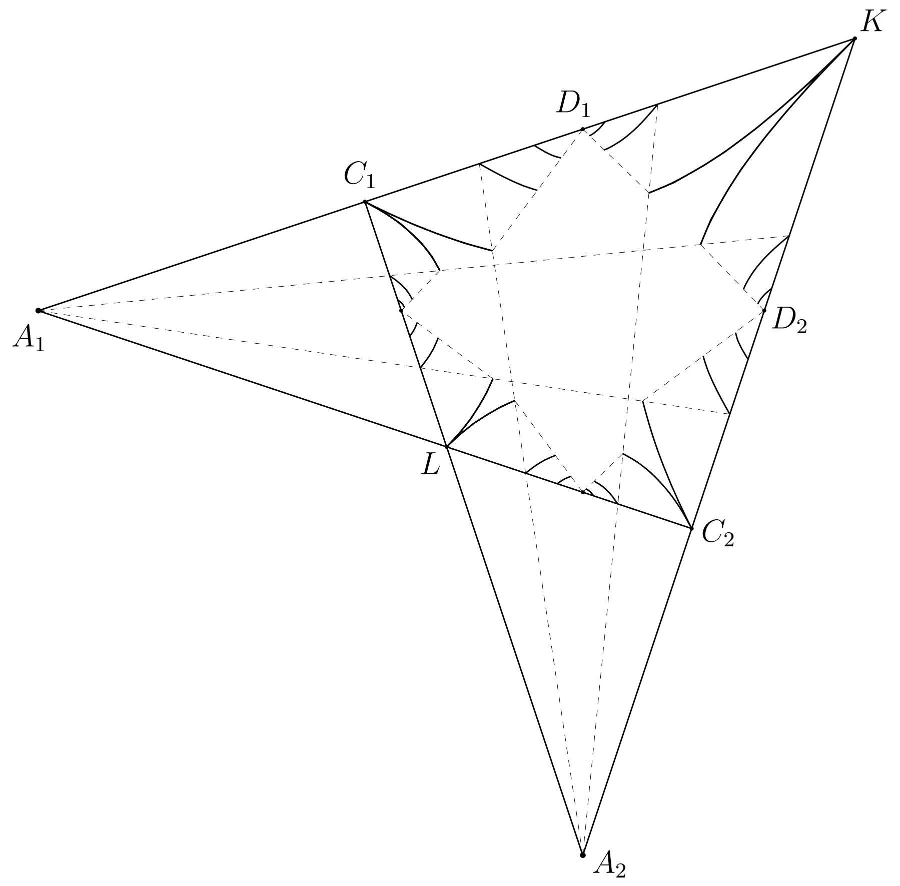


Figure 3: The construction with 8 sequences of arcs.

Similarly, the sequence c_0, c_1, $c_2, \dots$ of arcs of hyperbolas with foci at A_2 and D_1 is uniquely defined by the following conditions. (1) c_0 coincides with c_K^1. (2) The endpoints of the arc c_i, $i = 0, 1, 2, \dots$ are denoted by σ_i and χ_i; σ_i lies on the segment A_1D_1 and χ_i lies on the segment χ_0D_1. (3) σ_{i+1} lies on the extension of the segment χ_iA_2 (see Fig. 5).

Below we will need the following statement.

Lemma 1. *(a) Consider two different points F_1 and F_2, and let two rays from F_1 intersect two rays from F_2 at four points: e_1, e_2, h_1, h_2 (see Fig. 4). Draw two ellipses $\mathcal{E}_1$, $\mathcal{E}_2$ and two hyperbolas $\mathcal{H}_1$, $\mathcal{H}_2$ with foci F_1 and F_2 through e_1, e_2, h_1, h_2, respectively. (Recall that by hyperbola we mean the corresponding branch of hyperbola.) We claim that the point of intersection of $\mathcal{E}_1$ and $\mathcal{H}_1$, the point of intersection of $\mathcal{E}_2$ and $\mathcal{H}_2$, and F_2 are collinear.*

(b) Consider a ray from F_2 and two points u_1, u_2 on it. Draw the ellipse $\mathcal{E}_1$ and the hyperbola $\mathcal{H}_1$ with foci F_1, F_2 through u_1, and the ellipse $\mathcal{E}_2$ and the hyperbola $\mathcal{H}_2$ with the same foci F_1, F_2 through u_2. Take a ray from F_1 intersecting $\mathcal{E}_1$ and $\mathcal{H}_1$ at e_1 and h_1, respectively. Let the rays F_2e_1 and F_2h_1 intersect $\mathcal{E}_2$ and $\mathcal{H}_2$ at e_2 and h_2. We claim that the points e_2, h_2, and F_1 are collinear.

