

COPIES OF A ROOTED WEIGHTED GRAPH ATTACHED TO AN ARBITRARY WEIGHTED GRAPH AND APPLICATIONS*

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Abstract. The spectrum of the Laplacian, signless Laplacian and adjacency matrices of the family of the weighted graphs $\mathcal{R}\{\mathcal{H}\}$, obtained from a connected weighted graph \mathcal{R} on r vertices and r copies of a modified Bethe tree \mathcal{H} by identifying the root of the i -th copy of \mathcal{H} with the i -th vertex of \mathcal{R} , is determined.

Key words. Weighted graph, Generalized Bethe tree, Laplacian matrix, Signless Laplacian matrix, Adjacency matrix, Randić matrix.

AMS subject classifications. 05C50, 15A18.

1. Introduction. Let $\mathcal{G} = (V(\mathcal{G}), E(\mathcal{G}))$ be a simple undirected graph with vertex set $V(\mathcal{G}) = \{1, \dots, n\}$ and edge set $E(\mathcal{G})$. We assume that each edge $e \in E(\mathcal{G})$ has a positive weight $w(e)$. The adjacency matrix $A(\mathcal{G}) = (a_{i,j})$ of \mathcal{G} is the $n \times n$ matrix in which $a_{i,j} = w(e)$ if there is an edge e joining i and j and $a_{i,j} = 0$ otherwise. Let $D(\mathcal{G})$ be the diagonal matrix in which the diagonal entry $d_{i,i} = \sum_e w(e)$ where the sum is over all the edges e incident to the vertex i . The Laplacian matrix and the signless Laplacian matrix of \mathcal{G} are $L(\mathcal{G}) = D(\mathcal{G}) - A(\mathcal{G})$ and $Q(\mathcal{G}) = D(\mathcal{G}) + A(\mathcal{G})$, respectively. The matrices $L(\mathcal{G})$, $Q(\mathcal{G})$ and $A(\mathcal{G})$ are real and symmetric. From Geršgorin's theorem, it follows that the eigenvalues of $L(\mathcal{G})$ and $Q(\mathcal{G})$ are nonnegative real numbers. Since the rows of $L(\mathcal{G})$ sum to 0, $(0, \mathbf{e})$ is an eigenpair for $L(\mathcal{G})$, where \mathbf{e} is the all ones vector. Fiedler [9] proved that \mathcal{G} is a connected graph if and only

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if the second smallest eigenvalue of $L(\mathcal{G})$ is positive. This eigenvalue is called the algebraic connectivity of \mathcal{G} . The signless Laplacian matrix has recently attracted the attention of several researchers and some papers on this matrix are [2, 4, 5, 6, 7]. In this paper, $M(\mathcal{G})$ is one of the matrices $L(\mathcal{G})$, $Q(\mathcal{G})$ or $A(\mathcal{G})$. If $w(e) = 1$ for all $e \in E(\mathcal{G})$ then \mathcal{G} is an unweighted graph.

Let \mathcal{R} be a connected weighted graph on r vertices. Let v_1, v_2, \dots, v_r be the vertices of \mathcal{R} . As usual, $v_i \sim v_j$ means that v_i and v_j are adjacent. Let $\varepsilon_{i,j} = \varepsilon_{j,i}$ be the weight of the edge $v_i v_j$ if $v_i \sim v_j$, and let $\varepsilon_{i,j} = \varepsilon_{j,i} = 0$ otherwise. Moreover, for $i = 1, 2, \dots, r$, let $\varepsilon_i = \sum_{v_j \sim v_i} \varepsilon_{i,j}$. Let $\mathcal{R}\{\mathcal{H}\}$ be the graph obtained from \mathcal{R} and r copies of a rooted weighted graph \mathcal{H} by identifying the root of i -copy of \mathcal{H} with v_i .

EXAMPLE 1.1. If \mathcal{R} is the graph depicted in Figure 1.1 and \mathcal{H} is the graph

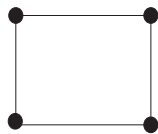


FIG. 1.1. The cycle C_4 .

depicted in Figure 1.2 then $\mathcal{R}\{\mathcal{H}\}$ is the graph depicted in Figure 1.3.

