Whittaker transform on distributions*

M.M. Rodrigues[‡] and N. Vieira[‡]

[‡] CIDMA - Center for Research and Development in Mathematics and Applications
Department of Mathematics, University of Aveiro
Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.
E-mail: mrodrigues@ua.pt, nloureirovieira@gmail.com

Abstract

The aim of this paper is to construct a testing function space equipped with the topology generated by the $L_{\nu,p}$-multinorm of the differential operator

$$B_x = -4x^2 \frac{d^2}{dx^2} - 1 + x^2 - \mu x,$$

where $\mu < \frac{1}{2}$, $\nu > 0$, $p \in [1, \infty[$, and its $k-$iterates B_x^k where $k = 0, 1, \ldots$, and $B_x^0 \phi = \phi$. We also introduce the correspondent dual space for the index Whittaker transform on distributions. The existence, uniqueness, imbedding and inversion properties are investigated.

Keywords: Testing-function spaces; Distributions; Index Whittaker transform; Whittaker functions; Special functions.

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1 Introduction

The Whittaker functions (which are solution of the so called Whittaker equation - see [3], Vol.1) have acquired an ever increasing significance due to their frequent use in applications of mathematics to physical and technical problems [1]. Moreover, they are closely related to the confluent hypergeometric functions which play an important role in various branches of applied mathematics and theoretical physics, for instance, fluid mechanics scalar and electromagnetic diffraction theory, atomic structure theory, input-output situations, storage-consumption situations in economic problems, and so on. This justifies the continuous effort in studying properties of these functions and in gathering information about them.

The aim of this paper is to define a testing-function space associated to the index Whittaker transform, which generalizes some known testing-function spaces (see [2, 4]). Furthermore, we will show that this space can be used to study the index Whittaker transform for distributions.

This paper formally follows the ideas presented in [9], and it is organized as follows: In Section 2, we recall some preliminaries results about index Whittaker transform and special functions. Section 3 is devoted to the definition of the testing-function space associated to our index transform. Finallly, we define the correspondent of the index Whittaker transform for distributions, then we study its existence, uniqueness and inversion properties on a manner to be found in [10].

2 Preliminaries

In this paper, we will consider the index Whittaker transform (see [6])

$$\mathcal{W}_\mu[f](x) = \int_{\mathbb{R}_+} \tau \left| \Gamma\left(\frac{1}{2} - \mu + i\tau\right)\right|^2 W_{\mu,i\tau}(x)\; f(\tau)\; d\tau, \tag{1}$$

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where $W_{\mu,i\tau}(x)$ is the Whittaker function with $x,\tau \in \mathbb{R}_+$, and $\mu < \frac{1}{2}$. The convergence of (1) is guaranteed in the sense of the norm in $L_2(\mathbb{R}_+, x^{-2}\, dx)$. Precisely, the operator

$$\mathcal{W}_\mu[f] : L_2\left(\mathbb{R}_+, \frac{\tau\pi^2}{\sinh(2\pi\tau)}\left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2\, d\tau\right) \to L_2(\mathbb{R}_+, x^{-2}\, dx)$$

is an isomorphism between these Hilbert spaces.

Let $\nu > 0$ and $f \in L_{\nu,p}(\mathbb{R}_+)$ with

$$||f||_{\nu,p} = \left(\int_{\mathbb{R}_+} |f(x)|^p\, x^{\nu p-1}\, dx\right)^{\frac{1}{p}} < \infty. \tag{2}$$

It is shown in [6] that (1) is a bounded mapping from the space $L_r(\mathbb{R}_+)$ into $L_{\nu,p}(\mathbb{R}_+)$, where $\nu > -\mu$, $p, r \in [1,\infty[$, and the parameters p and r have no dependence. Moreover, its inversion can be written in terms of the singular integral

$$f(\tau) = \frac{\sinh(2\pi\tau)}{\pi^2}\int_{\mathbb{R}_+} x^{-2}\, W_{\mu,i\tau}(x)\, \mathcal{W}_\mu[f](x)\, dx, \qquad \tau \in \mathbb{R}_+, \tag{3}$$

where the convergence of (3) is understood with respect to the norm in $L_2\left(\mathbb{R}_+, \frac{\tau\pi^2}{\sinh(2\pi\tau)}\, \left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2\, d\tau\right)$. The transformation (1) was extended in [7] to a certain class of distributions.