Its proof is given in Appendix. Note that another proof of Lemma using elegant geometric argument is provided by Pavel Dolgirev in [4].

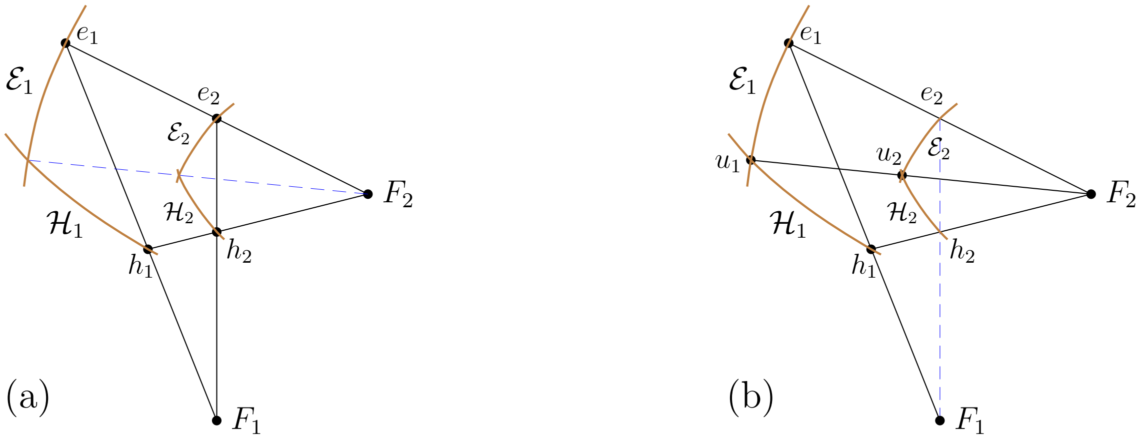


Figure 4:

Let us now prove by induction that the points χ_i, ν_i, and A_2 (and therefore the points $\sigma_{i+1},\ \chi_i,\ \nu_i,\ \lambda_{i+1},\ A_2$) are collinear (see Fig. 5). For $i = 0$ this statement is obvious, since the points χ_0, ν_0, A_2 lie on the line A_2H_1. Now assume that $\chi_{i-1},\ \nu_{i-1}$, and A_2 are collinear for $i \geq 1$, and prove that $\chi_i,\ \nu_i,\ A_2$ are collinear. Consider the dilation with the center at A_2 that takes D_1 to B_2. This dilation takes c_i to the arcs c_i' of hyperbolas with foci at A_2 and B_2. Let the endpoints of c_i' be σ_i' and χ_i'; the assumption of induction implies that the points $\sigma_i',\ \chi_{i-1}',\ \nu_{i-1},\ \lambda_i,\ A_2$ are collinear. We are going to prove that χ_i', ν_i, and A_2 are collinear.

The rays $A_2\sigma_{i-1}$ and $A_2\sigma_i$ intersect the rays B_2A_1 and $B_2\sigma_{i-1}'$ at four points: λ_{i-1}, $\lambda_i,\ \sigma_{i-1}',\ \sigma_i'$. The ellipses l_{i-1} and l_i contain the points λ_{i-1} and λ_i, and the hyperbolas c_{i-1}' and c_i' contain the points σ_{i-1}', and σ_i', therefore, according to the statement (a) of Lemma 1, the point of intersection of l_{i-1} and c_{i-1}', the point of intersection of l_i and c_i', and B_2 are collinear. Further, using additionally that the points ν_{i-1} and ν_i lie on a single ray from B_2, χ_{i-1}' and χ_i' lie on another ray from B_2, ν_{i-1} and χ_{i-1}' lie on a ray from A_2, and applying the statement (b) of Lemma, we finally conclude that the points χ_i', ν_i, and A_2 are collinear.

The resulting body is the union of sets $Q_L,\ Q_{C_1},\ Q_{C_2},\ Q_K$, and 8 sequences of arcs. Let us show that it is invisible from the points A_1 and A_2.