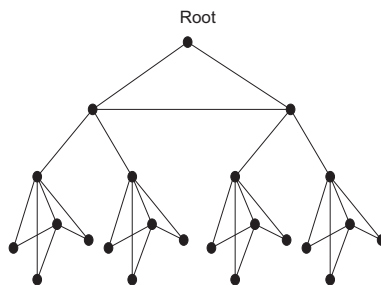


FIG. 1.2. A modified Bethe tree, \mathcal{H} , with four levels.

We recall that for a rooted graph the level of a vertex is one more than its distance from the root vertex. Let \mathcal{B} be a weighted generalized Bethe tree of $k > 1$ levels, that is, \mathcal{B} is a rooted tree in which vertices at the same level have the same degree and edges connecting vertices at consecutive levels have the same weight. Consider a nonempty subset $\Delta \subseteq \{1, 2, \dots, k-1\}$ and a family of graphs $F = \{\mathcal{G}_j : j \in \Delta\}$. For $j \in \Delta$, we assume that the edges of \mathcal{G}_j have weight u_j . Let $\mathcal{B}(F)$ be the graph obtained from \mathcal{B} and the graphs in F identifying each set of children of \mathcal{B} at level $k-j+1$ with the vertices of \mathcal{G}_j .

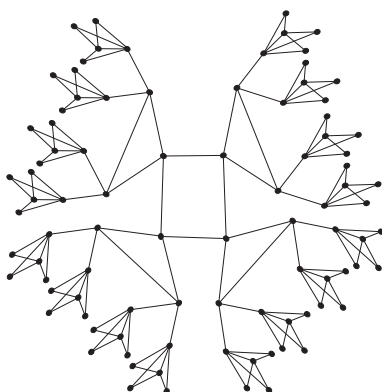


FIG. 1.3. The graph $\mathcal{R}\{\mathcal{H}\}$.

EXAMPLE 1.2. Let $\mathcal{B}(F)$ be the graph depicted in Figure 1.2. In this graph, \mathcal{B} is a generalized Bethe tree of $k = 4$ levels, $\Delta = \{1, 3\}$, $F = \{\mathcal{G}_1, \mathcal{G}_3\}$, where \mathcal{G}_1 is a star of 4 vertices and \mathcal{G}_3 is a path of 2 vertices.

In this paper, we derive a general result on the spectrum of $M(\mathcal{R}\{\mathcal{H}\})$. Using this result, we characterize the eigenvalues of the Laplacian matrix, including their multiplicities, of the graph $\mathcal{R}\{\mathcal{B}(F)\}$; and also of the signless Laplacian and adjacency matrices whenever the subgraphs in F are regular. They are the eigenvalues of symmetric tridiagonal matrices of order j , $1 \leq j \leq k$. In particular, the Randić eigenvalues are characterized.

Denote by $\sigma(C)$ the multiset of eigenvalues of a square matrix C .

2. A result on the spectrum of $M(\mathcal{R}\{\mathcal{H}\})$. Let E be the matrix of order $n \times n$ with 1 in the (n, n) -entry and zeros elsewhere. For $i = 1, 2, \dots, r$, let $d(v_i)$ be the degree of v_i as a vertex of \mathcal{R} and let n be the order of \mathcal{H} . Then $\mathcal{R}\{\mathcal{H}\}$ has rn vertices. We label the vertices of $\mathcal{R}\{\mathcal{H}\}$ as follows: for $i = 1, 2, \dots, r$, using the labels $(i-1)n+1, (i-1)n+2, \dots, in$, we label the vertices of the i -th copy of \mathcal{H} from the bottom to the vertex v_i and, at each level, from the left to the right. With this labeling, $M(\mathcal{R}\{\mathcal{H}\})$ is equal to

$$(2.1) \quad \begin{bmatrix} M(\mathcal{H}) + a\varepsilon_1 E & s\varepsilon_{1,2} E & \cdots & \cdots & s\varepsilon_{1,r} E \\ s\varepsilon_{1,2} E & \ddots & \ddots & & s\varepsilon_{2,r} E \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & M(\mathcal{H}) + a\varepsilon_{i,r-1} E & s\varepsilon_{r-1,r} E \\ s\varepsilon_{1,r} E & s\varepsilon_{2,r} E & \cdots & s\varepsilon_{r-1,r} E & M(\mathcal{H}) + a\varepsilon_r E \end{bmatrix},$$

where

$$(2.2) \quad s = \begin{cases} -1 & \text{if } M \text{ is the Laplacian matrix,} \\ 1 & \text{if } M \text{ is the signless Laplacian or adjacency matrix} \end{cases}$$

and

$$(2.3) \quad a = \begin{cases} 0 & \text{if } M \text{ is the adjacency matrix,} \\ 1 & \text{if } M \text{ is the Laplacian or signless Laplacian matrix.} \end{cases}$$