The Whittaker function is an eigenfunction of a second order differential operator (see [1], [3]-Vol.I). More precisely,

$$B_x W_{\mu,i\tau}(x) = 4\tau^2 W_{\mu,i\tau}(x), \tag{4}$$

where

$$B_x \equiv 4x^2\frac{d^2}{dx^2} + 1 - x^2 + 4\mu x. \tag{5}$$

From (4) and (5), it follows that the Whittaker function satisfies the following differential equation

$$\frac{d^2 w(x)}{dx^2} + \left(-\frac{1}{4} + \frac{\mu}{x} + \frac{\frac{1}{4}-\tau^2}{x^2}\right) w(x) = 0 \tag{6}$$

which has a regular singularity at zero and an irregular singularity at infinity. Up to scalars, $x \mapsto W_{\mu,i\tau}(x)$ is the unique function that decays as $x \to \infty$ (exponentially). This special function is analytic in μ,τ and x. Moreover, we have the following estimate [6]

$$\left|\Gamma\left(\frac{1}{2}-\mu+i\tau\right)\right|^2 |W_{\mu.i\tau}(x)| \leq x^\mu\, e^{-\frac{x}{2}}\, \Gamma^2\left(\frac{1}{2}-\mu\right), \qquad x,\tau > 0,\ \ \mu < \frac{1}{2}. \tag{7}$$

3 Testing-function space $\mathcal{B}_{\nu,p}$ and its dual

Consider the class $\mathcal{B}_{\nu,p}$ of smooth complex-valued functions $\phi(x)$ on $\mathbb{R}_+$ for which

$$\beta_{k,\nu,p}(\phi) = \beta_{0,\nu,p}\left(B_x^k\phi\right) = \left(\int_{\mathbb{R}_+}\left|B_x^k\phi\right|^p\, x^{\nu p-1}\, dx\right)^{\frac{1}{p}} \tag{8}$$

is finite for every $k \in \mathbb{N}_0$, with $1 \leq p < \infty$ and $\nu > 0$. Initially, we prove that the class $\mathcal{B}_{\nu,p}$ is a testing function space which is associated with the multinorm (8). In fact, $\mathcal{B}_{\nu,p}$ is a linear space, each $\beta_{k,\nu,p}$ is a seminorm, and $\beta_{0,\nu,p}$ is a norm on $\mathcal{B}_{\nu,p}$. We equip $\mathcal{B}_{\nu,p}$ as with the topology that is generated by $\{\beta_{k,\nu,p}\}_{k=0}^\infty$, and hence $\mathcal{B}_{\nu,p}$ is a countably multinormed space. This space contains the functions that generalizes those belonging to the spaces of type in [2, 4, 8]. $\mathcal{B}_{\nu,p}$ is a subspace of the space $L_{\nu,p}(\mathbb{R}_+)$ and convergence in $\mathcal{B}_{\nu,p}$ implies convergence in $L_{\nu,p}(\mathbb{R}_+)$.

Following [10], we establish the completeness of the space $\mathcal{B}_{\nu,p}$ in the following result.