It is enough to show invisibility from the point A_2; the invisibility from A_1 can be verified in a completely similar way. If the light ray emanating from A_2 does not belong to the angle C_1A_2K (which is the union of the angles C_1A_2O and OA_2K), it does not hit the body. It remains to consider the cases when it belongs to the angle C_1A_2O and to the angle OA_2K. We consider only the angle C_1A_2O, since the case of the angle OA_2K is completely similar.

The light ray reflects from an arc l_{i-1} ($i \geq 1$) and then goes along a straight line containing B_2. (In Fig. 6 the case $i = 1$ is shown.) Next it reflects from l_i and then goes along a line containing A_2. Then after reflection from the arc c_i it goes along a

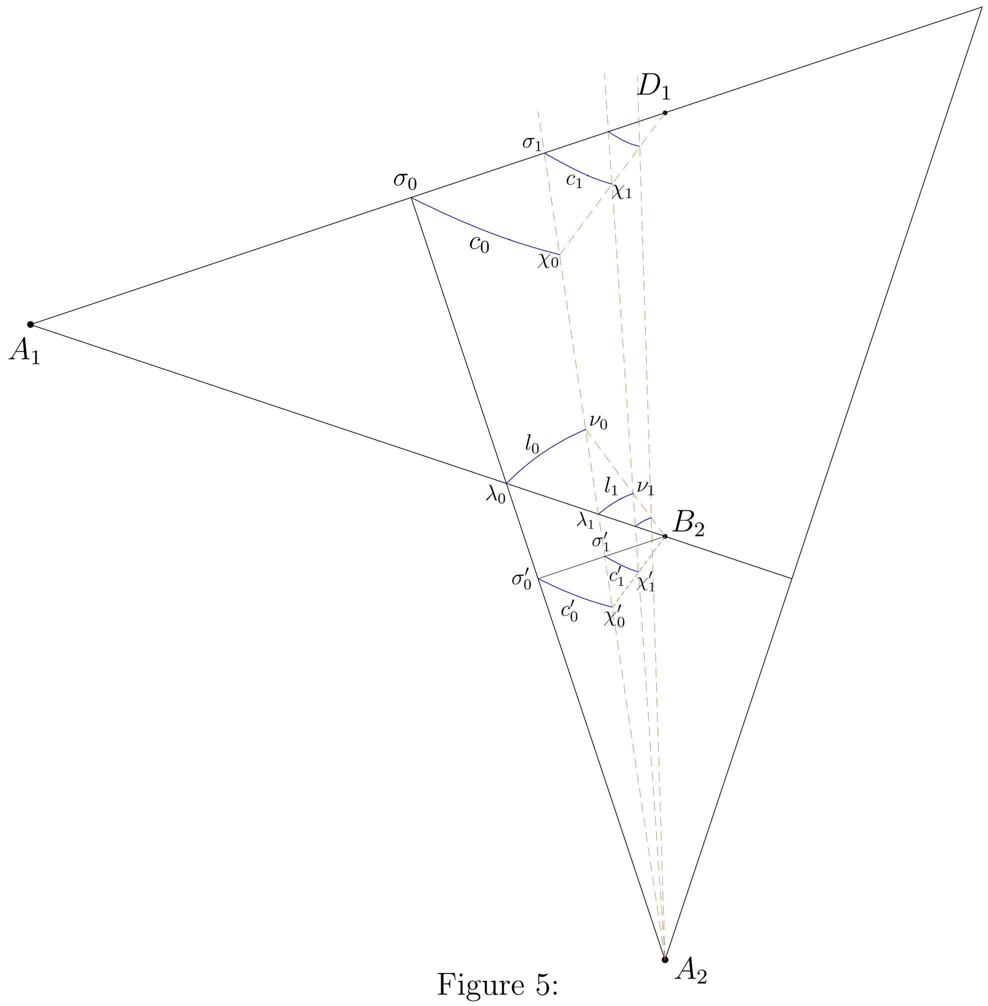


Figure 5:

line containing D_1, hits c_{i-1} and finally goes along a line containing A_2. (The points of reflection are denoted by $P_1,\ P_2,\ P_3,\ P_4$.) It remains to prove that this final line coincides with the initial one.

