Universidade de Aveiro Departamento de Matemática,
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Alexandrino Duarte Delgado

Modelos de otimização para a distribuição de combustíveis em curta distância marítima

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Optimization Models for a Short Sea Fuel Oil Distribution Problem

Dissertação apresentada à Universidade de Aveiro para cumprimento dos requesitos necessários à obtenção do grau de Doutor em Matemática e Aplicações, realizada sob a orientação científica de Agostinho Agra, Professor Auxiliar do Departamento de Matemática da Universidade de Aveiro e Marielle Christiansen, Professora Catedrática da Faculty of Social Sciences and Technology Management, Trondheim, Noruega.

Dissertation submitted to the University of Aveiro in fulfilment of the requirements for the degree of Doctor of Philosophy in Mathematics, under the supervision of Agostinho Agra, Associated Professor at the Department of Mathematics of the University of Aveiro and Marielle Christiansen, Full Professor at the Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology.

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To my parents Teófilo e Margarida for their unconditional love. To my kid Rafael who, as always, is my best inspiration and to my wife Cátia, whose love, support, and limitless patience made all of this possible.

o júri

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palavras-chave

Transporte marítimo; Programação inteira mista; Formulações estendidas, Desigualdades válidas; Heurísticas; Otimização estocástica

Resumo

O transporte marítimo é o principal meio de transporte de mercadorias em todo o mundo. Combustíveis e produtos petrolíferos representam grande parte das mercadorias transportadas por via marítima. Sendo Cabo Verde um arquipélago o transporte por mar desempenha um papel de grande relevância na economia do país.

Consideramos o problema da distribuição de combustíveis em Cabo Verde, onde uma companhia é responsável por coordenar a distribuição de produtos petrolíferos com a gestão dos respetivos níveis armazenados em cada porto, de modo a satisfazer a procura dos vários produtos. O objetivo consiste em determinar políticas de distribuição de combustíveis que minimizam o custo total de distribuição (transporte e operações) enquanto os níveis de armazenamento são mantidos nos níveis desejados.

Por conveniência, de acordo com o planeamento temporal, o problema é divido em dois sub-problemas interligados. Um de curto prazo e outro de médio prazo. Para o problema de curto prazo são discutidos modelos matemáticos de programação inteira mista, que consideram simultaneamente uma medição temporal contínua e uma discreta de modo a modelar múltiplas janelas temporais e taxas de consumo que variam diariamente. Os modelos são fortalecidos com a inclusão de desigualdades válidas. O problema é então resolvido usando um "software" comercial. Para o problema de médio prazo são inicialmente discutidos e comparados vários modelos de programação inteira mista para um horizonte temporal curto assumindo agora uma taxa de consumo constante, e são introduzidas novas desigualdades válidas. Com base no modelo escolhido são comparadas estratégias heurísticas que combinam três heurísticas bem conhecidas: "Rolling Horizon", "Feasibility Pump" e "Local Branching", de modo a gerar boas soluções admissíveis para planeamentos com horizontes temporais de vários meses.

Finalmente, de modo a lidar com situações imprevistas, mas importantes no transporte marítimo, como as más condições meteorológicas e congestionamento dos portos, apresentamos um modelo estocástico para um problema de curto prazo, onde os tempos de viagens e os tempos de espera nos portos são aleatórios. O problema é formulado como um modelo em duas etapas, onde na primeira etapa são tomadas as decisões relativas às rotas do navio e quantidades a carregar e descarregar e na segunda etapa (designada por sub-problema) são consideradas as decisões (com recurso) relativas ao escalonamento das operações. O problema é resolvido por um método de decomposição que usa um algoritmo eficiente para separar as desigualdades violadas no sub-problema.

keywords

Maritime transportation, Mixed integer programming; Extended formulations, Valid inequalities; Heuristics; Stochastic optimization

Abstract

Maritime transportation is a major mode of transportation of goods worldwide. Most of cargo of the maritime transport accounted for liquid cargo oil and petroleum products. As Cape Verde is an archipelago, maritime transportation is of great importance for the local economic activity.

We consider a fuel oil distribution problem where an oil company is responsible for the coordination of the distribution of oil products with the inventory management of those products at ports in order to satisfy the demands for the several oil products. The objective is to determine distribution policies that minimize the routing and operating costs, while inventory levels are maintained within given limits.

For convenience, the planning problem is divided into two related subproblems accordingly to the length of the planning horizon: A short-term and medium-term planning. For the short-term planning problem we discuss mathematical mixed integer programming models that combine continuous and discrete time measures in order to handle with multiple time windows and a daily varying consumption rate of the various oil products. These models are strengthened with valid inequalities. Then the problem is solved using a commercial software. For the second subproblem several mixed integer formulations are discussed and compared for a short time horizon, and assuming constant consumption rates and new valid inequalities are introduced. Then, based on the chosen model, we compare several heuristic strategies that combine the well-known Rolling Horizon, Feasibility Pump and Local Branching heuristics, in order to derive good feasible solutions for planning horizons of several months.

Finally, as weather conditions and ports congestion are very important in maritime transportation, we present a stochastic model for a short sea shipping problem, where traveling and waiting time are random. The problem is formulated as a two stage recourse problem, where in the first stage the routing and the load/unload quantities are defined, and in the second stage (subproblem) the scheduling of operations is determined. The problem is solved by a decomposition method that uses an efficient separation algorithm to include inequalities from the subproblem.

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Chapter 1

Introduction

Maritime transportation is the major mode of transportation of goods worldwide. In 2011 about 2.8 millions tons of liquid oil and petroleum products were transported by sea (UNCTAD, 2011). Between 1980 and 2011 the transport of oil products and their derivatives increased around 50% and represents about 32% of total seaborne trade.

Transportation planning has been studied extensively in the literature [7]. However, until the last two and half decades, relatively little work has been done on maritime transportation, when compared with others modes of transport. Nowadays we have witnessed a remarkable growth in scientific research on maritime transportation. Most of the published contributions in maritime transportation are based on real-life problems from the industry.

Operations Research is one of the most popular managerial decision science tools used in many industries. Routing and scheduling problems is an important area of Operations Research that has been widely studied in the past. For routing problems see [5, 6, 7, 8]. Most of the research on routing problems has been focused on land transportation. Research on ship routing and scheduling has been gaining attention. The first survey on ship routing and scheduling was presented by Ronen [26] in 1983. That work presents the differences between vehicle and ship routing and scheduling, and explains the reasons for the low attention to ship scheduling in the past. It also suggests a classification of ship routing and scheduling problems and models. Ten years later, in 1993, Ronen [28] presents a review on ship scheduling and related works in the decade 1982-1992, and identifies news challenges for future research. In 2004, Christiansen et al. [8] present a review on ship routing and scheduling during the last decade. A very complete survey on maritime transportation is given by Christiansen et al. [7]. For a more recent review within maritime transportation, combining routing and inventory management see Christiansen and Fagerholt [6]. Many applications within maritime inventory routing and extensions are presented as well as some examples of research contributions. A survey on ship routing and scheduling in the new millennium is present in [9].

This work was motivated by a real logistic problem occurring in the Cape Verde archipelago, where the distribution of several fuel oil products needs to be coordinated with the management of the inventory levels of those products. Cape Verde is located in the Atlantic Ocean, 460 Km from the African Coast across from Senegal, at the crossroads between Africa, Europe and America (see Figure 1.1). The country intends to take advantage of its privileged geographic position with regard to international maritime transport. The relatively small size of the land area contrasts with the extent of Exclusive Economic Zone of Cape Verde which corresponds to 800,000 Km^2 [33]. All the inhabited islands have ports that allow maritime access. The country has no known oil resources and it is entirely reliant on oil imports for its fuel oil supply. Most of the economic activities depend on fuel oil products. Therefore, fuel oil distribution is a vital activity for Cape Verde economy. Concerning the fuel oil distribution problem at Cape Verde,

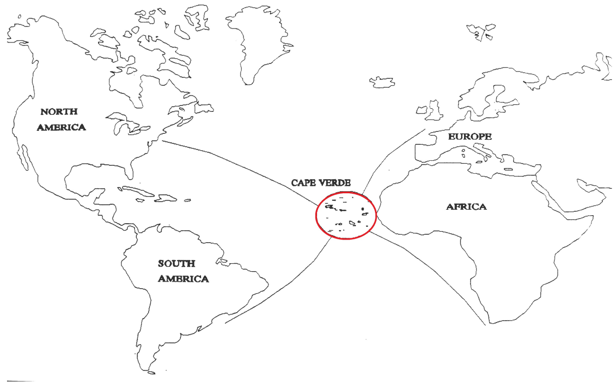

Figure 1.1: Geographic localization of Cape Verde.

there are several interrelated decisions, belonging to different levels of planning, that need to be taken on a regular basis, such us, redesign/optimize the inter-islands fuel oil distribution; expansion and construction of deposits; buy or charter news ships etc..

This thesis considers only the inter-islands fuel oil distribution problem. An oil company has both the responsibility for the transportation of the oil products and for the management of the inventories at the ports. Given known demand rates and in order to maintain the stock levels within desirable limits, a distribution plan shall include the ship routes, the quantity of each product to be loaded into each cargo tank in each ship visit to a supply port, the quantity of products to be unloaded from each ship to a destination port, and the schedule of the operations. In the literature, this problem can be categorized as the Short Sea Inventory Routing Problem. Since the objective of this thesis is to develop efficient optimization approaches for the oil distribution problem and these approaches are based on the mathematical models , we decided to give the title *Optimization Models for a Short Sea Fuel Oil Distribution Problem* to this thesis.

In order to define the scope of this thesis, we first make a brief review of modes of shipping operations and length of the planning horizon. The shipping industry is divided into three different modes of transportation: industrial, tramp and linear shipping (Lawrence, [22]). Christiansen et al. [8] describe in detail such division. In industrial shipping, the shipper owns the ships and aims to minimize the total shipping cost. In tramp shipping, a carrier engages in contracts with shippers to carry cargoes between specified ports within a specific time range. Linear shipping is a service that operates within a schedule and has a fixed routes with published schedule. In other words, industrial shipping may be compared with "owning a car", tramp shipping with "a taxi service" and linear shipping with "a bus service".

According to the length of the planning horizon, maritime transportation problems can be divided into three different planning levels: strategic, tactical, and operational (Christiansen et al. [7]). Strategic planning level may deal with a relatively long planning horizon of, say, 2 to 5 years. Tactical planning focuss on decisions that are medium-term, usually, from 2 month up to 1 year. Operational planning is concerned with the short-term decisions, with a planning horizon from a few hours to a few weeks. This thesis is focused on developing optimizations models for solving a short sea inventory routing problem, at the operational and tactical levels, faced by the industrial shipping oil company. Figure 1.2 shows the scope of the thesis for different planning levels and modes of transportation.

By convenience, the problem is divided into two related stages accordingly to the planning level: short-term (operational level) and medium-term (tactical level) plan.

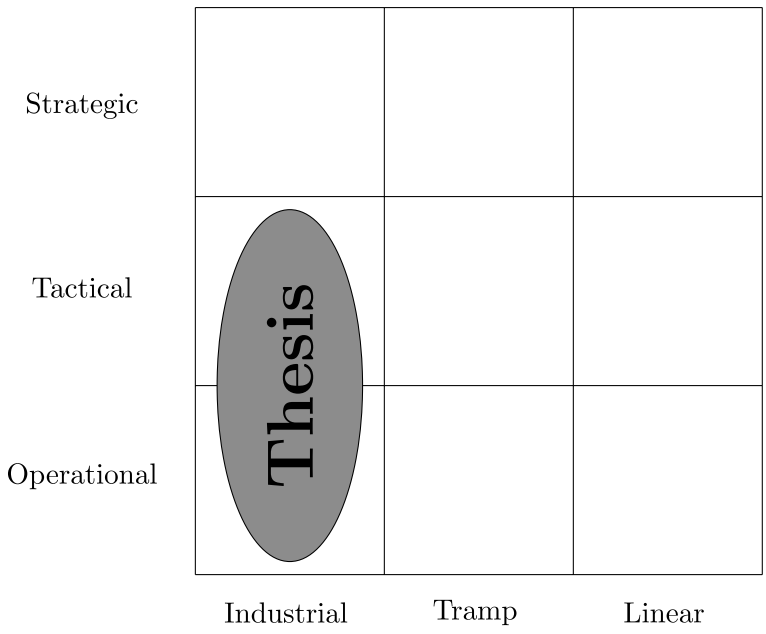

Figure 1.2: Matrix showing how this thesis may be classified according to planning level and mode of transportation

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The rest of the introduction is organized as follows. Section 1.1 introduces the reader shortly to maritime transportation and optimization concepts. The main characteristics of the Cape Verde fuel oil distribution problem, such as, distribution and consumption, ships, ports, weather conditions and time windows, are described in Section 1.2. Section 1.3 presents the purpose of this thesis. The contributions of this thesis to the research community and industry are given in Section 1.4. Finally, Section 1.5 includes some final remarks on future research.

1.1 Background

In this section, we provide some definitions and briefly explain some concepts that are used in this thesis. We split these definitions and concepts into two subsections: maritime transportation and optimization concepts.

1.1.1 Maritime Transportation Concepts

Here we introduce the reader to some basic concepts in maritime transportation. For a more complete introduction to the area, see Ronen ([26, 27, 28]) and Christiansen et al. [7].

Production/consumption can be deterministic, stochastic or a decision variable. In the deterministic production/consumption case the rate can be constant during the whole planning horizon or may vary from period to period. In the stochastic case, the production/consumption rate follows a given statistical distribution. Finally, the production/consumption rate can be a decision variable and by this an output from the model.

Routing is the sequencing of port visits to be made by the fleet of ships.

Inventory management is concerned with the management of the stock level of a set of products at given physical places, typically large tanks;

Inventory routing is concerned with the coordination of the inventory management of the stock levels of a set of products with the distribution of those products by a fleet of ships.

Scheduling is concerned with sequencing port visits and specifying time for the different activities on a ship's route.

Berth capacity is the space allocated to vessels at anchor or at a wharf.

Draft limits determine the minimum depth of water a ship can safely navigate. Draft limits in ports can thus prevent large or fully loaded ships to enter these ports.

Time windows are time limits imposed to port visits. We consider two types of time windows: those implied indirectly by inventory levels, capacities of ship tanks and consumption tanks (inventory time windows); and those imposed by port regulations (operating time windows). In the last case we may have multiple time windows for the same port visit. This is the case when a port is open for cargo operations every day from time A to time B. Time windows can either be hard, if they cannot be violated, or soft, when it is allowed to operate outside the time windows by paying an extra cost. A time window can be simultaneously hard on one side and soft on the other.

1.1.2 Integer programming concepts

Now we introduce some basic integer programming concepts. For details see [24].

A polyhedron in $\mathbb{R}^n$ is a set of the form $P := \{x \in \mathbb{R}^n : Ax \leq b\}$. A polyhedron P is called a formulation for $X \subseteq \mathbb{R}^p \times \mathbb{Z}^{n-p}$, if $X = P \cap (\mathbb{R}^p \times \mathbb{Z}^{n-p})$.

A mixed integer program (MIP) is the problem of minimizing or maximizing a linear objective function in the presence of linear constraints and integrality restrictions on a set of variables. To be more precise, a MIP can be written as $max\{cz + fy : Az + By \leq b, z \in \mathbb{R}^p, y \in \mathbb{Z}^{n-p}\}$.

If we exclude the integer variables y of the MIP we obtain a linear programming (LP) model $max\{cz : Az \leq b, z \in \mathbb{R}^n\}$. Also, if we remove the continuous variables x we have a pure integer programming (IP) model, $max\{fy : By \leq b, y \in \mathbb{Z}^n\}$.

A binary linear program BP is an IP model, where all variables y take binary values.

The linear relaxation (LR) of a MIP is the formulation obtained by relaxing the integrality constraints on the y variables, and it is given by $max\{cz+fy : Az+By \leq b, (z, y) \in \mathbb{R}^p \times \mathbb{R}^{n-p}\}$.

Given two formulations P_1 and P_2, defined in the same space for $X \subseteq \mathbb{R}^n$, we say that P_1 is tighter or stronger than P_2 if $P_1 \subset P_2$. Finding tight formulations is a key to solve many MIP problems. We follow two main approaches to tighten formulations. One is to use valid inequalities and the other is to use extended formulations. Next we explain these concepts.

An inequality $\alpha x \leq \beta$ is a *valid inequality* for X if $\alpha x \leq \beta$ for all $x \in X$.

Given a family of valid inequalities $\mathcal{F}$ and a formulation P for X, if P does not include all the inequalities in $\mathcal{F}$, then P can be strengthen by adding the family of inequalities to P, in order to obtain $\overline{P} = P \cap \{x \in \mathbb{R}^n : \alpha x \leq \beta, \forall(\alpha, \beta) \in \mathcal{F}\}$. The inequalities can either be added a priori to P or they can be added dynamically by solving the separation problem. Given $x^* \in \mathbb{R}^n$ the *separation problem* associated to $\mathcal{F}$, is to decide whether $x \in \overline{P}$ or not. If not, find an inequality $\alpha x \leq \beta$ in $\mathcal{F}$ that separates x^* from $\overline{P}$, that is, $\alpha x^* > \beta$.

The definition of extended formulation is not consensual. We define an extended formulation in relation to a given formulation $P \subseteq \mathbb{R}^n$ for X. Polyhedron $P_E \subseteq \mathbb{R}^{n+k}$ is an *extended formulation* for X if $X = \overline{P}_E \cap (\mathbb{Z}^p \times \mathbb{R}^{n-p})$, where $\overline{P}_E$ is the projection of $\overline{P}_E$ onto $\mathbb{R}^n$. Informally, an extended formulation is a formulation that uses additional variables in relation to a given formulation. Thus, the extended formulation is a polyhedron defined in a higher space.

Optimizations Algorithms

Here we describe the basic algorithms to optimize integer programs. The algorithms are stated for a minimization problem.

Branch-and-Bound Algorithm

Branch-and-bound (BB) is one of the principal exact solution techniques used in practice for solving mixed integer programming problems. This is a divide and conquer method. The algorithm divides the set of feasible solutions into smaller subsets. A tree of nodes is constructed where each node corresponds to a subproblem. There is a value called the incumbent, which is the value of the best feasible solution found, and therefore, is an upper bound of the value of the optimal solution. In the beginning, if no feasible solution is known, the incumbent is set to $+\infty$. At each tree node, the LP relaxation is solved. If the LP solution is integral the incumbent is updated and the tree node is pruned. If the LP is infeasible the node is also pruned since the corresponding subproblem is also infeasible. Similarly, if the value of the incumbent is less than the value of the LP solution, the node can be pruned since the optimal solution of the subproblem is worse than a known feasible solution. When the node is not pruned, a variable with fractional value in the LP solution is chosen and new subproblems are created by bounding the value of the chosen variable. If the set of subproblems is empty, the BB algorithm stops, and the optimal solution is found, otherwise, we need to branch and solve the resulting subproblems, recursively.

The branch-and-bound scheme is summarized in (Figure 1.3):

Cutting-Plane

The cutting-plane (CP) is a methodology used to approximate a MIP by a linear model. The cutting-plane algorithm starts by solving the linear relaxation of the original formulation (see Figure 1.4). If the solution is integral the process terminates. Otherwise, the separation routines associated to the LP solution are used to generate valid inequalities that cut off the LP solution. If some violated valid inequalities are found, they are added to the original formulation and the process continues iteratively until no violated inequality is obtained or other stopping criteria is verified. Then go to the branching phase.

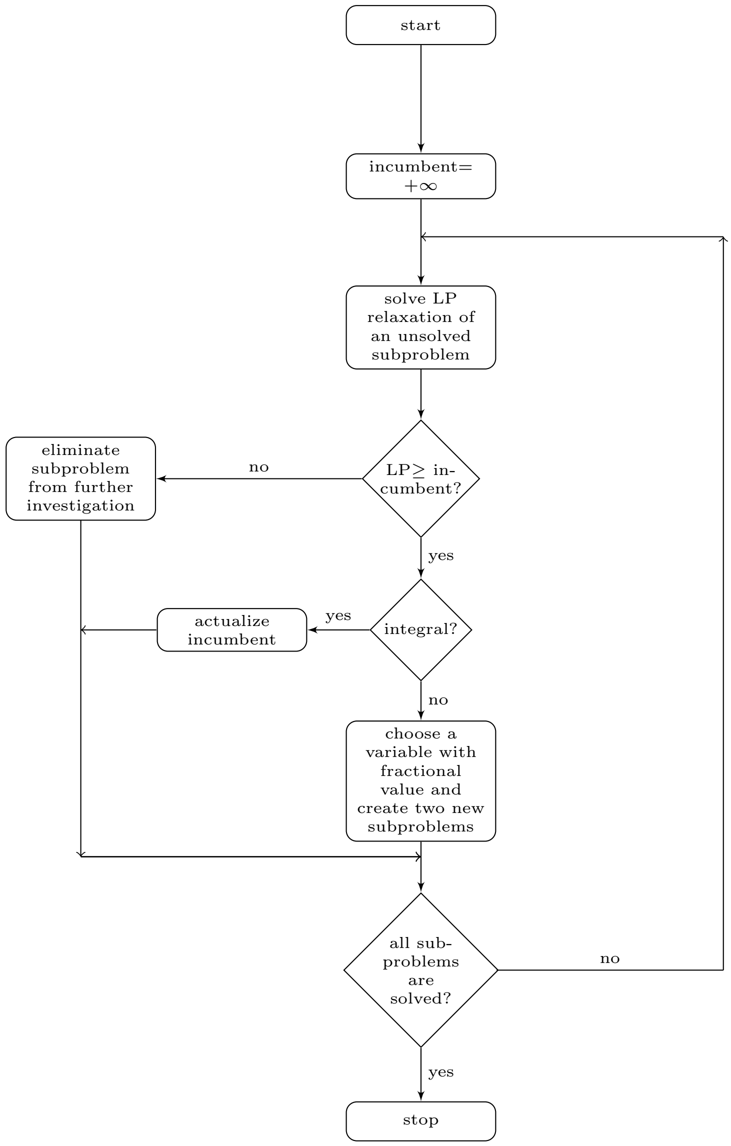

Figure 1.3: Branch and bound algorithm

Branch-and-Cut Algorithm

The idea of this algorithm is similar to the branch-and-bound algorithm (see Figure 1.5). Here a lower bound is determined by solving a linear relaxation of each subproblem strengthened with the inclusion valid of valid inequalities. The steps of a branch-and-cut (BC) algorithm are similar those of the branch and bound algorithm with one additional step in which valid inequalities are generated. Adding valid inequalities can strengthen the formulation which may produce better lower bounds resulting from the LP relaxation at each BB node. This will tend to reduce the number of enumerated nodes. On the other hand, the size of the LP model at each node increases and the time spent to obtain the LP relaxation tends to increase.

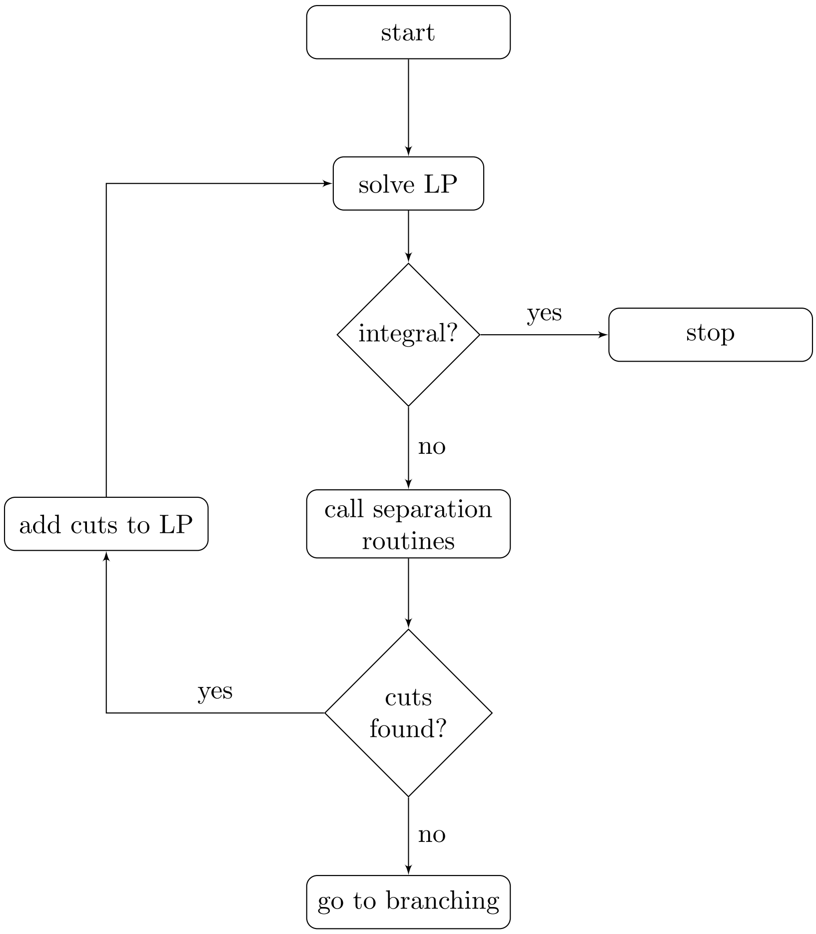

Figure 1.4: Cutting plane followed by BB algorithm

Heuristics solution methods in Maritime Transportation

Exact methods may not be able to solve complex routing and scheduling problems occurring in Maritime Transportation. In order to solve larger problem instances, heuristics approaches are frequently used.

Some heuristics are based on modifications or simplifications of a mathematical model which is then solved using an exact solution method, such as in [5]. Next we describe the most relevant heuristics to our work.

Rolling horizon heuristics have been employed in maritime transportation problem (see for instance Bredström and Rönnqvist [2]). The idea is decompose the planning horizon into sub-horizons and repetitively solve smaller and tractable mixed integer problems for the smaller sub-horizons, using an exact algorithm.

