Wave diffraction by wedges having arbitrary aperture angle†

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Abstract

The problem of plane wave diffraction by a wedge sector having arbitrary aperture angle has a very long and interesting research background. In fact, we may recognize significant research on this topic for more than one century. Despite this fact, up to now no clear unified approach was implemented to treat such a problem from a rigourous mathematical way and in a consequent appropriate Sobolev space setting. In the present paper, we are considering the corresponding boundary value problems for the Helmholtz equation, with complex wave number, admitting combinations of Dirichlet and Neumann boundary conditions. The main ideas are based on a convenient combination of potential representation formulas associated with (weighted) Mellin pseudo-differential operators in appropriate Sobolev spaces, and a detailed Fredholm analysis. Thus, we prove that the problems have unique solutions (with continuous dependence on the data), which are represented by the single and double layer potentials, where the densities are solutions of derived pseudo-differential equations on the half-line.

Keywords: wedge diffraction problem, Helmholtz equation, boundary value problem, potential operator, pseudo-differential operator, cone Sobolev

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1. Introduction

The problem of plane wave diffraction by wedge sectors counts already more than one century of research. Indeed, we may identify the classical works of Sommerfeld [67] and Poincaré [59] as the first ones where this type of problem was significantly tackled. There, the solution of the Helmholtz equation in an infinite wedge sector with Dirichlet and Neumann boundary conditions was studied by using the Sommerfeld integrals and separation of variables, respectively. Anyway, previous partial results can also be identified. This is the case of Macdonald [39] who already gave in 1895 a representation of the first and second Green's functions (i.e., electrostatic and velocity potentials) of the potential equation for a wedge of an arbitrary aperture angle. In fact, this was first considered only for angles of the form π/m, where m is a positive integer, and later (cf. [40]) Macdonald was able to obtain formulas for the two Green's functions of the Helmholtz equation for wedges with any aperture angle. However, Macdonald's method is not easy to follow when involving somehow conventional formalisms of nineteenth century.

Carslaw did also relevant work on the construction of appropriate potentials, by using the Sommerfeld's method, first for some wedges of particular aperture angles and then for arbitrary ones (cf. [3, 4, 5]).

In the last decades the mathematical analysis of wave diffraction problems by wedge configurations has been receiving increased attention. Consequently, we can identify a significant number of publications where such analysis was taken for particular cases of wedge amplitudes and/or boundary conditions (cf., e.g., [2, 7, 8, 9, 10, 11, 12, 13, 18, 21, 22, 33, 34, 35, 42, 48, 44, 45, 46, 49, 50, 55, 56, 58, 60, 66, 72]). However, none of these listed papers contain final solvability results for the general problems in a rigourous mathematical Sobolev space setting as is done in the present paper.

It is clear that one of the main difficulties in such analysis arises from the geometry of the domain in consideration. For some regions, the direct method of layer potentials works very well, allowing the well-posedness of the problems in appropriate Sobolev spaces and, in some cases, closed-form solutions. For smooth domains the list of publications is quite huge. Anyway,

we would like to refer here to some corresponding excellent works which present a somehow rather complete account of the theory in smooth domains, as is the case of the books by Colton and Kress [14], Courant and Hilbert [17], Hsiao and Wendland [26], Kress [37], McLean [41], and Taylor [69]. Moreover, among the non-smooth domains, general theories for Lipschitz domains are also available and can be tested in concrete corresponding boundary value problems. Related to this, we would like to refer the works of Costabel [15], Costabel and Stephan [16], Jerrison and Kenig [27, 28, 29], Kohr, Pintea and Wendland [31], Mitrea and Mitrea [51, 52], Mitrea and Taylor [53, 54], and Verchota [71].

We may say that the recent developments in problems of wave diffraction by non-smooth regions were certainly inspired by also somehow recent significant general results for boundary value problems in non-smooth domains. As representatives of the latter ones, we may also cite here the monographs [19, 25, 57, 62], as well as the pertinent work [36]. Here, Kondrat'ev's method is mainly based on the Mellin transform, already allowing information on the smoothness and asymptotic expansion of the solutions at the edges of the boundary angles.

The relevant work of Komech, Merzon and their collaborators [32, 33, 34, 35] must also be referred, where the so-called method of complex characteristics for elliptic equations in nonconvex angles is used. Typically, the crucial part of the method is played by the connection equation on the Riemann surface of complex characteristics of the given elliptic operator. Also, the limiting amplitude principle in the two-dimensional scattering of an incident plane harmonic wave by a wedge has recently been successfully applied, cf. [13, 49, 56].

In [18], the problem of wave diffraction by impenetrable wedges having arbitrary aperture angle was studied by means of the Wiener-Hopf method. This positively answered the important issue (that had been open for a long period) on the possibility of applying the Wiener-Hopf technique to this more complex geometrical problem of having wedges with arbitrary angles. However, no concern with the space setting was there presented. As a very significant result, it was obtained that the diffraction by an impenetrable wedge always reduces to a standard Wiener-Hopf factorization. For given impedance boundary conditions, explicit factorizations were derived which lead to consequent closed-form solutions.

The series of results obtained by Meister, Speck and their collaborators (cf., e.g., [10, 11, 12, 21, 22, 42, 48, 44, 45, 46, 58]) constitute a systematic

approach to a rigourous mathematical analysis of plane wave diffraction by wedges. They obtained important conclusions on both the well-posedness of the problems and consequent closed-form solutions, in appropriate Sobolev spaces, for a large number of particular cases of aperture angles and different types of boundary conditions. In particular, in [21, 22], the authors obtained the well-posedness for the so-called rational angles of the form $\pi m/n$, where m and n are natural numbers. This was done by using symmetry properties within certain Riemann surfaces. In addition, this was a somehow natural development of the previous work [58] where, by using also symmetry arguments and Sommerfeld potentials (resulting from special Sommerfeld problems which are explicitly solvable), the well-posedness and explicit solution in closed analytic form of the Dirichlet and Neumann problems for the Helmholtz equation in the non-convex and non-rectangular wedge with angle of $4\pi/3$ was obtained. However, the proposed method does not work for non-rational angles.

The authors of the present work have also previously considered several cases of problems of wave diffraction by wedges with particular aperture angles (cf. [7, 8, 9]) in which symmetry arguments, the potential method, and Wiener-Hopf and Hankel operators were combined in a successful way.

Having all this in mind, we note that a thorough justification in appropriate Sobolev spaces, in the Hadamard well-posedness sense, for the problems under analysis, has never been done.

Thus, in the present paper, we would like to consider problems of wave diffraction by wedges having arbitrary aperture angle, facing Dirichlet-Dirichlet, Neumann-Neumann and Dirichlet-Neumann boundary conditions, in a strict mathematical perspective where everything will be considered in appropriate Sobolev space settings. Thus, as a main result, we shall prove the unique existence of solution, and its continuous dependence on the data, for each of those classes of problems. Moreover, integral representations of the solutions are obtained in terms of the single and double layer potentials, where their densities are solutions of certain Mellin pseudo-differential equations on the half-line. In particular, this also opens the possibility of considering further studies on the solutions based on the obtained formulas – like the regularity and asymptotic behavior of the solutions near the edge of the corresponding cones. Thus, the present work, at the same time, unifies several past works and completes the existent open situations when considering the solvability of these classes of wedge wave diffraction problems for any aperture angle within Sobolev spaces (although, for the Dirichlet-Neumann

problems we only consider convex angles).

The paper is organized as follows: Section 2 is devoted to the presentation of the basic definitions, the problem formulation and the conclusion that we are dealing with classes of problems which admit at most one solution in the considered Sobolev spaces. Section 3 reports the use of potentials and their adjustment in a corresponding half-line setting which allows the construction of appropriate auxiliary operators that will appear in the solutions representation. In Section 4, the Mellin transform and weighted Sobolev spaces will be considered in order to obtain a reinterpretation of the problems in an operator theory language, and so that it will serve for wedges with any aperture angle. Then, in the last section, a detailed Fredholm analysis of the obtained operators is deduced and the consequent properties are transferred to the boundary value problems under consideration.