Consider the arcs $c'_{i-1},\ c'_i$ homothetic to $c_{i-1},\ c_i$ and extend the segment P_2P_3 of trajectory between l_i and c_i until the intersection P'_3 with c'_i (dashed line $P_2P'_3$ in the figure). Let P'_3 and P'_4 be the images of P_3 and P_4, respectively, under the dilation (recall that the dilation is centered at A_2 and takes D_1 to B_2). Next we take the straight line through P'_3 and B_2 and fix the point P'_4 of its intersection with c'_{i-1}. The segment $P'_3P'_4$ is a dashed line in the figure. We already know that the point of intersection of the hyperbola c'_{i-1} with the ellipse l_{i-1}, the point of intersection of the hyperbola c'_i with the ellipse l_i, and B_2 are collinear. The points $P_2,\ P'_3$, and A_2 are collinear, the points $P'_4,\ P'_3$, and B_2 are collinear, and the points $P_1,\ P_2$, and B_2 are also collinear; therefore, according to statement (b) of Lemma 1, the points $P'_1,\ P'_4$, and A_2 are also collinear. Thus, P'_4 lies on A_2P_1, and therefore, P_4 also lies on this line. This implies that the line A_2P_4 of final motion of the particle coincides with the line A_2P_1 of initial motion. Invisibility is proved.

The invisible body is shown in Fig. 7 (a). It is the union of four curvilinear quadrangles and 8 infinite sequences of curves of vanishing length. Notice that the body lies on one side of the line A_1A_2.

To obtain a body invisible from two points in a higher dimensional settings, it is