Theorem 3.1 *The space $\mathcal{B}_{\nu,p}$ is complete and consequently it is a Fréchet space.*

Proof: Consider a Cauchy sequence in $\mathcal{B}_{\nu,p}$, say $\{\phi_m\}_{m=1}^{\infty}$. For each k and some $\nu>0$, $\{\phi_m\}_{m=1}^{\infty}$ is a Cauchy sequence in $L_{\nu,p}(\mathbb{R}_+)$. Since $L_{\nu,p}(\mathbb{R}_+)$ is complete, there exists a function $\rho_k \in L_{\nu,p}(\mathbb{R}_+)$ which is the limit in $L_{\nu,p}(\mathbb{R}_+)$ of $\{B_x^k\phi_m\}_{m=1}^{\infty}$. We will prove that ρ_k is almost everywhere on $\mathbb{R}_+$ equal to $B_x^k\chi_0$, where $\chi_0\in\mathcal{B}_{\nu,p}$ is independent of k. Let $x_1>0$ a fixed point and x a variable point in $\mathbb{R}_+$. From (5), we obtain

$$x^2\frac{d^2}{dx^2}B_x^k\phi_m=\left(\frac{x^2-1}{4}-\mu x\right)B_x^k\phi_m-\frac{1}{4}B_x^{k+1}\phi_m.$$

Dividing by x^2 and integrating twice over the interval $[x_1,x]$ with z instead of x as variable of integration, we get

$$B_x^k\phi_m=\int_{x_1}^{x}\int_{x_1}^{x}\left[\left(\frac{1}{4}-\frac{1}{4z^2}-\frac{\mu}{z}\right)B_z^k\phi_m-\frac{1}{4z^2}B_z^{k+1}\phi_m\right]\,dz\,dz+b_m+a_m(x-x_1), \tag{9}$$

where $a_m=\left[\frac{d}{dx}B_x^k\phi_m\right]_{x=x_1}$ and $b_m=B_{x_1}^k\phi_m$ are constants. Now we study the interior integral in (9). Let $m_1,m_2\in\mathbb{N}$. By using Hölder and Minkowski inequalities on the interval $[x_1,x]$, we have

$$\begin{aligned}
&\left|\int_{x_1}^{x}\left[\left(\frac{1}{4}-\frac{1}{4z^2}-\frac{\mu}{z}\right)B_z^k(\phi_{m_1}-\phi_{m_2})-\frac{1}{4z^2}B_z^{k+1}(\phi_{m_1}-\phi_{m_2})\right]\,dz\right|\\
&\quad\le\frac{1}{4}\left|\int_{x_1}^{x}B_z^k(\phi_{m_1}-\phi_{m_2})\,dz\right|+\frac{1}{4}\left|\int_{x_1}^{x}\frac{1}{z^2}\,B_z^k(\phi_{m_1}-\phi_{m_2})\,dz\right|\\
&\qquad+|\mu|\left|\int_{x_1}^{x}\frac{1}{z}\,B_z^k(\phi_{m_1}-\phi_{m_2})\,dz\right|+\frac{1}{4}\left|\int_{x_1}^{x}\frac{1}{z^2}\,B_z^{k+1}(\phi_{m_1}-\phi_{m_2})\,dz\right|\\
&\quad\le\frac{1}{4}\left(\int_{x_1}^{x}z^{-\nu q+\frac{q}{p}}\,dz\right)^{\frac{1}{q}}\left(\int_{x_1}^{x}\left|B_z^k(\phi_{m_1}-\phi_{m_2})\right|^p z^{\nu p-1}\,dz\right)^{\frac{1}{p}}\\
&\qquad+\frac{1}{4}\left(\int_{x_1}^{x}z^{-\nu q-2}\,dz\right)^{\frac{1}{q}}\left(\int_{x_1}^{x}\left|B_z^k(\phi_{m_1}-\phi_{m_2})\right|^p z^{\nu p-2}\,dz\right)^{\frac{1}{p}}\\
&\qquad+|\mu|\left(\int_{x_1}^{x}z^{-\nu q-1}\,dz\right)^{\frac{1}{q}}\left(\int_{x_1}^{x}\left|B_z^k(\phi_{m_1}-\phi_{m_2})\right|^p z^{\nu p-1}\,dz\right)^{\frac{1}{p}}\\
&\qquad+\frac{1}{4}\left(\int_{x_1}^{x}z^{-\nu q-2}\,dx\right)^{\frac{1}{q}}\left(\int_{x_1}^{x}\left|B_z^{k+1}(\phi_{m_1}-\phi_{m_2})\right|^p z^{\nu p-2}\,dz\right)^{\frac{1}{p}},
\end{aligned} \tag{10}$$

where $q=\frac{p}{p-1}$. The first integrals in the products of the right-hand side of (10) are bounded by smooth functions on every interval I whose closure is compact in $\mathbb{R}_+$. The second integrals converge to zero when m_1 and m_2 tend to infinity, independently. This show that, the left-hand side of (10) converges to zero uniformly on every interval I.