Fix-and-relax is another decomposition heuristic used to generate feasible solutions. Fix-and-relax was originally described by Dillenberger et al. [10]. This heuristic procedure involves the solution of a series of partially relaxed MIPs, each one with a number of binary variables that is small enough to be quickly and optimally solved by an exact algorithm. As the series progresses, each set of binary variables is permanently fixed at their solution values, and the relaxed variables are reduced in number, eventually disappearing. As an example, Uggen et al. [31] uses a fix-and-relax heuristic to solve maritime inventory routing problems.

Local branching was originally described by Fischetti and Lodi [14]. This is an approach

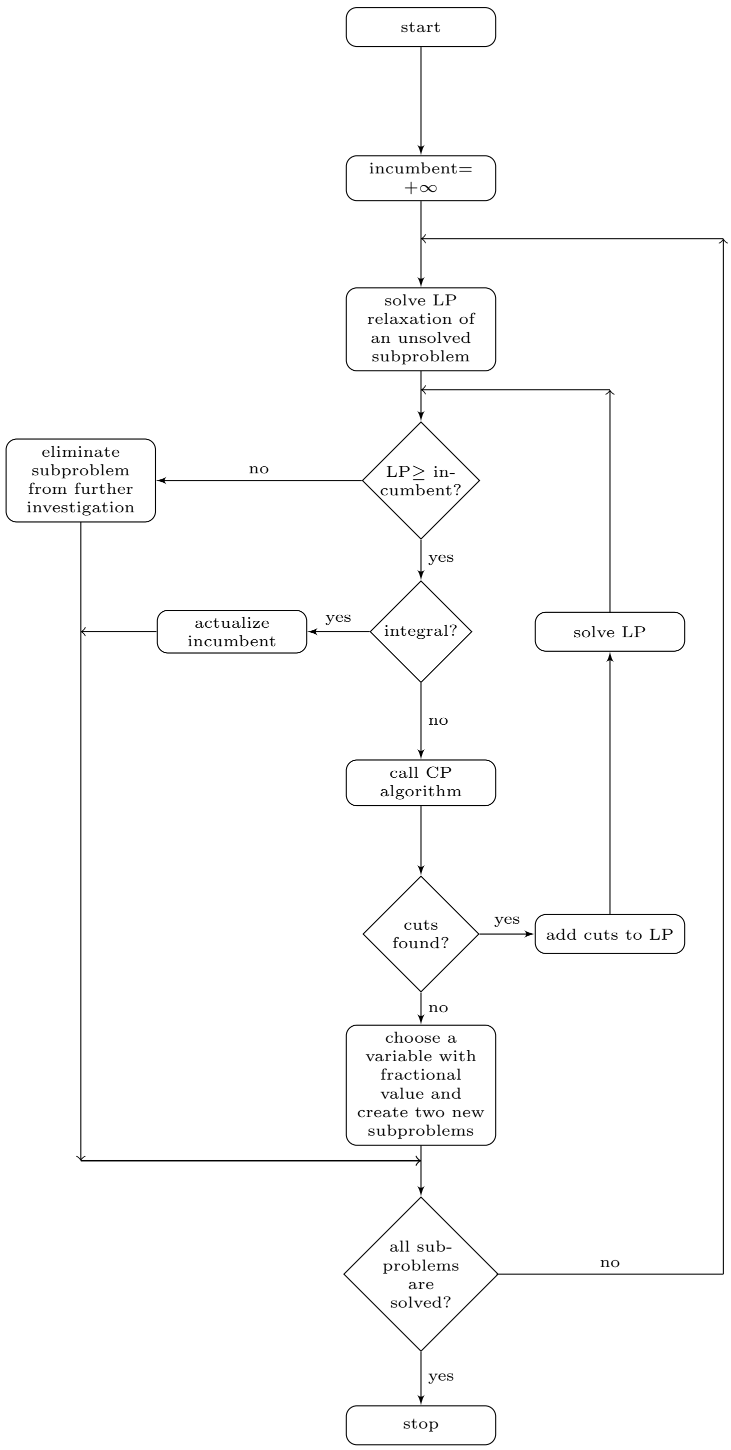

Figure 1.5: Branch and cut algorithm

to improve feasible solutions of hard mixed integer problems. Given a feasible solution, an additional constraint is added to the model to impose that only a limited number of binary variables can have a different value from the current solution. Then a local optimal, or near optimal solution is obtained using a MIP solver. Local branching can be regarded as a neighborhood search heuristic. For more details and application of neighborhood search algorithms see [3, 11, 12, 18, 19, 23, 29].

Feasibility pump is an heuristic scheme used to find feasible solutions for mixed integer problems. The heuristic starts to solve a LP relaxation and to obtain a linear solution x. If x is integral, then the algorithm stops. Otherwise, the solution x is rounded in order to obtain an integer solution $\bar{x}$. If $\bar{x}$ is feasible to the problem the algorithm stops. Otherwise, a new linear solution x that minimizes a distance function to $\bar{x}$ is found. The procedure is repeated until a feasible solution is found or other stopping criteria is reached. For more details of feasibility pump heuristics, see ([13],[15]).

1.2 Cape Verde fuel oil distribution problem

The geography of Cape Verde is a natural barrier to develop a fuel oil pipeline transportation infrastructure (see Figure 1.6) since the territorial dispersion and the long-depth sea makes prohibitive the use of pipelines. Fuel oil products can only be transported by ships.

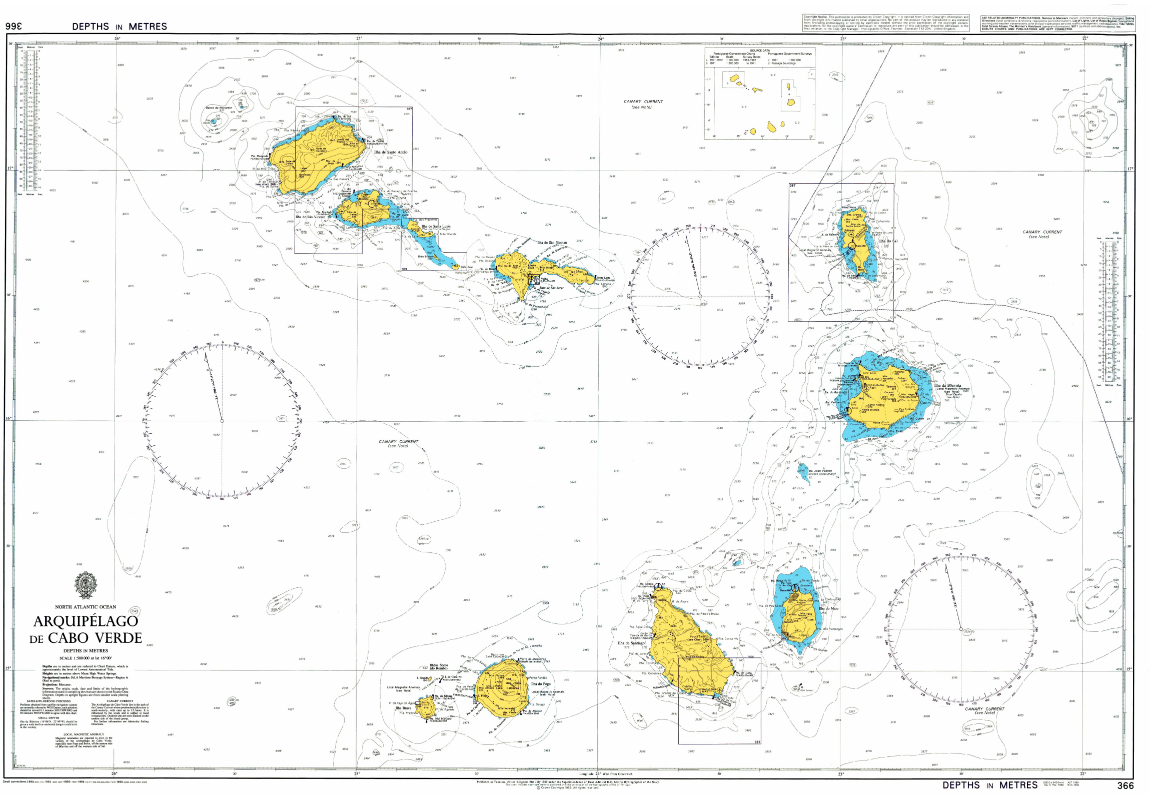

Figure 1.6: Nautical letter of Cape Verde archipelago.

Distribution and consumption

Since there is no production of fuel oil products and there are no refiners at Cape Verde, a company imports the fuel oil products (Diesel, Fuel, Gasoline and Jet). These products are transported by large tankers and delivered into large supply storage tanks located in specific islands. Diesel and Fuel are stored in S.Vicente, while Gasoline and Jet are stored in Sal. Considering the internal distribution problem, this means that S.Vicente and Sal are the origins of these products. The remaining islands are just consumers. From these origins fuel oil products are distributed to all islands, using a small heterogeneous fleet. Due to low consumption rates, Maio and Brava islands, are supplied by a different ship, since in this case, the consumption products are delivered by small containers. Not all islands consume all products. The products are stored in separate consumption storage tanks. These depots have a maximum and a minimum capacity. S.Vicente is a consumer of Gasoline and Jet, while Sal is a consumer of Diesel and Fuel. A ships may carry several products simultaneously (in separated compartments). A loading port can still be visited with some cargo on board.

Ships

The company uses chemical tankers with double hull. Each tank has a piping system which is independent from other tanks. Therefore, each tank can load a separate cargo without mixing any products. Tanks which are not properly cleaned of all cargo residue can adversely affect the purity of the next cargo loaded. Before cleaning the tanks, it is very important that they are properly ventilated and checked to be free of potentially explosive gases. Because of the high cost associated to the tanks cleaning, change-over of products carried in the same tank is allowed only in very restricted circumstances.

Figure 1.7: Principle structure of a double-hull tanker.

Ships have different number of compartments, with different capacities, and differ on speeds and operations costs. Ship speed is influenced by the hull form, engine, fuel economy, weather conditions, loading conditions and schedules. Speed can be determined indirectly by means of distance and time. Usually, we consider the average speed, using past information, in order to determine the sailing time between ports. During the last years the company operated with two or, during short time periods, three ships. The fleet has been changing, the two ships operating at the beginning of the thesis work are different from those operating today.

Ports

Ports impose physical limitations on the dimensions of the ships (ship draft, length and width). Currently there are no draft limit constraints. Ports also charge fees for their services.

An important aspect to schedule ship operations are time windows. These restrictions are established due to different reasons. Some ports can receive passenger ships, cargo ships and tanker ships, but not at the same time, due to safety conditions. That means that it is not possible to unload/load the product when passenger ships are operating. Hence, different time windows are established for each type of operation. Also, some ships cannot anchor in certain ports due to their length, and in that case the ship needs to anchor near the port to discharge the products. At those ports, where operations at night are allowed, after 6 p.m. the company pays an extra cost per hour. In these cases the time windows are hard at the start and soft at the end. Waiting time is permitted. In some ports different types of cargo ships are allowed to operate during the same time window. In those cases very often port congestion causes ship delays.

The company takes control of their tanks' inventory levels and safety stock at each port, ensuring that adequate service levels are maintained. Stock-out is completely forbidden under regular conditions. Time limits for each port visit can also be established implicitly by the inventory levels of the consumed products. These limits lead to inventory time windows which are of a different nature from operating time windows.

Weather conditions

Weather conditions are a major aspect in maritime transportation. The oil company looses many days/weeks per year because of the seiche phenomenon. A seiche is a standing wave in an enclosed or partially enclosed body of water. This phenomenon occurs in certain periods in Cape Verde and affects many important ports. During that period, it is very difficult or impossible to operate in those ports affected with this phenomenon. Additionally, bad weather conditions extend the traveling times between ports.

1.2.1 Current planning practice

In addition to the strategic planing, not considered in this thesis, concerned with decisions such as acquisition of ships, expansion of tank capacity, creation of new supply ports, etc., the company deals with two related planning problems.

A planning for six months is done based on the assumption of constant consumption rates. In this planning the company simulates the distribution plan for a given fleet and tank capacities, and ignore many details such as time windows. This planning is conducted to allow the company to evaluate the necessity of chartering ships or expand tanks, for instance.

For a shorter time horizon of 10-14 days, the company plans the distribution of fuel oil products as follows. Every Monday, the planner gathers information on the stock level at each consumption storage depot. This information is put in an excel sheet and, using that information, the company planner develops, manually, the distribution plan, which includes the ship routes, the quantity of each product that must be unloaded/loaded at each port in the corresponding route, and schedule of the operations. Usually, the design of the routes takes into account the ports urgency in order to avoid stock-outs. This planning task is very complex because in addition to the many known details that must be taken into account (ship speeds, estimated travel times between ports, safety rules, time windows, loading and unloading times, ship and tank capacities, etc.), the manager usually tries to gather additional information such as port occupancy in order to avoid expensive waiting times. This manual procedure has produced satisfactory results for many years. However, in recent years Cape Verde has a rapid economic growth and the fuel oil consumption has increased. This is the main reason for a deep and detailed study of the redesign of the distribution structure of fuel oil in Cape Verde.

1.3 Purpose and outline of the thesis

The purpose of this thesis is discussed in Section 1.3.1. Then, Sections 1.3.2-1.3.5, present a brief summary of each paper.

1.3.1 Purpose of the Thesis

The purpose of this thesis is to develop new optimization tools for inventory routing and scheduling problems found in industrial shipping. The tools are aimed to provide decision support at an operational and at a tactical planning level. The optimization tolls are based on mathematical models which were developed in close cooperation with a local company that needs to improve its decision support system. Although we have developed models for real-world planning, they are generic and may be used in other contexts as well.

This work is a result of the relationship between the local Shell company and the Cape Verde University. The company agreed to provide all the information and details. On the other hand, the research undertakes to maintain confidentiality and to develop and test, solutions approaches, using real data.

The author started this research work in October, 2008. During this period many meetings with the manager of the company occurred in order to clarify some details and formulate the problem as realistic as possible.

Following the company's approach, according to the length of the planing horizon, we study two related problems: a short and a medium-term operational planning problem. In both cases, we assume that the number of ships and capacities are fixed, and are sufficient to satisfy the total demand during the planning horizon. All ships and consumption depots are owned by the company. We consider the operation cost (loading/unloading at ports), sailing cost and penalty cost (when a ship operates after the end of time windows). The main objective of the company is to maintain the stocks levels within their limits over the planing horizon. In accomplishing this objective, the plans must (i) minimize the total cost comprised of the operational cost, sailing cost and penalty cost (ii) determine the ship routes, the schedule of operations and the quantity of each product to load and discharge while satisfying a set of constraints.

Usually, short-term planning addresses goals that can be obtained within a short period of time. According to the practice of the company, we consider a planning horizon of 12 days. The production and consumption rates vary during the planing horizon. Often the production/consumption is zero in many periods (days) and relatively high in the remaining periods. Another real aspect taken into account is scheduling. Multiple operating time windows (one for each day) are considered.

For the medium-term plan, a planning horizon of six months is considered. The production and consumption rates are assumed constant during this planing horizon. Time windows constraints are ignored, but inventory levels are taken into account, and minimum and maximum limits are considered. The output of the second stage (medium-term) can be used to define the input for the first one.

Uncertainty related to traveling times and times at ports is a major issue in maritime transportation. This issue is more relevant for the short-term planning since for longer planning horizons estimated values based on the past information, are considered. We also address a variant of a short-tem planning problem with stochastic traveling and port times.

The solution approach followed for each problem has essentially been the same. First we provide and discuss the mathematical formulation (mixed integer program), then we either optimize to optimality the problem for a set of real instances using a commercial solver, or we provide hybrid heuristics that use the solver as a black-box.

During this thesis, four scientific papers have been written and submitted to international journals. The first paper: *Mixed integer formulations for a short sea fuel oil distribution problem* considers a short-term fuel oil distributions problem with a planning horizon of two weeks, and has been published in *Transportation Science*. The second paper: *Discrete-time and continuous-time formulations for a maritime inventory routing problem* is dedicated to compare discrete-time versus continuous-time formulations for the constant demand rate. The third paper: *Hybrid heuristics for a maritime short sea inventory routing problem* studies heuristic solution approaches for a medium-term problem with a time horizon of 6 months. The last paper: *A maritime inventory routing problem with stochastic sailing and port times* models a stochastic short-term planning problem with constant demand rates and uncertain time parameters. Figure 1.8 shows the scope of each paper, according to the planning levels and transportation modes. Next we explain in more detail each one of these paper.

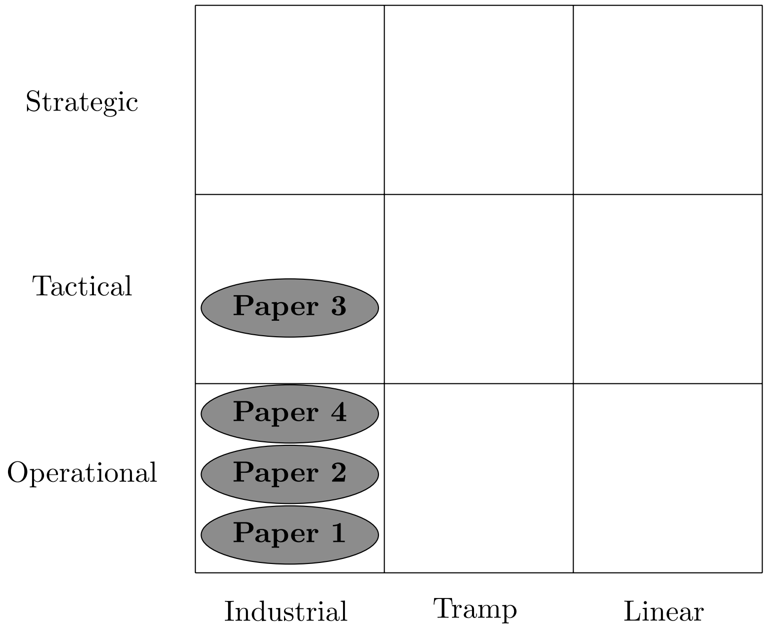

Figure 1.8: Purpose of each paper according to the planning levels and transportation modes

1.3.2 Paper 1: Mixed integer formulations for a short sea fuel oil distribution problem

In this article we consider the short-term fuel oil distribution problem. Since cleaning tanks is very expensive and time consuming, the company follows a policy where products have dedicated compartments, if possible. In this paper we consider both the undedicated and dedicated tank case.

Three mixed integer formulations are presented and discussed. Following the related literature (see Chistiansen [5] and Al-Khayyal and Hwang [1]) we first introduce an arc-load formulation. Then we introduce two new tighter formulations, an arc-load flow formulation (flow variables are assigned to ship arcs) and a multi-commodity formulation. All the formulations are strengthened by tightened linking constraints and inclusion of valid inequalities. These formulations were tested using real instances. Using the best formulation, all instances are solved to optimality. The average running time was less than one minute for undedicated compartments instances and less than 25 minutes for the dedicated case.

As consumption continues to grow and it is expected to continue to increase in the future, the company believes that in a near future the fleet size needs to increase. In this context, future scenarios considering larger consumption rates and new ships were tested. We have shown that most of those large instances can still be solved to optimality within a time limit of 3 hours.

1.3.3 Paper 2: Discrete-time and continuous-time formulations for a maritime inventory routing problem

In order to choose a good model for the constant consumption rate case, in the second paper, we conduct a study to compare several formulations and to test the impact of certain assumptions on the model and on the solution.

We provide two alternative mixed integer formulations: a discrete-time model adapted from the case where the consumption rates vary (the model combines a discrete and continuous time where the discrete time corresponds to an artificial discretization of the continuous time), and an event based model known as continuous-time formulation. For each alternative formulation we discuss two different extended formulations and inclusion of valid inequalities that allow us to reduce the linear gap of the two initial formulations. We also test the impact of limiting the number of visits to each port, and to impose minimum load/unload quantities, both on the model size and on the solution. In order to compare the proposed models accordingly to their size, linear gap and running time, a computational study based on real small-sized instances using a commercial software is conducted.

1.3.4 Paper 3: Hybrid heuristics for a maritime short sea inventory routing problem

In this paper, the medium-term fuel oil distribution problem is considered, assuming constant consumption rates and considering dedicated compartments in the ships. Many important aspects taken into account in the short-term problem are relaxed here or incorporated indirectly in the data. For instance, port operating time windows that are essential in the short-term plan are ignored here. On the other hand, other aspects such as safety stocks are more relevant for the medium-term planing.

Considering a planning horizon of 6 months, the tested instances become too large to be solved to optimality by a commercial software. Therefore, we develop several heuristic schemes that combine three well-known hybrid approaches that use the mixed integer programming solver as a black-box: Rolling Horizon (RH), Local Branching (LB) and Feasibility Pump (FP). In RH heuristics the planning horizon is split into smaller sub-horizons. Then, each limited and tractable mixed integer problem is solved to optimality. LB heuristics are used to search for local optimal solutions by restricting the number of binary variables that are allowed to change their value in the current solution. Feasibility Pump is a strategy to find initial feasible solutions for MIP problems. Based on the information of the second paper, we use the continuous-time model, tightened with valid inequalities, to solve each subproblem.

Computational testes were conducted to compare the different heuristic schemes. These tests show that the best strategy provides significant improvements when compared against the RH heuristic, which is one of the most popular heuristics for long time horizon problems.

1.3.5 Paper 4: A maritime inventory routing problem with stochastic sailing and port times

In the real problem, much of the input data needed for the planning is uncertain. Weather conditions have great influence on the traveling times. Port congestion is also a major problem the company has to deal with. In the past, the company lost many working days (measured in weeks per year) due to the waiting for ports to became vacant.

In this paper we develop a stochastic model where traveling times and waiting times are stochastic. We consider a constant consumption rate. The model can be regarded as a two-stage stochastic programming model with recourse where the first-stage consists of routing, loading and unloading decisions, and the second stage consists of scheduling decisions. The first stage decisions are fixed *a priori*, that is, before actual values of the uncertain parameters are revealed, while the second stage decisions can be adjusted to the stochastic parameters (traveling and waiting times). The model is solved using a decomposition approach similar to an L-shaped algorithm where optimality cuts are added dynamically. This solution process is embedded within the sample average approximation method. A computational study based on ten real-world instances shows the effectiveness of the solution method and the importance of considering a stochastic approach.

1.4 Contributions

This section presents the contributions of this thesis . The contributions are divided into three parts. In the first part, we discuss the contributions to the research community. The corresponding contributions to the industry are presented in the second parte. Finally, in the last part, an overview of the author contributions to each one of the papers that constitutes this thesis is given.

1.4.1 Thesis contribution to the research community

In this section we present the main contributions of the thesis to the research community. Research work in these papers have been presented at many different scientific conferences, including 14^{o} bi-annual congress on operation research, Seventh Triennial Symposium on Transportation Analysis, Workshop on Applied Combinatorial Optimization and the 23th, 24th and 25th European Conference on Operational Research. Here we summarize the main contributions of the thesis.

- We propose solution approaches to solve a real maritime inventory routing problem considering the short-term planning and the medium term-planning, separately. All the approaches are based on mixed integer models which are solved (exactly, for the short-term planning, and heuristically, for the medium-term planning) by a commercial software.

- For the short-term planning problem we also propose a stochastic model where traveling times and waiting times are random. The solution method combines the use of the sample average approximation method with a decomposition procedure resembling an L-shaped method. We provide computational evidence of the importance to use a stochastic model instead of a deterministic model.

- We introduce and compare different mixed integer models for each problem. These models are tightened using valid inequalities. To the best of our knowledge, some of the inequalities introduced in this thesis and some formulations have never been applied before in maritime transportation problems. Such models and inequalities can be easily used/adapted for similar maritime inventory routing problems.

- The hybrid approach proposed to provide reasonable solutions for the medium-term planning problem, that combines three well known heuristics (Rolling Horizon, Local Branching and Feasibility Pump) and uses a mixed integer programming solver as black-box, can also be easily applied in many other inventory routing problems.

A more detailed description of these contributions can be found in each one of the following Chapters.

1.4.2 Thesis contribution to the industry

Paper 1 solves to optimality the distribution problem of the company at the operational planning level. The solution approach followed allows for cost savings in relation to current practice. The model and the solution approach used in this paper can be easily adapted to other related maritime transportation problems.

Paper 3 presents different combinations of hybrid heuristic approaches to produce good quality solutions, within reasonable running times, for the distribution problem at a tactical planning level. This work allows the company do evaluate the capacity to meet the demands over large time horizons and to identify possible routing and scheduling patterns. These approaches combining rolling horizon, local branching and feasibility pump heuristics, may be used as a reference heuristic solution approach to other real problems.

Paper 4 solves the distribution problem at the operational planning level under uncertain traveling and waiting times. This paper reinforces the importance in the use of stochastic programming instead of a deterministic approach to solve these maritime transportation problems.

The study of models and derivation of inequalities conducted in all papers can be easily applied to other maritime transportation problems.

1.4.3 The author's contribution to each paper in the thesis

In this section the contributions to the four papers that constitute this thesis are presented. The contributions can be measured as follows: intellectual input, implementation, and writing. Intellectual input refers to identification and formulation of the planning problems, development of the mathematical models, valid inequalities and solution algorithms. Furthermore, implementation covers the data handling, coding and execution of the models, as well as analyzing the results. Finally, writing refers to the writing of the scientific papers.

Intellectual input: the problem arose from a meeting between the author, Agostinho Agra and the manager of the local Shell company. The author gave the main contributions to problem identification and definition of the important aspects of the problem. The development of the mathematical models of the four papers was done by Agostinho Agra, Marielle Christiansen and the author. The fourth paper had also the contribution of Lars Magnus Hvattum. The development of valid inequalities was done by Agostinho Agra and the author. The discussion of algorithm strategies was done with all the authors of each paper.