In this paper, the identity matrix of appropriate order is denoted by I and I_m denotes the identity matrix of order m . Furthermore, $|A|$ denotes the determinant of the matrix A and A^T is the transpose of A .

We recall that the Kronecker product [12] of two matrices $A = (a_{i,j})$ and $B = (b_{i,j})$ of sizes $m \times m$ and $n \times n$, respectively, is defined as the $(mn) \times (mn)$ matrix $A \otimes B = (a_{i,j}b_{i,j})$. Then, in particular, $I_n \otimes I_m = I_{nm}$. Some basic properties of the Kronecker product are $(A \otimes B)^T = A^T \otimes B^T$ and $(A \otimes B)(C \otimes D) = AC \otimes BD$ for matrices of appropriate sizes. Moreover, if A and B are invertible matrices then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.

THEOREM 2.1. *Let $\rho_1(\mathcal{R}), \rho_2(\mathcal{R}), \dots, \rho_r(\mathcal{R})$ be the eigenvalues of $M(\mathcal{R})$. Then*

$$(2.4) \quad \sigma(M(\mathcal{R}\{\mathcal{H}\})) = \cup_{s=1}^r \sigma(M(\mathcal{H}) + \rho_s(\mathcal{R})E).$$

Proof. From (2.1), it follows

$$M(\mathcal{R}\{\mathcal{H}\}) = I_r \otimes M(\mathcal{H}) + M(\mathcal{R}) \otimes E.$$

Let

$$V = [\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_{r-1} \quad \mathbf{v}_r]$$

be an orthogonal matrix whose columns $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ are eigenvectors corresponding to the eigenvalues $\rho_1(\mathcal{R}), \rho_2(\mathcal{R}), \dots, \rho_r(\mathcal{R})$, respectively. Then

$$\begin{aligned} (V \otimes I_n) M(\mathcal{R}\{\mathcal{H}\}) (V^T \otimes I_n) &= (V \otimes I_n) (I_r \otimes M(\mathcal{H}) + M(\mathcal{R}) \otimes E) (V^T \otimes I_n) \\ &= I_r \otimes M(\mathcal{H}) + (VM(\mathcal{R})V^T) \otimes E. \end{aligned}$$

We have

$$(VM(\mathcal{R})V^T) \otimes E = \begin{bmatrix} \rho_1(\mathcal{R}) & & & & \\ & \rho_2(\mathcal{R}) & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \rho_r(\mathcal{R}) \end{bmatrix} \otimes E$$

$$= \begin{bmatrix} \rho_1(\mathcal{R}) E & & & & \\ & \rho_2(\mathcal{R}) E & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & \rho_r(\mathcal{R}) E \end{bmatrix}.$$

Therefore,

$$(V \otimes I_n) M(\mathcal{R}\{\mathcal{H}\}) (V^T \otimes I_n) = \begin{bmatrix} M(\mathcal{H}) + \rho_1(\mathcal{R}) E & & & & \\ & M(\mathcal{H}) + \rho_2(\mathcal{R}) E & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & M(\mathcal{H}) + \rho_r(\mathcal{R}) E \end{bmatrix}.$$

Since $M(\mathcal{R}\{\mathcal{H}\})$ and $(V \otimes I_n) M(\mathcal{R}\{\mathcal{H}\}) (V^T \otimes I_n)$ are similar matrices, we conclude (2.4). \square

3. Application to a modified generalized Bethe tree with subgraphs at some levels. Let \mathcal{B} be a weighted generalized Bethe tree of k levels ($k > 1$). For $1 \leq j \leq k$, n_j and d_j are the number and the degree of the vertices of \mathcal{B} at the level $k - j + 1$, respectively. Thus, d_k is the degree of the root vertex (assumed greater than 1), $n_k = 1$, $d_1 = 1$ and n_1 is the number of pendant vertices. For $1 \leq j \leq k - 1$, let w_j be the weight of the edges connecting the vertices of \mathcal{B} at the level $k - j + 1$ with the vertices at the level $k - j$. We consider

$$\delta_j = \begin{cases} w_j & \text{if } j = 1, \\ (d_j - 1)w_{j-1} + w_j & \text{if } 2 \leq j \leq k - 1, \\ d_k w_{k-1} & \text{if } j = k. \end{cases}$$

