

SOME NEW CONSIDERATIONS ABOUT DOUBLE NESTED GRAPHS

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ABSTRACT. In the set of all connected graphs with fixed order and size, the graphs with maximal index are nested split graphs, also called threshold graphs. It was recently (and independently) observed in [F.K.Bell, D. Cvetković, P. Rowlinson, S.K. Simić, Graphs for which the largest eigenvalue is minimal, II, Linear Algebra Appl. 429 (2008)] and [A. Bhattacharya, S. Friedland, U.N. Peled, On the first eigenvalue of bipartite graphs, Electron. J. Combin. 15 (2008), #144] that double nested graphs, also called bipartite chain graphs, play the same role within class of bipartite graphs. In this paper we study some structural and spectral features of double nested graphs. In studying the spectrum of double nested graphs we rather consider some weighted nonnegative matrices (of significantly less order) which preserve all positive eigenvalues of former ones. Moreover, their inverse matrices appear to be tridiagonal. Using this fact we provide several new bounds on the index (largest eigenvalue) of double nested graphs, and also deduce some bounds on eigenvector components for the index. We conclude the paper by examining the questions related to main versus non-main eigenvalues.

1. INTRODUCTION

Let $G = (V(G), E(G))$ be an undirected simple graph, i.e. a finite graph without loops or multiple edges. $V(G)$ is its vertex set, while $E(G)$ its edge set. The order of G is denoted by ν ($= |V(G)|$), and its size by ϵ ($= |E(G)|$). We write $u \sim v$ whenever vertices $u, v \in V(G)$ are adjacent, and denote by uv the corresponding edge.

Given a graph G , $A(G)$ denotes the $(0, 1)$ -adjacency matrix of G . The polynomial $P(x; G) = \det(xI - A(G))$ is called the *characteristic polynomial* of G . Its roots comprise the *spectrum* of G , denoted by $\text{Sp}(G)$. Since $A(G)$ is symmetric, its spectrum is real, and in general it is a multiset containing ν non-necessarily distinct *eigenvalues*. So let

$$\lambda_1(G) \geq \lambda_2(G) \geq \dots \geq \lambda_\nu(G)$$

be the corresponding eigenvalues (given in non-increasing order). Recall, as well known, $\lambda_1(G) > \lambda_2(G)$ whenever G is connected. Further on, if not told otherwise, we will consider only connected graphs. The largest eigenvalue of G , denoted by $\rho(G)$, is called the *spectral radius* of G (or, for short, its *index*). For a given $\lambda \in \text{Sp}(G)$, $m(\lambda; G)$ denotes the *multiplicity* of λ in G . Since $A(G)$ is symmetric, the algebraic and geometric multiplicities of λ coincide. If $m(\lambda; G) = 1$, then λ is a *simple eigenvalue* of G .

The equation $A\mathbf{x} = \lambda\mathbf{x}$ is called the *eigenvalue equation* of A , or of a labelled graph G , if $A = A(G)$. For a fixed $\lambda \in \text{Sp}(G)$, a non-trivial solution $\mathbf{x} = (x_1, x_2, \dots, x_\nu)^T$ of the eigenvalue equation is a λ -eigenvector of a labelled graph G . In particular, if $\lambda = \rho(G)$,

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then the corresponding vector, with positive coordinates, is called a *principal eigenvector* of G . In the scalar form, for any $\lambda \in \text{Sp}(G)$, the eigenvalue equation reads:

$$\lambda x_u = \sum_{v \sim u} x_v,$$

where $u \in V(G)$. The null space of $A(G) - \lambda I$ is called the eigenspace of G and is denoted by $\mathcal{E}(\lambda; G)$. Note also that any $\mathbf{x} \in \mathcal{E}(\lambda; G)$ can be interpreted as a mapping $\mathbf{x} : V(G) \rightarrow \mathbb{R}$. So, for any $v \in V(G)$, $\mathbf{x}(v)$ and x_v can be identified, and considered as vertex weights (with respect to \mathbf{x}).

Finally, recall that an eigenvalue μ of G is *main* if $\mathcal{E}(\mu; G)$ is not orthogonal to \mathbf{j} , i.e. the all-1 vector; otherwise it is *non-main*.

For all other notions (and notation) from graph theory, including spectral graph theory, the reader is referred to the book [14]. The same notation will be adopted for matrices, by passing from the adjacency matrix to an arbitrary (symmetric) matrix.

In [2] graphs G for which the least eigenvalue is minimal among the connected graphs of prescribed order and size were investigated. It was shown that if G is incomplete then its least eigenvalue is simple, and G is either bipartite, or a join of two nested split graphs (with isolated vertices allowed, but not both being edge-free). In [3] the structural description of these graphs which are bipartite is given. They were called double nested graphs (or DNGs for short), and they also feature as maximal graphs for the largest eigenvalue in the same class of graphs. The same characterization was independently given in [4], in the same year. There, these graphs were recognized as bipartite chain graphs (known in some other contexts, but not too much in spectral ones).

The plan of the paper is as follows: after this introduction, we precise in Section 2 the structure of DNGs, and introduce various parameters relevant to them. We also list several important properties of DNGs (followed by the corresponding properties of nested split graphs or NSGs for short). In Section 3, we show that the index of any DNG, say G , is equal to the index of a certain nonnegative matrix of significantly smaller order than that of G . Several lower and upper bounds on the index of DNGs are obtained by estimating the index of such matrix. In Section 4, we show that the inverse of the matrix established in the previous section is tridiagonal, and exploit this fact for getting further bounds on the index of DNGs. In Section 5, we provide some bounds on the entries of a principal eigenvector of DNGs, and also consider if some eigenvalues of DNGs are main or non-main. The paper is ended with some concluding remarks.

