Chapter 11
Burst Erasure Correction of 2D convolutional codes*

Joan-Josep Climent, Diego Napp, Raquel Pinto, and Rita Simões

Abstract In this paper we address the problem of decoding 2D convolutional codes over the erasure channel. In particular, we present a procedure to recover bursts of erasures that are distributed in a diagonal line. To this end we introduce the notion of balls around a burst of erasures which can be considered an analogue of the notion of sliding window in the context of 1D convolutional codes. The main result reduces the decoding problem of 2D convolutional codes to a problem of decoding a set of associated 1D convolutional codes.

Key words 2D convolutional codes, erasure channel

Contents

Joan-Josep Climent
Departament d'Estadística i Investigació Operativa, Universitat d'Alacant, Ap. Correus 99, E-03080, Alacant, Spain
e-mail: jcliment@ua.es

Diego Napp · Raquel Pinto · Rita Simões
CIDMA – Center for Research and Development in Mathematics and Applications, Department of Mathematics, University of Aveiro, Campus Universitario de Santiago, 3810-193 Aveiro, Portugal
e-mail: {diego,raquel,ritasimoes}@ua.pt

* This work was partially supported by Spanish grant MTM2011-24858 of the Ministerio de Ciencia e Innovación of the Gobierno de España. The work of D. Napp, R. Pinto, and R. Simões was partially supported by Portuguese funds through the CIDMA - Center for Research and Development in Mathematics and Applications, and the Portuguese Foundation for Science and Technology ("FCT-Fundação para a Ciência e a Tecnologia"), within project PEst-UID/MAT/04106/2013.

1 Introduction

When transmitting over an erasure channel the symbol sent either arrive correctly or they are erased. Internet is an important instance of such a channel. One of the problems that arises in this channel is that some packets get lost and the receiver experience it as a delay on the received information. The solutions proposed to deal with this problem are commonly based on the use of block codes. However, in recent years, there has been an increased interest in the study of one-dimensional (1D) convolutional codes over the erasure channel [2, 8–10] as a possible alternative for the widely use of block codes. Due to their rich structure 1D convolutional codes have an interesting property called sliding window property that allows adaptation to the correction process to the distribution of the erasure pattern. In the recent paper [10] it has been shown how it is possible to exploit this property in order to easily recover erasures which are uncorrectable by any other kind of (block) codes. The codes proposed in this paper are codes with strong distance properties, called Maximal Distance Profile (MDP), reverse-MDP and complete-MDP, and simulations results have shown that they can decode extremely efficiently when compared to MDS block codes.

In the 1D case, if the received codeword is viewed as a finite sequence $v = (v_0, v_1, \ldots, v_\ell)$, then the sliding windows is given by selecting a subsequence of v, $(v_i, \ldots, v_{i+N})$, where $i, N \in \mathbb{N}$ depend on the erasure burst pattern. However, when considering two-dimensional (2D) convolutional codes [4–7, 11] the information is distributed in two dimensions and therefore there is not an obvious way to extend the idea of sliding window to the 2D case. In this work we propose several solutions for dealing with this problem by introducing the notion of *balls* around an erasure. We show that when considering these particular balls one reduces the problem of decoding 2D convolutional codes over the erasure channel to a problem related to decoding of 1D convolutional codes.

2 2D convolutional codes

In this section we recall the basic background on 2D finite support convolutional codes. Denote by $\mathbb{F}[z_1,z_2]$ the ring of polynomials in the two variables, z_1 and z_2, with coefficients in the finite field $\mathbb{F}$.

Definition 1. A 2D finite support convolutional code $\mathscr{C}$ of rate k/n is a free $\mathbb{F}[z_1,z_2]$-submodule of $\mathbb{F}[z_1,z_2]^n$ with rank k.