2. Formulation of the Problems and Uniqueness of Solutions

We use the notation $\mathcal{S}(\mathbb{R}^n)$ for the Schwartz space of all rapidly decreasing functions and $\mathcal{S}'(\mathbb{R}^n)$ for the dual space of tempered distributions on $\mathbb{R}^n$. The Bessel potential space $H^s = H^s(\mathbb{R}^n)$, with $s \in \mathbb{R}$, is formed by the elements $\varphi \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|\varphi\|_{H^s} = \|\mathcal{F}^{-1}(1+|\xi|^2)^{s/2} \cdot \mathcal{F}\varphi\|_{L_2(\mathbb{R}^n)}$$

is finite. As the notation indicates, $\|\cdot\|_{H^s}$ is a norm for the space $H^s(\mathbb{R}^n)$ which makes it a Banach space. Here, $\mathcal{F} = \mathcal{F}_{x\mapsto\xi}$ denotes the Fourier transformation in $\mathbb{R}^n$. For a given non-empty, open set $\mathcal{D} \subset \mathbb{R}^n$, we denote by $H^s_{\mathcal{D}} = H^s_{\mathcal{D}}(\mathbb{R}^n)$ the closed subspace of H^s whose elements have supports in $\overline{\mathcal{D}}$, and $H^s(\mathcal{D})$ denotes the space of generalized functions on $\mathcal{D}$ which have extensions into $\mathbb{R}^n$ that belong to H^s. The space $H^s_{\mathcal{D}}$ is endowed with the subspace topology, and on $H^s(\mathcal{D})$ we introduce the norm of the quotient space $H^s/H^s_{\mathbb{R}^n\setminus\overline{\mathcal{D}}}$. Thus $H^s(\mathcal{D}) = r_{\mathcal{D}}(H^s)$, where $r_{\mathcal{D}}$ denotes the restriction operator to $\mathcal{D}$. Finally, let us introduce the spaces $\widetilde{H}^s(\mathcal{D}) = r_{\mathcal{D}}H^s_{\mathcal{D}}$ with a norm naturally defined as

$$||u||_{\widetilde{H}^s(\mathcal{D})} := \inf_{\ell_0} ||\ell_0 u||_{H^s}.$$

Here $\ell_0 u$ stands for any extension of a distribution on $\mathcal{D}$ to a distribution in $H^s_{\mathcal{D}}$ (which is not unique for $s < -1/2$).

Throughout the paper we will use the notation

$$\mathbb{R}^n_\pm := \{x = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n : \pm x_n > 0\}.$$

Note that the spaces $H^0(\mathbb{R}^n_+)$ and $\widetilde{H}^0(\mathbb{R}^n_+)$ can be identified with $L_2(\mathbb{R}^n_+)$. For a comprehensive treatment of the introduced spaces we refer to [1, 41, 70].

Let $\Omega = \Omega_\alpha$ be a plane angle of magnitude α, $0 < \alpha < 2\pi$ with vertex at the origin and sides $S_{1,+} := \{(t, 0) : t \in \mathbb{R}_+\}$, $S_{2,+} := \{(t\cos\alpha, t\sin\alpha) : t \in \mathbb{R}_+\}$. Let further $\partial\Omega := S_{1,+} \cup S_{2,+} \cup \{(0, 0)\}$ and denote by n the unit exterior vector of Ω, which equals to $n_1 = (0, -1)^\top$ on $S_{1,+}$ and $n_2 = (-\sin\alpha, \cos\alpha)^\top$ on $S_{2,+}$.

We are interested in studying the problem of existence and uniqueness of an element $v \in H^{1+\varepsilon}(\Omega)$, $0 \le \varepsilon < 1/2$ such that

$$\left(\Delta + k^2\right) v = 0 \quad \text{in} \quad \Omega, \tag{2.1}$$

and v satisfies one of the three boundary conditions

$$[v]^+_{S_{1,+}} = g_1 \ \text{ on } S_{1,+}, \qquad [v]^+_{S_{2,+}} = g_2 \ \text{ on } S_{2,+}, \tag{2.2}$$

$$[\partial_n v]^+_{S_{1,+}} = f_1 \ \text{ on } S_{1,+}, \qquad [\partial_n v]^+_{S_{2,+}} = f_2 \ \text{ on } S_{2,+}, \tag{2.3}$$

$$[v]^+_{S_{1,+}} = g_1 \ \text{ on } S_{1,+}, \qquad [\partial_n v]^+_{S_{2,+}} = f_2 \ \text{ on } S_{2,+}, \tag{2.4}$$

where the wave number $k \in \mathbb{C} \setminus \mathbb{R}$ is given. In addition, Δ stands for the Laplace operator, and the Dirichlet and Neumann traces on $S_{j,+}$, $j = 1, 2$, are denoted by $[v]^+_{S_{j,+}}$ and $[\partial_n v]^+_{S_{j,+}}$, respectively. Note that the Dirichlet type condition can be understood in the trace sense, while the Neumann type condition is understood in the distributional sense, defined by means of Green's formula and duality arguments (cf. [41]). Finally, for $j = 1, 2$, the elements $g_j \in H^{1/2+\varepsilon}(S_{j,+})$ and $f_j \in H^{-1/2+\varepsilon}(S_{j,+})$ are arbitrarily given provided they satisfy the following compatibility conditions:

$$g_1 - \chi_* g_2 \in \widetilde{H}^{1/2+\varepsilon}(\mathbb{R}_+), \tag{2.5}$$

$$f_1 + \chi_* f_2 \in \widetilde{H}^{-1/2+\varepsilon}(\mathbb{R}_+); \tag{2.6}$$

here $\chi_* = \chi_{\alpha,*}$ denotes the pull back $(\chi_* u)(t) = u(\chi(t))$ of a function $\chi = \chi_\alpha : \mathbb{R}_+ = S_{1,+} \ni t \mapsto e^{i\alpha} t \in S_{2,+}$, i.e. $\chi_\alpha(t) = e^{i\alpha} t$, $t \in \overline{\mathbb{R}_+}$. Clearly, χ_* induces isomorphisms

$$\chi_* : H^s(S_{2,+}) \to H^s(\mathbb{R}_+) \qquad \text{and} \qquad \chi_* : \widetilde{H}^s(S_{2,+}) \to \widetilde{H}^s(\mathbb{R}_+)$$

for all $s \in \mathbb{R}$.

It is worth mentioning that from the compatibility conditions (2.5) and (2.6) it follows (cf. [16, 25]) that there exist unique elements $g \in H^{\frac{1}{2}+\varepsilon}(\partial\Omega)$ and $f \in H^{-\frac{1}{2}+\varepsilon}(\partial\Omega)$, respectively, such that

$$r_{S_{j,+}} g = g_j \quad \text{and} \quad r_{S_{j,+}} f = f_j, \quad j = 1, 2. \tag{2.7}$$

This observation allows us to state an equivalence between the boundary conditions (2.2) and

$$[v]^+_{\partial\Omega} = g \quad \text{on } \partial\Omega \tag{2.8}$$

for the Dirichlet problem and between the boundary conditions (2.3) and

$$[\partial_n v]^+_{\partial\Omega} = f \quad \text{on } \partial\Omega \tag{2.9}$$

for the Neumann problem. Here note also that, the compatibility condition (2.6) is an additional restriction only for $\varepsilon = 0$. Finally, let us mention that the case when $-1/2 < \varepsilon < 0$ will not be considered here. We are not aware of general uniqueness results for $-1/2 < \varepsilon < 0$ (apart from special cases on the geometry/angle and on the boundary conditions; cf. [21, 22]).

From now on we will refer to:

- Problem $\mathcal{P}_{\text{D-D}}$ as the one characterized by (2.1) and (2.8);
- Problem $\mathcal{P}_{\text{N-N}}$ as the one characterized by (2.1) and (2.9);
- Problem $\mathcal{P}_{\text{mixed}}$ as the one characterized by (2.1) and (2.4).

Theorem 2.1. *The problems $\mathcal{P}_{\text{D-D}}$, $\mathcal{P}_{\text{N-N}}$, and $\mathcal{P}_{\text{mixed}}$ have at most one solution.*

Proof. The proof is somehow standard and uses the Green's formula (being sufficient to consider the case $\varepsilon = 0$). Let R be a sufficiently large positive number and $B(R)$ be the open disk centered at the origin with radius R. Set $\Omega_R := \Omega \cap B(R)$. Note that the domain Ω_R has a piecewise smooth boundary S_R and denote by $n(x)$ the outward unit normal vector at the non-singular points $x \in S_R$.