Now, we return to the analysis of (9). Since $B_x^k\phi_m$ converges in $L_{\nu,p}(I)$ for every I, and the iterated integral in (10) converges uniformly on I as $m\to\infty$, we have that the sequence $\{\psi_m\}_{m=1}^{\infty}$, where $\psi_m(x)=b_m+a_m(x-x_1)$, converges in $L_{\nu,p}(I)$. Moreover, since the measure of the interval I is finite and $p\ge 1$, we conclude that $\phi_m(x)$ converges in $L_{\nu,2}(I)$. Further, the coefficients a_m, b_m tend to limits, say a and b. Consequently, $\phi_m(x)$ and therefore $B_x^k\phi_m$ converge uniformly on every I.

Denote by $\chi_k(x)$, the uniform limit of the sequence $\{B_x^k\phi_m\}_{m=1}^{\infty}$ which is a continuous function on I. Passing the limit in (9) when $m\to\infty$ we get

$$\chi_k(x)=\int_{x_1}^{x}\int_{x_1}^{x}\left[\left(\frac{1}{4}-\frac{1}{4z^2}-\frac{\mu}{z}\right)\chi_k(z)-\frac{1}{4z^2}\,\chi_{k+1}(z)\right]\,dz\,dz+b+a(x-x_1). \tag{11}$$

We conclude that $\chi_k(x)$ is a smooth function, and making the necessary differentiation in (11) we obtain $\chi_{k+1}=B_x\chi_k$ and thus $\chi_k(x)=B_x^k\chi_0$.

Since $\chi_k(x)$ is the uniform limit on every I of the sequence $\{B_x^k\phi_m\}_{m=1}^{\infty}$, and the limit of $\{B_x^k\phi_m\}_{m=0}^{\infty}$ in $L_{\nu,p}(\mathbb{R}_+)$ is $\rho_k(x)$, we deduce that $\rho_k(x)=\chi_k(x)$ almost everywhere on I. Hence $B_x^k\chi_0$ and $\rho_k(x)$ are in the same equivalence class of $L_{\nu,p}(\mathbb{R}_+)$. From (8), it follows that for every k and some ν, $\beta_{k,\nu,p}(\chi_0)=\beta_{0,\nu,p}(B_x^k\chi_0)<\infty$ and

$$\beta_{k,\nu,p}(\chi_0-\phi_m)=\beta_{0,\nu,p}(B_x^k\chi_0-B_x^k\phi_m)\to 0,\qquad m\to\infty.$$

■

We will denote by $\mathcal{D}(\mathbb{R}_+)$, $\mathcal{E}(\mathbb{R}_+)$ the usual spaces of testing functions [10]. We have that $\mathcal{D}(\mathbb{R}_+) \subset \mathcal{B}_{\nu,p}(\mathbb{R}_+) \subset \mathcal{E}(\mathbb{R}_+)$ and since $\mathcal{D}(\mathbb{R}_+)$ is dense in $\mathcal{E}(\mathbb{R}_+)$, we also have that $\mathcal{B}_{\nu,p}(\mathbb{R}_+)$ is also dense in $\mathcal{E}(\mathbb{R}_+)$. In the next result we show a representation formula for an element of $\mathcal{D}(\mathbb{R}_+)$.