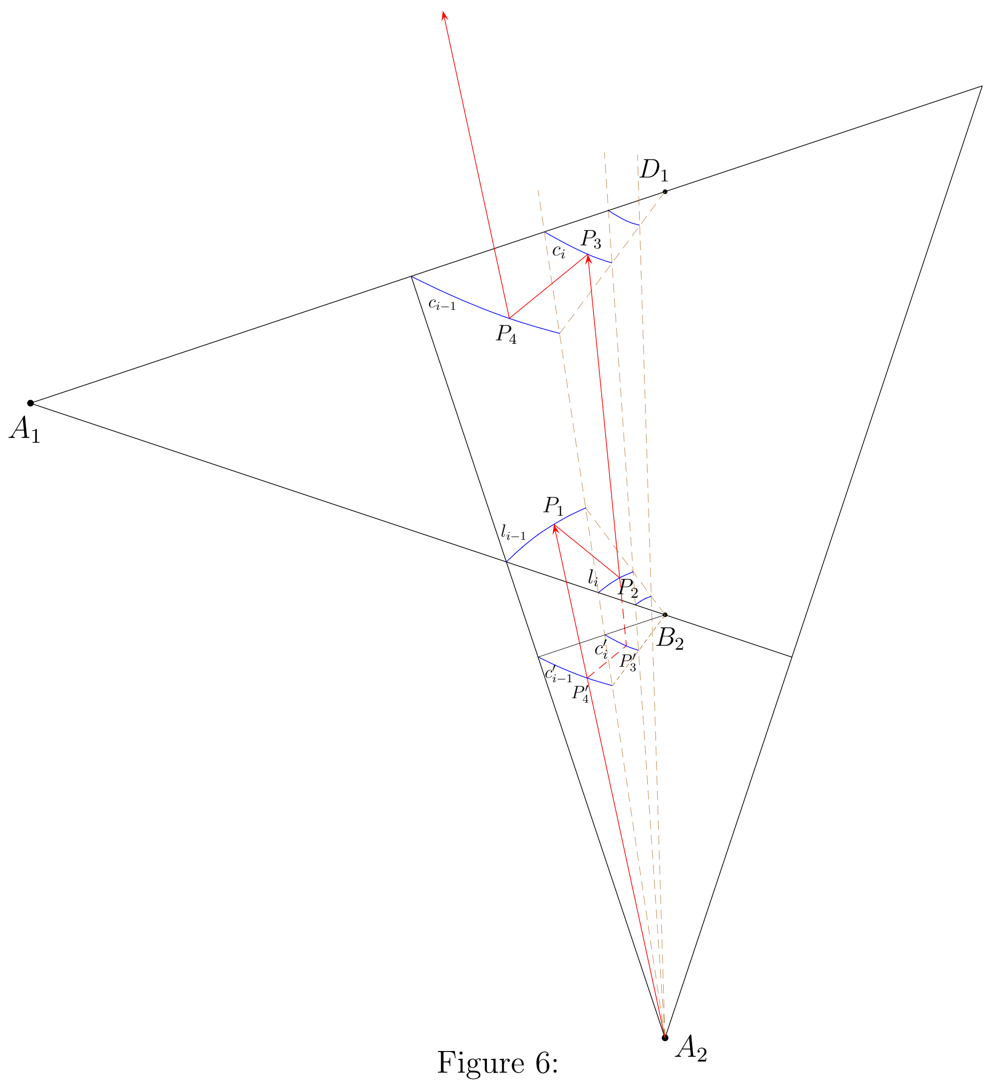


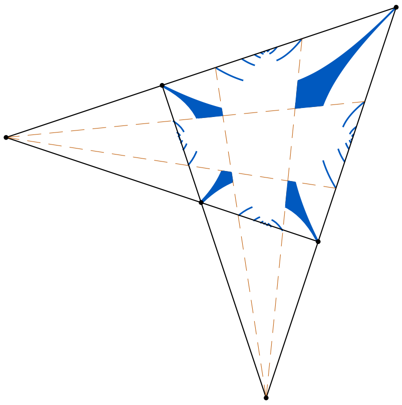
Figure 6:

sufficient to rotate the two-dimensional construction around the axis A_1A_2 (see Fig. 7 (b) for the illustration of the 3D case).

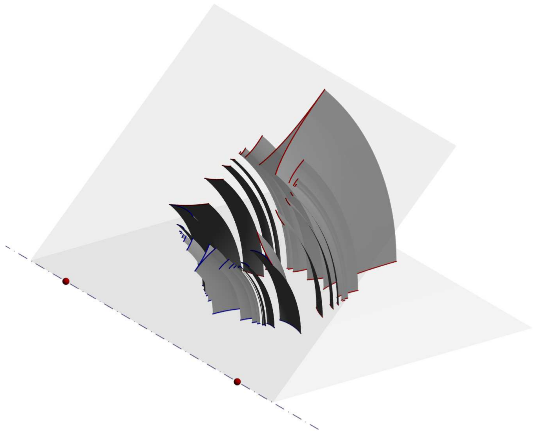
More precisely, take two different points A_1 and A_2 in $\mathbb{R}^d$, and let l be the straight line through these points. $\mathbb{R}^d$ is the union of 2D planes through l. Each plane is divided by l into two half-planes. Fix a 2D plane through l and take a body $\mathcal{B}$ in this plane invisible from A_1 and A_2. Take the half-plane Π containing $\mathcal{B}$. There exists a unique mapping $\Phi : \mathbb{R}^d \to \Pi$ such that the restriction of Φ on any 2D half-plane bounded by l is an isometry with the fixed points A_1 and A_2. Then the body $\Phi^{-1}(\mathcal{B}) \subset \mathbb{R}^d$ is invisible from A_1 and A_2.

It is not necessary to make a full rotation; for instance, in the 3D case a rotation by a certain angle (not necessary 360^0) results in an invisible body. In arbitrary dimension, let a domain $\mathcal{A} \subset \mathbb{R}^d$ be a union of 2D half-planes bounded by l, and let $\Phi\rfloor_{\mathcal{A}}$ be the restriction of Φ on $\mathcal{A}$. Then $\Phi\rfloor_{\mathcal{A}}^{-1}(\mathcal{B})$ is a body invisible from A_1 and A_2. We hence proved Theorem 1.

Remark 1. *The construction of the invisible body does not have to be symmetric. In fact, it is enough to choose four points H_1, M, H_2, N in the quadrangles $OB_1C_1D_1$, OD_1KD_2, $OD_2C_2B_2$, OB_2LB_1, respectively. We require that the straight lines H_1M and H_2N contain A_1, the lines MH_2 and NH_1 contain A_2, and each of the points H_1, M, H_2, N belongs to the exterior of both curves through C_1, K, C_2, or L, respectively. Moreover, we*



(a)



(b)

Figure 7: Bodies invisible from two points: a) 2D case, b) 3D case

require that each of the segments A_1N, A_2N intersects only one curve through L and that the analogous conditions related to the other three vertices of LC_1KC_2 hold.

To demonstrate the existence of a body invisible in one direction and from one point simultaneously, a similar construction may be used. The difference is that here one needs to use 4 parabolas, 2 ellipses, and 2 hyperbolas instead of 4 ellipses and 4 hyperbolas. The details of this modified construction are left to the reader.