Let u be a solution of the homogeneous problem. Then the first Green's identity for u and its complex conjugate $\bar{u}$ in the domain Ω_R, together with zero boundary conditions on S_R yields

$$\int_{\Omega_R} \Big[|\nabla u|^2 - k^2 |u|^2 \Big] dx = \int_{\partial B(R)\cap\Omega} [\partial_n u]^+ [\bar{u}]^+ \, dS_R \,. \tag{2.10}$$

From to the real and imaginary parts of the last identity, we obtain

$$\int_{\Omega_R} \Big[|\nabla u|^2 + (\Im\mathrm{m}\, k)^2 |u|^2\Big] dx = \Re\mathrm{e} \int_{\partial B(R)\cap\Omega} [\partial_n u]^+ [\bar{u}]^+ \, dS_R \,, \tag{2.11}$$

for $\Re\mathrm{e}\, k = 0$ and

$$-2(\Re\mathrm{e}\, k)(\Im\mathrm{m}\, k) \int_{\Omega_R} |u|^2 \, dx = \Im\mathrm{m} \int_{\partial B(R)\cap\Omega} [\partial_n u]^+ [\bar{u}]^+ \, dS_R \,, \tag{2.12}$$

for $\Re\mathrm{e}\, k \neq 0$. Recall that we consider the case $\Im\mathrm{m}\, k \neq 0$. Now, note that since $u \in H^1(\Omega)$ then there is a monotonic sequence of positive numbers $\{R_j\}$, such that $R_j \to \infty$ as $j \to \infty$ and

$$\lim_{j\to\infty} \int_{\partial B(R_j)\cap\Omega} [\partial_n u]^+ [\bar{u}]^+ \, dS_{R_j} = 0. \tag{2.13}$$

Indeed, first in (R, φ) polar coordinates we have

$$\int_{\partial B(R)\cap\Omega} [\partial_n u]^+ [\bar{u}]^+ \, dS_R = R \int_0^\alpha \partial_n u(R,\varphi)\, \bar{u}(R,\varphi) \, d\varphi.$$

Due to $u, \partial_n u \in L_2(\Omega)$ we have that the integrals

$$\int_0^\infty \left(R \int_0^\alpha |u(R,\varphi)|^2 d\varphi \right) dR \quad \text{and} \quad \int_0^\infty \left(R \int_0^\alpha |\partial_n u(R,\varphi)|^2 d\varphi \right) dR$$

are finite. This fact in particular implies that there exists a monotonic sequence of positive numbers $\{R_j\}$ such that $R_j \to \infty$ as $j \to \infty$ and

$$\int_0^\alpha |u(R_j,\varphi)|^2 d\varphi = \bar{o}\left(R_j^{-1}\right) \text{ and } \int_0^\alpha |\partial_n u(R_j,\varphi)|^2 d\varphi = \bar{o}\left(R_j^{-1}\right) \text{ as } j \to \infty.$$

Further, applying the Cauchy-Schwarz inequality for every R_j we get

$$\left| \int_0^\alpha \partial_n u(R_j,\varphi)\, \bar{u}(R_j,\varphi) \, d\varphi \right| \leq \int_0^\alpha |\partial_n u(R_j,\varphi)\, u(R_j,\varphi)| \, d\varphi$$

$$\leq \left(\int_0^\alpha |\partial_n u(R_j,\varphi)|^2 \, d\varphi \right)^{\frac{1}{2}} \left(\int_0^\alpha |u(R_j,\varphi)|^2 \, d\varphi \right)^{\frac{1}{2}} = \bar{o}\left(R_j^{-1}\right) \quad \text{as} \quad j \to \infty$$

and therefore we obtain (2.13).

Since the expressions under the integrals on the left side of the equalities in (2.11) and (2.12) are non-negative then we have that these integrals are monotonic with respect to R. This observation together with (2.13) implies

$$\int_\Omega \Big[|\nabla u|^2 + (\Im\mathrm{m}\, k)^2 |u|^2\Big] dx = \lim_{R\to\infty} \int_{\Omega_R} \Big[|\nabla u|^2 + (\Im\mathrm{m}\, k)^2 |u|^2\Big] dx = 0,$$

for $\Re\mathrm{e}\, k = 0$ and

$$\int_\Omega |u|^2\, dx = \lim_{R\to\infty} \int_{\Omega_R} |u|^2\, dx = 0,$$

for $\Re\mathrm{e}\, k \neq 0$.

Thus, it follows from the last two identities that $u = 0$ in Ω. □

3. Reduction to the Half-line

In the present section, we will start by recalling some results from potential theory. Then, we will construct operators that will help us in the analysis of the problems under study.

From now on, throughout the remaining part of the paper, we assume that $\Im\mathrm{m}\, k > 0$; the complementary case $\Im\mathrm{m}\, k < 0$ runs with obvious changes. Let us denote the standard fundamental solution of the Helmholtz equation (in two dimensions) by

$$\Phi(x) := -\frac{i}{4} H_0^{(1)}(k|x|)\ ,$$

where $H_0^{(1)}(k|x|)$ is the Hankel function of the first kind of order zero (cf. [14, §3.4]). Furthermore, we introduce the single and double layer potentials on S_j

$$\begin{aligned} V_j\psi(x) &= \int_{S_j} \Phi(x-y)\psi(y) d_y S_j\ , \quad x \notin S_j\ , \\ W_j\varphi(x) &= \int_{S_j} [\partial_{n_j(y)}\Phi(x-y)]\varphi(y) d_y S_j\ , \quad x \notin S_j\ , \end{aligned}$$

where $S_1 := \{(t,0) : t \in \mathbb{R}\}$, $S_2 := \{(t\cos\alpha, t\sin\alpha) : t \in \mathbb{R}\}$, $j = 1, 2$, and ψ, φ are density functions. Note that, for $j = 1$, sometimes we will write $\mathbb{R}$ instead of S_1. Let us first consider the operators V_1 and W_1.

Proposition 3.1 (cf. [7, 20, 26])**.** *The single and double layer potentials V_1 and W_1 are continuous operators*

$$V_1 : H^s(\mathbb{R}) \to H^{s+1+\frac{1}{2}}(\mathbb{R}^2_\pm), \qquad W_1 : H^{s+1}(\mathbb{R}) \to H^{s+1+\frac{1}{2}}(\mathbb{R}^2_\pm) \tag{3.1}$$

for all $s \in \mathbb{R}$.

Let us now recall some properties of the above introduced potentials. The following limit relations are well-known (cf. [7, 20, 26]):

$$[V_1\psi]^+_{\mathbb{R}} = [V_1\psi]^-_{\mathbb{R}}, \qquad [\partial_{n_1} V_1\psi]^\pm_{\mathbb{R}} = [\mp\tfrac{1}{2}I]\psi\,,$$
$$[W_1\varphi]^\pm_{\mathbb{R}} = [\pm\tfrac{1}{2}I]\varphi, \qquad [\partial_{n_1} W_1\varphi]^+_{\mathbb{R}} = [\partial_{n_1} W_1\varphi]^-_{\mathbb{R}},$$

where I denotes the identity operator. Clearly, analogous results hold true for the operators V_2 and W_2. Note that for the single and double layer potentials given on $\partial\Omega$

$$\begin{aligned} V\psi(x) &= \int_{\partial\Omega} \Phi(x-y)\psi(y)d_y\partial\Omega\,, \quad x \notin \partial\Omega\,, \\ W\varphi(x) &= \int_{\partial\Omega} [\partial_{n(y)}\Phi(x-y)]\varphi(y)d_y\partial\Omega\,, \quad x \notin \partial\Omega\,, \end{aligned}$$

we have

$$[V\psi]^+_{\partial\Omega} = [V\psi]^-_{\partial\Omega}, \qquad [\partial_n V\psi]^\pm_{\partial\Omega} =: [\mp\tfrac{1}{2}I + V_0]\psi\,,$$

$$[W\varphi]^\pm_{\partial\Omega} =: [\pm\tfrac{1}{2}I + W_0]\varphi, \quad [\partial_n W\varphi]^+_{\partial\Omega} = [\partial_n W\varphi]^-_{\partial\Omega}\,,$$

where

$$\begin{aligned} V_0\psi(z) &:= \int_{\partial\Omega} [\partial_{n(z)}\Phi(z-y)]\psi(y)\, d_y\partial\Omega\,, \quad z \in \partial\Omega\,, \\ W_0\varphi(z) &:= \int_{\partial\Omega} [\partial_{n(y)}\Phi(y-z)]\varphi(y)\, d_y\partial\Omega\,, \quad z \in \partial\Omega \end{aligned}$$

are the direct values of the operators $\partial_n V$ and W on $\partial\Omega$, respectively.

In the sequel we will need to consider operators on $\mathbb{R}_+$. Passing to the boundary we arrive to operators $r_{S_{1,+}}W_0$ and $r_{S_{2,+}}W_0$. To treat this type of operators by using the Mellin transform we decompose the integration over $\partial\Omega$ into the integrations over $S_{1,+}$ and $S_{2,+}$. Therefore, in what follows we

will use the single and double layer potentials not only in the above sense, when applied to elements defined in a full line or in the full boundary $\partial\Omega$, but also to elements defined just in a half-line. The mapping properties of these operators are characterized in Lemma 4.2, Theorems 4.3 and 5.2 below; see also [16, §1-2] for the corresponding details.

The following result is needed before we start applying the potential method to our problems.