A full column rank polynomial matrix $\widehat{G}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ whose columns constitute a basis for $\mathscr{C}$, i.e., such that

$$\begin{aligned}\mathscr{C} &= \operatorname{im}_{\mathbb{F}[z_1,z_2]} \widehat{G}(z_1,z_2)\\ &= \left\{ \widehat{v}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^n \mid \widehat{v}(z_1,z_2) = \widehat{G}(z_1,z_2)\widehat{u}(z_1,z_2) \text{ with } \widehat{u}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^k \right\},\end{aligned}$$

is called an *encoder* of $\mathscr{C}$. The elements of $\mathscr{C}$ are called *codewords*.

If the code $\mathscr{C}$ admits a right factor prime encoder [3], then it can be equivalently described using an $(n-k)\times n$ full rank polynomial matrix $\widehat{H}(z_1,z_2)$, called *parity-check matrix* of $\mathscr{C}$, as

$$\mathscr{C} = \ker_{\mathbb{F}[z_1,z_2]} \widehat{H}(z_1,z_2) = \left\{ \widehat{v}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^n \mid \widehat{H}(z_1,z_2)\widehat{v}(z_1,z_2) = 0 \right\}.$$

We denote by $\mathbb{N}_0$ the set of nonnegative integers, and define an ordering in $\mathbb{N}_0^2$ as

$$(a,b) \prec (c,d) \text{ if and only if } a+b < c+d, \text{ or } a+b = c+d \text{ and } b < d. \quad (1)$$

For a polynomial vector $\widehat{v}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^n$, we write

$$\widehat{v}(z_1,z_2) = v(0,0) + v(1,0)z_1 + v(0,1)z_2 + \cdots + v(0,\gamma)z_2^{\gamma} = \sum_{0\le a+b\le\gamma} v(a,b) z_1^a z_2^b,$$

(with $\gamma \geq 0$) and we define its support as the set

$$\operatorname{supp}(\widehat{v}(z_1,z_2)) = \{(a,b) \in \mathbb{N}_0^2 \mid v(a,b) \neq 0\}.$$

Moreover, we represent a polynomial matrix $\widehat{H}(z_1,z_2)$ as

$$\widehat{H}(z_1,z_2) = H(0,0) + H(1,0)z_1 + H(0,1)z_2 + \cdots + H(0,\delta)z_2^{\delta} = \sum_{0\le i+j\le\delta} H(i,j) z_1^i z_2^j, \quad (2)$$

where $H(i,j) \neq 0$ for some (i,j) with $i+j=\delta$. We call δ the degree of $\widehat{H}(z_1,z_2)$.

The *weight* of $\widehat{v}(z_1,z_2)$ is defined as

$$\operatorname{wt}(\widehat{v}(z_1,z_2)) = \sum_{(a,b)\in\mathbb{N}_0^2} \operatorname{wt}(v(a,b))$$

where $\operatorname{wt}(v(a,b))$ is the number of nonzero entries of $v(a,b)$ and the *distance* of a code is

$$\operatorname{dist}(\mathscr{C}) = \min\{\operatorname{wt}(\widehat{v}(z_1,z_2)) \mid \widehat{v}(z_1,z_2) \in \mathscr{C}, \text{ with } \widehat{v}(z_1,z_2) \neq 0\}.$$

We can expand the kernel representation

$$\widehat{H}(z_1,z_2)\widehat{v}(z_1,z_2) = \sum_{0\le a+b\le\gamma} \left[\sum_{0\le i+j\le\delta} H(i,j) v(a-i,b-j) \right] z_1^a z_2^b = 0$$

as

$$\mathbf{H}\mathbf{v} = 0 \quad (3)$$

where $\mathbf{H}$, for $\delta = 3$, and $\mathbf{v}$ are given in Figure 1, where O denotes the $(n-k)\times n$ zero matrix. To understand the structure of matrix $\mathbf{H}$, note that for $t = 0,1,2,\ldots$ in the columns corresponding to the block indices $\frac{t(t+1)}{2}+1, \frac{t(t+1)}{2}+2, \ldots, \frac{t(t+1)}{2}+t+1$ appear all the coefficient matrices of $\widehat{H}(z_1,z_2)$ ordered according to $\prec$ with the