Theorem 3.2 *Let $\phi \in \mathcal{D}(\mathbb{R}_+)$. Then ϕ can be represented by the Lebedev type integral*

$$\phi(x) = \lim_{\epsilon \to 0^+} \frac{1}{\pi^2} \int_{\mathbb{R}_+} \tau \sinh(2(\pi - \epsilon)\tau) \left| \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \right|^2 W_{\mu,i\tau}(x) \int_{\mathbb{R}_+} W_{\mu,i\tau}(y)\ \phi(y)\ \frac{dy}{y^2}\ d\tau, \tag{12}$$

where the limit is understood as a convergence in $\mathcal{B}_{\nu,p}$ with $\nu > 0$ and $p \geq 1$.

Proof: From relation (7) we establish, under conditions of lemma, the uniform convergence of the outward integral (12) on every compact interval $[x_0, X_0] \subset \mathbb{R}_+$. Thus, denoting by

$$\phi_\epsilon(x) = \frac{1}{\pi^2} \int_{\mathbb{R}_+} \tau \sinh(2(\pi - \epsilon)\tau) \left| \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \right|^2 W_{\mu,i\tau}(x) \int_{\mathbb{R}_+} W_{\mu,i\tau}(y)\ \phi(y)\ \frac{dy}{y^2}\ d\tau, \tag{13}$$

we may repeatedly differentiate under the integral sign

$$B_x^k \phi_\epsilon(x) = \frac{1}{\pi^2} \int_{\mathbb{R}_+} \tau \sinh(2(\pi - \epsilon)\tau) \left| \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \right|^2 B_x^k W_{\mu,i\tau}(x) \int_{\mathbb{R}_+} W_{\mu,i\tau}(y)\ \phi(y)\ \frac{dy}{y^2}\ d\tau.$$

Invoking (5), we integrate by parts the inner integral with respect to y, where integrated terms are vanishing since $\phi \in \mathcal{D}(\mathbb{R}_+)$. Thus

$$B_x^k \phi_\epsilon(x) = \frac{1}{\pi^2} \int_{\mathbb{R}_+} \tau \sinh(2(\pi - \epsilon)\tau) \left| \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \right|^2 W_{\mu,i\tau}(x) \int_{\mathbb{R}_+} W_{\mu,i\tau}(y)\ B_y^k \phi(y)\ \frac{dy}{y^2}\ d\tau. \tag{14}$$

Further, changing the order of integration in (14) by the Fubini theorem, we find

$$B_x^k \phi_\epsilon = \int_{\mathbb{R}_+} \mathcal{K}(x,y)\ B_y^k \phi(y)\ \frac{dy}{y^2}, \tag{15}$$

where

$$\mathcal{K}(x,y) = \frac{1}{\pi^2} \int_{\mathbb{R}_+} \tau \sinh(2(\pi - \epsilon)\tau) \left| \Gamma\left(\frac{1}{2} - \mu + i\tau\right) \right|^2 W_{\mu,i\tau}(x)\ W_{\mu,i\tau}(y)\ d\tau. \tag{16}$$

Now, we study integral (16) and see if it is convergent. Taking into account the following inequality for the Gamma function

$$\frac{1}{\left|\Gamma\left(\frac{1}{2} - \mu + i\tau\right)\right|} \leq \frac{\Gamma\left(\frac{1}{2} + \mu\right)}{\pi}\ e^{\pi\tau}, \tag{17}$$

inequality (7), relation (2.3.3.1) in [5] and the properties of the trigonometric hyperbolic function sinh, we conclude after some calculations that

$$|\mathcal{K}(x,y)| \leq \frac{C\ (x,y)^\mu\ e^{-\frac{x+y}{2}}\ \Gamma^2\left(\frac{1}{2} + \mu\right)\ \Gamma^4\left(\frac{1}{2} - \mu\right)}{4\pi^4\ (\epsilon + 2\pi)^2},$$

and so we have the convergent of the integral (17). The previous conclusion gives us the convergence of $B_x^k \phi_\epsilon$ to $B_x^k \phi$ with respect to the norm in $L_{\nu,p}(\mathbb{R}_+)$ with $\nu > 0$, $p \geq 1$, and when $\epsilon \to 0^+$. Thus, we derive

$$\beta_{k,\nu,p}(\phi_\epsilon - \phi) = \beta_{0,\nu,p}\left(B_x^k \phi_\epsilon - B_x^k \phi\right) \to 0, \qquad \epsilon \to 0^+.$$