Lemma 3.2. *Let $0 < s < 1$. Then, the operators*

$$\chi_* r_{S_{2,+}} W_1 \ell_0,\ r_{S_{1,+}} W_2 \ell_0 \chi_*^{-1} \quad : \quad \widetilde{H}^s(\mathbb{R}_+) \longrightarrow \widetilde{H}^s(\mathbb{R}_+) \tag{3.2}$$

are continuous and they are equal, i.e.,

$$\chi_* r_{S_{2,+}} W_1 \ell_0 = r_{S_{1,+}} W_2 \ell_0 \chi_*^{-1}. \tag{3.3}$$

Proof. From the mapping properties of the double layer potential (cf. (3.1)) and the restriction operator it follows that $\chi_* r_{S_{2,+}} W_1 : H^s_{\mathbb{R}_+} \to H^s(\mathbb{R}_+)$. Since for any $\varphi \in H^s_{\mathbb{R}_+}$ the function $W_1\varphi \in H^{s+\frac{1}{2}}(\Omega)$, for $x \in \Omega$, therefore its Dirichlet boundary data necessarily satisfy the compatibility condition (cf. [25]), i.e., $\chi_* r_{S_{2,+}} W_1 \varphi - r_{\mathbb{R}_+}\varphi \in \widetilde{H}^s(\mathbb{R}_+)$. Thus $\chi_* r_{S_{2,+}} W_1 \varphi \in \widetilde{H}^s(\mathbb{R}_+)$, for any $\varphi \in H^s_{\mathbb{R}_+}$. Similarly, we have $r_{S_{1,+}} W_2 \ell_0 \chi_*^{-1}\varphi \in \widetilde{H}^s(\mathbb{R}_+)$, for any $\varphi \in \widetilde{H}^s(\mathbb{R}_+)$.

To show (3.3), we compare the kernels of these integral operators. First note that

$$-\partial_{y_j}\left(-\frac{i}{4}H_0^{(1)}(k|x-y|)\right) = -\frac{ik}{4}\dot{H}_0^{(1)}(k|x-y|)\frac{x_j - y_j}{|x-y|}, \quad j = 1, 2,$$

where $\dot{H}_0^{(1)}$ denotes the ordinary derivative of the Hankel function, which equals to $-H_1^{(1)}$ cf. [24, 8.473(6)]. Therefore, taking $x = (\tau\cos\alpha, \tau\sin\alpha)$ and $y = (t, 0)$ for the kernel of the operator $\chi_* r_{S_{2,+}} W_1 \ell_0$, we obtain

$$\partial n_1(y)\Phi(x-y) = -\frac{ik}{4}\dot{H}_0^{(1)}\left(k\sqrt{\tau^2 - 2\tau t\cos\alpha + 1}\right)\frac{t\sin\alpha}{\sqrt{\tau^2 - 2\tau t\cos\alpha + 1}},$$

while taking $x = (\tau, 0)$ and $y = (t\cos\alpha, t\sin\alpha)$ for the kernel of operator

$r_{S_{1,+}} W_2 \ell_0 \chi_*^{-1}$ we get

$$\begin{aligned}
\partial n_2(y)\Phi(x-y) &= (-\sin\alpha\,\partial_{y_1} + \cos\alpha\,\partial y_2)\Phi(x-y) \\
&= -\frac{ik}{4}\dot{H}_0^{(1)}(k|x-y|)\left(\frac{\sin\alpha(x_1-y_1)}{|x-y|} - \frac{\cos\alpha(x_2-y_2)}{|x-y|}\right) \\
&= -\frac{ik}{4}\dot{H}_0^{(1)}\left(k\sqrt{\tau^2-2\tau t\cos\alpha+1}\right)\frac{t\sin\alpha}{\sqrt{\tau^2-2\tau t\cos\alpha+1}}.
\end{aligned}$$

Since the kernels of the integral operators $\chi_* r_{S_{2,+}} W_1 \ell_0$ and $r_{S_{1,+}} W_2 \ell_0 \chi_*^{-1}$ are the same they are equal. □

Notice that the range of indices s for which the operators in the lemma are continuous can be extended. However, the case $s \in (0,1)$ is sufficient for our proposes and therefore, for simplicity, our consequent results will also be formulated for this interval. In addition, for a matter of notation simplicity, from now on we will avoid to exhibit the notation of the zero extension operator in the appropriate places of the operators multiplication - considering that it is clear where this trivial operator is in action.

3.1. The Problem $\mathcal{P}_{D\text{-}D}$

Let us look for a solution of $\mathcal{P}_{D\text{-}D}$ problem in the following form

$$v(x) = W\varphi(x), \quad x \in \Omega, \tag{3.4}$$

where W is the double layer potential on $\partial\Omega$ and $\varphi \in H^{\frac{1}{2}+\varepsilon}(\partial\Omega)$ is an unknown function. Setting

$$\varphi_1 := r_{S_{1,+}}\varphi \quad \text{and} \quad \varphi_2 := r_{S_{2,+}}\varphi \tag{3.5}$$

we have that the unknown functions $\varphi_1 \in H^{\frac{1}{2}+\varepsilon}(S_{1,+})$ and $\varphi_2 \in H^{\frac{1}{2}+\varepsilon}(S_{2,+})$ satisfy the compatibility condition $\varphi_1 - \chi_*\varphi_2 \in H^{\frac{1}{2}+\varepsilon}_{\mathbb{R}_+}$. Clearly, the functions v is an element of the space $H^{1+\varepsilon}(\Omega)$ and satisfy (2.1) in Ω. Further, from the given boundary conditions on $S_{1,+}$ and $S_{2,+}$, we obtain (cf. (2.7)) the following equations

$$\frac{1}{2}r_{S_{1,+}}\varphi_1 + [W_2\varphi_2]^+_{S_{1,+}} = g_1$$

and

$$[W_1\varphi_1]^+_{S_{2,+}} + \frac{1}{2}r_{S_{2,+}}\varphi_2 = g_2.$$

Thus we get a system of equations with respect to φ_1 and φ_2

$$\begin{cases} r_{S_{1,+}}\varphi_1 + 2[W_2\varphi_2]^+_{S_{1,+}} &= 2g_1\,, \\ 2[W_1\varphi_1]^+_{S_{2,+}} + r_{S_{2,+}}\varphi_2 &= 2g_2\,. \end{cases} \tag{3.6}$$

The sum and the difference of the first equation and the pull back of the second equation in (3.6), due to (3.3), gives us that the system (3.6) is equivalent to the following system of equations on $\mathbb{R}_+$:

$$\begin{cases} r_{S_{1,+}}\phi_1 + 2\chi_* r_{S_{2,+}} W_1\phi_1 &= 2g_1 + 2\chi_* g_2\,, \\ r_{S_{1,+}}\phi_2 - 2\chi_* r_{S_{2,+}} W_1\phi_2 &= 2g_1 - 2\chi_* g_2\,, \end{cases} \tag{3.7}$$

where $\phi_1 := \varphi_1 + \chi_*\varphi_2$ and $\phi_2 := \varphi_1 - \chi_*\varphi_2$.

The solvability of the obtained system is equivalent to the solvability of both equations in (3.7). Thus we need to study the invertibility of operators (see also (2.5))

$$\mathcal{A}_+ : H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+), \tag{3.8}$$

and

$$\mathcal{A}_- : \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+), \tag{3.9}$$

where

$$\mathcal{A}_\pm\psi := r_{\mathbb{R}_+}\psi \pm 2\chi_* r_{S_{2,+}} W_1\psi. \tag{3.10}$$

Then φ_1 and φ_2 can be recovered by

$$\varphi_1 = (\phi_1 + \phi_2)/2, \quad \varphi_2 = \chi_*^{-1}((\phi_1 - \phi_2)/2).$$

Lemma 3.3. *The operators $\mathcal{A}_\pm$ in* (3.8), (3.9) *have trivial kernels, i.e.,*

$$\mathrm{dimKer}\mathcal{A}_\pm = 0.$$

Proof. It suffices to show that the system (3.6) has at most one pair (φ_1, φ_2) of solutions, i.e., the corresponding homogeneous system

$$\begin{cases} \frac{1}{2} r_{S_{1,+}}\varphi_1 + [W_2\varphi_2]^+_{S_{1,+}} &= 0\,, \\ [W_2\varphi_1]^+_{S_{2,+}} + \frac{1}{2} r_{S_{2,+}}\varphi_2 &= 0\,, \end{cases} \tag{3.11}$$

has only the trivial solution, which is an easy consequence of Theorem 2.1 and the exhibited limit relations of the potentials. Indeed, let (φ_1, φ_2) be a non-trivial solution of (3.11), thus $\varphi \not\equiv 0$ (cf. (3.4)). Then the function

$$v(x) = W\varphi(x), \quad x \in \Omega \cup (\mathbb{R}^2 \backslash \overline{\Omega}),$$

solves the Helmholtz equation in Ω with zero Dirichlet boundary conditions on $S_{1,+} \cup S_{2,+}$. Then due to Theorem 2.1 the function $v(x) \equiv 0$, $x \in \Omega$, and therefore its Neumann data are equal to zero. Moreover, v solves the Helmholtz equation in $\mathbb{R}^2 \backslash \overline{\Omega}$ with zero Neumann boundary conditions (since $\partial_n W$ is continuous) and therefore $v(x) \equiv 0$, $x \in \mathbb{R}^2 \backslash \overline{\Omega}$. This implies that zero Dirichlet data on $\partial \mathbb{R}^2 \backslash \overline{\Omega} = \partial\Omega$. Then we obtain $\varphi = [u]^+_{\partial\Omega} - [u]^-_{\partial\Omega} = 0$. Consequently the homogeneous system (3.7) has only trivial solutions, i.e., $\dim\mathrm{Ker}\mathcal{A}_\pm = 0$. $\square$