$$
\begin{array}{ccccccccccc}
H(0,0) & & & & & & & & & & \\
H(1,0) & H(0,0) & & & & & & & & & \\
H(0,1) & O & H(0,0) & & & & & & & & \\
H(2,0) & H(1,0) & O & H(0,0) & & & & & & & \\
H(1,1) & H(0,1) & H(1,0) & O & H(0,0) & & & & & & \\
H(0,2) & O & H(0,1) & O & O & H(0,0) & & & & & \\
H(3,0) & H(2,0) & O & H(1,0) & O & O & H(0,0) & & & & \\
H(2,1) & H(1,1) & H(2,0) & H(0,1) & H(1,0) & O & O & H(0,0) & & & \\
H(1,2) & H(0,2) & H(1,1) & O & H(0,1) & H(1,0) & O & O & H(0,0) & & \\
H(0,3) & O & H(0,2) & O & O & H(0,1) & O & O & O & H(0,0) & \\
 & H(3,0) & O & H(2,0) & O & O & H(1,0) & O & O & O & \ddots \\
 & H(2,1) & H(3,0) & H(1,1) & H(2,0) & O & H(0,1) & H(1,0) & O & O & \ddots \\
 & H(1,2) & H(2,1) & H(0,2) & H(1,1) & H(2,0) & O & H(0,1) & H(1,0) & O & \ddots \\
 & H(0,3) & H(1,2) & O & H(0,2) & H(1,1) & O & O & H(0,1) & H(1,0) & \ddots \\
 & & H(0,3) & O & O & H(0,2) & O & O & O & H(0,1) & \ddots \\
 & & & H(3,0) & O & O & H(2,0) & O & O & O & \ddots \\
 & & & H(2,1) & H(3,0) & O & H(1,1) & H(2,0) & O & O & \ddots \\
 & & & H(1,2) & H(2,1) & H(3,0) & H(0,2) & H(1,1) & H(2,0) & O & \ddots \\
 & & & H(0,3) & H(1,2) & H(2,1) & O & H(0,2) & H(1,1) & H(2,0) & \ddots \\
 & & & & H(0,3) & H(1,2) & O & O & H(0,2) & H(1,1) & \ddots \\
 & & & & & H(0,3) & O & O & O & H(0,2) & \ddots \\
 & & & & & & H(3,0) & O & O & O & \ddots \\
 & & & & & & H(2,1) & H(3,0) & O & O & \ddots \\
 & & & & & & H(1,2) & H(2,1) & H(3,0) & O & \ddots \\
 & & & & & & H(0,3) & H(1,2) & H(2,1) & H(3,0) & \ddots \\
 & & & & & & & H(0,3) & H(1,2) & H(2,1) & \ddots \\
 & & & & & & & & H(0,3) & H(1,2) & \ddots \\
 & & & & & & & & & H(0,3) & \ddots \\
 & & & & & & & & & & \ddots
\end{array}
$$

(a) Matrix $\mathbf{H}$

$$
\begin{array}{c}
v(0,0) \\ v(1,0) \\ v(0,1) \\ v(2,0) \\ v(1,1) \\ v(0,2) \\ v(3,0) \\ v(2,1) \\ v(1,2) \\ v(0,3) \\ \vdots
\end{array}
$$

(b) Vector $\mathbf{v}$

Fig. 1 Parity check matrix $\mathbf{H}$, for $\delta = 3$, and code word $\mathbf{v}$

particularity that the matrices $H(i,j)$, with $i+j=d$, for $d=0,1,2,\ldots,\delta-1$, are separated from the matrices $H(i,j)$, with $i+j=d+1$, by t zero blocks.