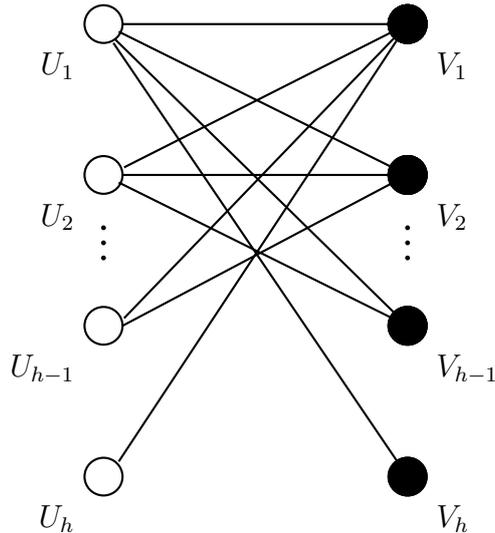
2. DOUBLE NESTED GRAPHS

In this section, we first precise the structure of connected DNGs (i.e. without isolated vertices). The vertex set of any DNG, say $G = (U, V; E)$, consists of two colour classes (or co-cliques). To specify the nesting property, both of colour classes are partitioned into h non-empty cells; so $U = \bigcup_{i=1}^h U_i$ and $V = \bigcup_{i=1}^h V_i$, respectively; all vertices in U_s ($s = 1, 2, \dots, h$) are joined (by cross edges) to all vertices in $\bigcup_{k=1}^{h+1-s} V_k$. Denote by $N_G(w)$ the set of neighbours of a vertex w . Hence, if $u' \in U_{s+1}$ and $u'' \in U_s$, $v' \in V_{t+1}$ and $v'' \in V_t$ then $N_G(u') \subset N_G(u'')$ and $N_G(v') \subset N_G(v'')$, and this makes precise the double nesting property.

If $m_s = |U_s|$ and $n_s = |V_s|$ ($s = 1, 2, \dots, h$), then G is denoted by

$$\text{DNG}(m_1, m_2, \dots, m_h; n_1, n_2, \dots, n_h).$$

We also write $\hat{m} = (m_1, m_2, \dots, m_h)$ and $\hat{n} = (n_1, n_2, \dots, n_h)$, and then, for short, $G = \text{DNG}(\hat{m}; \hat{n})$.

FIGURE 2.1. The structure of a double nested graph G .

We now introduce some notation to be used later on. Let

$$M_s = \sum_{i=1}^s m_i \quad \text{and} \quad N_t = \sum_{j=1}^t n_j, \quad \text{for } 1 \leq s, t \leq h.$$

Thus G is of order $\nu = M_h + N_h$, and size $\epsilon = \sum_{s=1}^h m_s N_{h+1-s}$. Observe that N_{h+1-s} is the degree of a vertex $u \in U_s$; the degree of a vertex $v \in V_t$ is equal to M_{h+1-t} .

We next define the following quantity:

$$e_s = \sum_{i=1}^s m_i N_{h+1-i},$$

the total number of cross edges with one end in $\bigcup_{k=1}^s U_k$.

Similarly, we can introduce further parameters if we exchange the roles of sets U and V . The parameters which arise in this way will be named by the letter f .

It is well-known that in the class of connected graphs of order n and size m , the graphs with largest index are the threshold graphs [25], also called nested split graphs, or NSGs for short (cf. [14]). For bipartite graphs they are known as bipartite chain graphs [4, 35] or double nested graphs (cf. [3]). Note that threshold and/or bipartite chain graphs admit isolated vertices, but in most of our considerations they will be ignored (since we are studying only connected graphs). In what follows we will mention some not too widely known facts about NSGs and DNGs in order to make this topic closer to readers with different backgrounds.

In [9] Brualdi and Solheid addressed NSGs as graphs whose adjacency matrix admits the step-wise form. This can be restated as: the vertices of such a graph can be ordered in such a way that whenever uv is an edge, then $u'v'$ is also an edge for all $u' \leq u$ and $v' \leq v$. The analogous result for DNGs tells that the incidence matrix between vertices of different colour classes admits a step-wise form. Then the upper right block of the adjacency matrix is in this form (the proof is trivial and omitted here). Taking into account that NSGs are split graphs, it is also worth mentioning that the deletion of all edges from the (maximal) clique turns the NSG into the DNG, and vice versa, if all edges missing in one (maximal) co-clique are added to the DNG (see, for example, [3]).