3.2. The Problem $\mathcal{P}_{N\text{-}N}$

The representation formula for any solution of the $\mathcal{P}_{N\text{-}N}$ problem suggests us to look for a solution as follows

$$v(x) = W\varphi(x) - Vf(x), \quad x \in \Omega,$$

where $\varphi \in H^{\frac{1}{2}+\varepsilon}(S_{1,+})$ is an unknown Dirichlet datum of v. Thus on $S_{1,+}$ we have (cf. (3.5))

$$[v]^+_{S_{1,+}} = \varphi_1 = \frac{1}{2}\varphi_1 + r_{S_{1,+}} W_2 \varphi_2 - r_{S_{1,+}} Vf,$$

which give us an equation $\varphi_1 - 2r_{S_{1,+}} W_2\varphi_2 = -2r_{S_{1,+}} Vf$. Similarly, on $S_{2,+}$ we obtain $\varphi_2 - 2r_{S_{2,+}} W_1\varphi_1 = -2r_{S_{2,+}} Vf$. Thus we equivalently reduce the $\mathcal{P}_{N\text{-}N}$ problem to the following system of equations

$$\begin{cases} \varphi_1 - 2r_{S_{1,+}} W_2\varphi_2 &= -2r_{S_{1,+}} Vf\,, \\ \varphi_2 - 2r_{S_{2,+}} W_1\varphi_1 &= -2r_{S_{2,+}} Vf\,. \end{cases} \tag{3.12}$$

Arguing as above we take the sum and the difference of the first equation and the pull back of the second equation in (3.12) and get the following equivalent system of equations on $\mathbb{R}_+$

$$\begin{cases} \phi_1 - 2\chi_* r_{S_{2,+}} W_1\phi_1 &= -2r_{S_{1,+}} Vf - 2\chi_* r_{S_{2,+}} Vf\,, \\ \phi_2 + 2\chi_* r_{S_{2,+}} W_1\phi_2 &= -2r_{S_{2,+}} Vf + 2\chi_* r_{S_{2,+}} Vf \end{cases} \tag{3.13}$$

where $\phi_1 := \varphi_1 + \chi_*\varphi_2 \in H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+)$ and $\phi_2 := \varphi_1 - \chi_*\varphi_2 \in \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+)$ due to (2.5). Note also that $-2r_{S_{2,+}} Vf + 2\chi_* r_{S_{2,+}} Vf \in \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+)$, cf. [25] or [16]. Thus we need to study the invertibility of operators

$$\mathcal{A}_- : H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+), \tag{3.14}$$

and

$$\mathcal{A}_+ : \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+) \longrightarrow \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+). \tag{3.15}$$

The functions φ_1 and φ_2 can be recovered by

$$\varphi_1 = (\phi_1 + \phi_2)/2, \quad \varphi_2 = \chi_*^{-1}((\phi_1 - \phi_2)/2).$$

The following lemma is a consequence of the uniqueness Theorem 2.1 due to equivalent reduction of the $\mathcal{P}_{N\text{-}N}$ problem to the system of equations (3.12).

Lemma 3.4. *The operators $\mathcal{A}_\pm$ in* (3.14), (3.15) *have trivial kernels, i.e.,*

$$\mathrm{dimKer}\mathcal{A}_\pm = 0.$$

Thus, having in mind the desired conclusions for Problem $\mathcal{P}_{D\text{-}D}$ and Problem $\mathcal{P}_{N\text{-}N}$, we realize that we need to study invertibility of the operators $\mathcal{A}_\pm$ in both spaces $H^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+)$ and $\widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+)$.

3.3. The Problem $\mathcal{P}_{mixed}$

In the present paper, for simplicity, we investigate the $\mathcal{P}_{mixed}$ problem only for $0 < \alpha < \pi$. The case $\pi < \alpha < 2\pi$ involves different operators which need additional investigation and, therefore, it will be considered in a forthcoming paper.

Let us first consider the following auxiliary Dirichlet problem for the plane angle $\Omega_{2\alpha}$ of magnitude 2α with the following boundary data

$$[u]^+_{S_{1,+}} = \tilde{g}_1 \quad \text{on } S_{1,+} \quad \text{and} \qquad [u]^+_{S^\star_{2,+}} = \chi^{-1}_{2\alpha,*}\,\tilde{g}_1 \quad \text{on } S^\star_{2,+},$$

where $\tilde{g}_1$ is an arbitrary element of $H^{\frac{1}{2}+\varepsilon}(S_{1,+})$,

$$S^\star_{2,+} := \{(t\cos 2\alpha, t\sin 2\alpha) : t \in \mathbb{R}_+\}$$

and $\chi^{-1}_{2\alpha,*}\,\tilde{g}_1 \in H^{\frac{1}{2}+\varepsilon}(S^\star_{2,+})$. As we will see below (cf. Theorem 5.3) this problem is uniquely solvable. It turns out that $[\partial_{n_2} u]_{S_{2,+}} = 0$. Indeed, since the problem is invariant under the rotation we may assume that $S_{2,+}$ coincides with the positive ordinate half-axis, which makes the problem symmetric with respect to the ordinate axis (of the Cartesian plane). This implies that $u(x_1, x_2)$ and $u(-x_1, x_2)$ are solutions of the same Dirichlet problems and due

to the uniqueness results they coincide. Therefore $[\partial_{n_2} u]^+_{S_{2,+}} = [-\partial_{x_1} u]^+_{S_{2,+}} = [\partial_{x_1} u]^+_{S_{2,+}}$, which gives us $[\partial_{n_2} u]_{S_{2,+}} = 0$.

From this observation we immediately have that the problem $\mathcal{P}_{mixed}$ has a unique solution which is represented as

$$v(x) = u(x) - 2V_2 \ell f_2(x), \quad x \in \Omega,$$

where u is a solution of $\mathcal{P}_{D\text{-}D}$ problem in the plane angle of magnitude 2α with the following Dirichlet data $\tilde{g}_1 := g_1 + 2[V_2 \ell f_2]^+_{S_{1,+}}$ on $S_{1,+}$ and $\tilde{g}_2 := \chi_*^{-1} \tilde{g}_1$ on $S^\star_{2,+}$; here $\ell f_2 \in H^{-\frac{1}{2}+\varepsilon}(S_2)$ is any fixed extension of the generalized function $f_2 \in H^{-\frac{1}{2}+\varepsilon}(S_{2,+})$.

4. Analysis of Associated Operators in Weighted Sobolev Spaces

Let

$$Mu(z) = \int_0^\infty t^{z-1} u(t) dt$$

be the Mellin transform on the half-axis $\mathbb{R}_+ \ni t$, first defined for functions $C_0^\infty(\mathbb{R}_+)$. For the inverse, we have

$$M^{-1} g(t) = (2\pi i)^{-1} \int_{\Gamma_\beta} t^{-z} g(z) dz$$

for some $\beta \in \mathbb{R}$, where $\Gamma_\beta = \{z \in \mathbb{C} : \Re\mathrm{e}\, z = \beta\}$ and $g(z) = Mu(z)$.

Define the space $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ for $s, \gamma \in \mathbb{R}$ to be the completion of $C_0^\infty(\mathbb{R}_+)$ with respect to the norm $||\langle z\rangle^s Mu(z)|_{\Gamma_{1/2-\gamma}}||_{L_2(\Gamma_{1/2-\gamma})}$, where $\langle z\rangle := (1 + |z|^2)^{1/2}$ and $L_2(\Gamma_\beta)$ is the space of square integrable functions with respect to $d\xi$, $\xi = \Im\mathrm{m}\, z$. Note that $\mathcal{H}^{0,\gamma}(\mathbb{R}_+) = t^\gamma L_2(\mathbb{R}_+)$. This definition shows that the weighted Mellin transform $M_\gamma : u \to Mu|_{\Gamma_{1/2-\gamma}}$ extends from $C_0^\infty(\mathbb{R}_+)$ to an isomorphism

$$M_\gamma : \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \langle z\rangle^{-s} L_2(\Gamma_{1/2-\gamma}).$$

Finally, we define the cone Sobolev spaces as a mixture between the spaces $\mathcal{H}^{s,\gamma}(\mathbb{R}_+)$ and $H^s(\mathbb{R}_+)$, namely,

$$\mathcal{K}^{s,\gamma}(\mathbb{R}_+) := \{\omega u + (1-\omega) v : u \in \mathcal{H}^{s,\gamma}(\mathbb{R}_+), v \in H^s(\mathbb{R}_+)\}$$

for a fixed cut-off function ω. Throughout the paper a function $\omega \in C^\infty(\overline{\mathbb{R}}_+)$ is called a cut-off function (with respect to $t = 0$) if $\operatorname{supp} \omega$ is bounded and $\omega \equiv 1$ near $t = 0$.