Appendix

Consider two rays emanating from F_2 and a ray emanating from F_1. Let the points of intersection of the first two ones with the third one be denoted by h and e. Let $\alpha = \measuredangle F_1F_2h$, $\beta = \measuredangle F_1F_2e$, $\gamma = \measuredangle hF_1F_2$. Draw an ellipse and a hyperbola with foci at F_1 and F_2 through e and h, respectively; let u be the point of their intersection and $\varphi = \measuredangle F_1F_2u$. We are going to show that φ depends only on α and β, and does not depend on γ (that is, does not depend on the choice of the ray from F_1). Thus the statement (a) of Lemma 1 will be proved. The proof of the statement (b) is completely similar and therefore is omitted here.

Denote $a_1 = F_1h$, $a_2 = F_1e$, $b_1 = F_2h$, $b_2 = F_2e$, $f = F_1F_2$, $c = F_2u$. By the focal property of hyperbola and ellipse we have

$$F_1h - F_2h = F_1u - F_2u \quad \text{and} \quad F_1e + F_2e = F_1u + F_2u,$$

whence

$$a_1 - b_1 = \sqrt{f^2 + c^2 - 2cf\cos\varphi} - c, \qquad a_1 + b_1 = \sqrt{f^2 + c^2 - 2cf\cos\varphi} + c,$$

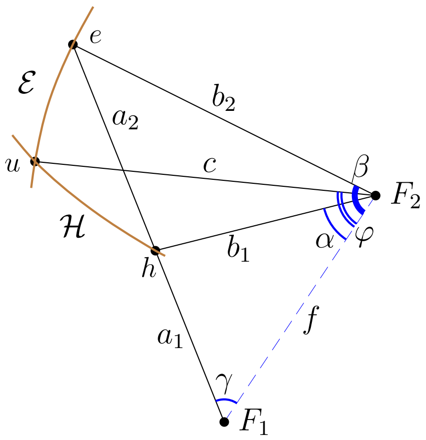


Figure 8:

and therefore,

$$\sqrt{f^2 + c^2 - 2cf\cos\varphi} = \frac{1}{2}(a_1 - b_1 + a_2 + b_2), \tag{1}$$

$$c = \frac{1}{2}(-a_1 + b_1 + a_2 + b_2). \tag{2}$$

Further, by sine law,

$$\frac{a_1}{\sin\alpha} = \frac{b_1}{\sin\gamma} = \frac{f}{\sin(\alpha+\gamma)}, \qquad \frac{a_2}{\sin\beta} = \frac{b_2}{\sin\gamma} = \frac{f}{\sin(\beta+\gamma)},$$

hence

$$a_1 = f\,\frac{\sin\alpha}{\sin(\alpha+\gamma)}, \quad b_1 = f\,\frac{\sin\gamma}{\sin(\alpha+\gamma)}, \quad a_2 = f\,\frac{\sin\beta}{\sin(\beta+\gamma)}, \quad b_2 = f\,\frac{\sin\gamma}{\sin(\beta+\gamma)},$$

and substituting these quantities in (1) and (2) one gets

$$f^2 + c^2 - 2cf\cos\varphi = \frac{f^2}{4}\left[\frac{\sin\alpha - \sin\gamma}{\sin(\alpha+\gamma)} + \frac{\sin\beta + \sin\gamma}{\sin(\beta+\gamma)}\right]^2,$$

$$c = \frac{f}{2}\left[\frac{\sin\gamma - \sin\alpha}{\sin(\alpha+\gamma)} + \frac{\sin\beta + \sin\gamma}{\sin(\beta+\gamma)}\right],$$

and thus,

$$\cos\varphi = \frac{A}{B},$$

where

$$B = 2cf = f^2\left[\frac{\sin\gamma - \sin\alpha}{\sin(\alpha+\gamma)} + \frac{\sin\beta + \sin\gamma}{\sin(\beta+\gamma)}\right],$$

$$A = f^2 + c^2 - \frac{f^2}{4}\left[\frac{\sin\alpha - \sin\gamma}{\sin(\alpha+\gamma)} + \frac{\sin\beta + \sin\gamma}{\sin(\beta+\gamma)}\right]^2 =$$

$$= f^2 \left\{ 1 + \frac{1}{4} \left[\frac{\sin\gamma - \sin\alpha}{\sin(\alpha+\gamma)} + \frac{\sin\beta + \sin\gamma}{\sin(\beta+\gamma)} \right]^2 - \frac{1}{4} \left[\frac{\sin\alpha - \sin\gamma}{\sin(\alpha+\gamma)} + \frac{\sin\beta + \sin\gamma}{\sin(\beta+\gamma)} \right]^2 \right\}.$$