Another possibility to characterize NSGs and DNGs is by the way of forbidden induced subgraphs. It is known that threshold graphs are characterized as being $\{2K_2, P_4, C_4\}$ -free graphs. On the other hand, DNGs are characterized as being $\{2K_2, C_3, C_5\}$ -free graphs. To see that, say $2K_2$ cannot appear in NSGs (or DNGs) as an induced subgraph, it suffices to use the “rotational rule” which is a simple strategy to increase the index of any graph. Namely, if G is a graph with extremal index and \mathbf{x} its principal eigenvector, then, if a vertex r is adjacent to s but non-adjacent to t , and if $\mathbf{x}(t) \geq \mathbf{x}(s)$, then $\lambda_1(G - rs + rt) > \lambda_1(G)$ (see, for example, [14] for more details). Now if $H = 2K_2$ is an induced subgraph of G , and if t is a vertex of H with maximal weight, r is a nonneighbour of t in H , while s is a neighbour of r in H , then by the above rule, we obtain that $\lambda_1(G - rs + rt) > \lambda_1(G)$, which is a contradiction. The same reasoning applies if $H = P_4$ or C_4 , and therefore the threshold graphs arise. On the other hand, if G is bipartite and $2K_2$ -free it is a bipartite chain graph, or equivalently a $\{2K_2, C_3, C_5\}$ -free graph. The nesting property arises due to $2K_2$. Indeed, then any two vertices in G of the same colour have comparable neighbourhoods (by inclusion), whence the nesting property arises, and the corresponding name (introduced in [3]).

In view of the above considerations, it immediately follows that the complement of any NSG (with isolated vertices allowed) is also an NSG, and the bipartite complement of any DNG (with isolated vertices allowed and assigned to colour classes) is also a DNG. Here, for bipartite complement, we only exchange edges and non-edges between vertices from different colour classes.

To summarize the above considerations, we next add a list of some remarkable facts about the classes of graphs being observed:

- (i) each NSG is uniquely determined by its vertex degree sequence; each DNG is uniquely determined by its vertex degree bi-sequences;
- (ii) vertices of the same degree (or same degree within each colour class) give rise to the cells of an equitable partition (moreover, they are also the orbits of the automorphism group); see [14] for more details);
- (iii) the algorithms for recognizing NSGs (or DNGs) are linear in $n + m$ (see [17]);
- (iv) vertices of the largest degree in NSGs (or of the largest degrees in DNGs within each colour class) are adjacent to all non-isolated vertices (resp. are adjacent to all non-isolated vertices from the other colour class).

3. BOUNDS ON THE INDEX OF DNGS

If G is a connected DNG of order ν and size ϵ , recall that ρ ($= \rho(G)$) is its index. Since $A(G)$ is a nonnegative and irreducible matrix, an eigenvector corresponding to the index can be taken to be positive; so it is a principal eigenvector. In [1] several lower and upper bounds on the index of DNGs were obtained by so called “eigenvalue technique” which is based on the approximations of the entries of the principal eigenvector. A good approximation of the principal eigenvector leads to a good estimation of the index using, for example, the Rayleigh principle. In this section we will obtain several bounds on the index of a DNG, but this time by a different approach, namely, by using the bounds on the largest eigenvalue of nonnegative matrices applied to divisor matrix of a DNG. It turns out that some of bounds can be obtained by either approaches, like those in [1, Propositions 4.1, 4.4 and 4.5]. The results of these two techniques, in general, are incomparable although numerical examples show that new ones in most of the cases are significantly better.

To start we first recall several bounds on the largest eigenvalue of nonnegative matrices (to be used later on), and next we define the concepts of the *equitable partition* and *divisor matrix*.

For a given $n \times n$ nonnegative matrix $A = (a_{ij})$, some bounds on the largest eigenvalue (or *Perron eigenvalue*, $\rho(A)$) are summarized below:

$$\rho(A) \geq \max a_{ii} \quad (\text{Frobenius [15]}).$$

Let R_i be the sum of the entries in row i of A . Setting $r = \min_i R_i$ and $R = \max_i R_i$, then

$$r \leq \rho(A) \leq R \quad (\text{Frobenius [15]})$$

$$(3.1) \quad \min_i \left(\frac{\sum_{j=1}^h a_{ij} R_j}{R_i} \right) \leq \rho(A) \leq \max_i \left(\frac{\sum_{j=1}^h a_{ij} R_j}{R_i} \right) \quad (\text{Minc [27]})$$

$$(3.2) \quad \min_i \sqrt{\frac{1}{R_i} \sum_{t=1}^h a_{it} \sum_{j=1}^h a_{tj} R_j} \leq \rho(A) \leq \max_i \sqrt{\frac{1}{R_i} \sum_{t=1}^h a_{it} \sum_{j=1}^h a_{tj} R_j} \quad (\text{Liu [24]}).$$

Another result is due to Wolkowitz and Styan [36] and it gives bounds on eigenvalues using the trace of a matrix.

Theorem 3.1 (Wolkowicz, Styan [36]). *Let $A = (a_{ij})$ be a real matrix of order $n \geq 1$ with real eigenvalues and let*

$$m = \frac{\text{tr} A}{n}, \quad s^2 = \frac{\text{tr} A^2}{n} - m^2.$$

Then

$$(3.3) \quad m + \frac{s}{\sqrt{n-1}} \leq \rho(A) \leq m + s\sqrt{n-1}$$

$$(3.4) \quad m - s\sqrt{n-1} \leq \lambda_n(A) \leq m - \frac{s}{\sqrt{n-1}}.$$

Given a graph G , the partition $\mathcal{D} = W_1 \cup W_2 \cup \dots \cup W_k$ of its vertex set is an *equitable partition* if every vertex in W_i has the same number of neighbours in W_j , say d_{ij} for all $i, j \in \{1, 2, \dots, k\}$. The matrix $D = (d_{ij})$ is called *divisor matrix* arising from \mathcal{D} . The greatest benefit of usage of divisor matrices stems from the fact that the characteristic polynomial of any divisor matrix divides the characteristic polynomial of the graph (for more details see [14, p. 85]).