Let us note that the space $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ is independent of the particular choice of ω. Each $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ can be endowed with a Banach space norm which is generated by a Hilbert space scalar product, for more details and properties of these spaces, we refer to [61, 62].

Now we shall formulate a result from [63, Section 2.3.1] which shows that the $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ spaces are a natural modification of the "usual" Sobolev spaces. Let $s \in \mathbb{R}$, $\kappa(s) := \max\{j \in \mathbb{N} : j < |s| - 1/2\}$, and

$$\mathcal{T}^s := \{\text{linear span of } t^j\omega(t) \text{ for } j = 0, \dots, \kappa(s)\}$$

for $s > 1/2$, $\mathcal{T}^s := \{0\}$ for $s \le 1/2$, ω a fixed cut-off function. Further set

$$\mathcal{D}^s := \{\text{linear span of } (d/dt)^j\delta_0 \text{ for } j = 0, \dots, \kappa(s)\}$$

for $s < -1/2$, $\mathcal{D}^s := \{0\}$ for $s \ge -1/2$, with δ_0 being the Dirac delta function at $t = 0$.

Proposition 4.1. *Let $s \in \mathbb{R}$, then there are canonical isomorphisms*

$$H^s(\mathbb{R}_+) \cong \begin{cases} \mathcal{K}^{s,s}(\mathbb{R}_+) + \mathcal{T}^s & \textit{for} \quad s \ge 0,\ s \ne 1/2 \bmod \mathbb{Z}, \\ \mathcal{K}^{s,s}(\mathbb{R}_+) & \textit{for} \quad s \le 0 \end{cases}$$

$$\widetilde{H}^s(\mathbb{R}_+) \cong \begin{cases} \mathcal{K}^{s,s}(\mathbb{R}_+) & \textit{for} \quad s \ge 0, \\ \mathcal{K}^{s,s}(\mathbb{R}_+) + \mathcal{D}^s & \textit{for} \quad s \le 0, s \ne 1/2 \bmod \mathbb{Z}. \end{cases}$$

The isomorphism for $H^s(\mathbb{R}_+)$ follows by identifying distributions on $\mathbb{R}_+$ and that for $\widetilde{H}^s(\mathbb{R}_+)$ by duality. The identifications are continuous (in both directions).

Further, for $a, b \in \mathbb{R}$ we set $S(a, b) := \{z \in \mathbb{C} : a < \Re\mathrm{e}\, z < b\}$ and denote by $\mathcal{M}^{-\infty}_{\mathcal{O}}(S(a, b))$ the subspace of all holomorphic functions $h(z)$ on $S(a, b)$ with $h|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ (the Schwartz space on the weight line Γ_β), for every β uniformly in $c \le \beta \le c'$ for every $a \le c \le c' \le b$.

For our proposes below let us show that

$$h(z) := \frac{\sin((\pi - \alpha)z)}{\sin(\pi z)} \in \mathcal{M}^{-\infty}_{\mathcal{O}}(S(-1, 1)). \tag{4.1}$$

Indeed, first note that the sine function is an entire function. The Weierstrass factorization

$$\sin(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2\,\pi^2}\right)$$

indicates that the function $h(z)$ has no poles at $z = 0$ and therefore it is holomorphic on $S(-1,1)$. Finally, since $-\pi < \pi - \alpha < \pi$ and

$$h(z) = \frac{e^{iz(\pi-\alpha)} - e^{-iz(\pi-\alpha)}}{e^{iz\pi} - e^{-iz\pi}}$$

we see that $h(z)$ tends exponentially to zero when $\Im\mathrm{m}\, z \to \pm\infty$, thus $h|_{\Gamma_\beta} \in \mathcal{S}(\Gamma_\beta)$ for every β uniformly in $c \leq \beta \leq c'$ for every $-1 \leq c \leq c' \leq 1$. Let $-\frac{1}{2} < \gamma < \frac{3}{2}$, then the weighted Mellin pseudo-differential operator

$$\mathrm{op}_M^\gamma(h) := M_\gamma^{-1} h(z) M_\gamma : \mathcal{H}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{H}^{s,\gamma}(\mathbb{R}_+)$$

is continuous for all $s \in \mathbb{R}$.

Lemma 4.2. *Let $0 < s < 1$, then the operator*

$$\chi_* r_{S_{2,+}} W_1 : \mathcal{K}^{s,s}(\mathbb{R}_+) \to \mathcal{K}^{s,s}(\mathbb{R}_+) \tag{4.2}$$

is continuous. Moreover, for any fixed cut-off functions ω_1 and ω_2 the operators

$$\begin{aligned} (1-\omega_1)\chi_* r_{S_{2,+}} W_1 \omega_2 &: \mathcal{K}^{s,s}(\mathbb{R}_+) \to \mathcal{K}^{s,s}(\mathbb{R}_+), \\ (1-\omega_1)\chi_* r_{S_{2,+}} W_1 (1-\omega_2) &: \mathcal{K}^{s,s}(\mathbb{R}_+) \to \mathcal{K}^{s,s}(\mathbb{R}_+), \\ \omega_1 \chi_* r_{S_{2,+}} W_1 (1-\omega_2) &: \mathcal{K}^{s,s}(\mathbb{R}_+) \to \mathcal{K}^{s,s}(\mathbb{R}_+) \end{aligned}$$

are compact.

Proof. The continuity result for the operator in (4.2) immediately follows from Lemma 3.2 and Proposition 4.1.

Further, since $\Im\mathrm{m}\, k > 0$ the function $r_{S_{2,+}} W_1 \varphi(x)$ exponentially tends to 0 as $|x| \to \infty$, for $x \in S_{2,+}$. Clearly, the same is true for the function $\chi_* r_{S_{2,+}} W_1 \varphi(t)$ when $t \to \infty$. Having in mind the isomorphism from Proposition 4.1 we get that the operators

$$(1-\omega_1)\chi_* r_{S_{2,+}} W_1 \omega_2 : \mathcal{K}^{s,s}(\mathbb{R}_+) \to \langle t \rangle^{-N} \mathcal{K}^{\infty,\infty}(\mathbb{R}_+)$$

and

$$(1-\omega_1)\chi_* r_{S_{2,+}} W_1 (1-\omega_2) : \mathcal{K}^{s,s}(\mathbb{R}_+) \to \langle t \rangle^{-N} \mathcal{K}^{\infty,\infty}(\mathbb{R}_+)$$

are continuous for any $N \in \mathbb{N}$. For the operator $\omega_1 \chi_* r_{S_{2,+}} W_1 (1-\omega_2)$ we have that at infinity it is identically zero due to the cut-off function ω_1, while

the presence of $(1-\omega_2)$ ensures the smoothness of the kernel. Therefore, we obtain the fact that the operator

$$\omega_1\chi_* r_{S_{2,+}} W_1(1-\omega_2) : \mathcal{K}^{s,s}(\mathbb{R}_+) \to \langle t\rangle^{-N}\mathcal{K}^{\infty,\infty}(\mathbb{R}_+)$$

is continuous. Then the final conclusions follow from the composition with the compact embedding of $\langle t\rangle^{-N}\mathcal{K}^{\infty,\infty}(\mathbb{R}_+)$ into $\mathcal{K}^{s,s}(\mathbb{R}_+)$. □

Theorem 4.3. *Let $0 < s < 1$ and ω_1, ω_2 be arbitrary cut-off functions, then the operator*

$$T := \chi_* r_{S_{2,+}} W_1 - \frac{1}{2}\omega_1 \mathrm{op}_M^s(h)\omega_2 \ : \mathcal{K}^{s,s}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{s,s}(\mathbb{R}_+) \tag{4.3}$$

is compact.