■

For $\nu_1 < \nu_2$ and $p_1 < p_2$, we have that $\mathcal{B}_{\nu_2,p_2}(\mathbb{R}_+) \subset \mathcal{B}_{\nu_1,p_1}(\mathbb{R}_+)$. Denote by $\mathcal{B}'_{\nu,p}$, the dual of $\mathcal{B}_{\nu,p}$, equipped with the weak topology. $\mathcal{B}'_{\nu,p}$ represents a Hausdorff locally convex space of distributions. From the previous imbedding relations, we obtain that $\mathcal{E}'(\mathbb{R}_+) \subset \mathcal{B}'_{\nu,p}$. Since $\mathcal{B}_{\nu,p} \subset L_{\nu,p}(\mathbb{R}_+)$, we imbed the dual space $L_{1-\nu,q}(\mathbb{R}_+)$ with $q = \frac{p}{p-1}$ into $\mathcal{B}'_{\nu,p}$ as a subspace of regular distributions. They act upon elements ϕ from $\mathcal{B}_{\nu,p}$ according to

$$\langle f, \phi \rangle := \int_{\mathbb{R}_+} f(x)\ \phi(x)\ dx. \tag{18}$$

From the fact that if $\{\phi_m\}_{m=1}^{\infty}$ converges in $\mathcal{B}_{\nu,p}$ to zero, we have that the linear functional (18) is continuous on $\mathcal{B}_{\nu,p}$. By the Hölder inequality

$$|\langle f, \phi_m \rangle| \leq \beta_{k,1-\nu,q}(f)\ \beta_{k,\nu,p}(\phi_m) \to 0, \qquad m \to \infty.$$

This imbedding of $L_{1-\nu,q}(\mathbb{R}_+)$ into $\mathcal{B}'_{\nu,p}$ is one-to-one. This will imply that $f = g$ almost everywhere on $\mathbb{R}_+$ (see [10]). From the general theory of continuous linear functionals on countably multinormed spaces follows that for each element $f \in \mathcal{B}'_{\nu,p}$ there exists a nonnegative integer r and positive constant C such that

$$|\langle f, \phi \rangle| \leq C \max_{0 \leq k \leq r} \beta_{k,\nu,p}(\phi), \tag{19}$$

for every $\phi \in \mathcal{B}_{\nu,p}$. Here r, C depends on f but not on ϕ.

4 The index Whittaker transform for distributions

The aim of this section is to introduce the index Whittaker transform for distributions $f \in \mathcal{B}'_{\nu,p}$. Namely, it is defined by

$$\mathcal{W}L[f](x) := \left\langle f(\cdot), \cdot \left| \Gamma\left(\frac{1}{2} - \mu + i\cdot\right)\right|^2 W_{\mu,i\cdot}(x) \right\rangle, \qquad \tau \in \mathbb{R}_+. \tag{20}$$

From the properties of the Whittaker function presented in the preliminaries, we conclude that $W_{\mu,i\tau}(x) \in L_{\nu,p}(\mathbb{R}_+)$, for $\nu > 0$. Moreover, it belongs to $\mathcal{B}_{\nu,p}$. Hence, for regular distributions $f \in L_{1-\nu,q}(\mathbb{R}_+)$ the index Whittaker transform $\mathcal{W}L[f]$ can be written in the form (18), which coincides with (1). Moreover, taking into account (19) and (7), we have

$$\begin{aligned} |\mathcal{W}L[f](x)| &\leq C \max_{0 \leq k \leq r} \beta_{k,\nu,p}(W_{\mu,i\tau}(x)) \\ &\leq \frac{C\ 4^r}{\left|\Gamma\left(\frac{1}{2} - \mu - i\tau\right)\right|^2} \left(\int_{\mathbb{R}_+} x^{\mu+\nu p-1}\ e^{\frac{x}{2}}\ dx\right)^{\frac{1}{p}} \max_{0 \leq k \leq r} \tau^{2k} \\ &\leq \frac{C_{f,\delta,\nu,p}}{\left|\Gamma\left(\frac{1}{2} - \mu - i\tau\right)\right|^2} \max\{1, \tau^{2k}\}, \qquad \nu > 0. \end{aligned} \tag{21}$$