Implementation: the author had the responsibility for the computer implementation of the models, while cooperating with the manager of the Shell company to gather the real data used as input of each paper. The computer implementation supervision and data analyzes was done by Agostinho Agra and Marielle Christiansen in all papers and Luidi Simonetti in the third paper and Lars Magnus Hvattum in the fourth paper.

The writing of all papers was mainly handled by Agostinho Agra, with very important contributions from Marielle Christiansen, with feedback and collaboration from the author and from Lars Magnus Hvattum in the fourth paper.

1.5 Further Research

There are many research directions that may be of interest to be followed in the future. Here we provide three such directions.

- Much research in the maritime transportation literature uses models based on path formulations. These modes are then solved by column generation based algorithms such as Branch-and-Cut-and-Price. Testing such models against our approaches would be an interesting line of research.

- An important subject of research for the company is the study of the strategic planning problem, in order to help the decision maker to define a tanks expansion policy, acquisition or chartering of ships, etc..

- In order to assess the quality of the feasible solutions provided by the hybrid approaches, described in Chapter 4, for the medium-term planning, is essential to provide good lower bounds. Because of the large size of the mixed integer model used, the linear relaxation model can hardly be used to obtain good lower bounds. An alternatively approach could be to test other relaxations, such as, the lagrangean relaxation.

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Paper I

A. Agra, M. Christiansen and A. Delgado :

Mixed integer formulations for a short sea fuel oil distribution problem

*Transportation Science, **47**, 108-124, (2013).*

Chapter 2

Mixed integer formulations for a short sea fuel oil distribution problem

Abstract

We consider a short sea fuel oil distribution problem occurring in the archipelago at Cape Verde. Here, an oil company is responsible for the routing and scheduling of ships between the islands such that the demand for various fuel oil products is satisfied during the planning horizon. Inventory management considerations are taken into account at the demand side, but not at the supply side. The ports have restricted opening hours each day, so multiple time windows are considered. In contrast to many other studies within ship routing and scheduling, considerable time is spent in the ports compared to at sea. Hence, the time in port is modeled in detail by incorporating both a variable (un)loading time and a set up time for loading different products in the same ports. A mathematical model of the problem is presented and it includes a combined continuous and discrete time horizon due to the multiple time windows and a daily varying consumption rate of the various products in the different ports. We discuss several strategies to improve the proposed model, such as tightening bounds, using extended formulations and including valid inequalities. The computational study shows that the real problem can be solved to optimality within reasonable time by the use of improved formulations based on a combination of such strategies.

Keywords: Maritime transportation, Inventory, Routing, Extended formulations, Valid inequalities.

2.1 Introduction

The inter-islands distribution of fuel oil is a real problem of Cape Verde, an archipelago with nine inhabited islands. Fuel oil products are imported and delivered to specific islands and stored in large supply storage tanks. From these islands, fuel oil products are distributed among all the inhabited islands using a small heterogeneous fleet of ships. These products are stored in consumption storage tanks. Some ports have both supply and consumption tanks (see Figure 2.1). For the inter-islands distribution problem we ignore two islands (with a circle in Figure 2.1) since they are supplied by another ship using a different technology.

The inter-island distribution plan consists of designing routes and schedules for the fleet of ships including determining the (un)loading quantity of each product at each port. This

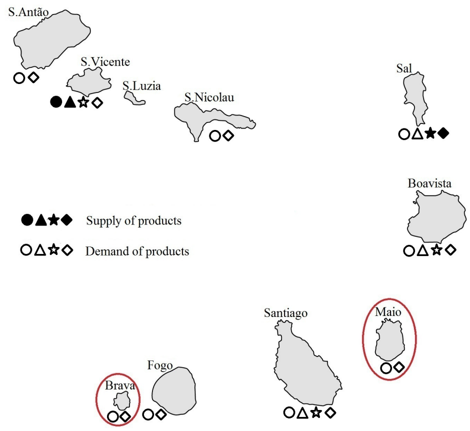

Figure 2.1: Supply and demand for fuel oil products at several islands in Cape Verde.

plan must satisfy (i) the demand of each product at each island per period, (ii) time window constraints for the port operations (loading/unloading), and (iii) the capacities of the ships, ports and depots. The total cost of the distribution plan is to be minimized, and includes sailing costs, a fixed cost for each operation and a penalty cost for violation of time windows.

We consider a short-term distribution problem with a planning horizon of twelve days. The input to this problem is the output of a medium-term plan for several weeks (few months). The demands correspond to the quantities to be delivered at each port per day determined in the medium-term plan. Hence, usually the demands at each port follow a pattern where the demands are zero for most periods and relatively large in the rest of the periods. By coordinating the distribution of all products in all ports during the planning horizon, it might be efficient to deliver the demand in periods prior to the specified period by the medium-term plan or in other quantities. This means that we need to keep track of the inventory level at the consumption storage tanks for all products in all ports. Storage capacities in supply and consumption tanks are taken into account in the medium-term planning. In the short-term plan considered here capacities in supply tanks can be ignored, since the global consumption of each product from all consumption tanks during the time horizon, is much smaller than the capacity of the supply tanks. However, for the consumption tanks the capacity of the tanks for a particular product can be less than the total demand over the planning horizon for that product. When solving our instances, we consider only inventory capacity bounds for the consumption tanks where the corresponding capacity can be lower than the total demand over the entire planning horizon. It is assumed that at most one ship can operate in each port at a given time period. During a port call for a ship, it is possible to load and unload different products. We assume that there is a fixed (un)loading time per unit product (un)loaded. This (un)loading time may vary for different products and different ports. In addition, there exists a considerable set up time between (un)loading different products due to coupling and decoupling of pipes between tanks in the ship and tanks in the port.

Most of the ports are closed during night and some ports have operational restrictions during certain periods of the day. This means that in each period (day), there may exist a time window for (un)loading. These time windows may vary from port to port. A ship cannot start to operate before the beginning of the time window. However, if the operation has begun inside the time window, it can be finished outside that time window, see Figure 2.2. In that case, an extra man-power cost considered as a penalty cost is incurred.

To transport the fuel products between the islands, the planners control two different ships, but a larger heterogeneous fleet is expected in the future. Each ship has a specified load capacity,

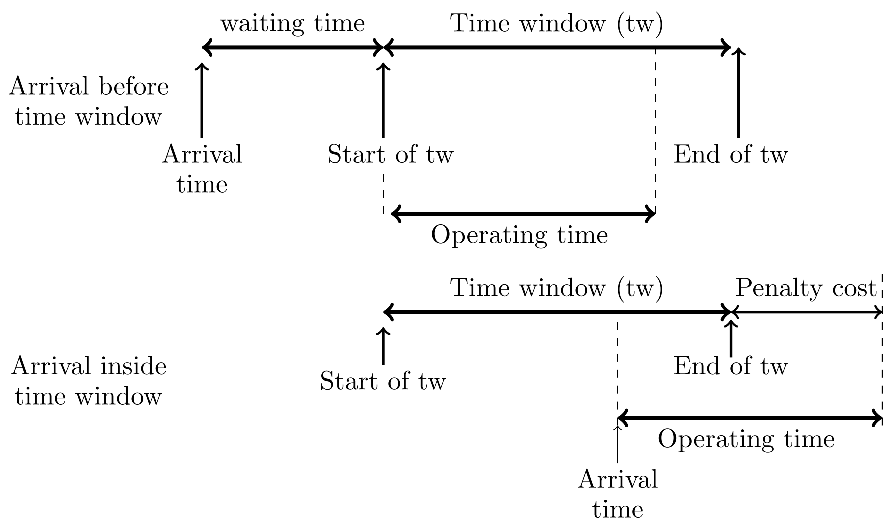

Figure 2.2: Time windows: operating time inside and outside (with penalty) of the time window, and waiting times.

fixed speed and cost structure. The setup and (un)loading times are independent of ship. The cargo hold of each ship is separated into several cargo tanks.

We consider two scenarios: the case where the allocation of the different fuel products into different cargo tanks is not considered and the case where there are dedicated tanks for families of products. The later case is the closest to reality (and later denoted as the Real Case) although in some situations it can be considered too restrictive since changes between families of products are possible. However, such changeovers are only allowed under exceptional circumstances. On the other hand, the first case can be regarded as a relaxation of the real situation. We focus on the first case and explain in a later section how to deal with the dedicated tanks case.

We consider a fixed cost associated with each operation at each port and associate a setup time to each operation (loading/unloading) as depicted in Figure 2.3.

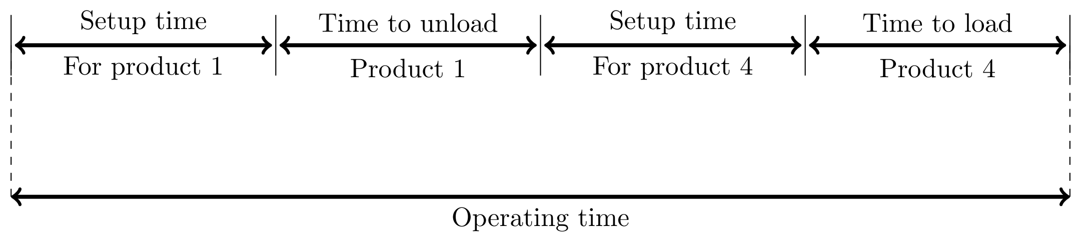

Figure 2.3: Schedule of operations: the ship unloads product 1 and then loads product 4. A setup time is required for each operation.

In this paper we present mixed integer formulations for the two cases of the short sea fuel oil distribution problem (SSDP) in Cape Verde and provide several strategies to improve the formulation applying techniques such as the use of extended formulations and the inclusion of strong valid inequalities. Based on an extensive computational study we propose improved formulations which can solve the tested instances based on real data to optimality within reasonable time.

The models proposed are based on the underlying real planning problem. However, some simplifications are made and some issues are omitted. Safety stocks are not explicitly considered, but could easily be taken into account by considering net stocks (the stock level minus the safety stock), see [3]. Fluctuating weather conditions are neglected, and very few contributions in the literature have so far focused on this issue within maritime transportation. In addition, we chose to make some assumptions; such as a maximum number of ships operating in the same port per day, no draft limits in the ports and assuming the described tank allocation policy. However, these simplification and assumptions should not prevent the planners from making valuable short-term decisions based on the SSDP models.

Although the paper is concerned with a real problem our contributions are also of interest for other maritime transportation problems. Besides the application, the contributions comprise (i) new models that include a combined continuous and discrete time horizon. These are fairly general models that deal with multiple time windows and a daily varying consumption rate of the various products. They can also be used for the constant consumption rates case as this is a particular case of the previous one; (ii) discussion of different strategies to strengthen the proposed models. The computational results also provide some insight into the relevance of each strategy.

We remark that the models presented here can also be used to solve larger instances. For instance, heuristic procedures based on the mathematical formulation such as rolling-horizon heuristics, relax-and-fix heuristics, etc. can be used to derive feasible solutions.

The rest of the paper is organized as follows: In Section 2.2 we give a brief literature review related to the planning problem and the formulation techniques considered. We have limit ourselves to the maritime transportation literature. Section 2.3 presents a mixed integer formulation of the SSDP in Cape Verde. Different strategies to improve the initial formulation are discussed in Section 2.4. These strategies include tightening bounds, including valid inequalities and deriving extended formulations. In Section 2.5 we discuss the real problem with dedicated tanks. We focus on the main differences between the two cases (with and without dedicated tanks) and explain how to adapt the results from the previous sections to the case with dedicated tanks. Section 2.6 is devoted to the results of an extensive computational study to compare different ways of combining the improving strategies, and to test our best strategy on the real case and on future scenarios. Finally, the main conclusions of this work follow in Section 2.7.

2.2 Related Literature

We have witnessed an increased interest in studying optimization problems within maritime transportation, see the reviews on maritime transportation, [9], and maritime inventory routing problems, [8]. Combined routing and inventory management within maritime transportation have been present in the literature the last one and a half decades only. [6] considers a supply chain for ammonia consisting of several facilities that either produce or consume ammonia and the transportation network between those facilities. Ammonia is produced and stored in inventories at given loading ports and transported at sea to inventories at unloading ports. Inventory capacities are defined in all ports. Here, the production and consumption rates are given and fixed during the planning horizon in all ports. The planning problem is to find routes and schedules for a fleet of ships that minimize the transportation costs without interrupting production or consumption at the storages. The overall problem is solved by a branch-and-price method in [10] and [11] and by a heuristic in [15]. Unlike the problem studied in [6], the short sea fuel oil distribution problem (SSDP) includes several products. Also [17] studies a maritime inventory routing problem that allowed for multiple products on board the ship and with dedicated compartments in the ship for various products. [4] give a mathematical formulation for such a problem where the products are assumed to require dedicated compartments in the ship. For this problem there exist inventory limits and production/consumption rates for each product in each port, just as for our SSDP. We include the product-compartment allocation case in Section 2.6.3. As for the SSDP, [4] include a fixed setup time for switching between (un)loading different products in a port. The maritime inventory routing problem described in [23] also includes multiple products. The underlying model focuses on the inventory management and not the routing part of the problem, as the model solution suggests shipment sizes that are assumed to be input for a cargo routing problem at a later stage.

Both [6] and [4] present continuous time models and introduce an index indicating the visit number to a particular port. For both models it is assumed that the production/consumption rate is fixed and constant during the planning horizon. In [16] and [23] discrete time models are developed to overcome the complicating factors with varying production and consumption rates. In the SSDP, the production inventory side is not considered, but we have varying consumption rates during the planning horizon.

In most studies in the literature, the inventory management is considered both at the production and consumption sites. However, [25] consider a liquefied natural gas inventory routing problem with just one large production port and no inventory management aspects considered at the consumption ports.

Most of the maritime transportation planning problems studied in the literature are within the deep sea segment, see [9]. However, we are considering a short sea distribution problem with relatively low-activity ports. Considerable time is spent in port, and some ports are closed during night. This corresponds to a problem with multiple time windows. Within maritime transportation, this is considered in [7] for a tramp ship scheduling problem without inventory management considerations. In contrast to the SSDP, no loading and unloading are possible after the end of the time window. This means that a ship might stay idle in port during night or in the weekend if it did not finish its service in the port opening hours. In order to avoid such idle times due to unexpected delays, the authors have introduced penalties for finishing the service in port just before the end of time windows. In this way, they expect that more robust schedules are designed. We can also find a few other contributions within maritime transportation where penalties are used in connection with time windows. In [14], the hard time windows are extended to soft ones. There the penalty costs occur outside the hard time windows. The work of [6] is extended in [12] to reduce the possibility of violating the inventory limits at the storages. Here another pair of soft inventory limits within the hard ones is introduced. This means that those soft inventory limits can be violated at a penalty, but it is not possible to exceed the storage capacity or be under the lower inventory limits. They show that the soft inventory constraints can be transformed into soft time windows.

Although the study of valid inequalities for mixed-integer sets and the derivation of extended formulations is currently receiving large attention with several applications to other mixed-integer problems, little work has been done in applying these techniques to maritime transportation problems. However, a few contributions already exist. [24] include valid inequalities in order to enhance the proposed formulations of an oil products transportation problem, and [21] develop valid inequalities within a column generation approach for a maritime inventory routing problem. Also, [16], include valid inequalities to improve the path-flow formulation presented for the liquefied natural gas inventory routing problem.

2.3 Mathematical Formulation

In this section, we describe a mathematical model for the SSDP without dedicated tanks.

The nature of the production and consumption rates affects the underlying model. If it is assumed that the production and consumption rates are fixed and constant during the planning horizon, then a mathematical model based on continuous time can be used (e.g. [4] and [6]). When the production and/or consumption rate is variable or fixed but varying during the planning horizon a discrete time model is applied (see [16] and [23]). The case of variable production and consumption rates is, of course, the most general one, but often the rates are fixed. In practice, the production and consumption rates are most often varying, although, in some applications, the simplification made by assuming a constant rate is acceptable.

In this paper we consider a combined continuous and discrete time horizon. The discrete time horizon corresponds to the continuous one divided into periods (corresponding to days). The discrete time horizon allows us to easily handle the multiple time windows and non-constant demand requests. The drawback of this approach is the large number of variables involved in the mathematical model.

In this formulation, the decision variables are written in lower case letters and the parameters and sets are written in upper case letters.

Indices

k	products;
i, j	ports;
v	ships;
i_v	initial port position of ship v;
m, n	time periods.

Sets

V	set of ships;
N	set of ports;
K	set of products;
M	set of periods. $M = \{1, \dots \mid M \mid\}$, where $\mid M \mid$ is the number of periods considered.

Parameters

T_{ijv}	time required by ship v to sail from port i to port j;
T^A_{im}	start of time window in period m at port i;
T^B_{im}	end of time window in period m at port i;
C^W_{ik}	fixed cost of operating (loading/unloading) product k at port i;
T^Q_{ik}	time required to (un)load one unit of product k at port i;
D_{imk}	demand of product k at port i in period m;
C_{ijv}	total transportation cost for ship v to sail from port i to port j;
V^{CAP}_v	total storage capacity of ship v;
U_{ik}	storage capacity of the depot for product k at port i;
W_{ik}	setup time required for operating product k at port i;
Q_{vk}	quantity of product k on board ship v at the beginning of the planning horizon;
C^P_{im}	penalty cost, per hour, for operating outside the time window at port i in period m;
J_{ik}	=1 if port i is a producer of product k; =-1 if port i is a consumer of product k; 0 otherwise.

Continuous Variables

t_{im} start time of operation at port i in time period m, $i \in N, m \in M$. We assume $t_{i_v 1} = 0, v \in V$;

t^E_{im} ending time of the operation that started during period m in port i, $i \in N$, $m \in M$ (these variables are not necessary for the model but they are useful to ease the reading);

p_{im} operating time outside the time window of period m at port i, $i \in N, m \in M$;

q_{imvk} amount of product k loaded onto or unloaded from ship v at port i in time period m, $i \in N, m \in M, v \in V, k \in K$. We assume $q_{imvk} = 0$ if $J_{ik} = 0$; or $m = 1, i \neq i_v$; or $m = 1, i = i_v, Q_{vk} = 0, J_{ik} = -1$;

l_{imvk} amount of product k onboard ship v when leaving port i after an operation that started in time period m, $i \in N, m \in M, v \in V, k \in K$. We assume $l_{i1vk} = 0$ if $i \neq i_v$;

s_{imk} stock level of product k at port i at the end of time period m, $i \in N, m \in M$, $k \in K$. (s_{i0k} is the stock level at the beginning of period 1).

Binary variables

x_{imjnv} 1 if ship v starts to operate at port i in period m and then sails from port i to port j and starts to operate at port j in perid n, 0 otherwise, $i, j \in N$, $m, n \in M, v \in V$. We assume $x_{imjnv} = 0$ if $m \geq n$; or $i = j$; or $m = 1, i \neq i_v$;

z_{imv} 1 if ship v ends its route at port i after an operation that started in time period m, 0 otherwise, $i \in N, m \in M, v \in V$;

o_{imvk} 1 if product k is loaded onto or unloaded from ship v at port i in time period m; 0 otherwise, $i \in N, m \in M, v \in V, k \in K$. We assume $o_{imvk} = 0$ if $J_{ik} = 0$; or $m = 1, i \neq i_v$; or $m = 1, i = i_v, Q_{vk} = 0, J_{ik} = -1$.

The MIP model for the SSDP:

$$\min \sum_{i,j \in N} \sum_{n,m \in M} \sum_{v \in V} C_{ijv} x_{imjnv} + \sum_{i \in N} \sum_{m \in M} \sum_{v \in V} \sum_{k \in K} C^W_{ik} o_{imvk} + \sum_{i \in N} \sum_{m \in M} C^P_{im} p_{im}, \tag{2.1}$$

subject to:

$$\sum_{j \in N} \sum_{n \in M} x_{i_v 1 jnv} + z_{i_v 1 v} = 1, \qquad \forall v \in V, \tag{2.2}$$

$$\sum_{j \in N} \sum_{n \in M} x_{jnimv} - \sum_{j \in N} \sum_{n \in M} x_{imjnv} - z_{imv} = 0, \quad \forall i \in N, m \in M, m > 1, v \in V, \tag{2.3}$$

$$\sum_{i \in N} \sum_{m \in M} z_{imv} = 1, \qquad \forall v \in V, \tag{2.4}$$

$$\sum_{j \in N} \sum_{n \in M} \sum_{v \in V} x_{jnimv} \leq 1, \qquad \forall i \in N, m \in M, \tag{2.5}$$

$$T^A_{im} \leq t_{im} \leq T^B_{im}, \qquad \forall i \in N, m \in M, \tag{2.6}$$

$$(t^E_{im} + T_{ijv} - t_{jn}) x_{imjnv} \leq 0, \qquad \forall i, j \in N, m, n \in M, v \in V, \tag{2.7}$$

$$p_{im} \geq t^E_{im} - T^B_{im}, \qquad \forall i \in N, m \in M, \tag{2.8}$$

$$t^E_{im} = t_{im} + \sum_{v\in V}\sum_{k\in K} W_{ik} o_{imvk} + \sum_{v\in V}\sum_{k\in K} T^Q_{ik} q_{imvk}, \qquad \forall i \in N, m \in M, \tag{2.9}$$
$$t_{im} \geq t^E_{i,m-1}, \qquad \forall i \in N, m \in M, m > 1 \tag{2.10}$$
$$x_{imjnv}(l_{imvk} + J_{jk} q_{jnvk} - l_{jnvk}) = 0, \qquad \forall i, j \in N, m, n \in M, v \in V, k \in K, \tag{2.11}$$
$$Q_{vk} + J_{i_v k} q_{i_v 1vk} - l_{i_v 1vk} = 0, \qquad \forall v \in V, k \in K, \tag{2.12}$$
$$q_{imvk} \leq V^{CAP}_v o_{imvk}, \qquad \forall i \in N, m \in M, v \in V, k \in K : J_{ik} \neq 0, \tag{2.13}$$
$$\sum_{k\in K} l_{imvk} \leq V^{CAP}_v \sum_{j\in N}\sum_{n\in M} x_{imjnv}, \qquad \forall i \in N, m \in M, v \in V, \tag{2.14}$$
$$s_{i,m-1,k} + \sum_{v\in V} q_{imvk} = D_{imk} + s_{imk}, \qquad \forall i \in N, m \in M, k \in K : J_{ik} = -1, \tag{2.15}$$
$$s_{imk} \leq U_{ik}, \qquad \forall i \in N, m \in M, k \in K : J_{ik} = -1, \tag{2.16}$$
$$x_{imjnv} \in \{0, 1\}, \qquad \forall i, j \in N, i \neq j, m, n \in M, m < n, v \in V, \tag{2.17}$$
$$z_{imv} \in \{0, 1\}, \qquad \forall i \in N, m \in M, v \in V, \tag{2.18}$$
$$o_{imvk} \in \{0, 1\}, \qquad \forall i \in N, m \in M, v \in V, k \in K, \tag{2.19}$$
$$q_{imvk}, l_{imvk} \geq 0, \qquad \forall i \in N, m \in M, v \in V, k \in K, \tag{2.20}$$
$$s_{imk} \geq 0, \qquad \forall i \in N, m \in M, k \in K, \tag{2.21}$$
$$t_{im}, t^E_{im}, p_{im} \geq 0, \qquad \forall i \in N, m \in M. \tag{2.22}$$

The objective function (2.1) is to minimize the cost (transportation cost, setup cost of operations and penalty cost).

The set of routing constraints (2.2)-(2.4) is defined under a network whose set of nodes is $\{(i, m) \in N \times M\}$. Hence, each node corresponds to a port-period pair. Constraints (2.2) ensure that ship v either departs from the initial port position i to another port j or it ends its route in port i ($z_{imv} = 1$). Constraints (2.3) are the flow conservation constraints for each port and each time period. That is, if ship v starts an operation in port i at period m, then either it must travel to another port j, or it finishes its route in port i. Constraints (2.4) ensure that ship v ends its route at some port. Constraints (2.5) guarantee that at most one ship v can operate in port i at a given time period. The time window constraints are given by (2.6). Constraints (2.7) ensure that if ship v sails from port i (after an operation started in period m) to port j (to initialize an operation in period n), then the operation at port j can only start after the end time of operation at port i plus the time required to travel from i to j. These constraints can be linearized as follows:

$$t^E_{im} + T_{ijv} - t_{jn} \leq B(1 - x_{imjnv}), \qquad \forall i, j \in N, m, n \in M, v \in V, \tag{2.23}$$

where $B = \max\{0, T^Q_{ik} V^{CAP}_v + T^B_{im} + T_{ijv} - T^A_{jn}\}$.

Constraints (2.8) enforce p_{im} to assume, at least, the value of the duration of operations outside the time windows. Notice that since the cost of p_{im} is positive, p_{im} assumes exactly the operation time violating the corresponding time window. Equations (2.9) define the end time of each operation. Constraints (2.10) ensure that, for each port and for each period, a ship can only start to operate if the operation of the previous period is already finished. Constraints (2.11) and (2.12) relate the quantity onboard to the quantity loaded and/or unloaded. Constraints (2.11) ensure that if ship v sails from port i (after an operation started in period m) to port j (to initialize an operation in period n), then the quantity of product k onboard at the departure from port j should be equal to the quantity onboard at departure from port i plus/minus the quantity loaded/unloaded from port j. Following [13], equations (2.11) can be linearized by replacing them with the following two sets of constraints :

$$\left[\begin{array}{c} l_{imvk} + J_{jk} q_{jnvk} - l_{jnvk} \\ +V_v^{CAP} x_{imjnv} \end{array} \right] \leq V_v^{CAP}, \qquad \forall i, j \in N, m, n \in M, v \in V, k \in K, \tag{2.24}$$

$$\left[\begin{array}{c} l_{imvk} + J_{jk} q_{jnvk} - l_{jnvk} \\ -V_v^{CAP} x_{imjnv} \end{array} \right] \geq -V_v^{CAP}, \qquad \forall i, j \in N, m, n \in M, v \in V, k \in K. \tag{2.25}$$

Equations (2.12) relate the quantity onboard with the quantity loaded/unloaded in the starting port. Constraints (2.13) ensure that if an operation occurs, that is, $q_{imvk} > 0$, then the setup variable o_{imvk} must be one. They also impose an upper bound on the quantity loaded/unloaded. Constraints (2.14) impose an upper bound on the quantity onboard. They also ensure that if the quantity onboard is positive, then the ship must travel to some other port. Constraints (2.15) are the inventory management balance constraints and, together with non-negativity constraints (2.20) and (2.21), ensure that the demand for each product at each port in each time period is satisfied. The storage capacity at each port of each product are given by constraints (2.16). Finally, (2.17)-(2.22) are the non-negativity and integrality constraints.