Proof. For (4.3) we have a decomposition $\chi_* r_{S_{2,+}} W = \omega_1\chi_* r_{S_{2,+}} W\omega_2 + T_1$, where

$$T_1 := \omega_1\chi_* r_{S_{2,+}} W(1-\omega_2) + (1-\omega_1)\chi_* r_{S_{2,+}} W\omega_2 + (1-\omega_1)\chi_* r_{S_{2,+}} W(1-\omega_2)$$

is a compact operator between the spaces $\mathcal{K}^{s,s}(\mathbb{R}_+)$; cf. Lemma 4.2. Further, for the points $x = (\tau\cos\alpha, \tau\sin\alpha) \in S_{2,+}$, $\tau \in \mathbb{R}_+$, let us write

$$r_{S_{2,+}} W\varphi(x) = r_{S_{2,+}} W_0\varphi(x) + T_2(x),$$

where

$$W_0\varphi(x_1,x_2) = \frac{1}{2\pi}\int_0^\infty (\partial_{n_1(t,0)}\ln)(|x_1-t, x_2|)\varphi(t)dt,$$

and

$$T_2(x_1,x_2) := \int_0^\infty (\partial_{n_1(t,0)} m)(|x_1-t, x_2|)\varphi(t)dt$$

with the function $m(x) := \Phi(x) - \frac{1}{2\pi}\ln(|x|)$, which has the following behavior as $|x| \to 0$ (cf. [24], [38])

$$m(|x|) = const + O(|x|^2\ln(|x|)),\ m'(|x|) = O(|x|\ln(|x|)),\ m''(|x|) = O(\ln(|x|)).$$

These estimates together with the cut-off functions ω_1 and ω_2 give us that the kernel of the operator $\omega_1\chi_* r_{S_{2,+}} T_2\omega_2$ is square integrable and therefore it is a compact operator between the spaces $\mathcal{K}^{s,s}(\mathbb{R}_+)$ as well as between the spaces $L_2(\mathbb{R}_+)$. Now it remains to show the equality

$$\chi_* r_{S_{2,+}} W_0 = \frac{1}{2}\mathrm{op}_M^s(h)$$

while $T = T_1 + \omega_1 T_2 \omega_2$.

We have

$$\begin{aligned}
\chi_* r_{S_{2,+}} W_0 \varphi(\tau) &= \frac{1}{2\pi} \int_0^\infty \frac{\tau \sin\alpha}{(\tau\cos\alpha - t)^2 + (\tau \sin\alpha)^2} \varphi(t) dt \\
&= \frac{\sin\alpha}{2\pi} \int_0^\infty \frac{\frac{\tau}{t}}{(\frac{\tau}{t})^2 - 2(\frac{\tau}{t})\cos\alpha + 1} \varphi(t) \frac{dt}{t} \\
&= \frac{1}{2} M_s^{-1}[h(z)(M_s\varphi)(z)] = \frac{1}{2}[\mathrm{op}_M^s(h)\varphi](\tau),
\end{aligned}$$

where

$$\begin{aligned}
h(z) &= \frac{\sin\alpha}{\pi} \int_0^\infty t^z \frac{t}{t^2 - 2t\cos\alpha + 1} \frac{dt}{t} \\
&= \frac{\sin\alpha}{\pi} \int_0^\infty \frac{t^z}{t^2 + 2t\cos(\pi - \alpha) + 1} dt \\
&= \frac{\sin((\pi - \alpha)z)}{\sin(\pi z)}
\end{aligned}$$

provided $-1 < \Re\mathrm{e}\, z < 1$, cf. [24, Formula 3.252.12]. □

5. Main Results

In this last section we will perform a Fredholm theory analysis of the previously derived operators, and this will generate the main conclusions for the problems under study.

Theorem 5.1. *For any aperture angle α, let ω_1, ω_2 be arbitrary cut-off functions, $s \in \mathbb{R}$ and $0 \leq \gamma \leq 1$. Then, the operators*

$$\mathcal{A}_\pm^0 := I \pm \omega_1 \mathrm{op}_M^\gamma(h)\omega_2 : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$$

are continuous. Moreover, they are Fredholm operators of index zero.

Proof. The continuity results follow from the properties of the spaces $\mathcal{K}^{s,\gamma}(\mathbb{R}_+)$ and the Mellin pseudo-differential operators $\mathrm{op}_M^\gamma(h)$ with the symbol h for $0 \leq \gamma \leq 1$, cf. [30, 62]. Moreover, it is well-known (cf. [23], [63], or [30, Section 2.1.9]) that the condition $1 \pm h(z) \neq 0$ for all $z \in \Gamma_{\frac{1}{2}-\gamma}$, cf. (4.1), implies that

$$I \pm \omega_1 \mathrm{op}_M^\gamma(h)\omega_2 : \mathcal{K}^{s,\gamma}(\mathbb{R}_+) \longrightarrow \mathcal{K}^{s,\gamma}(\mathbb{R}_+)$$

is Fredholm, and

$$\operatorname{ind}\left(I \pm \omega_1 \mathrm{op}_M^{\gamma}(h)\omega_2\right) = \frac{1}{2\pi}\mathbf{\Delta}\arg(1 \pm h(z))\Big|_{\Gamma_{\frac{1}{2}-\gamma}}\Big|_{\Im\mathrm{m}\, z=-\infty}^{\Im\mathrm{m}\, z=+\infty},$$

where the $\mathbf{\Delta}$ indicates the change of the arguments of $1 + h(z)$ when z runs from $\Im\mathrm{m}\, z = -\infty$ to $\Im\mathrm{m}\, z = +\infty$ on the line $\Gamma_{\frac{1}{2}-\gamma}$. Note that $\mathrm{Ker}\mathcal{A}_{\pm}^0$ and $\mathrm{CoKer}\mathcal{A}_{\pm}^0$ are independent of s.

The identity $|\sin z| = |\sin(\Re\mathrm{e}\, z) + i\sinh(\Im\mathrm{m}\, z)|$ implies

$$|\Re\mathrm{e}\, h(z)| \leq |h(z)| = \frac{|\sin((\pi-\alpha)\Re\mathrm{e}\, z) + i\sinh((\pi-\alpha)\Im\mathrm{m}\, z)|}{|\sin(\pi\Re\mathrm{e}\, z) + i\sinh(\pi\Im\mathrm{m}\, z)|} < 1 \quad (5.1)$$

provided $|\Re\mathrm{e}\, z| \leq \frac{1}{2}$. Indeed, for such z we have $|(\pi-\alpha)\Re\mathrm{e}\, z| < |\pi\Re\mathrm{e}\, z| \leq \frac{\pi}{2}$ and therefore $\sin^2((\pi-\alpha)\Re\mathrm{e}\, z) < \sin^2(\pi\Re\mathrm{e}\, z)$ while $\sinh^2((\pi-\alpha)\Im\mathrm{m}\, z) < \sinh^2(\pi\Im\mathrm{m}\, z)$, for all $\Im\mathrm{m}\, z \in \mathbb{R}$. The estimate (5.1) gives us

$$\Re\mathrm{e}\,(1 \pm h(z)) > 0, \quad \text{for all } z \in \Gamma_{\frac{1}{2}-\gamma},\ 0 \leq \gamma \leq 1. \quad (5.2)$$

Thus, the closed curve

$$\mathcal{C}_\gamma := \{1 \pm h(z) \in \mathbb{C} : z \in \Gamma_{\frac{1}{2}-\gamma}\} \cup \{(1;0)\} \in \mathbb{C}\backslash\{0\} \qquad (0 \leq \gamma \leq 1)$$

does not intersect the imaginary line due to (5.2), and therefore

$$\mathbf{\Delta}\arg(1 \pm h(z))\Big|_{\Gamma_{\frac{1}{2}-\gamma}}\Big|_{\Im\mathrm{m}\, z=-\infty}^{\Im\mathrm{m}\, z=+\infty} = 0.$$

This implies that operators $\mathcal{A}_{\pm}^0$ are Fredholm of index zero. □

Note that for the very special case of an angle aperture $\alpha = \pi$, the above closed curve $\mathcal{C}_\gamma$ degenerates to the particular case of the single point $(1;0) \in \mathbb{C}$. This in fact reflects the simplicity of the geometrical case $\alpha = \pi$, which coincides with the classical Sommerfeld situation of diffraction by an half-plane, for which the well-posedness and closed-form solution are well-known in a Sobolev space setting for a long time; cf. [6, 43, 47, 68].