In order to invert the transform (20), for each $\tau \in \mathbb{R}_+$, we will take into account the ideas presented in [6] and the fact that our index integral transform for distributions is coincident with (1) via (18). Hence, we start introducing the following auxiliar approximation operator

$$(I_\epsilon g)(\tau) = \frac{4\tau}{\pi^2} \left\langle g(\cdot), \cdot^{\epsilon-2} W_{\mu+\epsilon,i\tau}(\cdot) \right\rangle = \frac{4\tau}{\pi^2} \int_{\mathbb{R}_+} t^{\epsilon-2}\ W_{\mu+\epsilon,i\tau}(t)\ g(t)\ dt, \tag{22}$$

for which the following result holds (for its proof we refer to Theorem 2.3 in [6] where a similar integral is studied)

Theorem 4.1 *(cf. [6]) For $f \in \mathcal{B}'_{\nu,p}$ with $p \geq 1$, $\tau \in \mathbb{R}_+$ and $g(t) = \mathcal{W}L[f](t)$, the operator (22) has the form*

$$\begin{aligned} (I_\epsilon g)(\tau) &= \frac{4\tau}{\pi^2\ \Gamma(2\epsilon)} \left\langle f(\cdot), \cdot \left|\Gamma(\epsilon + (\cdot + \tau)i)\right|^2 \left|\Gamma(\epsilon + (\cdot - \tau)i)\right|^2 \right\rangle \\ &= \frac{4\tau}{\pi^2\ \Gamma(2\epsilon)} \int_{\mathbb{R}_+} \left|\Gamma(\epsilon + (y + \tau)i)\right|^2 \left|\Gamma(\epsilon + (y - \tau)i)\right|^2\ y\ f(y)\ dy \end{aligned}$$

From the previous result we get

Theorem 4.2 *(cf. [6]) For $f \in \mathcal{B}'_{\nu,p}$, with $p \geq 1$, and $g(x) = \mathcal{W}L[f](x)$. Then*

$$\lim_{\epsilon \to 0} (I_\epsilon g)(\tau) = \frac{4\tau}{\sinh(2\pi\tau)} \, |f(\tau) - f(-\tau)|,$$

where the convergence is understood in $\mathcal{D}'(\mathbb{R}_+)$, and

$$f(\tau) - f(-\tau) = \frac{\sinh(2\pi\tau)}{\pi^2} \int_{\mathbb{R}_+} x^{-2} \, W_{\mu,i\tau}(x) \, \mathcal{W}L[f](x) \, dx.$$

If f is an odd function then

$$2f(\tau) = f_1(\tau) = \frac{\sinh(2\pi\tau)}{\pi^2} \int_{\mathbb{R}_+} x^{-2} \, W_{\mu,i\tau}(x) \, \mathcal{W}L[f_1](x) \, dx.$$

From the previous two results we conclude that the inversion formula for the transformation (20) is given by

$$\begin{aligned} f_1(\tau) &= \frac{\sinh(2\pi\tau)}{\pi^2} \int_{\mathbb{R}_+} x^{-2} \, W_{\mu,i\tau}(x) \, \mathcal{W}L[f_1](x) \, dx \\ &= \frac{\sinh(2\pi\tau)}{\pi^2} \left\langle \mathcal{W}L[f_1](\cdot), \cdot^{-2} \, W_{\mu,i\tau}(\cdot) \right\rangle, \end{aligned} \tag{23}$$

where $f_1 \in \mathcal{B}'_{\nu,p}$.

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