Thus,

$$B \, \frac{\sin(\alpha+\gamma)\sin(\beta+\gamma)}{f^2} = \sin(\beta+\gamma)(\sin\gamma - \sin\alpha) + \sin(\alpha+\gamma)(\sin\beta + \sin\gamma),$$

$$A \, \frac{\sin(\alpha+\gamma)\sin(\beta+\gamma)}{f^2} = \sin(\alpha+\gamma)\sin(\beta+\gamma) + (\sin\gamma - \sin\alpha)(\sin\gamma + \sin\beta).$$

After some algebra one gets

$$B \, \frac{\sin(\alpha+\gamma)\sin(\beta+\gamma)}{f^2} = 2\sin\gamma\cos\frac{\alpha-\beta}{2} \left[\sin\left(\gamma + \frac{\alpha+\beta}{2}\right) + \sin\frac{\beta-\alpha}{2} \right],$$

$$A \, \frac{\sin(\alpha+\gamma)\sin(\beta+\gamma)}{f^2} = 2\sin\gamma\cos\frac{\alpha+\beta}{2} \left[\sin\left(\gamma + \frac{\alpha+\beta}{2}\right) + \sin\frac{\beta-\alpha}{2} \right],$$

and therefore,

$$\cos\varphi = \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}.$$

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References

[1] A. Aleksenko and A. Plakhov. *Bodies of zero resistance and bodies invisible in one direction.* Nonlinearity **22**, 1247-1258 (2009).

[2] K. Burdzy and T. Kulczycki. *Invisibility via reflective coating.* arXiv:1205.3735.

[3] X. Chen, Y. Luo, J. Zhang, K. Jiang, J. B. Pendry, and S. Zhang. *Macroscopic invisibility cloaking of visible light.* Nature Communications **2**, 176 (2011).

[4] P. Dolgirev. *Properties of confocal conics*, to appear in J. Classical Geometry.

[5] K. Falconer. *Fractal geometry: mathematical foundations and applications.* 2nd edition. John Wiley & Sons Ltd., 2003.

[6] A. Glutsyk and Y. Kudryashov. *No planar billiard possesses an open set of quadrilateral trajectories.* J. Modern Dynamics **6**, 287-326 (2012).

[7] E. Gutkin. *Billiard dynamics: An updated survey with the emphasis on open problems.* Chaos **22**, 026116 (2012).

[8] E. Lakshtanov, B. D. Sleeman, and B. Vainberg. *High frequency scattering by a classically invisible body.* SIAM J. Appl. Math. **72**, 646-669 (2012).

[9] D. Schurig, J. J. Mock, B. J. Justice, S. A. Cummer, J. B. Pendry, A. F. Starr, and D. R. Smith. *Metamaterial electromagnetic cloak at microwave frequencies.* Science **314**, 977-980 (2006).

[10] A. Plakhov. *Exterior billiards. Systems with impacts outside bounded domains.* Springer, New York, 2012.

[11] A. Plakhov and V. Roshchina. *Invisibility in billiards.* Nonlinearity **24**, 847-854 (2011).

[12] A. Plakhov and V. Roshchina. *Fractal bodies invisible in 2 and 3 directions.* Discr. Contin. Dynam. Syst.-A **33**, 1615-1631 (2013).

[13] A. Plakhov and V. Roshchina. *Invisibility in billiards is impossible in an infinite number of directions.* arXiv:1206.6965

[14] A. Plakhov and V. Roshchina. *Bodies invisible from one point.* arXiv:1112.6167

[15] J. B. Pendry, D. Schurig, and D. R. Smith. *Controlling electromagnetic fields.* Science **312**, 1780-1782 (2006).

[16] S. Tabachnikov, Personal communication (2008).

[17] M. Gharghi, C. Gladden, T. Zentgraf, Y. Liu, X. Yin, J. Valentine, and X. Zhang. *A carpet cloak for visible light.* Nano Lett. **11**, 2825-2828 (2011).

[18] B. Zhang, Y. Luo, X. Liu, and G. Barbastathis. *Macroscopic invisibility cloak for visible light.* Phys. Rev. Lett. **106**, 033901 (2011).