Suppose now that the vector $\widehat{v}(z_1,z_2)$ is transmitted through an erasure channel. Each one of the components of $\mathbf{v}$ is either received correctly or is considered erasure. Denote by $\mathscr{E}(\widehat{v}(z_1,z_2))$ and $\bar{\mathscr{E}}(\widehat{v}(z_1,z_2))$ the sets of indices in which there are erasures and there are not erasures, respectively, i.e.,

$$
\begin{aligned}
&\mathscr{E}(\widehat{v}(z_1,z_2)) = \left\{(a,b) \in \operatorname{supp}(\widehat{v}(z_1,z_2)) \mid \text{there is an erasure in } v(a,b)\right\}, \\
&\bar{\mathscr{E}}(\widehat{v}(z_1,z_2)) = \operatorname{supp}(\widehat{v}(z_1,z_2)) \setminus \mathscr{E}(\widehat{v}(z_1,z_2)).
\end{aligned}
$$

One can select the columns of the matrix in (3) that correspond to the coefficient of the erased elements to be the indeterminates of a new system. The rest of the columns in (3) will help us to compute the independent terms. The terms erasure and indeterminate are often used interchangeably. Hence, we denote by $\mathbf{H}_{\mathscr{E}}$ and $\mathbf{H}_{\bar{\mathscr{E}}}$ the submatrices of $\mathbf{H}$ whose block columns are indexed by $\mathscr{E}$ and $\bar{\mathscr{E}}$, respectively. Analogously, we denote $\mathbf{v}_{\mathscr{E}}$ and $\mathbf{v}_{\bar{\mathscr{E}}}$ to obtain $\mathbf{H}_{\mathscr{E}}\mathbf{v}_{\mathscr{E}} + \mathbf{H}_{\bar{\mathscr{E}}}\mathbf{v}_{\bar{\mathscr{E}}} = 0$. Note that as the channel is an erasure channel, $\mathbf{v}_{\bar{\mathscr{E}}}$, and therefore $\mathbf{H}_{\bar{\mathscr{E}}}\mathbf{v}_{\bar{\mathscr{E}}}$, is known. Hence, we obtain a system of linear nonhomogeneous equations

$$
\mathbf{H}_{\mathscr{E}}\mathbf{v}_{\mathscr{E}} = -\mathbf{H}_{\bar{\mathscr{E}}}\mathbf{v}_{\bar{\mathscr{E}}}, \tag{4}
$$

were the components of the vector $\mathbf{v}_{\mathscr{E}}$ that are considered to the indeterminates to be determined. Thus, in order to decode $\mathbf{v}_{\mathscr{E}}$ we need to solve system (4).

The next lemma shows the importance of the distance of a code when transmitting over the erasure channel.

Lemma 2. *Let* $\mathscr{C} = \ker_{\mathbb{F}[z_1,z_2]} \widehat{H}(z_1,z_2)$ *be given. The following are equivalent:*

1. $\mathrm{dist}(\mathscr{C}) \geq d$.
2. *Any* $d-1$ *erasures can be recovered.*
3. *Any* $d-1$ *columns of* $\mathbf{H}_{\mathscr{E}}$ *are linearly independent.*

In the context of 1D convolutional codes the analogous set of homogeneous equations of (3) is

$$\begin{bmatrix} H_0 & & & & \\ \vdots & \ddots & & & \\ H_\alpha & \cdots & H_0 & & \\ & \ddots & \vdots & \ddots & \\ & & H_\alpha & \cdots & H_0 \\ & & & \ddots & \vdots \\ & & & & H_\alpha \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_\gamma \end{bmatrix} = 0, \tag{5}$$

where $\mathscr{C} = \ker \widehat{H}(z)$ with $\widehat{H}(z) = H_0 + H_1 z + \cdots + H_\alpha z^\alpha$.

In this case every component of the received codeword $v = (v_0, v_2, \ldots, v_\gamma)$ depends on the previous α components. In order to find the values of a burst of erasures occurring in v, we can use the so-called *sliding window*, that is, we can select a suitable interval of consecutive components of v, say $(v_i, \ldots, v_{i+N})$, and solve the corresponding system of equations (see [10]).