Modeling a real problem implies to make assumptions and, in some cases, simplifications. In order to clarify our modeling options, we next present some observations related to the modeling issues.

(i) Considering the real operation (load/unload) times, a ship can start to operate in one period and finish the operations in that same period or in the next one. For the definition of variables we use the starting period.

(ii) We consider a penalty cost by violation of the time window. That penalty is considered during the operating time outside the time window where the operation started. However, it is in theory possible that a ship finishes the operations during the next time window. In this case we also penalize the operating time occurring in the next time window because this is not a desirable solution.

(iii) We impose that at most one ship can operate in each port per period. Since there is a large uncertainty with docking operations in maritime transportation and since we are considering a small fleet consisting of two ships, it is not desirable to schedule two ships at the same port in the same period.

(iv) The following set of constraints

$$o_{imvk} \leq \sum_{j \in N} \sum_{n \in M} x_{jnimv}, \qquad \forall i \in N, m \in M, m > 1, v \in V, k \in K, \tag{2.26}$$

ensure that if there is an operation involving ship v at a port i during a period m, then port i must belong to a ship route. These inequalities are not necessary in the model since the fixed cost associated with the x_{jnimv} variables are positive. However, we will include these inequalities in order to derive strong formulations for the model.

The "basic" MIP model for the SSDP consists of (2.1)-(2.6), (2.8)-(2.10), (2.12)-(2.26) and will be denoted the B-SSDP.

2.4 Formulation improvements

In this section we explore some directions to derive stronger models, which means models whose linear relaxations are tighter than the original one. Deriving stronger models may lead to better bounds which can be useful to reduce the number of nodes in a branch and bound-based scheme.

We consider different types of improvements. The first one consists of the tightening of bounds. The second one is based on reformulations of the model with the inclusion of additional variables (extended formulations). We propose an arc-load flow formulation and an arc-load multi-commodity formulation. The last improvement is related to the inclusion of valid inequalities. These inequalities are based on inequalities derived for simple mixed-integer sets arising from relaxations from the set of feasible solutions of B-SSDP.

2.4.1 Tighter bounds

Here we explain how to tighten certain constraints. Basically, for certain constraints, we replace the upper bound given by the capacity of the ship, V_v^{CAP}, by the total amount of fuel the ship can carry in order to satisfy the remaining demand.

Constraints (2.13), (2.14), (2.24) and (2.25) can be replaced, respectively, by the following constraints:

$$q_{imvk} \leq A_{imvk} o_{imvk}, \qquad \forall i \in N, m \in M, v \in V, k \in K : J_{ik} \neq 0, \tag{2.27}$$

$$\sum_{k \in K} l_{imvk} \leq A_{imv} \sum_{j \in N} \sum_{n \in M} x_{imjnv}, \qquad \forall i \in N, m \in M, v \in V, \tag{2.28}$$

$$l_{imvk} + J_{jk} q_{jnvk} - l_{jnvk} + A_{imjnvk} x_{imjnv} \leq A_{imjnvk},$$
$$\forall i, j \in N, m, n \in M, v \in V, k \in K, \tag{2.29}$$

$$l_{imvk} + J_{jk} q_{jnvk} - l_{jnvk} - \overline{A}_{imjnvk} x_{imjnv} \geq -\overline{A}_{imjnvk},$$
$$\forall i, j \in N, m, n \in M, v \in V, k \in K, \tag{2.30}$$

where, for all $i \in N, m \in M, v \in V, k \in K$,

$$A_{imvk} = \begin{cases} \min\{V_v^{CAP}, \sum_{n \in M: n > m} \sum_{u \in N: u \neq i} D_{unk}\}, & \text{if } J_{ik} = 1; \\ \min\{V_v^{CAP}, \sum_{n \in M: n \geq m} D_{ink}\}, & \text{if } J_{ik} = -1, \end{cases}$$

for all $i, j \in N, m, n \in M, v \in V, k \in K$, $A_{imjnvk} = min\{V_v^{CAP}, \sum_{t \in M: t > m} \sum_{u \in N: u \neq j} D_{utk}\}$,

$$\overline{A}_{imjnvk} = \begin{cases} \min\{V_v^{CAP}, \sum_{t \in M: t > n} \sum_{u \in N, u \neq j} D_{utk}\}, & \text{if } J_{jk} \neq -1; \\ \min\{V_v^{CAP}, \sum_{t \in M: t \geq n} \sum_{u \in N} D_{utk}\}, & \text{if } J_{jk} = -1, \end{cases}$$

and for all $i \in N, m \in M, v \in V$, $A_{imv} = \min\{V_v^{CAP}, \sum_{u \in N: u \neq i} \sum_{n \in M: n > m} \sum_{k \in K} D_{unk}\}$.

2.4.2 Extended formulations

In this section we propose two extended formulations. The new set of variables introduced in each formulation provides additional information about the solution. That information is essential to derive tighter inequalities. In the first extended formulation the new variables indicate the amount of each product carried along each arc, that is, the new variables associate a flow, for each product, to each arc in the ship path. The second formulation is a classical multi-commodity reformulation of the first extended formulation where the flow on each arc is disaggregated accordingly to its destination.

Arc-load flow reformulation

One of the weaknesses of the B-SSDP model is the set of constraints (2.14) since even if $\sum_{k\in K} l_{imvk} = V_v^{CAP}$ there can occur solutions with several fractional values of x_{imjnv}. In order to strengthen the model we introduce new variables, denoted by arc-load flow variables, and use these variables to decompose variables l_{imvk}. Instead of considering the amount of fuel of each product onboard the ship when it is leaving a port, the new variables indicate also the next port for where that fuel is being transported to. With these new variables we will replace (2.14) by stronger inequalities (inequalities that imply (2.14), and once they are included in the model the corresponding linear relaxation feasible set becomes tighter). Let us define the non-negative arc-load flow variables $f_{imjnvk}, i,j \in N, n,m \in M$ as the amount of product k that ship v transports from port i, after an operation that started in period m, to port j in order to start an operation in period n. We assume $f_{imjnvk} = 0$ whenever $x_{imjnv} = 0$.

The two sets of variables l_{imvk} and f_{imjnvk} can be related using the following equations

$$l_{imvk} = \sum_{j\neq i}\sum_{n>m} f_{imjnvk}, \qquad \forall i \in N, m \in M, v \in V, k \in K, \tag{2.31}$$

Using the arc-load flow variables we can replace constraints (2.11), (2.12) and (2.14) by

$$\sum_{i\neq j}\sum_{m<n} f_{imjnvk} + J_{jk} q_{jnvk} = \sum_{i\neq j}\sum_{m>n} f_{jnimvk},$$
$$\forall j \in N, n \in M : n > 1, v \in V, k \in K, \tag{2.32}$$
$$Q_{vk} + J_{i_v k} q_{i_v 1vk} - \sum_{j\neq i_v}\sum_{n>1} f_{i_v 1jnvk} = 0, \qquad \forall v \in V, k \in K, \tag{2.33}$$
$$\sum_{k\in K} f_{imjnvk} \leq V_v^{CAP} x_{imjnv}, \qquad \forall i,j \in N, m,n \in M, v \in V, \tag{2.34}$$

respectively. Adding constraints (2.34) for j and n we obtain

$$\sum_{j\neq i}\sum_{n>m}\sum_{k\in K} f_{imjnvk} \leq V_v^{CAP} \sum_{j\neq i}\sum_{n>m} x_{imjnv}.$$

Replacing $\sum\limits_{j\neq i}\sum\limits_{n>m} f_{imjnvk}$ by l_{imvk} we obtain (2.14). Hence constraints (2.14) can be obtained by aggregating constraints (2.34). Thus, the model using the arc-load flow variables should provide better bounds based on the linear relaxation. The drawback of this model is the huge number of continuous variables. For instances with higher dimension than those we tested, this reformulation can be of no use.

As in the previous subsection, the constant V_v^{CAP} can in some cases be replaced by a tighter bound.

Notice that with the inclusion of variables f_{imjnvk}, variables q_{jnvk} can be eliminated from the model using equations (2.32) and (2.33), that is, setting

$$q_{jnvk} = J_{jk}\left(\sum_{i\neq j}\sum_{m>n} f_{jnimvk} - \sum_{i\neq j}\sum_{m<n} f_{imjnvk}\right), \qquad \forall j \in N, n \in M : n > 1, v \in V, k \in K, \tag{2.35}$$

and

$$q_{i_v 1vk} = J_{i_v k}\left(\sum_{j\neq i_v}\sum_{n>1} f_{i_v 1jnvk} - Q_{vk}\right), \qquad \forall v \in V, k \in K. \tag{2.36}$$

We denote the arc-load flow model by F-SSDP. The F-SSDP includes constraints (2.1)-(2.6), (2.8)-(2.10), (2.13), (2.15)-(2.22), (2.23)-(2.26), (2.32)-(2.34).

Multi-commodity reformulation

A classical way to derive tighter models for flow formulations, as the arc-load flow formulation presented in the previous section, is to use multi-commodity formulations. The idea is to disaggregate the flow on each arc into different flows, one for each possible destination. Here, by destination we mean a port-period pair. With this reformulation it is possible to derive tighter models. From the practical point of view however the number of variables can be prohibitive when solving real problems.

Let us introduce the non-negative multi-commodity arc-load flow variables γ_{imjnvk}^{ut}, $i, j, u \in N, m, n, t \in M, k \in K$ as the amount of product k that ship v transports from port i, after an operation that started in period m, to port j in an operation starting in period n to be delivered at port u in period t. We assume $\gamma_{imjnvk}^{ut} = 0$ if $x_{imjnv} = 0$. These variables can be related with the arc-load flow variables throughout the following equations

$$f_{imjnvk} = \sum_{u \neq i} \sum_{t \geq n} \gamma_{imjnvk}^{ut}, \qquad \forall i, j, u \in N, m, n, t \in M, v \in V, k \in K. \tag{2.37}$$

The tightening of the F-SSDP model can be obtained by replacing constraints (2.34) by

$$\sum_{k \in K} \gamma_{imjnvk}^{ut} \leq \min\{V_v^{CAP}, \sum_{l \in M, l \geq t} \sum_{k \in K} D_{ulk}\} x_{imjnv}, \qquad \forall i, j, u \in N, m, n, t \in M, v \in V. \tag{2.38}$$

The multi-commodity flow model obtained from F-SSDP by replacing (2.34) with (2.38) and including (2.37) will be denoted by MF-SSDP. Of course the arc-flow variables f_{imjnvk} can be eliminated from that model using (2.37).

We note here that different multi-commodity arc-flow formulations could be derived. For instance, instead of considering the amount of product k delivered at port u in period t, one could consider the amount of product k to be consumed at port u in period t.

2.4.3 Valid inequalities

One approach to derive a stronger model is to include valid inequalities for the set of feasible solutions X. In order to derive valid inequalities we consider simpler substructures that result from relaxations of our formulation. Valid inequalities for the set of feasible solutions of these relaxations are also valid for X. We focus on deriving only those inequalities with great impact on the integrality gap reduction. For each family of inequalities we consider the separation problem and tune the separation algorithms.

In Section 2.4.3 we develop two types of inequalities based on the inventory constraints, while inequalities based on fixed charge flow sets are developed in Section 2.4.3. Finally, some strong inequalities for the F-SSDP are defined in Section 2.4.3.

Inequalities based on the inventory constraints

Here we consider valid inequalities for X derived from well-known valid inequalities for inventory lot-sizing sets obtained when considering constraints (2.5), (2.13), (2.15), (2.19), (2.20), (2.21), (2.26).

First we introduce valid inequalities for the set of feasible solutions based on the well-known (ℓ, S) inequalities derived for lot-sizing problems (see [22]). In order to do that we first consider the following set obtained from constraints (2.5), (2.13), (2.15), (2.19), (2.20), (2.21), (2.26):

$$s_{i,m-1,k} + \sum_{v \in V} q_{imvk} = D_{imk} + s_{imk}, \qquad \forall i \in N, m \in M, k \in K : J_{ik} = -1, \tag{2.39}$$

$$q_{imvk} \leq V_v^{CAP} o_{imvk}, \qquad \forall i \in N, m \in M, v \in V, k \in K, \tag{2.40}$$

$$\sum_{v \in V} o_{imvk} \leq 1, \qquad \forall i \in N, m \in M, k \in K, \tag{2.41}$$

$$s_{imk}, q_{imvk} \geq 0, \qquad \forall i \in N, m \in M, v \in V, k \in K, \tag{2.42}$$

$$o_{imvk} \in \{0, 1\}, \qquad \forall i \in N, m \in M, v \in V, k \in K. \tag{2.43}$$

Constraints (2.41) are implied by (2.5) and (2.26).

The set of solutions satisfying constraints (2.39)-(2.43) can be separated for each port i and each product k. By fixing a port i and a product k (and removing the corresponding indices, for simplicity), we obtain:

$$s_{m-1} + \sum_{v \in V} q_{mv} = D_m + s_m, \qquad \forall m \in M, \tag{2.44}$$

$$q_{mv} \leq V_v^{CAP} o_{mv}, \qquad \forall m \in M, v \in V, \tag{2.45}$$

$$\sum_{v \in V} o_{mv} \leq 1, \qquad \forall m \in M, \tag{2.46}$$

$$s_m, q_{mv} \geq 0, \qquad \forall m \in M, v \in V, \tag{2.47}$$

$$o_{mv} \in \{0, 1\}, \qquad \forall m \in M, v \in V. \tag{2.48}$$

The set of solutions satisfying (2.44)-(2.48), denoted by X^{LS}, is closely related to the feasible set of capacitated lot-sizing problems (see [22]). The polyhedral structure of related sets has been intensively study in the past. In [22] it is given a very complete and insightful survey of these studies.

Consider $y_m = \sum_{v \in V} o_{mv}$ and $x_m = \sum_{v \in V} q_{mv}$. From (2.46) and (2.48) it follows that $y_m \in \{0, 1\}$. Let $C = \max\{V_v^{CAP} : v \in V\}$. Hence the following set, denoted by X^{CLS} is a relaxation of X^{LS} :

$$s_{m-1} + x_m = D_m + s_m, \qquad \forall m \in M, \tag{2.49}$$

$$x_m \leq C y_m, \qquad \forall m \in M, \tag{2.50}$$

$$s_m, x_m \geq 0, \qquad \forall m \in M, \tag{2.51}$$

$$y_m \in \{0, 1\}, \qquad \forall m \in M. \tag{2.52}$$

Set X^{CLS} is the feasible set of the well-known single-item constant capacitated lot-sizing problem (see [22]). For the instances based on real data that we consider in this paper, in general, the demand of each product at each island over the time period does not exceed the capacity of the smallest ship. Hence, for these instances, constants V_v^{CAP} in (2.45) can be seen as a large positive constant and therefore X^{CLS} can be regarded as the incapacitated single-item lot-sizing problem. In this case the set of well-known (ℓ, S) inequalities defined for all $\ell \in M$, $S \subseteq \{1, \ldots, \ell\}$,

$$s_{r-1} + \sum_{j \in \{r, \ldots, \ell\} \setminus S} x_j + \sum_{j \in S} (\sum_{i=j}^{\ell} D_i) y_j \geq \sum_{i=r}^{\ell} D_i, \tag{2.53}$$

where $r = \min\{i \in S\}$, suffice to describe the convex hull of X^{CLS}.

By writing these inequalities in the original variables we obtain the following proposition.

Proposition 2.4.1. *For each $i \in N$, $\ell \in M$, $S \subseteq \{1,\dots,\ell\}, k \in K$ the inequality (ℓ, S)*

$$s_{i,r-1,k} + \sum_{n \in \{r,\dots,\ell\} \setminus S} \sum_{v \in V} q_{invk} + \sum_{m \in S} (\sum_{n=m}^{\ell} D_{ink}) \sum_{v \in V} o_{imvk} \geq \sum_{n=r}^{\ell} D_{ink}, \tag{2.54}$$

where $r = \min\{i \in S\}$, is valid for X.

Example 2.4.1. Let $N = \{1,2,\dots,7\}$, $M = \{1,2,\dots,12\}$, $V = \{1,2\}$, $K = \{1,2,3,4\}$. Fix port $i = 5$, product $k = 3$ (supposing $J_{53} = -1$) and consider the demand

$$(0, 300, 0, 0, 0, 0, 0, 0, 900, 0, 0, 700)$$

for twelve periods at that island for that product. Letting $\ell = 12$ and $S = \{8, 10\}$, then the following inequality is valid for X :

$$s_{573} + \sum_{n \in \{9,11,12\}} \sum_{v=1}^{2} q_{5nv3} + 1600(o_{5813} + o_{5823}) + 700(o_{5,10,1,3} + o_{5,10,2,3}) \geq 1600.$$

This inequality states that the demand during periods 8 to 12 must be satisfied either from an unloading operation in period 8, $1600(o_{5813} + o_{5823})$, or from a combination of an unloading operation in period 10, $700(o_{5,10,1,3} + o_{5,10,2,3})$, the stock from period 7, s_{573}, and the unload quantity in periods $\{9, 11, 12\}$, $\sum_{n \in \{9,11,12\}} \sum_{v=1}^{2} q_{5nv3}$.

For the few cases (a product-port pair) where the total demand of a product in a given port is greater than the capacity of a ship, inequalities are still valid, however they may no longer define the convex hull of X^{CLS}.

The family of inequalities (ℓ, S) includes an exponential number of inequalities. As we describe in Section 2.6 we only use a small number of these inequalities.

Mixed-integer rounding (MIR) is a very powerful technique to derive strong valid inequalities for mixed integer sets, see [18]. The well-known MIR-inequalities (see [19]) can be stated as follows.

Proposition 2.4.2. *Let $Y = \{(s, y) \in \mathbb{R}_+ \times \mathbb{Z} : s + ay \geq d\}$. The inequality $s \geq r(\lceil d/a \rceil - y)$ is valid for Y, where $r = d - (\lceil d/a \rceil - 1)a$.*

Next we apply this proposition to derive valid inequalities for each model, B-SSDP, F-SSDP and MF-SSDP. In order to do that we must define mixed-integer sets of the form of Y that result from relaxation of the set of feasible solutions of each formulation.

First we consider equations (2.15). For each port i and product k such that $J_{ik} = -1$, aggregating equations (2.15) for all periods in M, and using nonnegativity of $s_{i,|M|,k}$ we obtain:

$$s_{i0k} + \sum_{m \in M} \sum_{v \in V} q_{imvk} \geq \sum_{m \in M} D_{imk}. \tag{2.55}$$

For each $S \subseteq M$ and $v' \in V$ (2.55) can be written as

$$s_{i0k} + \sum_{m \in M \setminus S} \sum_{v \in V} q_{imvk} + \sum_{m \in S} \sum_{v \in V : v \neq v'} q_{imvk} + \sum_{m \in S} q_{imv'k} \geq \sum_{m \in M} D_{imk}. \tag{2.56}$$

Using (2.40), it follows that

$$s_{i0k} + \sum_{m \in M \setminus S} \sum_{v \in V} q_{imvk} + \sum_{m \in S} \sum_{v \in V : v \neq v'} q_{imvk} + V_{v'}^{CAP} \sum_{m \in S} o_{imv'k} \geq \sum_{m \in M} D_{imk}. \tag{2.57}$$

Let $s = s_{i0k} + \sum_{m \in M \setminus S} \sum_{v \in V} q_{imvk} + \sum_{m \in S} \sum_{v \in V : v \neq v'} q_{imvk}$, $y = \sum_{m \in S} o_{imv'k}$, $a = V_{v'}^{CAP}$, $d = \sum_{m \in M} D_{imk}$. Applying Proposition 2.4.2, we obtain the following result.

Proposition 2.4.3. *For each $i \in N, S \subseteq M, v' \in V, k \in K$ such that $J_{ik} = -1$, the following MIR inequality*

$$s_{i0k} + \sum_{m \in M \setminus S} \sum_{v \in V} q_{imvk} + \sum_{m \in S} \sum_{v \in V : v \neq v'} q_{imvk} + r \sum_{m \in S} o_{imv'k} \geq r \left\lceil \frac{\sum_{m \in M} D_{imk}}{V_{v'}^{CAP}} \right\rceil, \tag{2.58}$$

where $r = \sum_{m \in M} D_{imk} - \left(\left\lceil \frac{\sum_{m \in M} D_{imk}}{V_{v'}^{CAP}} \right\rceil - 1 \right) \cdot V_{v'}^{CAP}$, *is valid for X.*

Example 2.4.2. Continuing Example 2.4.1, let $V_1^{CAP} = 1500$, $V_2^{CAP} = 2000$. Considering $S = \{1, 2, 3, 4, 5, 6\}$, $v' = 1$, then the following inequality is valid for X :

$$s_{503} + \sum_{m=7}^{12} \sum_{v=1}^{2} q_{5mv3} + \sum_{m=1}^{6} q_{5m23} + 400 \sum_{m=1}^{6} o_{5m13} \geq 800.$$

This inequality states that either the number of unload operations of ship 1 (at port 5 for product 3) during the first six periods is at least two (the minimum number of unload operations from this ship necessary to satisfy all the demand) or else, if there is only one unload operation from ship 1 during the first six periods, then the unloaded quantity from the other ship (ship 2) and from ship 1 during period 7 to period 12 must be at least $r = 400$, that is, the total demand (1900) minus the capacity of ship 1, 1500.

Inequalities based on fixed charge flow sets

Here we introduce valid inequalities based on the number of ship visits to a set of ports during a given time horizon. We develop and present the family of inequalities for F-SSDP, but similar results can be derived for B-SSDP and MF-SSDP.

Let $\overline{D}(S, L, Q)$ denote the total demand for the subset of products $Q \subseteq K$, in ports $S \subseteq N$, such that $J_{ik} \neq 1$, for all $i \in S, k \in Q$, (S does not contain any supply port of products in Q), during the time horizon $L = \{1, \ldots, \ell\} \subseteq M$, with $\ell \geq 2$. Hence, $\overline{D}(S, L, Q) = \sum_{i \in S} \sum_{n \in L} \sum_{k \in Q} D_{ink}$. Let $D(S, L, Q)$ denote the amount of demand in $\overline{D}(S, L, Q)$ that must be transported from ports in $N \setminus S$, that is, $D(S, L, Q) = \overline{D}(S, L, Q) - \sum_{v \in V : i_v \in S} \sum_{k \in Q} Q_{vk} - \sum_{i \in S} \sum_{k \in Q} s_{i0k}$.

For each $Q \subseteq K$, $S \subseteq N$, $L = \{1, \ldots, \ell\} \subseteq M$, such that S does not contain any supply port of products in Q, define the following subset $X(S, L, Q)$:

$$\sum_{v \in V} \sum_{i \in N \setminus S} \sum_{j \in S} \sum_{m \in L} \sum_{n \in L, n > m} \sum_{k \in Q} f_{imjnvk} \geq D(S, L, Q), \tag{2.59}$$

$$\sum_{k \in Q} f_{imjnvk} \leq V_v^{CAP} x_{imjnv}, \forall i \in N \setminus S, j \in S, m, n \in L, v \in V, \tag{2.60}$$

$$f_{imjnvk} \geq 0, \forall i \in N \setminus S, j \in S, m, n \in L, v \in V, k \in Q, \tag{2.61}$$

$$x_{imjnv} \in \{0, 1\}, \forall i \in N \setminus S, j \in S, m, n \in L, v \in V. \tag{2.62}$$

In order to verify that $X(S, L, Q)$ can be obtained as relaxation of X, we consider the aggregation of constraints (2.15) over the sets S, L, Q :

$$\sum_{j \in S} \sum_{k \in Q : J_{jk} = -1} s_{j0k} + \sum_{v \in V} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q : J_{jk} = -1} q_{jnvk} = \overline{D}(S, L, Q) + \sum_{j \in S} \sum_{k \in Q : J_{jk} = -1} s_{j\ell k}.$$

Since variables $s_{j\ell k}$ are nonnegative, it follows that

$$\sum_{j\in S}\sum_{k\in Q:J_{jk}=-1} s_{j0k}+\sum_{v\in V}\sum_{j\in S}\sum_{n\in L}\sum_{k\in Q:J_{jk}=-1} q_{jnvk}\geq \overline{D}(S,L,Q). \tag{2.63}$$

Using (2.35) and (2.36), then we obtain

$$\sum_{j\in S}\sum_{k\in Q:J_{jk}=-1} s_{j0k}+\sum_{v\in V:j_v\in S}\sum_{k\in Q:J_{jk}=-1}\left(Q_{vk}-\sum_{i\neq j_v}\sum_{n>1} f_{j_v1invk}\right)$$
$$+\sum_{v\in V}\sum_{j\in S}\sum_{n\in L,n>1}\sum_{k\in Q:J_{jk}=-1}\left(\sum_{i\neq j}\sum_{m<n} f_{imjnvk}-\sum_{i\neq j}\sum_{m>n} f_{jnimvk}\right)\geq \overline{D}(S,L,Q) \tag{2.64}$$

$$\Leftrightarrow \sum_{v\in V}\sum_{j\in S}\sum_{n\in L}\sum_{k\in Q:J_{jk}=-1}\sum_{m<n}\left(\sum_{i\in N\setminus S} f_{imjnvk}+\sum_{i\in S\setminus\{j\}} f_{imjnvk}\right)$$
$$\geq D(S,L,Q)+\sum_{v\in V}\sum_{j\in S}\sum_{n\in L}\sum_{k\in Q:J_{jk}=-1}\sum_{m>n}\left(\sum_{i\in N\setminus S} f_{jnimvk}+\sum_{i\in S\setminus\{j\}} f_{jnimvk}\right). \tag{2.65}$$

Constraints (2.59) are implied by (2.65) since

$$\sum_{v\in V}\sum_{j\in S}\sum_{n\in L}\sum_{k\in Q:J_{jk}=-1}\sum_{m\in L:m<n}\sum_{i\in S\setminus\{j\}} f_{imjnvk}=\sum_{v\in V}\sum_{j\in S}\sum_{n\in L}\sum_{k\in Q:J_{jk}=-1}\sum_{m\in L:m>n}\sum_{i\in S\setminus\{j\}} f_{jnimvk},$$

and using nonnegativity of f^k_{jnimv}.