We observe that δ_j is the sum of the weights of the edges of \mathcal{B} incident with a vertex at the level $k - j + 1$ and if $w_1 = w_2 = \dots = w_{k-1} = 1$ then $\delta_j = d_j$ for $j = 1, 2, \dots, k$. Let $m_j = \frac{n_j}{n_{j+1}}$ for $j = 1, 2, \dots, k - 1$. Then

$$m_j = d_{j+1} - 1 \quad (1 \leq j \leq k - 2), \\ d_k = n_{k-1} = m_{k-1}.$$

Note that m_j is the cardinality of each set of children at the level $k - j + 1$.

Let $M_j = M(\mathcal{G}_j)$. From now on, we assume that $\mu_1(M_j), \dots, \mu_{m_j}(M_j)$ are the eigenvalues of M_j and \mathbf{e}_{m_j} is an eigenvector for $\mu_{m_j}(M_j)$, that is,

$$(3.1) \quad M_j \mathbf{e}_{m_j} = \mu_{m_j}(M_j) \mathbf{e}_{m_j}.$$

Proof. For $\mathcal{H} = \mathcal{B}(F)$, from Theorem 2.1, we have

$$(3.3) \quad \sigma(M(\mathcal{R}\{\mathcal{B}(F)\})) = \cup_{s=1}^r \sigma(M(\mathcal{B}(F)) + \rho_s(\mathcal{R})E).$$

Let $1 \leq s \leq r$. By linearity on the last column, we have

$$|\lambda I - M(\mathcal{B}(F)) - \rho_s(\mathcal{R})E| = |\lambda I - M(\mathcal{B}(F))| - \rho_s(\mathcal{R}) \left| \lambda I - \widetilde{M(\mathcal{B}(F))} \right|.$$

Using Theorem 3.4 and Lemma 3.5, $|\lambda I - M(\mathcal{B}(F)) - \rho_s(\mathcal{R})E|$ has the form

$$(D_k(\lambda) - \rho_s(\mathcal{R})D_{k-1}(\lambda)) \prod_{j \in \Omega - \Delta} (D_j(\lambda))^{n_j - n_{j+1}} \prod_{j \in \Delta} \prod_{i=1}^{m_j - 1} (D_{j,i}(\lambda))^{n_{j+1}}.$$

Now, by linearity on the last column, we have

$$|\lambda I - Y(s)| = |\lambda I - X_k| - \rho_s(\mathcal{R})|\lambda I - X_{k-1}| = D_k(\lambda) - \rho_s(\mathcal{R})D_{k-1}(\lambda).$$

Therefore,

$$|\lambda I - M(\mathcal{B}(F)) - \rho_s(\mathcal{R})E| = Y(s) \prod_{j \in \Omega - \Delta} (D_j(\lambda))^{n_j - n_{j+1}} \prod_{j \in \Delta} \prod_{i=1}^{m_j - 1} (D_{j,i}(\lambda))^{n_{j+1}}.$$

Applying (3.3), $|\lambda I - M(\mathcal{R}\{\mathcal{B}(F)\})|$ becomes

$$\prod_{s=1}^r Y(s) \prod_{j \in \Omega - \Delta} (D_j(\lambda))^{r(n_j - n_{j+1})} \prod_{j \in \Delta} \prod_{i=1}^{m_j - 1} (D_{j,i}(\lambda))^{rn_{j+1}}. \quad \square$$

4. On the Laplacian, signless Laplacian and adjacency eigenvalues of $\mathcal{R}\{\mathcal{B}(F)\}$. For each j , let

$$\mu_1(\mathcal{G}_j) \geq \mu_2(\mathcal{G}_j) \geq \dots \geq \mu_{m_j - 1}(\mathcal{G}_j) \geq \mu_{m_j}(\mathcal{G}_j) = 0,$$

and let

$$\mu_1(\mathcal{R}) \geq \mu_2(\mathcal{R}) \geq \dots \geq \mu_r(\mathcal{R}) = 0$$

be the Laplacian eigenvalues of \mathcal{G}_j and \mathcal{R} , respectively.