In view of the above definition, for any $G = DNG(m_1, \dots, m_h; n_1, \dots, n_h)$ the partition

$$\mathcal{D} = U_1 \cup U_2 \cup \dots \cup U_h \cup V_1 \cup V_2 \cup \dots \cup V_h$$

of its vertex set is an equitable partition, since every vertex in U_i (V_i) has the same number of neighbours in V_j (U_j), for all $i, j \in \{1, 2, \dots, h\}$. Let $D = A(\mathcal{D})$ be its divisor matrix

and

$$(3.8) \quad \rho \geq \sqrt[4]{\max \left\{ \frac{1}{M_h} \sum_{i=1}^h m_i f_{h+1-i}^2, \frac{1}{N_h} \sum_{i=1}^h n_i e_{h+1-i}^2 \right\}}.$$

Proof. By [24, Corollary 3.2] applied to NM , we obtain

$$\min_i \sqrt{\frac{1}{R_i} \sum_{t=1}^h b_{it} \sum_{j=1}^h b_{tj} R_j} \leq \rho(NM) \leq \max_i \sqrt{\frac{1}{R_i} \sum_{t=1}^h b_{it} \sum_{j=1}^h b_{tj} R_j}.$$

The maximum and minimum are attained for $i = 1$ and $i = h$, respectively, and the upper bound is unique. This leads to the stated formulas. \square

Proposition 3.3. *Let G be a connected DNG,*

$$p = \frac{\epsilon}{h} \quad \text{and} \quad q = \frac{\sum_{i=1}^h m_i (\sum_{j=1}^{i-1} m_j (N_{h+1-i})^2 + \sum_{j=i}^h m_j (N_{h+1-j})^2)}{h}.$$

Then

$$(3.9) \quad p + \sqrt{\frac{q-p^2}{h-1}} \leq \rho(G)^2 \leq p + \sqrt{(h-1)(q-p^2)}.$$

Proof. We first obtain that $\text{tr}(NM) = \epsilon$ and $\text{tr}(NM)^2 = \sum_{i=1}^h m_i (\sum_{j=1}^{i-1} m_j (N_{h+1-i})^2 + \sum_{j=i}^h m_j (N_{h+1-j})^2)$. Next we apply (3.3) to NM bearing in mind (3.5). \square

Remark 3.1. To give the better insight in quality of our bounds we provide the following numerical examples. All computational results are obtained using *Mathematica*. We start from a small DNG, $G = \text{DNG}(1, 2, 3, 2; 2, 1, 3, 1)$ arbitrarily chosen (as in [1]) and then we consider graphs obtained from G by multiplying exactly one parameter by 10, 100 or 1000.

Example 3.1. a) a DNG with $\hat{m} = (10, 2, 3, 2)$, $\hat{n} = (2, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
9.25705	9.2822	9.28851	9.30104
-0.27 %	0	0.068 %	0.20 %

b) a DNG with $\hat{m} = (1, 20, 3, 2)$, $\hat{n} = (2, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
11.3321	11.4962	11.499	11.5042
-1.43 %	0	0.024 %	0.07 %

c) a DNG with $\hat{m} = (1, 2, 30, 2)$, $\hat{n} = (2, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
10.1119	10.1293	10.1387	10.1507
-0.17 %	0	0.092 %	0.21 %

d) a DNG with $\hat{m} = (1, 2, 3, 20)$, $\hat{n} = (2, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
7.39546	7.41839	7.45484	7.49946
-0.31 %	0	0.49 %	0.11 %

e) a DNG with $\hat{m} = (1, 2, 3, 2)$, $\hat{n} = (20, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
12.9658	12.9704	12.9725	12.976
-0.036 %	0	0.016 %	0.04 %

f) a DNG with $\hat{m} = (1, 2, 3, 2)$, $\hat{n} = (2, 10, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
8.62725	8.81015	8.81315	8.82801
-2.08 %	0	0.034 %	0.2 %

g) a DNG with $\hat{m} = (1, 2, 3, 2)$, $\hat{n} = (2, 1, 30, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
9.98951	10.026	10.0432	10.0594
-0.363 %	0	0.17 %	0.33 %

h) a DNG with $\hat{m} = (1, 2, 3, 2)$, $\hat{n} = (2, 1, 3, 10)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
5.13609	5.35218	5.4036	5.51567
-4.04 %	0	0.96 %	3.05 %

i) a DNG with $\hat{m} = (100, 2, 3, 2)$, $\hat{n} = (2, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
26.7457	26.7467	26.747	26.7476
-0.0034 %	0	0.0012 %	0.0036 %

j) a DNG with $\hat{m} = (1, 200, 3, 2)$, $\hat{n} = (2, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
34.7433	34.8119	34.812	34.8122
-0.197 %	0	0.0003 %	0.0009 %

k) a DNG with $\hat{m} = (1, 2, 3, 200)$, $\hat{n} = (2, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
20.3055	20.3057	20.3084	20.3119
-0.0006 %	0	0.013 %	0.03 %

l) a DNG with $\hat{m} = (1, 2, 3, 2000)$, $\hat{n} = (2, 1, 3, 1)$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
63.3406	63.3406	63.3407	63.3408
$-6.3 \times 10^{-7}\%$	0	$1.5 \times 10^{-6}\%$	$3.4 \times 10^{-6}\%$