Sets $X(S,L,Q)$ have been intensively studied in the past (e.g. [20]). Although some computational tests have been conducted using valid inequalities derived from these sets we focus here only on valid inequalities for a set obtained from aggregation of these $X(S,L,Q)$ sets. There are two main reasons for this choice: (i) the commercial software used is able to include flow cover inequalities, which are known to be important to strengthening the gap for sets of type $X(S,L,Q)$; (ii) preliminary computational results showed that the inequalities we introduce below provided larger reduction of the integrality gap.

We aggregate the arc-load flow variables corresponding to each ship, that is

$$Y_v=\sum_{i\in N\setminus S}\sum_{j\in S}\sum_{m\in L}\sum_{n\in L}\sum_{k\in Q} f_{imjnvk}, \forall v\in V,$$

and aggregate the corresponding arc variables:

$$X_v=\sum_{i\in N\setminus S}\sum_{j\in S}\sum_{m\in L}\sum_{n\in L}\sum_{k\in Q} x_{imjnv}, \forall v\in V.$$

Variables Y_v indicate the load transported from ports in $N\setminus S$ to ports in S during the time horizon L by ship v, while X_v denotes the number of times ship v visits a port in S coming from a port not in S during the time horizon L (see Figure 2.4).

Let D denote the total demand for the subset of products Q, in ports S, during the time horizon L, that must be transported from ports in $N\setminus S$, that is, $D=D(S,L,Q)$. Hence, the following mixed integer set is a relaxation of the set of the feasible solutions:

$$\left\{(Y,X)\in\mathbb{R}^{|V|}_+\times\mathbb{Z}^{|V|}_+:\sum_{v\in V} Y_v\geq D, Y_v\leq V^{CAP}_v X_v, v\in V\right\}.$$

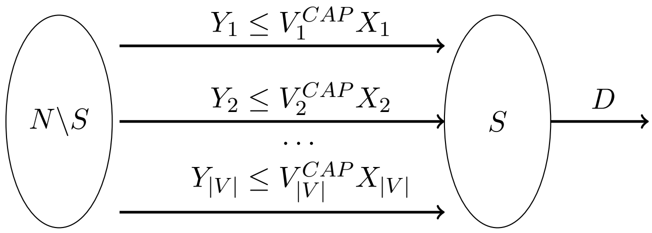

Figure 2.4: A fixed charge flow set.

In our case there are only two ships, that is $\mid V \mid= 2$. Then we obtain the following aggregated model, denoted by XY^2, with two continuous and two integer variables:

$$\left\{(Y_1, Y_2, X_1, X_2) \in \mathbb{R}_+^2 \times \mathbb{Z}_+^2 : Y_1 + Y_2 \geq D, Y_1 \leq V_1^{CAP} X_1, Y_2 \leq V_2^{CAP} X_2\right\}.$$

For each valid inequality for XY^2,

$$\alpha_1 X_1 + \alpha_2 X_2 + \beta_1 Y_1 + \beta_2 Y_2 \geq \alpha,$$

we obtain a valid inequality,

$$\alpha_1 \sum_{i \in N \backslash S} \sum_{m \in L} \sum_{j \in S} \sum_{n \in L} x_{imjn1} + \alpha_2 \sum_{i \in N \backslash S} \sum_{m \in L} \sum_{j \in S} \sum_{n \in L} x_{imjn1}$$

$$+\beta_1 \sum_{i \in N \backslash S} \sum_{m \in L} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q} f_{imjn2k} + \beta_2 \sum_{i \in N \backslash S} \sum_{m \in L} \sum_{j \in S} \sum_{n \in L} \sum_{k \in Q} f_{imjn2k} \geq \alpha, \tag{2.66}$$

for X. This model XY^2 is closely related to the models studies in [1]. The purpose of this paper is not to provide full polyhedral description for XY^2, but only to identify those valid inequalities with large impact on the gap.

It is easy to verify that facet-defining inequalities for the convex hull of

$$X^2 = \{(X_1, X_2) \in \mathbb{Z}_+^2 : V_1^{CAP} X_1 + V_2^{CAP} X_2 \geq D\}$$

are also facet-defining inequalities for the convex hull of XY^2.

In general the convex hull of X^2, ($conv(X^2)$) contains non-trivial facet-defining inequalities, that is, facets that are not defined by $X_1 \geq 0, X_2 \geq 0, V_1^{CAP} X_1 + V_2^{CAP} X_2 \geq D$. For a polyhedral description of $conv(X^2)$ see [1]. Such facet-defining inequalities were already used in [26] for a locomotive assignment problem. Since V_1^{CAP} and V_2^{CAP} are large and D is at most 3 or 4 times the smallest coefficient, it is easy to see that $V_1^{CAP} X_1 + V_2^{CAP} X_2 \geq D$ does not define a facet of $conv(X^2)$ and that $conv(X^2)$ has one or two facet-defining inequalities (this is not the general case of integer sets with two variables).

Example 2.4.3. *Consider the following set with* $V_1^{CAP} = 1500$, $V_2^{CAP} = 2000$, *and a demand of* $D = 6098$. *From figure 2.5 we can see that* $conv(X^2)$ *has two non-trivial facet defining inequalities:* $X_1 + X_2 \geq 4$ *and* $X_1 + 2X_2 \geq 5$.

This family of inequalities on the x_{jnimv} variables proved to be very important on solving our instances. We call these inequalities Nvisits-inequalities since they are written on the aggregation of x_{jnimv} variables, thus, on the number of visits to a subset of ports.

Families of inequalities similar to the Nvisits-inequalities have been used in other maritime transportation problems, see for example [21].

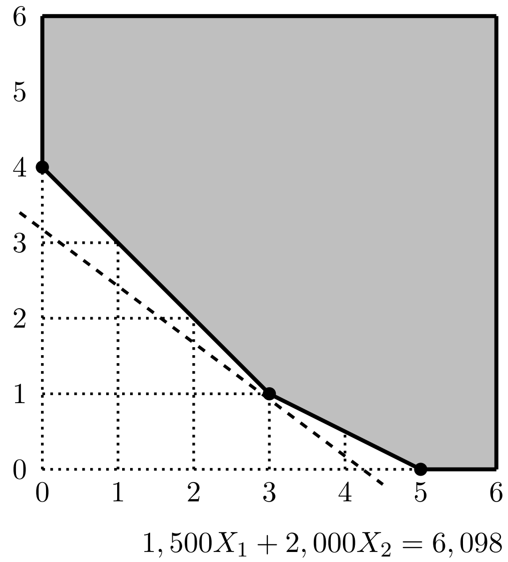

Figure 2.5: Facet defining inequalities for $\mathrm{conv}(\{(X_1, X_2) \in \mathbb{Z}_+^2 : 1,500X_1 + 2,000X_2 \geq 6,098\})$.

Another important family of inequalities that define facets for XY^2 is the MIR inequalities

$$Y_v \geq r\left(\left\lceil \frac{D}{V_v^{CAP}} \right\rceil - X_v\right), \forall v \in V,$$

where $r = D - \left(\left\lceil \frac{D}{V_v^{CAP}} \right\rceil - 1\right) V_v^{CAP}$. Some preliminary tests have shown that these MIR inequalities were ineffective in reducing the integrality gap and, therefore, we ignore them in the computational results.

Strong inequalities

The fourth family of inequalities, called strong inequalities (see [5] for the use of these inequalities on a related problem), are considered only for the F-SSDP and MF-SSDP. The strong inequalities for the F-SSDP are defined as follows:

$$f_{imjnvk} \leq \min\left\{V_v^{CAP}, \sum_{u \in N} \sum_{t \in M, t \geq n} D_{utk}\right\} x_{imjnv}, \qquad \forall i,j \in N, n,m \in M, v \in V, k \in K. \tag{2.67}$$

For the MF-SSDP, the strong inequalities are defined as follows:

$$\gamma_{imjnvk}^{ut} \leq \min\{V_v^{CAP}, \sum_{l \in M, l \geq t} D_{ulk}\} x_{imjnv}, \qquad \forall i,j,u \in N, n,m,t \in M, v \in V, k \in K. \tag{2.68}$$

Since $\min\left\{V_v^{CAP}, \sum_{u \in N} \sum_{t \in M, t \geq n} D_{utk}\right\} \geq \min\left\{V_v^{CAP}, \sum_{l \in M, l \geq t} D_{ulk}\right\}$, the strong inequalities are tighter for the MF-SSDP.

The huge number of inequalities in these families makes the use of a separation algorithm necessary in order to choose only a small number of cuts to be included.

2.5 Dedicated tanks case

In this section we consider the tank allocation policy followed by the company. Changing from a dirty product to a cleaner one imposes a major cleaning operation that is time consuming and expensive. In order to avoid such major changeovers the company dedicates tanks to families of products in such a way that the changeover time and cost between products of the same family can be neglected. The families of products do not depend on the ship. Unpredictable situations, such as bad weather conditions, may force the company to change this policy in order to satisfy the demands.

The three models discussed for the non dedicated tanks problem B-SSDP, F-SSDP and MF-SSDP can be adapted to handle the real case with dedicated tanks. Next we give the changes in F-SSDP, since this was the model that provided best result for the case without dedicated tanks. The new model will be denoted by F-SSDP-DC.

We denote by S_v^C the set of compartments of ship v. For each compartment $c \in S_v^C$, we define its capacity as $V_v^{CAP^c}$ and define the set of products that c can transport as K_v^c. Parameter Q_{vk}^c denotes the quantity of product k in compartment c of ship v at the beginning of the planning horizon.

When a family has more than one product, we need to specify the compartment where the product is transported for each continuous variable f_{imjnvk} and q_{imvk}. In order to do that we define the new continuous nonnegative variables f_{imjnvk}^c as the amount of flow f_{imjnvk} transported from compartment c, and q_{imvk}^c as the amount of product k loaded onto or unloaded in compartment c of ship v at port i in time period m.

In order to prevent the transportation of more than one product of the same family in the same tank, we define the new binary variables χ_{imjnvk}^c indicating whether ship v transports product k in compartment c when sailing from port i, after an operation that started in period m, to port j and starts to operate at port j in period n.

The F-SSDP-DC model is obtained from the model F-SSDP by replacing the constraints (2.13), (2.32), (2.33) and (2.15) with

$$q_{imvk}^c \leq V_v^{CAP^c} o_{imvk}, \qquad \forall i \in N, m \in M, v \in V, c \in S_v^C, k \in K_v^c : J_{ik} \neq 0, \tag{2.69}$$

$$\sum_{i \neq j} \sum_{m<n} f_{imjnvk}^c + J_{jk} q_{jnvk}^c = \sum_{i \neq j} \sum_{m>n} f_{jnimvk}^c,$$

$$\forall j \in N, n \in M : n > 1, v \in V, c \in S_v^C, k \in K_v^c, \tag{2.70}$$

$$Q_{vk}^c = \sum_{j \neq i_v} \sum_{n>1} f_{i_v 1 jnvk}^c - J_{i_v k} q_{i_v 1 vk}^c, \qquad \forall v \in V, c \in S_v^C, k \in K_v^c, \tag{2.71}$$

$$s_{i,m-1,k} + \sum_{v \in V} \sum_{c \in S_v^C} q_{imvk}^c = D_{imk} + s_{imk}, \qquad \forall i \in N, m \in M, k \in K : J_{ik} = -1, \tag{2.72}$$

and replacing (2.34) with

$$f_{imjnvk}^c \leq V_v^{CAP^c} \chi_{imjnvk}^c, \qquad \forall i, j \in N, m, n \in M, v \in V, c \in S_v^C, k \in K_v^c, \tag{2.73}$$

$$\sum_{k \in K_v^c} \chi_{imjnvk}^c \leq x_{imjnv}, \qquad \forall i, j \in N, n, m \in M, v \in V, c \in S_v^C. \tag{2.74}$$

By aggregation of variables f_{imjnvk}^c and q_{imvk}^c, a feasible solution of F-SSDP from every feasible solution of F-SSDP-DC can be constructed. The converse is not true, since F-SSDP-DC can be feasible when F-SSDP is infeasible. From the computational point of view, this might lead to larger branch and bound trees when using F-SSDP-DC for those instances where finding a feasible solution is difficult. Also, F-SSDP-DC is larger than F-SSDP.

On the other side, the lower bounds based on the linear relaxation of F-SSDP-DC are in general better than those obtained with F-SSDP. This comes from the fact that the coefficients on the linking constraints are smaller since the capacity of the ships is replaced by the capacity of the tanks.

2.6 Computational results

In this section we present some computational results using 20 instances based on real data from Cape Verde. We tested different models resulting from different ways of combining the improving strategies. All computations were performed using the optimization software Xpress Optimizer Version 20.00.05 with Xpress Mosel Version 3.0.0, on a computer with processor Intel Core 2 Duo, CPU 2.2GHz, with 4GB of RAM. We consider the present real sized problem consisting of 4 products, 7 islands and 2 ships and a planning horizon of 12 periods (days).

In Section 2.6.1 we provide a computational comparison of the different models tightened with valid inequalities. Then, in Section 2.6.2, we test the best model with larger size instances in order to evaluate its performance on hypothetical future scenarios with increase of demand requirements and number of ships. Finally, in Section 2.6.3, the model F-SSDP-DC is tested on the real case where tanks are dedicated to families of products, and the results are compared with results obtained from the real plans established by the company.

2.6.1 Model comparison

For the B-SSDP model we tested the inclusion of three families of inequalities: (ℓ, S), MIR and Nvisits inequalities. We tested separately the inclusion of cuts from each of the families. Table 2.1 reports the results of these tests. For each case, identified in the first line of the table, we present the integrality gap (Gap), the number of cuts (Cuts) added, and the time (Time) in seconds to solve the problem. Here, the gap is defined as $GAP = \frac{Opt.Value - LowerBound}{Opt.Value} \times 100$. We also tested the inclusion of all cuts (last three columns in Table 2.1). In each case the cuts were introduced only at the root node. First we solve the linear relaxation, then we add cuts and finally we execute the branch and bound using the default options. The value LowerBound used to compute the GAP is the lower bound at the root node after the inclusion of these cuts. All the models with exception of the original one, without cuts, include the tightening of bounds. For each instance we consider a time limit of 3 hours. The asterisk in Instance 5 In Table 2.1 means that this instance was not solved within the time limit using the B-SSDP, the B-SSDP with (ℓ, S) and the B-SSDP with MIR inequalities. In the last line of the table we include the corresponding average value of all the 20 instances.

For the (ℓ, S) inequalities we use the separation procedure described in [22]. For the MIR inequalities we include all cuts (inequalities violated by the linear relaxation solution) from this family while the improvement in the bound is greater than 1%. For the Nvisits, the separation algorithm includes all cuts from those inequalities where either S or $N \setminus S$ is a singleton.

Since the transportation cost is the most relevant cost for the optimal value, the Nvisit-inequalities are the ones that provide highest reduction of the gap as these cuts consider explicitly the routing variables. The number of cuts introduced from this family is usually very small. Hence as we can see from Table 2.1 this family alone is the one that provides best results. The gap is half of the original gap. Only the approach that includes cuts from all families provides better results in terms of gap but not on the time. In this last approach the set of Nvisit cuts are introduced first, then the (ℓ, S) are introduced and, finally, the MIR cuts are introduced.

Table 2.1: Computational tests for the B-SSDP.

	B-SSDP		**B-SSDP+(l,S)**			**B-SSDP +MIR**			**B-SSDP+NVISITS**			**B-SSDP+All**		
Int.	Gap	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time
1	72,9	903	63,2	26	121	57,2	77	49	36,2	11	78	28,6	114	35
2	86,9	1701	77,9	37	2393	68,1	135	2356	53,2	10	337	42,9	182	526
3	88,1	382	64,1	15	389	78,1	200	237	56,9	11	169	40,5	226	214
4	77,5	33	67,7	3	8	59,1	33	15	15,1	8	7	13,8	44	6
5*	85,5	10800	71,9	4	10801	71,9	30	10801	34,8	10	568	34,1	44	752
6	79,2	184	49,1	9	64	57,9	238	64	36,4	6	33	30,3	253	22
7	75,7	9	35,1	18	6	32,5	107	3	25,2	6	3	11,5	131	4
8	84,8	134	77,3	30	58	73,6	43	55	46,4	11	48	45,1	84	74
9	90,9	19	63,0	6	15	72,3	79	9	61,2	9	11	58,5	94	12
10	84,4	35	66,6	53	34	67,6	160	34	31,1	10	8	27,3	223	8
11	77,2	10	41,1	5	5	32,4	97	5	23,4	3	4	18,4	105	7
12	86,4	64	69,1	13	39	81,7	176	29	52,1	13	22	48,7	202	25
13	88,6	6841	75,5	28	4403	74,5	184	5326	56,4	5	1717	55,7	217	2494
14	83,1	6	62,6	4	3	60,7	34	2	31,3	8	1	29,0	46	1
15	91,2	71	62,9	27	43	79,6	219	44	69,0	10	73	53,1	256	42
16	90,1	52	70,3	22	29	77,4	184	23	54,6	4	25	52,5	210	20
17	84,1	71	72,9	20	40	75,1	80	51	40,0	6	28	37,0	106	31
18	89,3	13	53,6	3	4	64,2	54	3	19,2	5	2	18,3	62	2
19	89,3	28	76,1	19	21	78,2	124	15	63,7	11	13	62,1	154	27
20	84,0	35	66,0	28	30	67,8	238	30	31,4	11	15	28,4	277	12
Av.	**84,5**	1069,55	64,3	18,5	925,3	66,5	124,6	957,6	**41,9**	8,4	158,1	**36,8**	151,5	**215,7**

With this approach the average gap is reduced from 84.5% to 36.8%. These gaps do not include the cuts from Xpress.

For the F-SSDP the results are presented in Table 2.2. For this model we also present the results for the family of strong inequalities (denoted by SI). In this case only the cuts corresponding to the greatest violation are introduced. The approach including all types of cuts follow the sequence of cuts: Nvisits, (ℓ, S), Strong inequalities and MIR. This approach proved to be the best one by reducing the average gap to 16.2% and by reducing the computational time to less than one minute on average, and always below 10 minutes. When the B-SSDP was used, one instance was not solved within 3 hours. Again, notice that the gaps reported do not include the cuts introduced by Xpress using the default options.

For the MF-SSDP the results are presented in Table 2.3. As expected, the best bounds were obtained with MF-SSDP, the tightest formulation. However, the size of the model leads to poor running times.

In Table 2.4 we provide an overview of the average results obtained with the three models. The line Nodes gives the average number of nodes of the branch and bound algorithm.

2.6.2 Future scenarios: larger size instances

To test the model that performed better, F-SSDP, on larger instances we created two artificial future scenarios where the demands as well as the number of ships are increased. One scenario with three ships and where demands are 1.5 times the current demands, and another scenario with four ships and where the demands are doubled. The results are given in Table 2.5. For each scenario, identified by the number of ships ($\mid V \mid= 3$ and $\mid V \mid= 4$), we provide the integrality gap (Gap-I), the gap given by Xpress at the end of the running time, limited to three hours (Gap-E), and the running time (Time). In the four ships case some instances become infeasible because of the port activity restrictions that impose a maximum of one ship operating at each port per time period.

In order to derive Nvisits inequalities for the three and four ships cases we first generate a Nvisits inequality for each subset of X^2 of two variables obtained considering two ships. Then the Nvisits inequality is lifted using the subadditive lifting function ω_3 given in [2].

To improve the running times we also adapted the branching strategy presented in [26] (see also [3]). We establish high priority for branching on the variables representing the number of ship visits to each port. This strategy proved to be very effective.

We can see from Table 2.5 that 17 instances were solved to optimality for the three ships case, and only 4 were solved for the four ships case, within the limit of three hours.

Table 2.2: Computational tests for the F-SSDP.

	F-SSDP		**F-SSDP+(l,S)**			**F-SSDP +MIR**			**SSDP+NVISITS**			**SSDP+SI**			**SSDP+All**		
Inst.	Gap	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time
1	33.9	64	24.1	9	82	23.5	70	33	20.3	8	44	29.5	1	65	16.3	78	28
2	45.5	688	29.5	23	284	33.7	137	336	32.9	9	300	38.0	7	446	21.3	161	150
3	56.5	120	42.5	19	51	30.0	160	81	40.2	9	53	52.8	1	74	7.1	308	46
4	25.0	14	14.4	2	11	14.4	28	8	13.4	5	10	19.4	1	15	7.3	52	5
5	40.0	1736	32.4	4	1726	31.8	42	797	18.3	7	93	33.3	1	1494	15.9	82	180
6	45.4	57	29.6	5	36	27.1	92	25	28.4	6	41	38.6	1	26	18.2	37	25
7	44.5	8	7.7	14	7	16.0	126	3	16.4	6	4	15.4	1	12	3.1	90	4
8	62.2	119	43.2	30	56	52.1	59	46	41.3	10	53	54.1	1	63	35.4	83	37
9	68.3	36	46.4	6	19	36.2	104	12	46.1	9	18	50.8	1	20	28.1	128	9
10	28.7	16	19.3	23	11	21.2	200	8	15.5	7	6	28.1	1	15	10.7	165	6
11	53.0	21	20.8	9	14	20.9	156	6	20.5	4	9	28.9	1	15	5.0	116	7
12	46.7	37	28.3	20	24	38.5	211	21	31.3	10	13	46.5	1	29	22.0	199	14
13	52.8	1592	39.5	4	825	43.6	36	855	39.7	5	750	44.8	1	1147	32.5	49	549
14	59.7	8	30.5	2	5	30.5	17	2	18.7	5	2	44.5	1	9	7.6	61	2
15	64.7	34	35.3	31	32	30.6	394	22	53.7	10	29	60.3	1	35	16.5	243	24
16	50.2	71	30.3	27	40	38.7	197	41	31.0	5	53	35.1	6	63	19.1	129	46
17	37.9	80	28.4	8	73	29.2	206	43	20.2	6	33	31.7	1	35	17.1	115	35
18	64.9	13	29.0	2	7	20.6	34	4	11.1	6	3	49.6	1	13	6.2	53	3
19	47.0	40	39.3	11	20	36.0	119	18	40.0	7	18	33.3	1	35	25.9	206	11
20	39.9	24	18.7	23	19	27.9	317	14	19.6	12	10	26.3	1	21	8.8	229	13
Av.	**48.3**	238.9	29.5	13.6	167.1	30.1	135.3	118.8	**27.9**	7.3	77.1	38.0	1.6	181.6	**16.2**	129.2	**59.7**

Table 2.3: Computational tests for the MF-SSDP.

	MF-SSDP		**MF-SSDP+(l,S)**			**MF-SSDP +MIR**			**MF-SSDP +NV**			**MF-SSDP +SI**			**MF-SSDP +All**		
Inst.	Gap	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time	Gap	Cuts	Time
1	22.7	100	21.1	17	115	22.2	17	142	17.9	7	62	22.4	1	86	17.3	14	100
2	29.3	4613	26.6	21	4095	26.6	65	4555	27.0	5	5755	25.9	11	2836	19.8	47	1445
3	15.6	328	15.6	17	213	15.6	6	274	10.8	6	176	15.6	5	268	6.8	14	168
4	16.5	16	14.4	16	12	14.4	26	14	8.7	6	11	16.5	1	12	7.3	93	9
5	29.2	4315	29.2	0	4448	29.2	10	4166	18.4	4	481	29.1	2	3846	13.5	30	294
6	27.6	9	27.6	0	7	27.6	0	11	21.4	4	11	27.6	1	11	21.4	26	6
7	8.3	5	8.3	3	5	8.3	21	7	8.2	1	11	8.3	2	12	7.8	5	6
8	34.1	135	34.1	16	117	34.1	8	155	33.0	5	180	30.7	3	190	27.0	25	129
9	25.6	19	25.6	0	18	25.6	7	23	25.6	4	31	24.5	3	31	19.5	39	17
10	17.4	11	15.6	26	11	15.3	59	12	13.1	5	30	17.4	1	22	11.0	42	11
11	17.4	8	8.2	44	10	8.2	26	12	17.4	1	11	15.3	5	17	4.9	72	11
12	26.2	39	26.2	16	44	26.2	42	31	23.6	6	25	25.8	2	47	16.8	21	19
13	37.8	1173	37.8	9	1280	37.8	14	1250	37.8	2	2347	37.8	2	2023	36.0	60	1384
14	28.4	2	28.4	25	2	28.4	39	3	7.6	5	2	25.9	3	3	6.8	7	2
15	12.8	53	12.8	10	62	12.8	35	63	12.8	2	54	12.7	1	58	12.4	93	82
16	28.1	56	28.1	0	53	28.1	20	65	26.5	1	66	26.9	2	130	22.8	87	70
17	29.6	129	27.3	37	87	24.7	32	118	21.5	3	126	29.2	2	103	19.2	145	104
18	20.1	2	20.1	0	3	20.1	0	3	5.0	5	6	20.1	1	3	3.6	10	2
19	34.9	26	34.5	24	22	34.4	40	31	34.8	3	30	34.6	1	34	26.7	90	29
20	13.0	23	12.8	10	15	12.8	60	30	11.4	6	24	12.9	2	25	8.3	14	11
Av.	23.7	553.1	22.7	14.6	531.0	22.6	26.4	548.3	19.1	4.1	472.0	23.0	2.6	487.85	15.5	46.7	195.0

Table 2.4: Summary of general information from the three models.