In the 2D case each component of $\mathbf{v}$, say $v(a,b)$, depends on components which support lie in the *triangle* $\{(a-i, b-j) \mid 0 \leq i+j \leq \delta\}$, where δ is the degree of $\widehat{H}(z_1,z_2)$ of the given 2D code $\mathscr{C} = \ker_{\mathbb{F}[z_1,z_2]} \widehat{H}(z_1,z_2)$. It is not straightforward to extend the notion of the sliding window in this context in order to correct burst of 2D erasures. A particular case is treated in the following section.

3 Decoding burst of erasures on lines

It is well-known that a phenomena observed in many channels modeled via the erasure channel is that errors tend to occur in bursts. This point is important to keep in mind when designing codes which are capable of correcting many errors over the erasure channel. In this preliminary work we aim at decoding burst of erasures that are distributed in a diagonal. We present a notion that can be considered as the analogue of the notion of sliding window, called *ball around a burst of erasures*, that will reduce the problem of decoding a 2D convolutional code to the problem of decoding a set of associated 1D convolutional codes.

Let us first suppose that the set of erasures of $\widehat{v}(z_1,z_2)$ contains a burst of erasures which support lie in a diagonal, i.e., given by

$$\mathscr{E}'(\widehat{v}(z_1,z_2)) = \{(r+t,s),(r+t-1,s+1),\ldots,(r,s+t)\} \subset \mathscr{E}(\widehat{v}(z_1,z_2)). \tag{6}$$

Hence, equation (3) can be divided as

$$\mathbf{H}_{\mathscr{E}'}\mathbf{v}_{\mathscr{E}'} = -\mathbf{H}_{\bar{\mathscr{E}}'}\mathbf{v}_{\bar{\mathscr{E}}'} \tag{7}$$

where $\mathbf{H}_{\mathscr{E}'}$ and $\mathbf{H}_{\bar{\mathscr{E}}'}$ are submatrices of $\mathbf{H}$ whose block columns are indexed by $\mathscr{E}'(\widehat{v}(z_1,z_2))$ and $\bar{\mathscr{E}}'(\widehat{v}(z_1,z_2)) = \operatorname{supp}(\widehat{v}(z_1,z_2)) \setminus \mathscr{E}'(\widehat{v}(z_1,z_2))$, respectively, and $\mathbf{v}_{\mathscr{E}'}$ and $\mathbf{v}_{\bar{\mathscr{E}}'}$ are defined accordingly. If no confusion arises we use $\mathscr{E}$ and $\mathscr{E}'$ for $\mathscr{E}(\widehat{v}(z_1,z_2))$ and $\mathscr{E}'(\widehat{v}(z_1,z_2))$, respectively.

Definition 3. Let $\mathscr{E}'$ be given with $(r^f,s^f) = (r+t,s)$ and $(r^\ell,s^\ell) = (r,s+t)$ being the first and last position (ordered by $\prec$) in this set. We define $\delta+1$ different *balls* around $\mathscr{E}'$ as

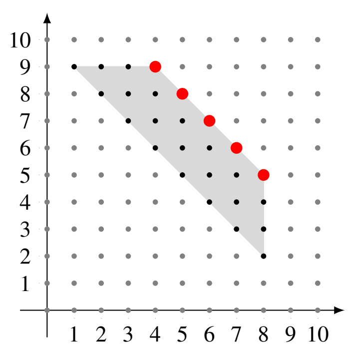
$$\Omega_{j,\delta}(\mathscr{E}') = \left\{(a,b) \mid a \le r^f + j,\ b \le s^\ell + j,\ r^f + s^f + j - \delta \le a+b \le r^f + s^f + j\right\}$$

for $j = 0,1,2,\ldots,\delta$.