The same DNGs were considered in [1]. It turns that the previous bounds and the new ones are generally incomparable, but in some situations (see items (d), (e) and (f) above), the new ones are significantly superior. This fact can be better illustrated by the following example (both ad hoc chosen):

Example 3.2. a) a DNG($\hat{m}; \hat{n}$) with

$$\hat{m} = \hat{n} = (1, 1, 1, 1, 117, 1, 6, 120, 1, 1, 4100, 1, 9990, 19500).$$

Prop. 3.2	ρ	Prop. 3.3	Prop. 3.2
206.163	223.243	268.07	295.484
-7.65063%	0	20.0801%	32.3599%

The next table provides bounds obtained in [1] (Here Prop. 4.1, 4.2, 4.3, 4.4 and 4.5 denote propositions from [1]).

$$(4.1) \quad A = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix}.$$

By the Choleski decomposition, we have that $A = R_0^T R_0$. For $i = 1, 2, \dots$ let

$$A_i = \begin{pmatrix} a_1^{(i)} & b_1^{(i)} & & \\ b_1^{(i)} & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1}^{(i)} \\ & & b_{n-1}^{(i)} & a_n^{(i)} \end{pmatrix}$$

and

$$A_{i+1} = R_i R_i^T$$

with

$$R_i = \begin{pmatrix} p_1^{(i)} & d_1^{(i)} & & \\ & \ddots & \ddots & \\ & & \ddots & d_{n-1}^{(i)} \\ & & & p_n^{(i)} \end{pmatrix},$$

where

$$\begin{aligned} (p_1^{(i)})^2 &= a_1^{(i)} \\ (d_k^{(i)})^2 &= \frac{(b_k^{(i)})^2}{(p_k^{(i)})^2}, \quad \text{for } k = 1, 2, \dots \\ (p_k^{(i)})^2 &= a_k^{(i)} - (d_{k-1}^{(i)})^2, \quad \text{for } k = 2, 3, \dots \end{aligned}$$

Then we apply the following result:

Theorem 4.1 (Golub [16]). *There is an eigenvalue of the Jacobi matrix A defined in (4.1) in the interval*

$$\left[a_k^{(i)} - \sigma_k^{(i)}, a_k^{(i)} + \sigma_k^{(i)} \right]$$

where

$$(\sigma_k^{(i)})^2 = (b_k^{(i)})^2 + (b_{k-1}^{(i)})^2$$

with $b_0^{(i)} = b_n^{(i)} = 0$.

As it was observed by Golub [16], if the intervals are non-overlapping, then the previous bounds are smaller than those obtained by the Geršgorin Circle Theorem.

Observe also that the eigenvalues of the nonsingular real tridiagonal matrices

$$\begin{pmatrix} a_1 & b_1 & & \\ c_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & c_{n-1} & a_n \end{pmatrix}, \quad \begin{pmatrix} a_1 & 1 & & \\ b_1 c_1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & b_{n-1} c_{n-1} & a_n \end{pmatrix},$$

and

$$\begin{pmatrix} a_1 & \sqrt{b_1 c_1} & & & \\ \sqrt{b_1 c_1} & \ddots & \ddots & & \\ & \ddots & \ddots & & \\ & & \sqrt{b_{n-1} c_{n-1}} & & \\ & & & \sqrt{b_{n-1} c_{n-1}} & a_n \end{pmatrix},$$

with $b_i c_i > 0$, are the same (for the matrices we are considering this holds).

Example 4.1. Let $G = DNG(1, 2, 3, 2; 2, 1, 3, 1000)$. Then

$$NM = \begin{pmatrix} 1006 & 12 & 9 & 4 \\ 6 & 12 & 9 & 4 \\ 3 & 6 & 9 & 4 \\ 2 & 4 & 6 & 4 \end{pmatrix}$$

and its inverse is

$$(NM)^{-1} = \begin{pmatrix} 1/1000 & -1/1000 & & & \\ -1/2000 & 1003/6000 & -1/6 & & \\ & -1/9 & 4/9 & -1/3 & \\ & & -1/2 & 3/4 & \end{pmatrix}.$$

The next table provides bounds for $\rho(G)$ after 10 and 15 iterations of LR method.

$i = 10$	$i = 15$	ρ	$i = 10$	$i = 15$
31.7188976944	31.7192132566	31.7192132748	31.7195288644	31.7192132930

We conclude this section by the following observation, which is deduced from the fact that the spectrum of any tridiagonal (symmetric) matrix is comprised only of simple eigenvalues, i.e. of multiplicity one.

Proposition 4.2. *If G is a connected DNG then all $2h$ eigenvalues of its divisor matrix are simple.*

Remark 4.1. A similar result was proved in the case of threshold graphs. It reads that in the spectra of any threshold graph all eigenvalues except 0 and 1 are simple (see [19]).