	B-SSDP	B-SSDP+ALL	F-SSDP	F-SSDP+ALL	MF-SSDP	MF-SSDP+ALL
Gap (%)	84.5	36.6	48.3	16.2	23.7	15.5
Nodes	50562.4	16622.6	4301.7	853.9	2307.7	665.9
Time (sec.)	1069.6	215.7	238.9	59.7	472	195

Table 2.5: Computational results for the F-SSDP with 3 and 4 ships

	$\|V\| = 3$			$\|V\| = 4$		
Inst.	Gap-I	Gap-E	Time	Gap-I	Gap-E	Time
1	22.9	0.0	309	24.6	1.1	10800
2	36.4	17.0	10800	34.9	21.4	10800
3	36.7	7.4	10800	30.2	12.9	10800
4	17.0	0.0	72	17.1	0.0	2724
5	23.1	0.0	1411	19.0	6.1	10800
6	19.8	0.0	11	35.5	0.0	686
7	35.8	0.0	119	-	-	-
8	37.1	0.0	195	25.5	0.0	863
9	40.3	0.0	103	32.3	0.9	1037
10	14.7	0.0	454	-	-	-
11	31.0	0.0	651	44.2	7.7	10800
12	14.9	0.0	89	-	-	-
13	22.9	0.0	1697	18.3	1.0	5027
14	20.0	0.0	7	22.7	0.0	21
15	36.4	0.0	5351	32.8	9.5	10800
16	30.1	0.0	2408	25.8	6.5	10800
17	33.7	6.1	10800	29.5	18.3	10800
18	21.3	0.0	11	20.5	1.0	208
19	33.1	0.0	893	31.4	8.6	10800
20	26.2	0.0	2790	-	-	-
Av.	27.7	1.5	2449.6	27.0	4.9	5552.15

2.6.3 Real case: tanks are dedicated to families of products

The F-SSDP-DC is tested on instances based on the real case of dedicated tanks. Three families of products were considered. Two of them with one product only: the fuel (the dirtiest product) and jet (the cleanest product), and one family with two products, gasoline and diesel.

Since the linear relaxation of F-SSDP-DC provides better lower bounds for this case than those provided by F-SSDP for the non dedicated tanks case, the impact of the inclusion of valid inequalities in F-SSDP-DC is lower than the impact of the inclusion of valid inequalities in F-SSDP. Hence we give the results only for the most relevant valid inequalities, the Nvisits inequalities. The computational results are reported in Table 2.6.

Table 2.6: Computational results for the dedicated tanks case.

	F-SSDP-DC			**F-SSDP-DC+ NVISITS**			
Instance	Gap	Nodes	Time	Gap	Nodes	Cuts	Time
1	26.0	82273	3840	25.7	26653	7	1444
2	29.4	1585	42	8.8	418	7	24
3	26.5	27957	1881	22.4	25680	5	1651
4	25.4	5752	2534	12.7	1746	11	979
5	31.2	35918	1602	10.2	27283	9	1334
6	32.3	23530	2257	4.7	21436	12	1775
7	32.2	59081	6272	20.7	28580	8	2716
8	24.6	22775	2387	23.0	21106	4	2282
9	26.8	1969	97	3.7	1775	6	74
10	22.3	20800	2651	13.4	8077	8	1856
11	30.0	21345	2687	25.9	19257	6	1873
12	35.2	13057	584	21.6	8634	5	351
Av.	28.5	26336.8	2236.2	16.1	16993.9	7.3	1454.0

In this case we tested 12 instances, compared them with the real plans followed by the company and verified an average gain in the cost (not reported in the table) of approximately 15%.

We can observe from Table 2.6 that, as expected, the average integrality gap is slightly lower in this case of dedicated tanks. Conversely, the running times are larger. In average the running time is less than 25 minutes.

Finally, Table 2.7 provides a general overview of the average size of the models tested.

Table 2.7: Average size of the tested models.

Model	$\|V\|$	Binary var.	Continuous var.	Total var.	Constraints
B-SSDP	2	6344	1920	8264	53163
F-SSDP	2	6344	23712	30056	22503
MF-SSDP	2	6344	577824	584168	43003
F-SSDP	3	9516	30076	39592	33397
F-SSDP	4	12688	39896	52584	44291
F-SSDP-DC	2	28184	37704	65888	123268

2.7 Conclusions

We developed a mixed integer model, B-SSDP, for the short sea fuel oil distribution problem occurring in Cape Verde. The model applies a combined discrete and continuous time horizon in order to take the varying demands and multiple time windows into account.

Both cases with and without dedicated ship tanks for families of products were considered. In order to efficiently solve the instances considered, we tested different approaches to improve the B-SSDP. In particular, we compared the B-SSDP with two extended formulations, an arc-load flow formulation F-SSDP, using additional variables indicating the amount of each type of fuel oil products each ship transports between each pair of ports and a multi-commodity formulation MF-SSDP. We also tightened the constraints and tested the inclusion of cuts from different families of inequalities. Separation algorithms were used such that we could include few inequalities from those inequality families with high impact on the integrality gap reduction.

The extended formulation, F-SSDP, with tighter bounds, combined with the approach of using a small subset of inequalities from each family proved to be the best option. It allowed us to solve all tested instances within reasonable time.

The models introduced are new and can also be used in other maritime transportation problems. Several of the types of cuts presented here have not been developed for maritime transportation problems previously in the literature, and they can easily be used when solving other real maritime inventory routing problems. We have shown how we can transform exiting valid inequalities in the literature to maritime inventory routing problems.

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Paper II

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Discrete time and continuous time formulations for a short sea inventory routing problem

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Chapter 3

Discrete time and continuous time formulations for a short sea inventory routing problem

Abstract

We consider a fuel oil distribution problem where an oil company is responsible for the routing and scheduling of ships between ports such that the demand for various fuel oil products is satisfied during the planning horizon. The consumption rates are given and assumed to be constant. We provide two alternative mixed integer formulations: a discrete time model adapted from the case where the production/consumption rates are varying and a classical continuous time formulation. We discuss different extended formulations and valid inequalities that allow us to reduce the linear gap of the two initial formulations. A computational study comparing the various models accordingly to their size, linear gap and running time, was conducted based on real small-size instances, using a commercial software.

Keywords: Maritime transportation, Discrete time and continuous time formulations, Extended formulations, Valid inequalities.

3.1 Introduction

Maritime transportation is a major mode of transportation of goods worldwide. The importance of this mode of transportation is obvious for the long distance transportation of cargoes but it is also crucial in local economies where the sea is the natural link between the local developed regions, such as countries formed by archipelagoes. When a company has the responsibility of coordinating the transportation of goods with the inventories at the ports, the underlying planning problem is a maritime inventory routing problem. Such problems are very complex. Usually modest improvements in the supply chain planning can translate into significant cost savings.

In this chapter we consider a real maritime inventory routing problem occurring in the archipelago of Cape Verde. An oil company is responsible for the inventory management of different oil products, such as, diesel, gasoline, fuel and jet, in several tanks located in the main islands. Fuel oil products are imported and delivered to specific islands and stored in large supply storage tanks. From these islands, fuel oil products are distributed among all the inhabited islands using a small heterogeneous fleet of ships. These products are stored in consumption storage tanks. Some ports have both supply tanks for some products and consumption tanks of other products. Not all islands consume all products.

Consumption rates are assumed to be given and constant. Typically the consumption rates are forecasted. Hence, safety stocks must be considered. Additionally, the storage tanks have limited capacity. Therefore, the level of each product in each tank must always be kept between a given lower level, determined by the safety stock, and an upper level, determined by the tank capacity. As the capacity of the supply tanks is very large when compared to the total demand over the horizon, we omit the inventory aspects for these tanks.

To transport fuel oil products between the islands, the planners control a small heterogeneous fleet. Each ship has a specified load capacity, fixed speed and cost structure. The cargo hold of each ship is separated into several cargo tanks. The products cannot be mixed, and cleaning operations to change between products on the same tank should be avoided. Therefore we assume that the ships have dedicated tanks for each product. Each port can receive at most one ship at a time, and in some ports there exists a minimum time interval between the departure of one ship and the arrival of the next ship.

Given the initial stock levels at the consumption tanks, the initial ship position (which can be a point at sea) and the quantities on board each ship, the inter-island distribution plan consists of designing routes and schedules for the fleet of ships including determining the number of visits to each port and the (un)loading quantity of each product at each visit to each port. This plan must satisfy the safety stocks of each product at each island, and the capacities of the ships and tanks. The transportation and operation costs of the distribution plan are to be minimized. This problem is called a Short Sea Inventory Routing Problem (SSIRP). Short sea stands for sea transportation between ports located in the same geographical area, in contrast to deep sea which is typically transportation between continents.

We have witnessed an increased interest in studying optimization problems within maritime transportation. See the reviews on maritime transportation; [13, 14, 15]. Combined routing and inventory management within maritime transportation have been present in the literature the last one and a half decades only; see [7] and [11]. These problems are often called Maritime Inventory Routing Problems (MIRPs). Most of the published MIRP contributions are based on real cases from the industry, see for the single product case [10, 17, 18, 19, 21] and for the multiple product case [6, 12, 25, 26, 28, 29, 30].

In [6, 10, 28], the production and/or consumption rates are considered given and fixed during the planning horizon. For those problems event based models are used where an index indicating the visit number to a particular port is added to most of the variables. These event based models are known as time continuous models [15]. In [1, 18, 19, 20, 21, 25, 26, 29] time discrete models are developed to capture the complicating factors with varying production and consumption rates.

The most related problems to the SSIRP given here are presented in [2] and [6]. In [2] it is considered a variant of this SSIRP for short-term planning with demand orders, that is, amounts of oil products that must be delivered within a given time period. These orders are determined from the initial stock levels and the consumption rates. Typically, demand orders lead to a problem with varying demands where demands are zero for most time periods and a large amount for a few periods. Several key issues taken into account in the short-term problem, such as port operating time windows for each time period, are relaxed here or incorporated indirectly in the data. Otherwise, the problems considered originate from the same company in the same region. In [6], a problem similar to the SSIRP is considered with constant consumption rates. However, in [6] only a continuous model is considered. In both papers the products have dedicated compartments in the ships.

Just recently the study of valid inequalities has been incorporated in MIRPs. In [27] valid inequalities are included in order to enhance the proposed formulations to an oil product transportation problem, and in [24] valid inequalities are developed within a column generation approach for a maritime inventory routing problem. Also, in [19] valid inequalities are derived for a single-product maritime inventory routing, which are used within a branch-price-and-cut algorithm. In [21], valid inequalities are included to improve the formulation presented for the liquefied natural gas inventory routing problem. Finally, [29] presents valid inequalities for MIRPs including several practical constraints for solving problems in different shipping segments. Comparison of different formulations in conjunction with valid inequalities have been used in [1] and [2].

As discussed in both [7] and [29], most combined maritime routing and inventory management problems described in the literature have particular features and characteristics, and tailor-made methods are developed to solve the problems. These methods are often based on heuristics or decomposition techniques. The choice of these solution approaches might be explained by the high complexity of real MIRPs and the possibility to utilize the special structure of the problem. However, the constant hardware development combined with the theoretical advances in optimization techniques have produced optimization solvers capable of handling increasingly larger instances. Currently, it is possible to obtain optimal or near optimal solutions to small real instances occurring in maritime transportation problems using commercial solvers. See [2] for the case of Cape Verde, and [1, 22, 27, 29].

Mathematical formulations, and related discussion, for MIRPs have received some attention during the last decades, see for instance, [1, 2, 4, 6, 9, 19, 29]. However, comparison of different formulations for a given MIRP has just been considered in a few studies so far; see e.g. [1, 2, 20]. Such studies are of crucial relevance when planning to solve a problem or subproblems (embedded in a more general solution approach) using commercial solvers. The SSIRP considered here offers an interesting test bed for a computational study of different formulations. In this chapter we discuss and compare different mathematical formulations for the SSIRP, some of them sharing the characteristics of well-known and widely used formulations. Therefore, although the problem presented here is a particular maritime inventory routing problem, the formulations discussed and compared are of interest to other related maritime inventory routing problems as well.

In addition to the common approach (see [6] and [10]) that consists of using event based models (known as continuous time models), we introduce a model that combines a discrete and continuous time where the discrete time corresponds to an artificial discretization of the continuous time. This model is similar to the one given in [2] for SSIRP with time varying consumption rates. For each approach, following [2] (see also [1] for a completely discrete model), we develop an arc-load formulation and two extended formulations. Arc-load formulations are the most used formulations in MIRPs, see [6, 10, 29]. The extended formulations use new sets of variables that provide additional information about the solution. That information is essential to derive a tighter model, that is, to derive a model whose linear relaxation is closer to the optimal solution than the linear relaxation of the arc-load model. Similar extended formulations have been extensively used for other problems, such as lotsizing and network flow problems. In MIRPs they have been used in [1, 2] for problems with time varying consumption rates. To the best of our knowledge, the two extended formulations introduced for the event time model, and the formulations resulting from adaptation to the constant rate problem of models including time discretization, are new for MIRPs.

We provide a comparison of the two approaches and the three different formulations for each approach using as criteria the size of the models, the integrality gaps, the number of branch and bound nodes, and the running time to solve the instances. All formulations are strengthened with valid inequalities and tightening of constraints. As in [1, 2], computational experiments indicate that the best performances are obtained using extended formulations based on sets of variables that associate flows to the ship arcs (called arc-load flow models). This conclusion is highly relevant since, as mentioned above, most MIRPs have been modeled using arc-load formulations which are *dominated* (both theoretically, considering the integrality gap, and computationally, considering the running times) by the arc-load flow models.

The real test instances are of small size which allow us to use a commercial software to solve them to optimality. However, it should be remarked that the tested models have a structure that is well suited for solving instances with longer planning horizons than those considered here. For instance, the underlying models can be used as subproblem of heuristic procedures when solving larger problems. In [4], instances are heuristically solved for time horizons of several months using a rolling horizon heuristic where the planning horizon is split into smaller sub-horizons. Then, repeatedly, a limited and tractable problem (which is much related to the one considered in this paper) is solved for the shorter sub-horizons using a commercial software.

The remaining of this chapter is organized as follows. Section 3.2 presents arc-load discrete time and arc-load continuous time formulations. Extended formulations are discussed in Section 3.3. In Section 3.4 we discuss how the formulations can be tightened with valid inequalities. The computational study is reported in Section 3.5. Conclusions and final remarks are presented in Section 3.6. A glossary of problem and model acronyms is given in Appendix A.

3.2 Mathematical Formulations

In this section we introduce two distinct arc-load formulations. It is mainly the network structure that differs in the two formulations. Since a ship can visit the same port several times during the planning horizon, one needs to define the ship visits to each port unambiguously. One approach consists of adapting a *discrete time model* of the SSIRP by performing a discretization of the time to overcome the complicating factor of handling the multiple visits to each port. The other approach is to consider an ordering of the visits, and introduce an index indicating the visit number to a particular port. Hence, each network node corresponds to an event. This approach corresponds to the *continuous time formulation*. The first network is in general larger than the second one. However while the first network can have only cycles within each time period, the second one includes many cycles.

First we introduce a discrete time formulation. This type of formulations is usually used in problems with time varying consumption rates. These problems differ from the constant consumption rate on the consumption type (constant or varying) and on the inventory constraints. While for the constant case inventory bounds (safety stocks and upper bound capacity) must be satisfied during all time horizon, for the varying case inventory bounds need to be guaranteed at the end of time periods only.

First, we introduce the SSIRP formulation for the time varying consumption rates problem and call it the Basic Arc-Load Discrete Time formulation with time varying consumption (BD-SSIRP-V). Then we explain the changes of the formulation for the problem with constant consumption rates, and call it BD-SSIRP. In both models, the time is discretized into time periods. A node in the underlying network is described by the port and time period. The time discretization needs to be appropriately chosen. The time unit should be simultaneously large enough to accommodate the duration of a full ship operation, and fine enough as certain constraints can only be ensured over the entire period or at the end of each time period. For example, restricting the number of operating vessels in a port can only be enforced over the entire time period, and constraints such as inventory capacity, are only enforced at the end of

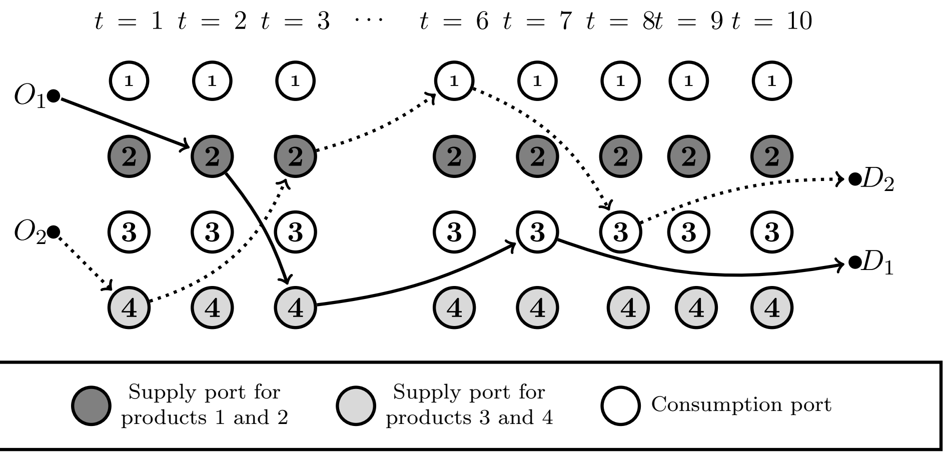

Figure 3.1: Example of two routes with a discrete time network. Each column corresponds to a period and each horizontal layer corresponds to a port.

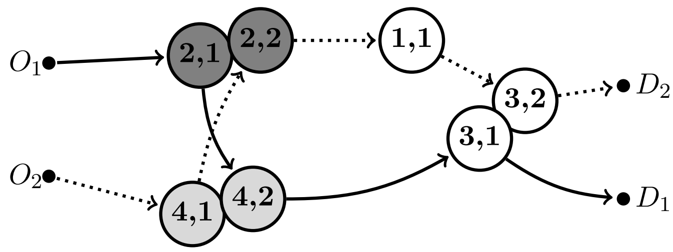

Figure 3.2: Example of ship routes where each node represents a visit. The first label indicates the port and the second label indicates the visit. Each arc type represents the path of a different ship.

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each time period. In addition, the consumption rate needs to be constant within a time period in the case of time varying consumption. Demand rates and consumption rates could be used interchangeably, but we use consumption rates throughout the chapter. An example of the ship routes in a feasible solution is depicted in Figure 3.1. Ship 1 sails from its origin to port 2. Then it starts to operate in period 2 at port 2. Further on, it sails to port 4 and starts to operate in period 3 at port 4. Then it sails to port 3 and starts to operate at port 3 in period 7. Finally the ship sails to the destination. Observe that the period that defines a visit is the period at which the ship starts to operate.

The second formulation is called the Basic Arc-Load Continuous Time formulation (BC-SSIRP) and has been used by several authors when the consumption rates are constant during the planning horizon, see for instance [6] and [10]. For each port, we define a sequence of events associated with the vessel arrivals. Each event is represented by a pair: (port, order of the arrival). Ship paths are illustrated in Figure 3.2. For instance, ship 2 leaves origin O_2 and sails to port 4 (for the first visit to this port), then sails to port 2 (for the second visit to this port, since the first visit was made by ship 1), and sails to port 1 for its first visit. Finally, the ship sails to port 3 (for the second visit to port 3 since the first visit was made by ship 1) before it ends at its destination.

3.2.1 Arc-Load Discrete Time Formulations

In this section we present the basic arc-load models BD-SSIRP-V and BD-SSIRP for the time varying consumption and constant consumption, respectively. The finite time horizon is divided into a discrete number of periods. A ship path is defined as a sequence of pairs (port, period) representing the nodes of the network. The period that defines a visit is the period in which the ship starts to operate. Waiting, operating and traveling times are considered in a continuous time measure. First we introduce the model BD-SSIRP-V. Then we adapt this formulation for the constant consumption rate case.

SSIRP with time varying consumption rates

The BD-SSIRP-V is similar to the formulation introduced in [2], but some of the problem specific details are skipped. To the best of our knowledge, the model has never been used for constant consumption rates. In this model, all variables will have a superscript D to indicate the discrete time model.

The presentation of the formulation is split into the following parts: routing constraints, loading and unloading constraints, time constraints and inventory constraints. The objective function is presented at the end.

Routing constraints:

Let V denote the set of ships. Each ship $v \in V$ must depart from its initial position (in the beginning of the planning horizon) that can be in a port or a point at sea. The set of ports is denoted by N and the set of periods is denoted by T.

For the routing we define the following binary variables: x^D_{itjuv} is equal to 1 if ship v starts to operate at port i in period t and then sails from port i to port j and starts to operate at port j in period u; and 0 otherwise, while x^D_{oitv} indicates whether ship v sails directly from its initial position to port i to start an operation in period t or not. x^D_{oitv} could have been included in x^D_{itjuv}, but is introduced to ease the reading. Variable z^D_{itv} is 1 if ship v ends its route at port i after an operation that started in time period t; and 0 otherwise, and z^D_{ov} is 1 if ship v ends its route at the origin (it is not used) and 0 otherwise. Variable w^D_{itv} is 1 if ship v visits port i in period t; and 0 otherwise. Finally, y^D_{it} is 1 if some ship visits port i in period t; 0 otherwise. Variables x^D_{itjuv} are not defined for $t > u$. For ease of notation we include them in the model assuming they are zero. We allow them to be positive if $t = u$, that means a ship can visit two ports in succession in the same time period. We also assume $x^D_{itjuv} = 0$ if $i = j$.

The routing constraints are as follows:

$$\sum_{i \in N} \sum_{t \in T} x^D_{oitv} + z^D_{ov} = 1, \qquad \forall v \in V, \tag{3.1}$$

$$w^D_{itv} - \sum_{j \in N} \sum_{u \in T} x^D_{juitv} - x^D_{oitv} = 0, \quad \forall v \in V, i \in N, t \in T, \tag{3.2}$$

$$w^D_{itv} - \sum_{j \in N} \sum_{u \in T} x^D_{itjuv} - z^D_{itv} = 0, \quad \forall v \in V, i \in N, t \in T, \tag{3.3}$$

$$\sum_{v \in V} w^D_{itv} = y^D_{it}, \qquad \forall i \in N, t \in T, \tag{3.4}$$

$$x^D_{itjuv} \in \{0, 1\}, \qquad \forall v \in V, i, j \in N, t, u \in T, \tag{3.5}$$

$$x^D_{oitv}, w^D_{itv}, z^D_{itv} \in \{0,1\}, \qquad \forall v \in V, i \in N, t \in T, \tag{3.6}$$

$$y^D_{it} \in \{0,1\}, \qquad \forall i \in N, t \in T, \tag{3.7}$$

$$z^D_{ov} \in \{0,1\}, \qquad \forall v \in V. \tag{3.8}$$

Constraints (3.1) ensure that ship v either departs from its initial position to port i in period t or it is not used. Constraints (3.2) and (3.3) are the flow conservation constraints ensuring that a ship arriving at a port also leaves that port by either visiting another port or ending its route. Equations (3.4) guarantee that at most one ship can operate at port i in a given time period.
Constraints (3.5)-(3.8) define the variables as binary.

Loading and unloading:

Let K represent the set of products and K_v represent the set of products that ship v can transport. Not all ports consume all products. Parameter J_{ik} assumes value 1 if port i is a supplier of product k; -1 if port i is a consumer of product k, and 0 if i is neither a consumer nor a supplier of product k. The quantity of product k on board ship v at the beginning of the planning horizon is given by Q_{vk}. C_{vk} is the capacity of the compartment of ship v dedicated for product k. The minimum and maximum discharge quantities of product k are given by $\underline{Q}_{ik}$ and $\overline{Q}_{ik}$, respectively.

In order to model the loading and unloading constraints we define the following binary variables: o^D_{itvk} is equal to 1 if product k is loaded onto or unloaded from ship v at port i in time period t, and 0 otherwise; and the following continuous variables: q^D_{itvk} is the amount of product k loaded onto or unloaded from ship v at port i in time period t, l^D_{itvk} is the amount of product k on board ship v when leaving from port i after an operation that started in time period t. For ease of notation, variables o^D_{itvk}, such that $J_{ik} = 0$, are included in the model and assumed to be zero.