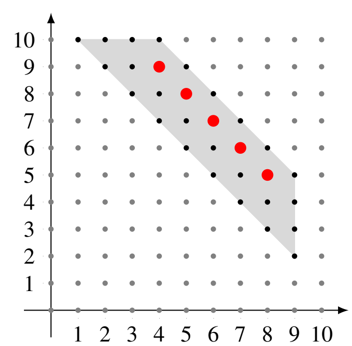
Example 4. Consider the burst of erasures given by

$$\mathscr{E}' = \{(8,5),(7,6),(6,7),(5,8),(4,9)\},$$

then, $(r^f,s^f) = (8,5)$ and $(r^\ell,s^\ell) = (4,9)$ and for $\delta = 3$. Figure 2 shows the set $\Omega_{0,3}(\mathscr{E}')$ and $\Omega_{1,3}(\mathscr{E}')$.



(a) $\Omega_{0,3}(\mathscr{E}')$



(b) $\Omega_{1,3}(\mathscr{E}')$

Fig. 2 Balls around the erasure given by $\mathscr{E}' = \{(8,5),(7,6),(6,7),(5,8),(4,9)\}$

By definition, the vector $\mathbf{v}_{\mathscr{E}'}$ contains a burst of erasures in a diagonal and $\mathbf{v}_{\bar{\mathscr{E}}'}$ may contain erasures as well. Depending on the structure of $\mathbf{H}_{\mathscr{E}'}$ and $\mathbf{H}_{\bar{\mathscr{E}}'}$ these errors may appear together in some of the equations of (7).

The following result gives a criterion to determine *some* sets of equations that involve *only* erasures in $\mathbf{v}_{\mathscr{E}'}$. The solution of such system would produce the desired decoding of $\mathbf{v}_{\mathscr{E}'}$.

Theorem 5. *Let $\mathscr{C} = \ker_{\mathbb{F}[z_1,z_2]} \widehat{H}(z_1,z_2)$, δ the degree of $\widehat{H}(z_1,z_2)$ and let $\mathscr{E}$ be the support of the erasures and $\mathscr{E}'$ be the support of a burst of erasures distributed on a diagonal line of a codeword $\widehat{v}(z_1,z_2)$. If $\mathscr{E}'$ are the only erasures in $\Omega_{j,\delta}(\mathscr{E}')$, i.e., if*

$$\mathscr{E} \cap \Omega_{j,\delta}(\mathscr{E}') = \mathscr{E}',$$

then, there exists a subsystem of (7) such that

$$\mathbf{H}^j_{\mathscr{E}'} \mathbf{v}_{\mathscr{E}'} = a_j \tag{8}$$

where a_j is a subvector of $\mathbf{H}_{\bar{\mathscr{E}}'} \mathbf{v}_{\bar{\mathscr{E}}'}$ that does not contain any erasures, and

$$\mathbf{H}^j_{\mathscr{E}'} = \begin{bmatrix} H(j,0) & & & & & \\ H(j-1,1) & H(j,0) & & & & \\ H(j-2,2) & H(j-1,1) & H(j,0) & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ H(0,j) & H(1,j-1) & H(2,j-2) & & & \\ & H(0,j) & H(1,j-1) & \cdots & H(j,0) & \\ & & \ddots & \ddots & & \ddots \\ & & & & H(0,j) & H(1,j-1) \\ & & & & & H(0,j) \end{bmatrix}, \text{ for } j = 0,1,\ldots,\delta$$

is a $(n-k)(t+1) \times n(t+1)$ submatrix of $\mathbf{H}_{\mathscr{E}'}$.

The structure of the matrices $\mathbf{H}^j_{\mathscr{E}'}$ have the same structure as the matrices in (5) which appear in the decoding problem of 1D convolutional codes, see [1, 8] for more details, and therefore the solution of (8) is analogous to the decoding problem of 1D convolutional codes.