Any two vertices u, v in U_i (or V_j), for $1 \leq i, j \leq h$, are *duplicate*, i.e. they have the same open neighbourhood. It is known that any pair of duplicate vertices gives rise to an eigenvector of G for eigenvalue 0 defined as follows: all its entries are zero except those corresponding to u and v which can be taken to be 1 and -1 , or vice versa. Thus any collection with k mutually duplicate vertices gives rise to $k - 1$ linearly independent eigenvectors for 0. Therefore we are able to construct $\sum_{i=1}^h (m_i - 1) + \sum_{i=1}^h (n_i - 1) = \nu - 2h$ linearly independent eigenvectors of the eigenspace of 0. Since these vectors are orthogonal to all ones vector \mathbf{j} , it follows that 0 is a non-main eigenvalue in the spectrum of any DNG. Notice that the remaining $2h$ eigenvalues are all different from 0. Moreover, by summarizing the previous observations we obtain:

Theorem 4.3. *The spectrum of a connected DNG($m_1, \dots, m_h; n_1, \dots, n_h$) consists of $2h$ distinct nonzero eigenvalues (determined by divisor matrix of G) and of 0 with multiplicity $\nu - 2h$.*

5. MISCELLANEOUS

In this section we consider the properties of the entries of the principal eigenvector of DNG. We also approximate principal eigenvector in order to obtain one lower bound on the index of DNG more, by the application of Rayleigh's principle. Besides this we discuss the question whether or not the nonzero eigenvalues of DNGs are main or non-main.

5.1 Bounds on the entries of the principal eigenvector of DNG

We will denote by a_i and b_j the entries of the principal eigenvector corresponding to the vertices in U_i and V_j , respectively. In [1] some properties of the entries of the principal eigenvector of any DNG were studied. For example, it was shown that $a_1 > a_2 > \dots > a_h$ as well as $b_1 > b_2 > \dots > b_h$. Here we provide bounds (lower and upper) on the maximal entry of the principal eigenvector. Moreover, we will use the fact that the sum of the squares of the entries of the principal eigenvector whose norm is 1 corresponding to the one colour class of any bipartite graph is equal to one half of its norm (see [12]). By the inequality between arithmetic and quadratic mean we obtain

$$\frac{\sum_{i=1}^h n_i b_i}{N_h} \leq \sqrt{\frac{\sum_{i=1}^h n_i b_i^2}{N_h}} = \sqrt{\frac{1}{2N_h}}.$$

Hence,

$$(5.1) \quad \rho a_1 = \sum_{i=1}^h n_i b_i \leq \sqrt{\frac{N_h}{2}},$$

i.e., $a_1 \leq \sqrt{\frac{N_h}{2\rho^2}}$. Since the maximal entry of the principal eigenvector is equal to a_1 or b_1 we obtain

$$\max\{a_1, b_1\} \leq \sqrt{\frac{\max\{N_h, M_h\}}{2\rho^2}}.$$

The equality is attained if and only if $h = 1$ and $M_1 = N_1$ which is equivalent to G being complete bipartite regular graph K_{M_1, M_1} . For each $1 \leq i \leq h$, we have

$$\rho a_i \leq \sum_{j=1}^{h+1-i} n_j b_j \leq N_{h+1-i} b_1,$$

and therefore

$$m_i \rho^2 a_i^2 \leq m_i N_{h+1-i}^2 b_1^2.$$

Summing up over all i , we obtain

$$\frac{1}{2} \rho^2 \leq b_1^2 \sum_{i=1}^h m_i N_{h+1-i}^2,$$

which implies

$$b_1 \geq \frac{\rho}{\sqrt{2 \sum_{i=1}^h m_i N_{h+1-i}^2}}.$$

Hence,

$$\max\{a_1, b_1\} \geq \frac{\rho}{\sqrt{2 \min\{\sum_{i=1}^h m_i N_{h+1-i}^2, \sum_{i=1}^h n_i M_{h+1-i}^2\}}}.$$

Equality occurs if and only if G is a complete bipartite graph (so when $h = 1$). Similarly, as in (5.1) we obtain

$$a_i \leq \frac{1}{\rho} \sqrt{\frac{N_{h-i+1}}{2}}, \quad b_i \leq \frac{1}{\rho} \sqrt{\frac{M_{h-i+1}}{2}}.$$

We use these inequalities in the next proposition to approximate the eigenvector of DNG corresponding to ρ .

Proposition 5.1. *If G is a connected DNG, then*

$$(5.2) \quad \rho \geq \frac{\sum_{i=1}^h m_i \sqrt{N_{h+1-i}} \sum_{j=1}^{h-i+1} n_j \sqrt{M_{h-j+1}}}{\epsilon}.$$

Proof. Let $\mathbf{y} = (y_1, y_2, \dots, y_\nu)^T$ be a vector (whose components are indexed by the vertices of G), and let $y_u = \frac{1}{\rho} \sqrt{\frac{N_i}{2}}$ if $u \in U_i$ ($1 \leq i \leq h$), or otherwise, if $v \in V_j$ let $y_v = \frac{1}{\rho} \sqrt{\frac{M_j}{2}}$ ($1 \leq j \leq h$). If we now use Rayleigh's principle and substitute in the Rayleigh quotient the vector \mathbf{y} as defined above, we arrive easily at the required inequality. \square

Example 5.1. From the following list of numerical examples we can conclude that the previous bound gives in some cases significant improvements (see cases (b), (f) and (h)).