The loading and unloading constraints are given by:

$$x^D_{itjuv}(l^D_{itvk} + J_{jk}q^D_{juvk} - l^D_{juvk}) = 0, \quad \forall v \in V, i, j \in N, t, u \in T, k \in K_v, \tag{3.9}$$

$$x^D_{oitv}(Q_{vk} + J_{ik}q^D_{itvk} - l^D_{itvk}) = 0, \quad \forall v \in V, i \in N, t \in T, k \in K_v, \tag{3.10}$$

$$l^D_{itvk} \le C_{vk} \sum_{j \in N} \sum_{u \in T} x^D_{itjuv}, \quad \forall v \in V, i \in N, t \in T, k \in K_v, \tag{3.11}$$

$$q^D_{itvk} \le C_{vk} o^D_{itvk}, \quad \forall v \in V, i \in N, t \in T, k \in K_v : J_{ik} = 1, \tag{3.12}$$

$$\underline{Q}_{ik} o^D_{itvk} \le q^D_{itvk} \le \min\{C_{vk}, \overline{Q}_{ik}\} o^D_{itvk}, \quad \forall v \in V, i \in N, t \in T, k \in K_v : J_{ik} = -1, \tag{3.13}$$

$$\sum_{k \in K_v} o^D_{itvk} \ge w^D_{itv}, \quad \forall v \in V, i \in N, t \in T, \tag{3.14}$$

$$o^D_{itvk} \le w^D_{itv}, \quad \forall v \in V, i \in N, t \in T, k \in K_v, \tag{3.15}$$

$$l^D_{itvk}, q^D_{itvk} \ge 0, \quad \forall v \in V, i \in N, t \in T, k \in K_v, \tag{3.16}$$

$$o^D_{itvk} \in \{0,1\}, \quad \forall v \in V, i \in N, t \in T, k \in K_v. \tag{3.17}$$

Constraints (3.9) and (3.10) relate the quantity on board to the quantity loaded and/or unloaded. Constraints (3.9) ensure that if ship v sails from port i (after an operation started in period t) to port j (to initialize an operation in period u), then the quantity of product k on board at the departure from island j should be equal to the quantity on board at departure from port i plus (resp. minus) the quantity loaded (resp. unloaded) from j. Equations (3.10) relate the quantity on board with the quantity loaded and/or unloaded in the starting position. Constraints (3.11) impose an upper bound on the quantity on board. They also ensure that if the quantity on board is positive than the ship must travel to some other port. Constraints (3.12) ensure that if an operation occurs at a loading port, that is, $q^D_{itvk} > 0$, than the setup variable o^D_{itvk} must be one. They also impose an upper bound on the quantity loaded. Constraints (3.13) impose lower and upper limits on the unload quantities, respectively. Constraints (3.14) ensure that if ship v starts an operation at port i in time period t, then at least one product must be (un)loaded. Constraints (3.15) ensure that if ship v (un)loads one product at port i in period t, then w^D_{itv} must be one. The nonnegativity requirements (3.16) are given for the variables representing the load on board and the (un)loading quantity. Finally, the formulation involves binary requirements (3.17) on the operating variables.

Constraints (3.9) and (3.10) are non-linear. Following [16], equations (3.9) can be linearized by replacing them with the following two sets of constraints:

$$l^D_{itvk} + J_{jk}q^D_{juvk} - l^D_{juvk} + C_{vk}x^D_{itjuv} \leq C_{vk}, \qquad \forall v \in V, i, j \in N, t, u \in T, k \in K_v, \tag{3.18}$$

$$l^D_{itvk} + J_{jk}q^D_{juvk} - l^D_{juvk} - C_{vk}x^D_{itjuv} \geq -C_{vk}, \qquad \forall v \in V, i, j \in N, t, u \in T, k \in K_v, \tag{3.19}$$

and equations (3.10) can be replaced by:

$$Q_{vk} + J_{ik}q^D_{itvk} - l^D_{itvk} + C_{vk}x^D_{oitv} \leq C_{vk}, \qquad \forall v \in V, i \in N, t \in T, k \in K_v, \tag{3.20}$$

$$Q_{vk} + J_{ik}q^D_{itvk} - l^D_{itvk} - C_{vk}x^D_{oitv} \geq -C_{vk}, \qquad \forall v \in V, i \in N, t \in T, k \in K_v. \tag{3.21}$$

Time constraints:

We have chosen a discrete time formulation consisting of few time periods compared to most existing discrete time formulations in the literature, see e.g. [1] and [19]. In order to account for the time aspects correctly we consider a continuous time measure in addition to the discrete time. In comparison to other discrete time MIRP formulations [1], we do not need to explicitly define binary waiting variables and by this avoid the symmetry problem that many such models have.

We define the following parameters: T^Q_{ik} is the time required to load/unload one unit of product k at port i; T^S_{ik} is the set up time required to operate product k at port i. Parameter T_{ijv} is the traveling time between port i and j by ship v; T^O_{iv} indicates the traveling time required by ship v to sail from its initial port position to port i; T^B_i is the minimum interval between the departure of one ship and the next arrival at port i. Finally, $\overline{T}$ is the length of the time horizon.

We define the nonnegative continuous variables t^D_{it} as the start time of the operation at port i in time period t, and t^{ED}_{it} as the end time of the operation that started during period t in port i. The time constraints are as follows,

$$t^{ED}_{it} \geq t^D_{it} + \sum_{v \in V}\sum_{k \in K_v} T^S_{ik}o^D_{itvk} + \sum_{v \in V}\sum_{k \in K_v} T^Q_{ik}q^D_{itvk}, \quad \forall i \in N, t \in T, \tag{3.22}$$

$$t^D_{it} - t^{ED}_{i(t-1)} \geq T^B_i y^D_{it}, \quad \forall i \in N, t \in T : t > 1, \tag{3.23}$$

$$t^{ED}_{it} + T_{ijv} - t^D_{ju} \leq \overline{T}(1 - x^D_{itjuv}), \quad \forall v \in V, i, j \in N, t, u \in T, \tag{3.24}$$

$$\sum_{v \in V} T^O_{iv}x^D_{oitv} \leq t^D_{it}, \quad \forall i \in N, t \in T, \tag{3.25}$$

$$t - 1 \leq t^D_{it} \leq t, \quad \forall i \in N, t \in T, \tag{3.26}$$

$$t^D_{it}, t^{ED}_{it} \geq 0, \quad \forall i \in N, t \in T. \tag{3.27}$$

Equations (3.22) define the end time of each operation. Notice the end time can be greater than the starting time plus the set up times and the time for the (un)load operations. This accounts the possibility of a ship to wait between (un)loadings. Constraints (3.23) impose a minimum interval between two consecutive visits at port i. Constraints (3.24) ensure that if ship v sails from port i (after an operation started in period t) to port j (to initialize an operation in period u), then the operation at port j can only start after the end time of operation at port i plus the time required to travel from i to j. Constraints (3.25) ensure that if ship v travels from its initial position to port i to start an operation in period t, then the starting time at port i can only occur after the traveling time. Constraints (3.26) link the continuous with the discrete time measures and constraints (3.27) define the sign of the continuous time variables.

When time windows are considered they can be easily included in the model. For instance, if the start of an operation at port i in period t is restricted to a time window $[A_{it}, B_{it}]$ then it suffices to replace constraints (3.26) by $A_{it} \leq t^D_{it} \leq B_{it}$.

Inventory constraints:

Inventory constraints are considered for each unloading port i ($J_{ik} = -1$). D_{itk} indicates the demand or consumption of product k at port i in period t. For each product k at a consumption port i, the minimum stock level is given by $\underline{S}_{ik}$ and the maximum stock level (tank capacity) is given by $\overline{S}_{ik}$. S^O_{ik} denotes the initial stock level of product k in port i.

The nonnegative continuous variables s^D_{itk} indicate the stock level of product k in port i at the end of period t. The inventory constraints are as follows:

$$s^D_{i(t-1)k} + \sum_{v \in V} q^D_{itvk} - s^D_{itk} = D_{itk}, \quad \forall i \in N, t \in T, k \in K : J_{ik} = -1, \tag{3.28}$$

$$s^D_{i0k} = S^O_{ik}, \quad \forall i \in N, k \in K : J_{ik} = -1, \tag{3.29}$$

$$\underline{S}_{ik} \leq s^D_{itk} \leq \overline{S}_{ik}, \qquad \forall i \in N, t \in T, k \in K : J_{ik} = -1. \tag{3.30}$$

Constraints (3.28) are the inventory balance constraints. These constraints together with the bounds ensure that the demand for each product at each port in each time period is satisfied. Constraints (3.29) define the initial stock levels. The upper and lower bounds on the stock levels are ensured by constraints (3.30).

Objective Function:

The objective function is to minimize the costs (transportation and setup costs):

$$Min \sum_{v \in V} \sum_{i,j \in N} \sum_{t,u \in T} C_{ijv} x^D_{itjuv} + \sum_{v \in V} \sum_{i \in N} \sum_{t \in T} C_{oiv} x^D_{oitv} + \sum_{v \in V} \sum_{i \in N} \sum_{t \in T} \sum_{k \in K_v} C^O_{ik} o^D_{itvk} \tag{3.31}$$

where C_{ijv} is the total transportation cost for ship v to sail from port i to port j, C_{oiv} is the cost for ship v to sail from its origin to port i, and C^O_{ik} is the fixed cost of operating (load/unload) product k at port i.

The basic arc-load discrete time formulation with time varying consumption rates, BD-SSIRP-V, is given by (3.1)-(3.8), (3.11)-(3.31). Even though the model includes discrete time and continuous time variables we call it a discrete time formulation.

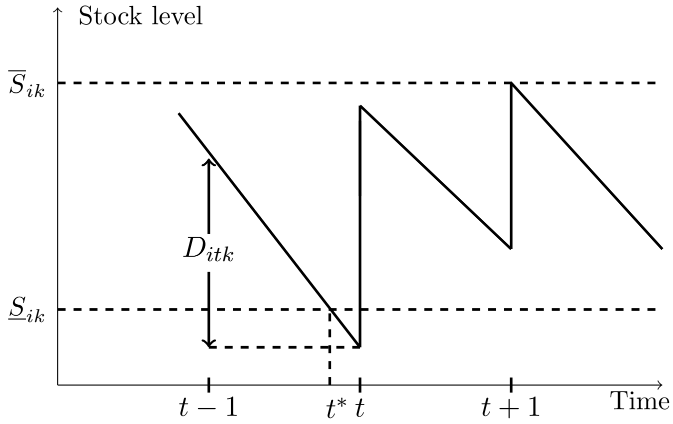

Figure 3.3: Delivery must occur no later than t^* in SSIRP, and it must occur no later than t in the SSIRP-V.

SSIRP with constant consumption rates

In this section we consider the variant of the SSIRP where constant consumption rates are assumed. The two related problems occur in two different planning problems. The time varying consumption problem occurs when a set of orders are given. Each order corresponds to a quantity of an oil product that must be delivered into a specific port and has a deadline to be satisfied. The constant consumption rate is normally assumed when the planners are considering longer time horizons. In this case the consumption rates correspond to the estimated consumption rates from real data. In order to model the constant rate case we can adapt the discrete time formulation.

In the BD-SSIRP-V the safety stock is guaranteed at the end of each period only. These ends of periods are artificially established. Hence, by choosing a different discretization the model will guarantee the stock level at different times. As depicted in Figure 3.3 it may happen that the stock level goes below the minimum stock level in the middle of a period. This situation should not be allowed in the constant rate case, SSIRP, where the safety stock must be satisfied at any time in the interval $[0, \overline{T}]$.

In order to prevent such a situation to occur, while keeping a chosen discretization, we add the following constraints

$$s^D_{i(t-1)k} - D_{itk}(t^D_{it} - t + 1) \geq \underline{S}_{ik}, \quad \forall i \in N, t \in T, k \in K : J_{ik} = -1. \tag{3.32}$$

The left hand side of (3.32) measures the stock level at the start time of the operation which is the stock level at the beginning of the period minus the consumption until the start of the operation. These levels should be above the safety stock levels.

Similarly, in order to prevent stock to go above the tank capacity at the end of a discharge operation, we add the following constraints

$$s^D_{i(t-1)k} - D_{itk}(t^{ED}_{it} - t + 1) + \sum_{v \in V} q^D_{itvk} \leq \overline{S}_{ik}, \quad \forall i \in N, t \in T, k \in K : J_{ik} = -1. \tag{3.33}$$

Constraints (3.32) and (3.33) could also have been added to BD-SSIRP-V if it is important to ensure that the inventory levels are within the limits during the time horizon.

The basic arc-load discrete time formulation with constant consumption rates, BD-SSIRP, is given by (3.1)-(3.8), (3.11)-(3.27), (3.28)-(3.33).

3.2.2 Arc-Load Continuous Time Formulation

In this section we present the basic arc-load continuous time formulation, BC-SSIRP, for the case with constant consumption rates.

In the BD-SSIRP we discretized the time such that in each period at most one visit could occur at each port. Here, we present an alternative formulation where port events, here called port visits, are distinguished by the order of the visit. This type of formulation was used in [6] and [10].

Again, we divide the set of constraints of the formulation into the following parts: routing constraints, loading and unloading constraints, time constraints and inventory constraints.

Contrary to the discrete model that combines both discrete and continuous time, this model uses continuous time only.

For each port we consider an ordering of the visits according to the time of the visit. The ship paths are defined on a network where the nodes are represented by a pair (i, m), where i is the port and m is the visit number at port i.

For this formulation only the new notation is introduced.

Routing constraints:

Each possible port visit is denoted by the pair (i, m) representing the m^{th} visit to port i. Direct ship movements (arcs) from port visit (i, m) to port visit (j, n) are represented by (i, m, j, n).

We define S^A as the set of possible port visits (i, m), S^A_v as the set of possible port visits made by ship v, and set S^X_v as the set of all possible movements (i, m, j, n) of ship v.

For the routing we define the following binary variables: x^C_{imjnv} is equal to 1 if ship v sails from port visit (i, m) directly to port visit (j, n); and 0 otherwise, x^C_{oimv} indicates whether ship v sails directly from its initial position to port visit (i, m) or not, w^C_{imv} is 1 if ship v visits port i at arrival (i, m); and 0 otherwise, z^C_{imv} is equal to 1 if ship v ends its route at port visit (i, m); and 0 otherwise, and y^C_{im} indicates whether a ship is visiting port arrival (i, m) or not.

The routing constraints are as follows:

$$\sum_{(i,m)\in S^A_v} x^C_{oimv} + z^C_{ov} = 1, \qquad \forall v \in V, \tag{3.34}$$

$$w^C_{imv} - \sum_{(j,n)\in S^A_v} x^C_{jnimv} - x^C_{oimv} = 0, \qquad \forall v \in V, (i, m) \in S^A_v, \tag{3.35}$$

$$w^C_{imv} - \sum_{(j,n)\in S^A_v} x^C_{imjnv} - z^C_{imv} = 0, \qquad \forall v \in V, (i, m) \in S^A_v, \tag{3.36}$$

$$\sum_{v\in V} w^C_{imv} = y^C_{im}, \qquad \forall (i, m) \in S^A, \tag{3.37}$$

$$y^C_{i(m-1)} - y^C_{im} \geq 0, \qquad \forall (i, m) \in S^A : m > 1, \tag{3.38}$$

$$x^C_{oimv}, w^C_{imv}, z^C_{imv} \in \{0, 1\}, \qquad \forall v \in V, (i, m) \in S^A_v, \tag{3.39}$$

$$x^C_{imjnv} \in \{0, 1\}, \qquad \forall v \in V, (i, m, j, n) \in S^X_v, \tag{3.40}$$

$$y^C_{im} \in \{0, 1\}, \qquad \forall (i, m) \in S^A, \tag{3.41}$$

$$z^C_{ov} \in \{0, 1\}, \qquad \forall v \in V. \tag{3.42}$$

Equations (3.34) ensure that each ship departs from its initial position and sails towards another port or the ship is not used. Equations (3.35) and (3.36) are the flow conservation constraints, ensuring that a ship arriving at a port also leaves that port by either visiting another port or ending its route. Constraints (3.37) ensure that each port visit (i,m) is made at most once. Constraints (3.38) state that if port i is visited m times, then it must also have been visited $m-1$ times. Constraints (3.39)-(3.42) define the variables as binary.

Loading and unloading:

In order to model the loading and unloading constraints, we define the following binary variables: o^C_{imvk} is equal to 1 if product k is loaded onto or unloaded from ship v at port visit (i,m), and 0 otherwise. In addition, we define the following continuous variables: q^C_{imvk} is the amount of product k (un)loaded at port visit (i,m) and l^C_{imvk} is the amount of product k on board ship v when leaving from port visit (i,m).

The loading and unloading constraints are given by:

$$x^C_{imjnv}(l^C_{imvk} + J_{jk}q^C_{jnvk} - l^C_{jnvk}) = 0, \qquad \forall v \in V, (i,m,j,n) \in S^X_v, k \in K_v, \tag{3.43}$$

$$x^C_{oimv}(Q_{vk} + J_{ik}q^C_{imvk} - l^C_{imvk}) = 0, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v, \tag{3.44}$$

$$l^C_{imvk} \leq C_{vk} \sum_{(j,n)\in S^A_v} x^C_{imjnv}, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v, \tag{3.45}$$

$$q^C_{imvk} \leq C_{vk}o^C_{imvk}, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v : J_{ik} = 1, \tag{3.46}$$

$$\underline{Q}_{ik}o^C_{imvk} \leq q^C_{imvk} \leq \min\{C_{vk}, \overline{Q}_{ik}\}o^C_{imvk}, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v : J_{ik} = -1, \tag{3.47}$$

$$\sum_{k\in K_v} o^C_{imvk} \geq w^C_{imv}, \qquad \forall v \in V, (i,m) \in S^A_v, \tag{3.48}$$

$$o^C_{imvk} \leq w^C_{imv}, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v, \tag{3.49}$$

$$l^C_{imvk}, q^C_{imvk} \geq 0, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v, \tag{3.50}$$

$$o^C_{imvk} \in \{0,1\}, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v. \tag{3.51}$$

Equations (3.43) determine the quantity of product k on board ship v when the ship sails from port visit (i,m) to port visit (j,n). These constraints can be linearized as follows:

$$l^C_{imvk} + J_{jk}q^C_{jnvk} - l^C_{jnvk} + C_{vk}x^C_{imjnv} \leq C_{vk}, \qquad \forall v \in V, (i,m,j,n) \in S^X_v, k \in K_v, \tag{3.52}$$

$$l^C_{imvk} + J_{jk}q^C_{jnvk} - l^C_{jnvk} - C_{vk}x^C_{imjnv} \geq -C_{vk}, \qquad \forall v \in V, (i,m,j,n) \in S^X_v, k \in K_v. \tag{3.53}$$

Constraints (3.44) are similar to (3.43) and determine the load on board the ship for the first ship visit. These constraints can be replaced by the following linear constraints:

$$Q_{vk} + J_{ik}q^C_{imvk} - l^C_{imvk} + C_{vk}x^C_{oimv} \leq C_{vk}, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v, \tag{3.54}$$

$$Q_{vk} + J_{ik}q^C_{imvk} - l^C_{imvk} - C_{vk}x^C_{oimv} \geq -C_{vk}, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v. \tag{3.55}$$

The ship capacity constraints are given by (3.45). Constraints (3.46) impose an upper bound on the quantity loaded at the supply port. Constraints (3.47) impose lower and upper limits on the unload quantities. Constraints (3.48) ensure that if ship v makes port visit (i,m), then at least one product must be (un)loaded. Constraints (3.49) ensure that if ship v (un)loads one product at visit (i,m), then w^C_{imv} must be one. Constraints (3.50)-(3.51) are the non-negativity and integrality constraints.

Time constraints:

Given the start time and end time variables, t_{im}^C and t_{im}^{EC} at port visit (i,m), the time constraints can be written as:

$$t_{im}^{EC} \geq t_{im}^C + \sum_{v \in V} \sum_{k \in K_v} T_{ik}^Q q_{imvk}^C + \sum_{v \in V} \sum_{k \in K_v} T_{ik}^S o_{imvk}^C, \qquad \forall (i,m) \in S^A, \tag{3.56}$$

$$t_{im}^C - t_{i(m-1)}^{EC} - T_i^B y_{im}^C \geq 0, \qquad \forall (i,m) \in S^A : m > 1, \tag{3.57}$$

$$t_{im}^{EC} + T_{ijv} - t_{jn}^C \leq \overline{T}(1 - x_{imjnv}^C), \qquad \forall v \in V, (i,m,j,n) \in S_v^X, \tag{3.58}$$

$$\sum_{v \in V} T_{iv}^O x_{oimv}^C \leq t_{im}^C, \qquad \forall (i,m) \in S^A, \tag{3.59}$$

$$t_{im}^C, t_{im}^{EC} \geq 0, \qquad \forall (i,m) \in S^A. \tag{3.60}$$

Constraints (3.56) define the end time of service of port visit (i,m). Constraints (3.57) impose a minimum interval between two consecutive visits at port i. Constraints (3.58) relate the end time of port visit (i,m) to the start time of port visit (j,n) when ship v sails directly from port (i,m) to (j,n). Constraints (3.59) ensure that if ship v travels from its initial position directly to port visit (i,m), then the start time is at least the traveling time between the two positions. Constraints (3.60) define the continuous time variables.

Single time windows for each visit can be introduced in a similar way as for the discrete case. However in case the time windows are associated with open hours at ports then new variables are necessary to model multiple time windows.

Inventory constraints:

The inventory constraints are necessary to ensure that the inventory levels are kept within the corresponding bounds and to link the inventory levels to the (un)loading quantities.

We define parameter R_{ik} as the consumption rate of product k at port i (that is, $D_{itk} = R_{ik}, \forall t \in T$), and define the nonnegative continuous variables s_{imk}^C and s_{imk}^{EC} indicating the inventory levels at the start and at the end of port visit (i,m), respectively. The inventory constraints are as follow:

$$s_{i1k}^C = S_{ik}^O - R_{ik} t_{i1}^C, \qquad \forall i \in N, k \in K : J_{ik} = -1, \tag{3.61}$$

$$s_{imk}^{EC} = s_{imk}^C + \sum_{v \in V} q_{imvk}^C - R_{ik}(t_{im}^{EC} - t_{im}^C), \qquad \forall (i,m) \in S^A, k \in K : J_{ik} = -1, \tag{3.62}$$

$$s_{imk}^C = s_{i(m-1)k}^{EC} - R_{ik}(t_{im}^C - t_{i(m-1)}^{EC}), \qquad \forall (i,m) \in S^A : m > 1, k \in K : J_{ik} = -1, \tag{3.63}$$

$$\underline{S}_{ik} \leq s_{imk}^C, s_{imk}^{EC} \leq \overline{S}_{ik}, \qquad \forall (i,m) \in S^A, k \in K : J_{ik} = -1. \tag{3.64}$$

Equations (3.61) calculate the inventory level of each product at the first visit. Equations (3.62) calculate the invrntory level of each product when the service ends at port visit (i,m). Similarly, equations (3.63) relate the inventory level at the start of port visit (i,m) to the inventory level at the end of port visit $(i,m-1)$. The upper and lower bounds on the inventory levels are ensured by constraints (3.64).

It remains to ensure that the inventory levels at the end of the planning horizon is within the inventory limits. We discuss two options. The following set of constraints was used in [22].

$$\underline{S}_{ik} \leq s^{EC}_{imk} - R_{ik}(\overline{T} - t^{EC}_{im})(y^C_{im} - y^C_{i(m+1)}) \leq \overline{S}_{ik}, \qquad \forall (i, m+1) \in S^A, k \in K : J_{ik} = -1.$$

We can see that t^{EC}_{im} is the end time of the last visit to port i if and only if $y^C_{im} - y^C_{i(m+1)} = 1$. This set of constraints is nonlinear and can be linearized as in [22]. However we omit the linearization process here, because we will follow the approach used in [10], to handle the stock level at the end of the planning horizon. Consider the following set of constraints where $\overline{\mu}_i$ is an upper bound on the number of visits to port i.

$$\underline{S}_{ik} \leq s^{EC}_{i\overline{\mu}_i k} + R_{ik}(\overline{T} - t^{EC}_{i\overline{\mu}_i}) \leq \overline{S}_{ik}, \qquad \forall i \in N, k \in K : J_{ik} = -1, \tag{3.65}$$

Here the end time of the last possible visit is given by $t^{EC}_{i\overline{\mu}_i}$.

Objective function:

The objective is to minimize the total routing and operating cost:

$$\sum_{v \in V} \sum_{(i,m,j,n) \in S^X_v} C_{ijv} x^C_{imjnv} + \sum_{v \in V} \sum_{(i,m) \in S^A_v} C_{oiv} x^C_{oimv} + \sum_{v \in V} \sum_{(i,m) \in S^A_v} \sum_{k \in K_v} C^O_{ik} o^C_{imvk} \tag{3.66}$$

The basic arc-load time continuous formulation with constant consumption rates, BC-SSIRP, is defined by (3.34)-(3.42), (3.45)-(3.66).

3.2.3 Comparison of the Discrete Time and Continuous Time Models

Here we discuss the two models regarding their integrality gaps, size, and level of information provided.

Integrality gaps

Although the definition of the variables in the time discrete model is different from the definition of variables in the time continuous model, we can easily see that the two mathematical models are very similar. In fact, removing the inventory constraints from both models and constraints (3.38) from the BC-SSIRP, the mathematical expressions of both models is similar. The unique difference is that variables x^D_{itjuv} are defined for all $u \geq t$ while x^C_{imjnv} are defined for all m and n. As a consequence the linear relaxation of the discrete model BD-SSIRP without inventory constraints should provide bounds at least as good as those provided by the linear relaxation of the continuous model BC-SSIRP without inventory constraints.

To compare theoretically the complete models (with inventory variables) is not a straightforward task since one needs to relate the two sets of variables. Here we only provide an experimental comparison. This study is conducted in Section 3.5 and shows that the bounds provided by the two models are the same for the tested instances, which reinforces our comment on the similarity of the models. The computational study also shows that the integrality gaps of BD-SSIRP and BC-SSIRP are very large. In the two following sections we improve these formulations by deriving tighter extended formulations (Section 3.3) and by including valid inequalities (Section 3.4). The ideas used in those improvements are similar for both types of formulations.

Size of the models

The size of the models is determined by the number of x (routing) variables since this number establishes the bound for the number of variables and constraints. Contrary to the discrete model, where the number of routing variables is well defined for a particular discretization, in the continuous case this number depends on the maximum number of visits to each port i, $\overline{\mu}_i$. These upper bounds $\overline{\mu}_i$ can be computed using the minimum (un)loading quantities $\underline{Q}_i^k$ and the time constraints. However, usually the quantities $\underline{Q}_i^k$ are not imposed by any real limit but to avoid a "large" number of visits. Our experience showed that the maximum number of visits can be set to a minimum number of visits (computed in Section 3.4) plus a constant: one, two or three, depending on the port activity. For larger increases of $\overline{\mu}_i$, only the running time increases, see Section 3.5.