Theorem 5.2. *The operators* (3.10)

$$\mathcal{A}_{\pm} : \widetilde{H}^s(\mathbb{R}_+) \longrightarrow \widetilde{H}^s(\mathbb{R}_+)$$

and

$$\mathcal{A}_{\pm} : H^s(\mathbb{R}_+) \longrightarrow H^s(\mathbb{R}_+)$$

are invertible for all $\frac{1}{2} - \delta < s < 1$, *where* $\delta > 0$ *is sufficiently small .*

Proof. The invertibility of the operators $\mathcal{A}_\pm$ in $\widetilde{H}^s(\mathbb{R}_+)$ spaces is a direct consequence of Theorem 5.1 together with Proposition 4.1, Theorem 4.3, Lemma 3.3, and Lemma 3.4 for $\frac{1}{2} \leq s < 1$, while for the case $\frac{1}{2} - \delta < s < \frac{1}{2}$ it follows from the classical result of Shneiberg, cf. [64, 65], which states that if an operator A is bounded on a complex interpolation scale $\{X\}_{0\leq\theta\leq 1}$ and it is invertible on an individual space X_{θ_0}, $0 < \theta_0 < 1$, then it is also invertible on X_θ, for $|\theta - \theta_0| < \delta$, where $\delta > 0$ is sufficiently small. Indeed, setting $X_0 := \widetilde{H}^{\frac{1}{2}-\varepsilon}(\mathbb{R}_+)$, $X_1 := \widetilde{H}^{\frac{1}{2}+\varepsilon}(\mathbb{R}_+)$ (see also Lemma 3.2), we have that the operator $\mathcal{A}_\pm$ is invertible on X_{θ_0}, for the $\theta_0 = \frac{1}{2}$ space, i.e, on the $\widetilde{H}^{\frac{1}{2}}(\mathbb{R}_+)$ space, and therefore it is invertible for $\frac{1}{2} - \delta < s < \frac{1}{2}$ too.

For the second result let us mention that for $s \in (\frac{1}{2} - \delta, \frac{1}{2})$ the spaces $\widetilde{H}^s(\mathbb{R}_+)$ and $H^s(\mathbb{R}_+)$ are isomorphic, therefore if we prove the result for $s \in (\frac{1}{2}, 1)$ then the result for the hole range will follow by interpolation. The mapping properties of the potential operators we have continuity result in $H^s(\mathbb{R}_+)$ spaces for the operators $\mathcal{A}_\pm$ as well as for the operators $\omega_1 \mathrm{op}_M^s(h)\omega_2$, for arbitrary cut-off functions ω_1 and ω_2 with the property $\omega_2\omega = \omega$, where ω is a fixed cut-off function from $\mathcal{T}^s$, cf. Proposition 4.1. Now setting

$$\tilde{\omega} := \omega_1 \mathrm{op}_M^s(h)\omega_2\omega,$$

then it follows by the calculus of residues that, cf. [16],

$$\tilde{\omega}(0) = \frac{\alpha - \pi}{\pi}.$$

This implies that $\tilde{\omega} \in \mathcal{T}^s$. Therefore for the fixed functions

$$\tilde{\omega}_\pm := \mathcal{A}_\pm \omega$$

we have

$$\tilde{\omega}_+(0) = \frac{\alpha}{\pi}, \qquad \tilde{\omega}_-(0) = \frac{2\pi - \alpha}{\pi}.$$

Further, for an arbitrary element $u = u_0 + \lambda\omega \in H^s(\mathbb{R}_+)$, with $u_0 \in \widetilde{H}^s(\mathbb{R}_+)$ and $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} \mathcal{A}_\pm u &= \mathcal{A}_\pm u_0 + \lambda \mathcal{A}_\pm \omega = \mathcal{A}_\pm u_0 + \tilde{\omega}_\pm \\ &= \mathcal{A}_\pm u_0 + \lambda(\tilde{\omega}_\pm - \tilde{\omega}_\pm(0)\omega) + \lambda\tilde{\omega}_\pm(0)\omega = \tilde{u}_0 + \tilde{\lambda}\omega, \end{aligned}$$

where $\tilde{u}_0 = \mathcal{A}_\pm u_0 + \lambda(\tilde{\omega}_\pm - \tilde{\omega}_\pm(0)\omega) \in \widetilde{H}^s(\mathbb{R}_+)$ and $\tilde{\lambda} = \lambda\tilde{\omega}_\pm(0)$. Due to the invertibility of the operators $\mathcal{A}_\pm$ in $\widetilde{H}^s(\mathbb{R}_+)$ spaces the obtained relations can

be also written as

$$\lambda = \frac{\tilde{\lambda}}{\tilde{\omega}_{\pm}(0)}, \quad u_0 = \mathcal{A}_{\pm}^{-1}(\tilde{u}_0 - \lambda(\tilde{\omega}_{\pm} - \tilde{\omega}_{\pm}(0)\omega)) \in \widetilde{H}^s(\mathbb{R}_+),$$

which show that the operators $\mathcal{A}_{\pm}$ in $H^s(\mathbb{R}_+)$ spaces are bijective. □

Due to a direct combination of the results obtained above we have now the main conclusions of the present work for the problems in consideration.

Theorem 5.3. *If $0 \leq \varepsilon < 1/2$, then the problem $\mathcal{P}_{D\text{-}D}$ has a unique solution which is represented as*

$$v(x) = W\varphi(x), \quad x \in \Omega,$$

where the functions $\varphi_1 = r_{S_{1,+}}\varphi \in H^{\frac{1}{2}+\varepsilon}(S_{2,+})$ and $\varphi_2 = r_{S_{2,+}}\varphi \in H^{\frac{1}{2}+\varepsilon}(S_{2,+})$ are unique solutions of the system of equations (3.6)*, namely,*

$$\varphi_1 = \mathcal{A}_{+}^{-1}(g_1 + \chi_* g_2) + \mathcal{A}_{-}^{-1}(g_1 - \chi_* g_2)$$

and

$$\varphi_2 = \chi_*^{-1}\left(\mathcal{A}_{+}^{-1}(g_1 + \chi_* g_2) - \mathcal{A}_{-}^{-1}(g_1 - \chi_* g_2)\right),$$

where $\mathcal{A}_{\pm}^{-1}$ denote the inverse operators of $\mathcal{A}_{\pm}$, respectively.

Theorem 5.4. *If $0 \leq \varepsilon < 1/2$, then the problem $\mathcal{P}_{N\text{-}N}$ has a unique solution which is represented as*

$$v(x) = W\varphi(x) - Vf(x), \qquad x \in \Omega,$$

where the functions $\varphi_1 = r_{S_{1,+}}\varphi \in H^{\frac{1}{2}+\varepsilon}(S_{2,+})$ and $\varphi_2 = r_{S_{2,+}}\varphi \in H^{\frac{1}{2}+\varepsilon}(S_{2,+})$ are unique solutions of the system of equations (3.12)*. Namely,*

$$\varphi_1 = \mathcal{A}_{+}^{-1}(r_{S_{2,+}}Vf - 2\chi_* r_{S_{2,+}}Vf) - \mathcal{A}_{-}^{-1}(r_{S_{1,+}}Vf + \chi_* r_{S_{2,+}}Vf)$$

and

$$\varphi_2 = -\chi_*^{-1}\left(\mathcal{A}_{+}^{-1}(r_{S_{2,+}}Vf - 2\chi_* r_{S_{2,+}}Vf) + \mathcal{A}_{-}^{-1}(r_{S_{1,+}}Vf + \chi_* r_{S_{2,+}}Vf)\right),$$

where $\mathcal{A}_{\pm}^{-1}$ denote the inverse operators of $\mathcal{A}_{\pm}$, respectively.

Theorem 5.5. *Let $0 < \alpha < \pi$ and $0 \le \varepsilon < 1/2$. Then the problem $\mathcal{P}_{\text{mixed}}$ has a unique solution which is represented as*

$$v(x) = u(x) - 2V_2\ell f_2(x), \quad x \in \Omega_\alpha,$$

where $u(x)$ is a solution of $\mathcal{P}_{\text{D-D}}$ problem in the plane angle $\Omega_{2\alpha}$ of magnitude 2α which is represented with the help of the double layer potential on $\partial\Omega_{2\alpha}$ as follows

$$u(x) = W\varphi(x), \quad x \in \Omega_{2\alpha};$$

here the functions $\varphi_1 = r_{S_{1,+}}\varphi \in H^{\frac{1}{2}+\varepsilon}(S_{2,+})$ and $\varphi_2 = r_{S^*_{2,+}}\varphi \in H^{\frac{1}{2}+\varepsilon}(S^*_{2,+})$ are unique solutions of the system of equations (3.6), namely,

$$\varphi_1 = 2\mathcal{A}_+^{-1}\tilde{g}_1 \quad \text{and} \quad \varphi_2 = 2\chi_{2\alpha,*}^{-1}\mathcal{A}_+^{-1}\tilde{g}_1, \tag{5.3}$$

where $\mathcal{A}_\pm^{-1}$ are the inverse operators of $\mathcal{A}_\pm$, respectively, $\tilde{g}_1 := g_1 + 2[V_2\ell f_2]^+_{S_{1,+}}$, for some fixed extension $\ell f_2 \in H^{-\frac{1}{2}+\varepsilon}(S_2)$ of the generalized function $f_2 \in H^{-\frac{1}{2}+\varepsilon}(S_{2,+})$ and $S^\star_{2,+} = \{(t\cos 2\alpha, t\sin 2\alpha) : t \in \mathbb{R}_+\}$.

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