It was shown in [10] that there exist a type of 1D convolutional codes, called (reverse or complete) MDP, that perform particularly well over the erasure channel. This together with Theorem 5 suggest that in order to construct a 2D convolutional code $\mathscr{C} = \ker_{\mathbb{F}[z_1,z_2]} \widehat{H}(z_1,z_2)$ with good decoding properties one can construct a parity-check matrix $\widehat{H}(z_1,z_2) = \sum_{0 \le a+b \le \delta} H(a,b) z_1^a z_2^b$ such that the associated 1D convolutional codes are given by $\mathscr{C}^{(j)} = \ker_{\mathbb{F}[z]} \widehat{H}^{(j)}(z)$ with $\widehat{H}^{(j)}(z) = H_0^{(j)} + H_1^{(j)} z + \cdots + H_\nu^{(j)} z^\nu$ and $H_k^{(j)} = H(j-k,k)$, for $k = 0,1,\ldots,j$ are MDP.

4 Conclusions

In this paper we have proposed a method to recover erasures $\mathscr{E}'$ in a 2D (finite support) convolutional code that are distributed in a diagonal line in the 2D plane.

We have shown that if $\mathscr{E}'$ does not have more erasures *close* (meaning in a ball centered around $\mathscr{E}'$) then it is possible to consider $\mathscr{E}'$ as a burst of erasures of a set of 1D convolutional codes. Decoding these 1D convolutional codes would immediately imply the recovery of the $\mathscr{E}'$.

This procedure is far from solving all the possible erasure patterns but it represents the first step toward the development of an effective approach to solve more general patterns of erasures.

References

1. Almeida, P., Napp, D., Pinto, R.: A new class of superregular matrices and MDP convolutional codes. Linear Algebra and its Applications **439**, 2145–2157 (2013). doi:10.1016/j.laa.2013.06.013
2. Arai, M., Yamamoto, A., Yamaguchi, A., Fukumoto, S., Iwasaki, K.: Analysis of using convolutional codes to recover packet losses over burst erasure channels. In: Proceedings of the 2001 Pacific Rim International Symposium on Dependable Computing, pp. 258–265. IEEE, Seoul, Korea (2001). doi:10.1109/PRDC.2001.992706
3. Charoenlarpnopparut, C.: Applications of Gröbner bases to the structural description and realization of multidimensional convolutional code. Science Asia **35**, 95–105 (2009). doi:10.2306/scienceasia1513-1874.2009.35.095
4. Climent, J.J., Napp, D., Perea, C., Pinto, R.: A construction of MDS 2D convolutional codes of rate $1/n$ based on superregular matrices. Linear Algebra and its Applications **437**, 766–780 (2012). doi:10.1016/j.laa.2012.02.032
5. Fornasini, E., Valcher, M.E.: Algebraic aspects of two-dimensional convolutional codes. IEEE Transactions on Information Theory **40**(4), 1068–1082 (1994). doi:10.1109/18.335967
6. Jangisarakul, P., Charoenlarpnopparut, C.: Algebraic decoder of multidimensional convolutional code: constructive algorithms for determining syndrome decoder and decoder matrix based on Gröbner basis. Multidimensional Systems and Signal Processing **22**(1–3), 67–81 (2011)
7. Napp, D., Perea, C., Pinto, R.: Input-state-output representations and constructions of finite support 2D convolutional codes. Advances in Mathematics of Communications **4**(4), 533–545 (2010). doi:10.3934/amc.2010.4.533
8. Tomás, V.: Complete-mdp convolutional codes over the erasure channel. Ph.D. thesis, Departamento de Ciencia de la Computación e Inteligencia Artificial, Universidad de Alicante, Alicante, España (2010)
9. Tomás, V., Rosenthal, J., Smarandache, R.: Reverse-maximum distance profile convolutional codes over the erasure channel. In: A. Edelmayer (ed.) Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems (MTNS2010), pp. 2121–2127 (2010)
10. Tomás, V., Rosenthal, J., Smarandache, R.: Decoding of convolutional codes over the erasure channel. IEEE Transactions on Information Theory **58**(1), 90–108 (2012). doi:10.1109/TIT.2011.2171530
11. Weiner, P.A.: Multidimensional convolutional codes. Ph.D. thesis, Department of Mathematics, University of Notre Dame, Indiana, USA (1998)