G	$\rho(G)$	Prop.5.1
(a) $DNG(10, 2, 3, 2; 2, 1, 3, 1)$	9.2822	9.21942
(b) $DNG(1, 20, 3, 2; 2, 1, 3, 1)$	11.4962	11.4239
(c) $DNG(1, 2, 30, 2; 2, 1, 3, 1)$	10.1293	9.92248
(d) $DNG(1, 2, 3, 20; 2, 1, 3, 1)$	7.41839	7.22697
(e) $DNG(1, 2, 3, 2; 20, 1, 3, 1)$	12.9704	12.91231
(f) $DNG(1, 2, 3, 2; 2, 10, 3, 1)$	8.8105	8.70397
(g) $DNG(1, 2, 3, 2; 2, 1, 30, 1)$	10.026	9.77286
(h) $DNG(1, 2, 3, 2; 2, 1, 3, 10)$	5.35218	5.28049
(i) $DNG(100, 2, 3, 2; 2, 1, 3, 1)$	26.7467	26.711568
(j) $DNG(1, 200, 2; 2, 1, 3, 1)$	34.8119	34.77855

5.2 Main and non-main eigenvalues of DNGs

In [34] it was proved that all eigenvalues of any threshold graph other than 0, 1 are main. However the analogous statement in the case of DNGs is not true. Still, there are some constraints. As we have seen in Section 4 for DNGs 0 is always non-main eigenvalue with multiplicity $\nu - 2h$. The spectrum of any DNG is symmetric with respect to 0 and, therefore, the characteristic polynomial has the form $\phi_G(\lambda) = \lambda^{\nu-2h} \varphi_G(\lambda) \varphi_G(-\lambda)$, for some real polynomial φ_G . So, if some real number μ is in the spectra of G , then $-\mu$ is also an eigenvalue. Moreover the following holds:

Proposition 5.2. *Let G be a DNG and let $\mu, -\mu \in Sp(G) \setminus \{0\}$, then μ and $-\mu$ cannot be both non-main eigenvalues.*

Proof. Assume on the contrary that μ and $-\mu$ are both non-main. If $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ is the eigenvector of G corresponding to μ (here \mathbf{y} and \mathbf{z} denote the subvectors of \mathbf{x} with entries corresponding to $\cup_{i=1}^h U_i$ and $\cup_{i=1}^h V_i$, respectively), then $\bar{\mathbf{x}} = (\mathbf{y}, -\mathbf{z})$ is the eigenvector corresponding to $-\mu$. Additionally, by a_i (resp., b_j) we denote the entries of \mathbf{x} corresponding to the vertices in U_i (resp., V_j). From the eigenvalue equations we have

$$(5.3) \quad \mu a_i = n_1 b_1 + n_2 b_2 + \cdots + n_{h+1-i} b_{h+1-i},$$

$$(5.4) \quad \mu b_j = m_1 a_1 + m_2 a_2 + \cdots + m_{h+1-j} a_{h+1-j}.$$

The conditions $\mathbf{x}\mathbf{j}^T = 0$ and $\bar{\mathbf{x}}\mathbf{j}^T = 0$ together with (5.3)-(5.4) applied to μ and $-\mu$ imply $m_1 a_1 + m_2 a_2 + \cdots + m_h a_h = 0$ and $n_1 b_1 + n_2 b_2 + \cdots + n_h b_h = 0$ and therefore $\mu a_1 = 0, \mu b_1 = 0$ (for $i = j = 1$). Since $\mu \neq 0$ we obtain $a_1 = b_1 = 0$. Now, by (5.3)-(5.4), for $i = j = h$, we obtain $a_h = b_h = 0$. By a similar reasoning, we obtain $a_2 = b_2 = a_{h-1} = b_{h-1} = 0$, etc. In conclusion $\mathbf{x} = 0$, which is a contradiction. \square

We next show, by examples, that the remaining two possibilities can occur. For this aim we take that μ is the largest eigenvalue of G (so it is main). Then $-\mu$ is non-main if $G = DNG(1, 1; 1, 1)$, while main if $G = DNG(1, 1; 1, 2)$, as required.

Yet, another thing deserves to be mentioned in this context. Recall first, that two distinct eigenvalues of a graph are algebraic conjugate if and only if they share the same minimal polynomial over rationals. In [13, p. 188] it was proved that if μ_i and μ_j are algebraic conjugate eigenvalues of a graph G then μ_i is a main eigenvalue of a graph G if and only if μ_j is a main eigenvalue of G . If it happens that μ and $-\mu$ are algebraic conjugate and annihilate the polynomial $p(\lambda) = \lambda^2 - c$ for some rational number c then they have to be both main. This, for example, happens when $G = P_3 = DNG(2; 1)$. For this graph $\sqrt{2}$ and $-\sqrt{2}$ are both main eigenvalues.

6. CONCLUDING REMARKS

In this paper we have discussed several properties on the spectra of double nested graphs. Most of them came as a generalization of the analogous properties of the spectra of nested split graphs. It turned out that many of them hold in both cases although there are some exceptions as for the main/non-main eigenvalues. The paper can be also seen as a bridge between theory on the spectra of nonnegative matrices and graphs. The new bounds on the index of DNGs are obtained in quite elegant way. The proofs are significantly shorter and more self-contained. We believe that the application of this technique can provide a new insight in other classes of graphs as well.

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REFERENCES

- [1] M. Anđelić, C.M. da Fonseca, S.K. Simić, D.V. Tošić, On bounds on the index of a double nested graphs, *Linear Algebra Appl.* 435 (2011), 2475-2490.
- [2] F.K. Bell, D. Cvetković, P. Rowlinson, S.K. Simić, Graphs for which the least eigenvalue is minimal, I, *Linear Algebra Appl.* 429(2008), 234-241.