Applying Theorem 3.7 to the determination of the Laplacian spectrum of \mathcal{G} , we obtain:

THEOREM 4.1. *The Laplacian spectrum of $\mathcal{G} = \mathcal{R}\{\mathcal{B}(F)\}$ is*

$$(4.1) \quad \sigma(L(\mathcal{G})) = \left(\cup_{j \in \Omega - \Delta} \sigma(U_j)^{r(n_j - n_{j+1})} \right) \cup \left(\cup_{j \in \Delta} \cup_{i=1}^{m_j - 1} \sigma(U_{j,i})^{rn_{j+1}} \right) \cup \left(\cup_{s=1}^r \sigma(W(s)) \right)$$

where, for $j = 1, \dots, k - 1$, U_j is the $j \times j$ leading principal submatrix of the $k \times k$ matrix $U_k = X_k$ except for the diagonal entries which in U_k are

$$\delta_1, \delta_2, \dots, \delta_{k-1}, \delta_k;$$

and, for $i = 1, 2, \dots, m_j - 1$, $U_{j,i} = X_{j,i}$ except for the diagonal entries which in $U_{j,i}$ are

$$\delta_1, \delta_2, \dots, \delta_{j-1}, \delta_j + \mu_i(\mathcal{G}_j);$$

and, for $s = 1, 2, \dots, r$, $W(s) = Y(s)$ except for the diagonal entries which in $W(s)$ are

$$\delta_1, \delta_2, \dots, \delta_{k-1}, \delta_k + \mu_s(\mathcal{R}).$$

The multiplicities of the eigenvalues of $L(\mathcal{G})$ are considered as in Theorem 3.7.

Proof. It must be noted that, since $M(\mathcal{G}) = L(\mathcal{G})$, we have $a = 1$,

$$S_j = \begin{cases} \delta_j I_{m_j} + L(\mathcal{G}_j) & \text{if } j \in \Delta, \\ \delta_j I_{m_j} & \text{if } j \notin \Delta \end{cases} \quad \text{and} \quad S_k = \delta_k I_r + L(\mathcal{R}).$$

Therefore, $L(\mathcal{G}_j) \mathbf{e}_{m_j} = \mathbf{0} = 0 \mathbf{e}_{m_j}$ and the Laplacian spectrum of \mathcal{G} is given by Theorem 3.7, replacing the matrices X_j , $X_{j,i}$, and $Y(s)$ by the matrices U_j , $U_{j,i}$ and $W(s)$, respectively. \square

As above, let $\mathcal{G} = \mathcal{R}\{\mathcal{B}(F)\}$. We apply now Theorem 3.7 to find the eigenvalues of $Q(\mathcal{G})$ and $A(\mathcal{G})$ whenever each \mathcal{G}_j is a regular graph of degree r_j . For convenience, the signless Laplacian eigenvalues and adjacency eigenvalues are denoted in increasing order. Let

$$q_1(\mathcal{G}) \leq q_2(\mathcal{G}) \leq q_3(\mathcal{G}) \leq \dots \leq q_{m-1}(\mathcal{G}) \leq q_m(\mathcal{G})$$

and

$$\lambda_1(\mathcal{G}) \leq \lambda_2(\mathcal{G}) \leq \dots \leq \lambda_{m-1}(\mathcal{G}) \leq \lambda_m(\mathcal{G})$$

be the eigenvalues of the signless Laplacian matrix and adjacency matrix of any graph \mathcal{G} , respectively. If \mathcal{G} is a regular graph of degree t and order m in which the edges have a weight equal to u then $Q(\mathcal{G}) \mathbf{e}_m = 2tue_m$ and $A(\mathcal{G}) \mathbf{e}_m = tue_m$. In this case, we may write $\lambda_m(\mathcal{G}) = tu$ and $q_m(\mathcal{G}) = 2tu$.