- [3] F.K. Bell, D. Cvetković, P. Rowlinson, S.K. Simić, Graphs for which the least eigenvalue is minimal, II, *Linear Algebra Appl.* 429(2008), 2168-2179.
- [4] A. Bhattacharya, S. Friedland, U.N. Peled, On the first eigenvalue of bipartite graphs, *Electron. J. Combin.* 15 (2008), no. 1, #144.
- [5] A. Brauer, The theorems of Ledermann and Ostrowski on positive matrices, *Duke Math. J.* 24(1957), 265-274.
- [6] A. Brauer, I.C. Gentry, Bounds for the greatest characteristic root of an irreducible nonnegative matrix. II, *Linear Algebra and Appl.* 13 (1976), 109-114.
- [7] A. Brauer, I.C. Gentry, Bounds for the greatest characteristic root of an irreducible nonnegative matrix, *Linear Algebra and Appl.* 8 (1974), 105-107.
- [8] R.A. Brualdi, A.J. Hoffman, On the spectral radius of $(0, 1)$ -matrices, *Linear Algebra Appl.* 65(1985), 133-146.
- [9] R.A. Brualdi, E.S. Solheid, On the spectral radius of connected graphs, *Publ. Inst. Math. (Beograd)*, 39(1986), 45-54.
- [10] Y-F. Chen, H-L. Fu, I-J. Kim, E. Stehr, B. Watts, On the largest eigenvalues of bipartite graphs which are nearly complete, *Linear Algebra Appl.* 432(2010), 606614.
- [11] V. Chvátal, P.L. Hammer, Aggregation of inequalities in integer programming, *Ann. of Discrete Math.* 1(1977), 145-162.
- [12] S. Cioabă, A necessary and sufficient eigenvector condition for a connected graph to be bipartite, *Electronic J. Linear Algebra*, 20(2010), 351-353.
- [13] D. Cvetković, P. Rowlinson, S. Simić, *Eigenspaces of graphs*, Cambridge University Press, New York, 1997.
- [14] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge University Press, New York, 2010.
- [15] G. Frobenius, Über Matrizen aus nicht negativen Elementen, *S.-B. Deutsch. Akad. Wiss. Berlin*, v. 1912, pp. 456-477.
- [16] G.H. Golub, Bounds for eigenvalues of tridiagonal symmetric matrices computed by the LR method, *Math. Comp.* 16(1962) 438-445.
- [17] P. Heggernes, D. Kratsch, Linear-time certifying recognition algorithms and forbidden induced subgraphs, *Nordic J. Comput.*, Vol. 14(1-2)(2007), 87-108.
- [18] P. Henrici, The quotient-difference algorithm, *Nat. Bur. Standards Appl. Math. Ser.* 49(1958), 23-46.
- [19] D. Jacobs, V. Trevisan, F. Tura, Locating the eigenvalues of threshold graphs, *Linear Algebra Appl.* 439 (2013), 2762-2773.
- [20] L.Yu. Kolotilina, Upper bound for the second largest eigenvalue of symmetric nonnegative matrices, *J. Math. Sci. (N. Y.)* 191 (2013), no. 1, 75-88.
- [21] L.Yu. Kolotilina, On the sharpness of two-sided bounds for the Perron root and of the related eigenvalue inclusions sets, *Linear Algebra Appl.* 429(2008), 2521-2539.
- [22] L.Yu. Kolotilina, Bounds and inequalities for the Perron root of a nonnegative matrix, *J. Math. Sci. (N. Y.)* 121(2004), 2481-2507.
- [23] W. Ledermann, Bounds for the greatest latent roots of a positive matrix, *J. London Math. Soc.* 25 (1950) 265-268.
- [24] S. Liu, Bounds for the greatest characteristic root of a nonnegative matrix, *Linear Algebra Appl.* 239(1996), 151-160.
- [25] N.V.R. Mahadev, U.N. Peled, *Threshold graphs and related topics*, Annals of Discrete Mathematics, 56, North-Holland Publishing Co., Amsterdam, 1995.
- [26] J.K. Merikoski, R. Kumar, AND R. A. Rajput, Upper bounds for the largest eigenvalue of a bipartite graph, *Electronic J. Linear Algebra*, 26(2013), 168-176.
- [27] H. Minc, *Nonnegative Matrices*, John Wiley & Sons, New York, 1988.
- [28] M. Marcus, H. Minc, *A Survey of Matrix Theory and Matrix Inequalities*, Dover Publications, Inc., New York, 1992.
- [29] A. Ostrowski, Bounds for the greatest latent root of a positive matrix, *J. London Math. Soc.* 27(1952), 253-256.
- [30] A. Ostrowski, H. Schneider, Bounds for the maximal characteristic root of a nonnegative irreducible matrix, *Duke Math. J.* 27(1960), 547-553.
- [31] D. Powers, Graph partitioning by eigenvectors, *Linear Algebra Appl.* 117 (1988), 121-133.
- [32] H. Rutishauser, Solution of eigenvalue problems with the LR-transformation, *Nat. Bur. Standards Appl. Math.* 49(1958), 47-81.

- [33] H. Rutishauser, Über eine kubisch konvergente Variante der LR-Transformation, Z. Angew. Math. Mech. 40(1958), 49-54.
- [34] I. Sciriha, S. Farrugia, On the spectrum of treshold graphs, ISRN Discrete Mathematics, vol. 2011, Article ID 108509, 21 pages, 2011.
- [35] M. Yannakakis. Node deletion problems on bipartite graphs. SIAM J. Comput. 10(1981), 310-327.
- [36] H. Wolkowitz, G. P. H. Styan, Bounds for eigenvalues using traces, Linear Algebra Appl., 29 (1980), 471-506.

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