THEOREM 4.2. *If for each $j \in \Delta$ the graph \mathcal{G}_j is a regular graph of degree r_j and $r_j = 0$ whenever $j \notin \Delta$, then the signless Laplacian spectrum of $\mathcal{G} = \mathcal{R}\{\mathcal{B}(F)\}$ is*

$$\sigma(Q(\mathcal{G})) = \left(\bigcup_{j \in \Omega - \Delta} \sigma(V_j)^{r(n_j - n_{j+1})} \right) \cup \left(\bigcup_{j \in \Delta} \bigcup_{i=1}^{m_j-1} \sigma(V_{j,i})^{rn_{j+1}} \right) \cup \left(\bigcup_{s=1}^r \sigma(U(s)) \right)$$

where, for $j = 1, 2, 3, \dots, k-1$, V_j is the $j \times j$ leading principal submatrix of $V_k = X_k$ except for the diagonal entries which in V_k are

$$\delta_1 + 2r_1u_1, \delta_2 + 2r_2u_2, \dots, \delta_{k-1} + 2r_{k-1}u_{k-1}, \delta_k;$$

and, for $i = 1, 2, \dots, m_j - 1$, $V_{j,i} = X_{j,i}$ except for the diagonal entries which in $V_{j,i}$ are

$$\delta_1 + 2r_1u_1, \delta_2 + 2r_2u_2, \dots, \delta_{j-1} + 2r_{j-1}u_{j-1}, \delta_j + q_i(\mathcal{G}_j);$$

and, for $s = 1, 2, \dots, r$, $U(s) = Y(s)$ except for the diagonal entries which in $U(s)$ are

$$\delta_1 + 2r_1u_1, \delta_2 + 2r_2u_2, \dots, \delta_{k-1} + 2r_{k-1}u_{k-1}, \delta_k + q_s(\mathcal{R}).$$

The multiplicities of the eigenvalues of $Q(\mathcal{G})$ must be considered as in Theorem 3.7.

Proof. For $M(\mathcal{G}) = Q(\mathcal{G})$, we have $a = 1$,

$$S_j = \begin{cases} \delta_j I_{m_j} + Q(\mathcal{G}_j), \\ Q(\mathcal{G}_j) = 0 \text{ if } j \notin \Delta \end{cases} \quad \text{and} \quad S_k = \delta_k I_r + Q(\mathcal{R}),$$

$Q(\mathcal{G}_j) \mathbf{e}_{m_j} = 2r_j u_j \mathbf{e}_{m_j}$ if $j \in \Delta$ and $r_j = 0$ for $j \notin \Delta$. From Theorem 3.7, we obtain that the set of eigenvalues of $Q(\mathcal{G})$ is given replacing the matrices X_j , $X_{j,i}$, and Y_s by the matrices V_j , $V_{j,i}$ and $U(s)$, respectively. \square

THEOREM 4.3. *If for each $j \in \Delta$ the graph \mathcal{G}_j is a regular graph of degree r_j and $r_j = 0$ whenever $j \notin \Delta$, then the adjacency spectrum of $\mathcal{G} = \mathcal{R} \{ \mathcal{B}(\mathbf{F}) \}$ is*

$$\sigma(A(\mathcal{G})) = \left(\bigcup_{j \in \Omega - \Delta} \sigma(T_j)^{r(n_j - n_{j+1})} \right) \cup \left(\bigcup_{j \in \Delta} \bigcup_{i=1}^{m_j-1} \sigma(T_{j,i})^{r n_{j+1}} \right) \cup \left(\bigcup_{s=1}^r \sigma(R(s)) \right)$$

where, for $j = 1, 2, 3, \dots, k-1$, T_j is the $j \times j$ leading principal submatrix of $T_k = X_k$ except for the diagonal entries which in T_k are

$$r_1u_1, r_2u_2, \dots, r_{k-1}u_{k-1}, 0;$$

and, for $i = 1, 2, \dots, m_j - 1$, $T_{j,i} = X_{j,i}$ except for the diagonal entries which in $T_{j,i}$ are

$$r_1u_1, r_2u_2, \dots, r_{j-1}u_{j-1}, \lambda_i(\mathcal{G}_j);$$

and, for $s = 1, 2, \dots, r$, $R(s) = Y(s)$ except for the diagonal entries which in $R(s)$ are

$$r_1u_1, r_2u_2, \dots, r_{k-1}u_{k-1}, \lambda_s(\mathcal{R}).$$

The multiplicities of the eigenvalues of $Q(\mathcal{G})$ must be considered as in Theorem 3.7.

Proof. For $M(\mathcal{G}) = A(\mathcal{G})$, we have $a = 0$,

$$S_j = \begin{cases} A(\mathcal{G}_j), \\ A(\mathcal{G}_j) = 0 \text{ if } j \notin \Delta \end{cases} \quad \text{and} \quad S_k = A(\mathcal{R}),$$

and $A(\mathcal{G}_j) \mathbf{e}_{m_j} = r_j \mathbf{e}_{m_j}$ if $j \in \Delta$ and $r_j = 0$ for $j \notin \Delta$. Then, from Theorem 3.7, we conclude that the set of eigenvalues of $A(\mathcal{G})$ is obtained replacing the matrices X_j , $X_{j,i}$, and $Y(s)$ by the matrices T_j , $T_{j,i}$ and $R(s)$, respectively. \square

5. On the Randić eigenvalues of $\mathcal{R}\{\mathcal{B}(F)\}$. Let \mathcal{H} be a simple connected graph with n vertices v_1, v_2, \dots, v_n . Denote by $d(v_1), d(v_2), \dots, d(v_n)$ the degrees of v_1, v_2, \dots, v_n , respectively. The Randić matrix of \mathcal{H} is the square matrix of order n whose (i, j) -entry is equal to

$$\frac{1}{\sqrt{d(v_i) d(v_j)}}$$

if v_i and v_j of \mathcal{H} are connected and 0 otherwise [1]. The Randić eigenvalues of \mathcal{H} are the eigenvalues of the Randić matrix of \mathcal{H} . The purpose of this section is to determine the Randić eigenvalues of $\mathcal{G} = \mathcal{R}\{\mathcal{B}(F)\}$ when each \mathcal{G}_j is regular of degree r_j and \mathcal{R} is a connected regular graph of degree p on r vertices v_1, v_2, \dots, v_r .

Keep in mind that $r_j = 0$ if $j \notin \Delta$. As usual $v_i \sim v_j$ means that v_i and v_j are adjacent. Observe that, for $i = 1, 2, \dots, r$, the degree of v_k as a vertex of $\mathcal{R}\{\mathcal{B}(F)\}$ is $d_k + p$. The Randić matrix of $\mathcal{R}\{\mathcal{B}(F)\}$ is the adjacency matrix of the weighted graph in which the edges joining the vertices at the level $j + 1$ with the vertices at the level j of $\mathcal{B}(F)$ have weights

$$(5.1) \quad w_{k-j} = \frac{1}{\sqrt{(d_{k-j+1} + r_{k-j+1})(d_{k-j} + r_{k-j})}} \quad (2 \leq j \leq k-1),$$

$$w_{k-1} = \frac{1}{\sqrt{(d_k + p)(d_{k-1} + r_{k-1})}},$$

the edges of graph \mathcal{G}_j have weights

$$(5.2) \quad u_j = \begin{cases} \frac{1}{d_j + r_j} & \text{if } j \in \Delta, \\ 0 & \text{if } j \notin \Delta, \end{cases}$$

and the weights of the edge $v_i v_l$ of \mathcal{R} are

$$(5.3) \quad \varepsilon_{i,l} = \varepsilon_{l,i} = \begin{cases} \frac{1}{p + d_k} & \text{if } v_i \sim v_l, \\ 0 & \text{otherwise.} \end{cases}$$

THEOREM 5.1. *If for each $j \in \Delta$ the graph \mathcal{G}_j is a regular graph of degree r_j and the graph \mathcal{R} is a regular graph of degree p then the Randić spectrum of $\mathcal{G} = \mathcal{R}\{\mathcal{B}(F)\}$*

is

$$\left(\bigcup_{j \in \Omega - \Delta} \sigma(T_j)^{r(n_j - n_{j+1})} \right) \cup \left(\bigcup_{j \in \Delta} \bigcup_{i=1}^{m_j-1} \sigma(T_{j,i})^{rn_{j+1}} \right) \cup \left(\bigcup_{s=1}^r \sigma(R(s)) \right)$$

in which the matrices $T_j, T_{j,i}$ and $R(s)$ are those of Theorem 4.3 with the weights indicated in (5.1), (5.2) and (5.3). The eigenvalues multiplicities must be considered as in Theorem 3.7.

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