We can also eliminate some routing variables x_{itjuv}^D from the discrete model. Since the maximum distance between two ports is short in the underlying real short sea inventory routing problem, we can eliminate variables where $u >> t$. In Section 3.5, we present computational experiments to evaluate the impact of the objective function, the size of the model, and on the running time of these restrictions on the variables.

Information provided

The solution of each model provides different information. However, the solution from one model can easily be converted into a solution of the other. In the discrete formulation, the information of the period in which the visits occur is given by the time variables t_{it}^D as well as the routing variables x_{itjuv}^D, while in the continuous model this information is provided only by the time variables. This difference allows us to relate the routing aspects directly to the inventory in the discrete models. As we will see in Section 3.4, this property can be used to tighten the discrete model.

3.3 Linear Relaxations and Extended Formulations

In this section we discuss some of the weaknesses of the arc-load formulations and introduce two extended formulations for each type of model (discrete time and continuous time). We consider only the SSIRP with constant consumption rates.

In Figure 3.4 we present a fractional solution of the arc-load continuous time model that illustrates the weaknesses of the arc-load formulations.

As we can see from the example, the fractional solution does not guarantee the equilibrium of the flow on board the ship. Both ships unload products that they do not transport. For instance, ship 2 unloads 50 units at port 1 and these units are never loaded. Next we justify how such solutions can occur. First notice that the unique link between the load on board the ship and the path of the ship is established at the nodes. Additionally, the link is established through constraints (3.18)-(3.21) in BD-SSIRP and through constraints (3.52)-(3.55) in the BC-SSIRP. These linking constraints are known to be very weak. It is therefore possible to get, in a linear fractional solution, an unload operation when the ship has nothing on board. Consider the BC-SSIRP case, and suppose $J_{jk} = -1$. If $0 < x_{imjnv}^C < 1$ and $l_{imvk}^C = l_{jnvk}^C = 0$, then the unload quantity q_{jnvk}^C of product k can be positive. More specifically $0 \leq q_{jnvk}^C \leq C_{vk} \min\{x_{imjnv}^C, 1 - x_{imjnv}^C\}$.

Also, as expected, each ship follows multiple fractional paths.

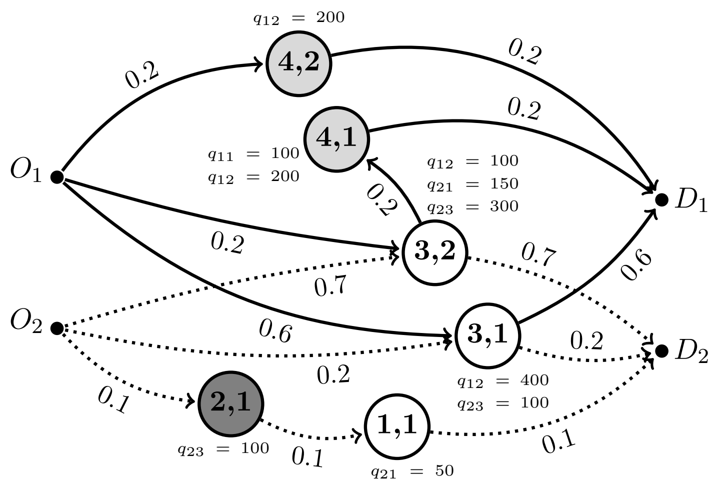

Figure 3.4: Example of an optimal solution of the linear relaxation of the BC-SSIRP. The quantities q_{vk} next to node (i, m) represent the quantity of product k unloaded by ship v in the m^{th} visit to port i. In this solution there are no loadings. The arc labels represent the values of the corresponding arc-variables. Dark arcs represent ship 1 and dashed arcs represent ship 2. We assume $Q_{vk} = 0$, $\forall v \in V, k \in K_v$.

In order to avoid some of the drawbacks of the arc-load formulations, we propose two extended formulations for each approach. The new set of variables introduced in each formulation provides additional information about the solution. That information will be essential to derive tighter models. All the formulations presented in the chapter are compact. In general, the linear relations of the extended formulations lead to better bounds but are harder (considering the computational effort) to obtain. When using such formulations in a branch and bound scheme, the number of tree nodes tends to be less than in the case where a smaller formulation is used. However, the time spent in each node is usually greater.

In the first extended formulation, new variables indicating the amount of each product carried along an arc are introduced. These new variables can be seen as defining the flow of individual products along the chosen paths resulting from the routing variables for each ship. The second extended formulation can be seen as a classical multi-commodity reformulation of the first extended formulation where the flow variables additionally indicate the destination of each product along the chosen paths.

3.3.1 Arc-Load Flow Reformulations

In this section we introduce new arc-load flow variables that indicate the amount of each product carried along each arc. These flow variables allow us to assign a flow of each product to the ship path. In this way we can prevent fractional solutions as the one depicted in Figure 3.4.

Discrete time reformulation

Next, we present the arc-load flow discrete time formulation with constant consumption rates (FD-SSIRP). Let us define f^D_{itjuvk}, as the amount of product k that ship v transports from port i, after an operation that started in period t, to port j in order to start an operation in period u. For ease of notation, when $x^D_{itjuv} = 0$, variables f^D_{itjuvk} are included in the model and set to zero.

Let f^D_{oitvk} denote the amount of product k that ship v transports from its initial port position to port i in period t.

The two sets of variables l^D_{itvk} and f^D_{itjuvk} can be related using the following equations

$$l^D_{itvk} = \sum_{j \neq i} \sum_{u \geq t} f^D_{itjuvk}, \quad \forall v \in V, i \in N, t \in T, k \in K_v, \tag{3.67}$$

Constrains (3.9), (3.10) and (3.11) can be replaced by constraints

$$f^D_{ojuvk} + \sum_{i \neq j} \sum_{t \leq u} f^D_{itjuvk} + J_{jk} q^D_{juvk} = \sum_{i \neq j} \sum_{t \geq u} f^D_{juitvk}, \qquad \forall v \in V, j \in N, u \in T, k \in K_v, \tag{3.68}$$

$$f^D_{oitvk} = Q_{vk} x^D_{oitv}, \qquad \forall v \in V, i \in N, t \in T, k \in K_v, \tag{3.69}$$

$$f^D_{itjuvk} \leq C_{vk} x^D_{itjuv}, \qquad \forall v \in V, i, j \in N, t, u \in T, k \in K_v, \tag{3.70}$$

$$f^D_{itjuvk} \geq 0, \qquad \forall v \in V, i, j \in N, t, u \in T, k \in K_v. \tag{3.71}$$

The flow conservation constraints are given by equations (3.68). Equations (3.69) determine the amount of product k on board ship v at departure from the initial position. Constraints (3.70) are the variable upper bound constraints. They relate the flow variable f^D_{itjuvk} to the routing variables x^D_{itjuv} and, together with the nonnegativity constraints (3.71) impose bounds on the flow variables.

The FD-SSIRP formulation is defined by (3.1)-(3.8), (3.12)-(3.17), (3.22)-(3.33), (3.68)-(3.71).

Adding constraints (3.70) for j and u we obtain

$$\sum_{j \in N} \sum_{u \in T} f^D_{itjuvk} \leq C_{vk} \sum_{j \in N} \sum_{u \in T} x^D_{itjuv}.$$

Using (3.67) we obtain (3.11). Hence constraints (3.11) can be obtained by aggregating constraints (3.70). Thus, the linear relaxation of FD-SSIRP should provide better bounds than the linear relaxation of BD-SSIRP. The drawback of this model is that it increases the size by adding a large number of continuous variables and constraints.

Notice that with the inclusion of variables f^D_{itjuvk}, variables q^D_{juvk} can be eliminated from the model using equations (3.68) and (3.69), that is, setting

$$q^D_{juvk} = J_{jk} \left(\sum_{i \in N} \sum_{t \in T} f^D_{juitvk} - \sum_{i \in N} \sum_{t \in T} f^D_{itjuvk} \right), \qquad \forall v \in V, j \in N, u \in T, k \in K_v. \tag{3.72}$$

Continuous time reformulation

Here we define a similar flow model for the continuous time formulation, denoted by FC-SSIRP. Let f^C_{imjnvk} denote the amount of product k that ship v transports from port visit (i, m) to port visit (j, n) and f^C_{ojnvk} as the amount of product k that ship v transports from its initial position to port visit (j, n).

Using these additional variables, constraints (3.43)-(3.45) can be replaced by the following set of constraints:

$$f^C_{ojnvk} + \sum_{(i,m)\in S^A_v} f^C_{imjnvk} + J_{jk} q^C_{jnvk} = \sum_{(i,m)\in S^A_v} f^C_{jnimvk}, \qquad \forall\, v \in V, (j,n) \in S^A_v, k \in K_v, \tag{3.73}$$

$$f^C_{oimvk} = Q_{vk} x^C_{oimv}, \qquad \forall v \in V, (i,m) \in S^A_v, k \in K_v, \tag{3.74}$$

$$f^C_{imjnvk} \le C_{vk} x^C_{imjnv}, \qquad \forall v \in V, (i,m,j,n) \in S^X_v, k \in K_v, \tag{3.75}$$

$$f^C_{imjnvk} \ge 0, \qquad \forall v \in V, (i,m,j,n) \in S^X_v, k \in K_v. \tag{3.76}$$

Constraints (3.73) ensure the equilibrium of product k on board ship v. Equations (3.74) determine the quantity on board when ship v sails from its initial port position to port visit (i,m). Constraints (3.75) link the new flow variables to the arc variables and impose an upper bound on the capacity of the compartment of ship v dedicated to carry product k.

The arc-load flow continuous time formulation with constant consumption rates, FC-SSIRP, is defined by (3.34)-(3.42), (3.46)-(3.51), (3.56)-(3.66), (3.73)-(3.76).

Similar to the discrete case, the linear relaxation of FC-SSIRP can be shown to be tighter than the linear relaxation of BC-SSIRP. In Figure 3.5 we illustrate the optimal solution of the linear relaxation of FC-SSIRP for the same example as the one depicted in Figure 3.4. We can see that the fractional solution satisfies the equilibrium of the flow along each fractional ship path.

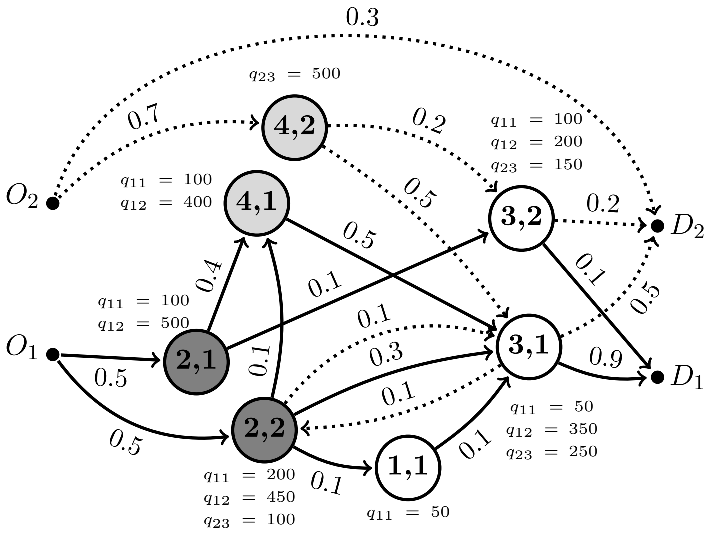

Figure 3.5: Optimal solution of the linear relaxation of FC-SSIRP for the example used in Figure 3.4. In this solution all unloaded products are previously loaded. The quantities q_{vk} represent the quantity of product k loaded (if $k \in \{1,2\}$ and $i = 2$, or $k = 3$ and $i = 4$) or unload (in the remaining cases) by ship v.

3.3.2 Multi-Commodity Reformulations

A multi-commodity reformulation of a flow formulation can be obtained by disaggregating the flow on each arc according to its destination. In general, such types of formulations lead to better bounds.

Multi-commodity discrete time reformulation

In this section we define the multi-commodity discrete time formulation with constant consumption rates (MD-SSIRP). By adding new indices to the flow variables indicating the destination of the flow, we construct the non-negative multi-commodity arc-load flow variables $v^D_{itjuvkpe}$, representing the amount of product k that ship v transports from port i, after an operation that started in period t, to port j for an operation starting in period u to be delivered at port p in period e.

These variables are nonnegative

$$v^D_{itjuvkpe} \geq 0, \qquad \forall v \in V, i, j, p \in N, t, u, e \in T, k \in K_v : J_{pk} = -1, \tag{3.77}$$

and can be related to the arc-load flow variables through the following equations,

$$f^D_{itjuvk} = \sum_{p \neq i} \sum_{e \geq u} v^D_{itjuvkpe}, \qquad \forall v \in V, i, j \in N, t, u \in T, k \in K_v. \tag{3.78}$$

The tightening of FD-SSIRP can be obtained by replacing constraints (3.70) with

$$v^D_{itjuvkpe} \leq min\{C_{vk}, \overline{Q}_{pk}\} x^D_{itjuv}, \qquad \forall v \in V, i, j, p \in N, t, u, e \in T, k \in K_v : J_{pk} = -1. \tag{3.79}$$

The MD-SSIRP can be obtained from the FD-SSIRP by replacing (3.70) with (3.77)-(3.79). Of course the arc-load flow variables f^D_{itjuvk} can be eliminated from the model using (3.78).

Multi-commodity continuous time flow reformulation

Now we define a similar multi-commodity flow formulation for the continuous time model, denoted by MC-SSIRP. We define $v^C_{imjnvkpl}$ as the amount of product k destined to port visit (p, l), which is transported from port visit (i, m) to port visit (j, n) using ship v. These variables are nonnegative,

$$v^C_{imjnvkpl} \geq 0, \qquad \forall v \in V, (i, m, j, n) \in S^X_v, (p, l) \in S^A_v, k \in K_v : J_{pk} = -1, \tag{3.80}$$

and can be related to the arc-load flow variables by the following equations

$$f^C_{imjnvk} = \sum_{(p,l) \in S^A_v : J_{pk} = -1} v^C_{imjnvkpl}, \qquad \forall v \in V, (i, m, j, n) \in S^X_v, k \in K_v. \tag{3.81}$$

The tightening of the FC-SSIRP can be obtained by replacing constraints (3.75) by

$$v^C_{imjnvkpl} \leq min\{C_{vk}, \overline{Q}_{pk}\} x^C_{imjnv}, \forall v \in V, (i, m, j, n) \in S^X_v, (p, l) \in S^A_v, k \in K_v : J_{pk} = -1. \tag{3.82}$$

The MC-SSIRP can be obtained from the FC-SSIRP by replacing (3.75) with (3.80)-(3.82). Of course the arc-load flow variables f^C_{imjnvk} can be eliminated from the model using (3.81).

3.4 Tightening the Models

The formulations discussed in Sections 3.2 and 3.3 can be strengthened by including valid inequalities and by tightening some constraints. The ideas employed in these improvements are similar for both types of formulations. However, the discrete model embeds time specific information in the network structure that makes the model more amenable for tightening and preprocessing. We discuss only the case with constant consumption rates. The inequalities used in this paper impose either a minimum number of visits to ports or a minimum number of (un)loads. Similar valid inequalities have been used in related papers for constant rate case and for the non constant consumption rates case; see for the last case [1, 2, 19, 21, 29]. When consumptions are not constant during time, inequalities based on lot-sizing relaxations have been used, see [1, 2, 19].

3.4.1 Valid Inequalities

Here we discuss valid inequalities for the models derived in the previous sections. These inequalities allow us to reduce the integrality gap of the proposed models. Hence, although the linear relaxations tend to become more time consuming to solve with the inclusion of these cuts, the reduction of the integrality gap tends to reduce the number of nodes in a branch and bound scheme. The gain in the reduced size of the branch and bound tree compensates the time increase required to obtain the dual bound at each node.

Here we just discuss a type of valid inequalities that impose visits to ports. These visits are forced by the inventory levels combined with the consumption rates. First we consider the discrete time models BD-SSIRP, FD-SSIRP, and MD-SSIRP.

For each unloading (consumption) port $i \in N$ and product k, $J_{ik} = -1$, let

$$ND_{ik} = max\{\overline{T} \times R_{ik} - S^O_{ik} + \underline{S}_{ik}, \quad 0\}$$

denote the net consumption or demand over the time horizon. If $0 < ND_{ik} < \underline{Q}_{ik}$, then the net demand can be increased to the minimum load quantity: $ND_{ik} = \underline{Q}_{ik}$. The minimum number of visits at port i for unloading product k is given by

$$\underline{\lambda}_{ik} = \left\lceil \frac{ND_{ik}}{\overline{Q}_{ik}} \right\rceil.$$

Hence, the following inequalities are valid

$$\sum_{v \in V} \sum_{j \in N} \sum_{u \in T} \sum_{t \in T} x^D_{juitv} \geq \underline{\lambda}_{ik}, \qquad \forall i \in N, k \in K : J_{ik} = -1, \tag{3.83}$$

$$\sum_{v \in V} \sum_{t \in T} o^D_{itvk} \geq \underline{\lambda}_{ik}, \qquad \forall i \in N, k \in K : J_{ik} = -1. \tag{3.84}$$

These inequalities can be generalized for each period $t \in T$, as follows. We split the time horizon into two periods, one from 0 to the end of period t and the other from t to the end of the time horizon. Let

$$ND^0_{itk} = t \times R_{ik} - S^O_{ik} + \underline{S}_{ik},$$

be the net consumption until the end of period t and let

$$\underline{ND}^{\overline{T}}_{itk} = (\overline{T} - t + 1) \times R_{ik} - \overline{S}_{ik} + \underline{S}_{ik},$$

be an underestimation of the net consumption from the end of period t until the end of the time horizon. Define

$$\epsilon^0_{i0tk} = \left\lceil \frac{ND^0_{itk}}{\overline{Q}_{ik}} \right\rceil,$$

and

$$\epsilon^{\overline{T}}_{itk} = \left\lceil \frac{\underline{ND}^{\overline{T}}_{itk}}{\overline{Q}_{ik}} \right\rceil,$$

as a lower bound on the number of visits to port i. Then the following inequalities are valid

$$\sum_{u \in T|u \leq t} \sum_{j \in N} \sum_{e \in T} \sum_{v \in V} x^D_{jeiuv} \geq \epsilon^0_{i0tk}, \qquad \forall i \in N, t \in T, k \in K : J_{ik} = -1, \tag{3.85}$$

$$\sum_{u \in T|u \leq t} \sum_{v \in V} o^D_{iuvk} \geq \epsilon^0_{itk}, \qquad \forall i \in N, t \in T, k \in K : J_{ik} = -1, \tag{3.86}$$

$$\sum_{u \in T|u > t} \sum_{j \in N} \sum_{e \in T} \sum_{v \in V} x^D_{jeiuv} \geq \epsilon^{\overline{T}}_{itk}, \qquad \forall i \in N, t \in T, k \in K : J_{ik} = -1, \tag{3.87}$$

$$\sum_{u \in T|u > t} \sum_{v \in V} o^D_{iuvk} \geq \epsilon^{\overline{T}}_{itk}, \qquad \forall i \in N, t \in T, k \in K : J_{ik} = -1. \tag{3.88}$$

In order to ensure that if ship v unloads product k at port i in period t, then there must exist a route of ship v passing through port i at period t, the following inequalities can be added.

$$o^D_{itvk} \leq \sum_{j \in N} \sum_{u \in T} x^D_{juitv}, \qquad \forall v \in V, i \in N, t \in T, k \in K_v : J_{ik} = -1. \tag{3.89}$$

Inequalities (3.89) coupled with constraints (3.84) imply (3.83). This is no longer true if we consider in (3.83) the aggregated demand (consumption) of a subset of consumption ports instead of the demand of port i only.

In the underlying real planning problem, the inventory bounds are usually not tight for the loading ports. Hence, the minimum number of departures can be estimated using the total demand supplied by those ports. In the real problem, each product has a single origin, so the demand of that product must be satisfied either from that port or from the quantity in the ship tanks at the beginning of the time horizon.

For each product $k \in K$ and loading port $i \in N$ ($J_{ik} = 1$), let

$$ND_{ik} = \sum_{j \in N|J_{jk}=-1} (\overline{T} \times R_{jk} - S^O_{ik} + \underline{S}_{ik}) - \sum_{v \in V} Q_{vk},$$

denote the demand (consumption) in excess of what is available on board the ship in the beginning of the planning horizon. The minimum number of loadings of product k at port i is given by

$$\underline{\lambda}_{ik} = \left\lceil \frac{ND_{ik}}{max\{C_{vk} : v \in V\}} \right\rceil.$$

Hence, the following inequalities are valid

$$\sum_{v \in V} \sum_{j \in N} \sum_{u \in T} \sum_{t \in T} x^D_{juitv} \geq \underline{\lambda}_{ik}, \qquad \forall i \in N, k \in K : J_{ik} = 1, \tag{3.90}$$

$$\sum_{v \in V} \sum_{t \in T} o^D_{itvk} \geq \underline{\lambda}_{ik}, \qquad \forall i \in N, k \in K : J_{ik} = 1. \tag{3.91}$$

As done for the consumption ports, we can derive inequalities for each period u for the loading ports as well; see (3.85) - (3.89). We omit these inequalities here.

Observe that a lower bound on the total number of visits to port $i \in N$ can be given by

$$\underline{\mu}_i = max\{\underline{\lambda}_{ik} : k \in K\}. \tag{3.92}$$

Hence, the following inequalities are valid:

$$\sum_{t \in T} y_{it}^D \geq \underline{\mu}_i, \qquad \forall i \in N. \tag{3.93}$$

Now we consider the continuous models BC-SSIRP, FC-SSIRP and MC-SSIRP. Here we can only impose a minimum number of visits during the planning horizon since the order of the visits does not provide information about the time for start of service at the visit. Inequalities (3.83) - (3.84) for the consumption ports and (3.90)-(3.91) for the loading/production ports can be written for the continuous case as follows:

$$\sum_{v \in V} \sum_{(j,n) \in S_v^A} \sum_{m \in \{1,\ldots,\overline{\mu}_i\}} x_{jnimv}^C \geq \underline{\lambda}_{ik}, \qquad \forall i \in N, k \in K, \tag{3.94}$$

$$\sum_{v \in V} \sum_{m \in \{1,\ldots,\overline{\mu}_i\}} o_{imvk}^C \geq \underline{\lambda}_{ik}, \qquad \forall i \in N, k \in K. \tag{3.95}$$

In the continuous time case, the lower bound on the number of visits can be imposed by the inequality

$$y_{i\underline{\mu}_i}^C = 1, \qquad i \in N. \tag{3.96}$$

3.4.2 Tightening constraints

Now we consider another approach to strengthen the models by tightening the linking constraints. The linking constraints relate the continuous variables to the binary variables. Improving these constraints can lead to reductions in the integrality gap and in running times. We focus on formulations for the constant consumption rate case only.

First we consider the tightening of constraints (3.24) for the discrete model and (3.58) for the continuous model, linking time variables with routing variables. The main idea is to aggregate the routing variables for v since the time variables do not depend on the particular ship v. Consider the time constraints (3.24) for the discrete model. These inequalities can be replaced by the following ones

$$t_{it}^{ED} + \sum_{v \in V} T_{ijv} x_{itjuv}^D - t_{ju}^D \leq \overline{T}(1 - \sum_{v \in V} x_{itjuv}^D), \quad \forall i, j \in N, t, u \in T.$$

When time windows are established to time events

$$A_{it} \leq t_{it}^D \leq B_{it}, \qquad \forall i \in N, t \in T,$$
$$A_{it}^E \leq t_{it}^{ED} \leq B_{it}^E, \qquad \forall i \in N, t \in T,$$

then, constraints (3.24) can be replaced by inequalities

$$t_{it}^{DE} - t_{ju}^D + (B_{it}^E + T_{ijv} - A_{ju}) x_{itjuv} \leq B_{it}^E - A_{ju}, \qquad \forall v \in V, i, j \in N, t, u \in T.$$

These inequalities can be further strengthened as follows (see Proposition 1 in [5]):

$$t_{it}^{DE} - t_{ju}^D + \sum_{v \in V} max\{0, B_{it}^E + T_{ijv} - A_{ju}\} x_{itjuv} \leq B_{it}^E - A_{ju}, \forall i, j \in N, t, u \in T. \tag{3.97}$$

Constraints (3.25) establish time windows for t_{it}^D. For t_{it}^{ED} we assume $A_{it}^E = t - 1$ and $B_{it}^E = t + 1$ since an operation takes at most one time period (day).

For the continuous models, constraints (3.58) can be strengthened in a similar way. We omit the details here. The major difference is related to the computation of time windows $[A_{im}, B_{im}]$

for t^C_{im}, and $[A^E_{im}, B^E_{im}]$ for t^{EC}_{im}. First we set $A_{im} = A^E_{im} = 0$ and $B_{im} = B^E_{im} = T$. By reducing the widths of these time windows we strengthen the resulting inequality. However, since we are dealing with multiple ships, multiple products, and all supply ports also act as demand ports of other products, it is hard to derive tight time windows. Additionally, some preliminary results showed that small improvements in the widths of time windows do not lead to any practical gain.

Next we consider another tightening which use information of the demands to tighten the linking coefficients. For instance, consider inequalities (3.13) in model BD-SSIRP. The unload quantity at period t can be additionally limited by the remaining consumption at that port. That is,

$$q^D_{itvk} \le \min\{C_{vk}, \overline{Q}_{ik}, A\} o^D_{itvk}, \quad \forall v \in V, i \in N, t \in T, k \in K_v : J_{ik} = -1, \tag{3.98}$$

where $A = \max\{R^k_i(\overline{T} - t + 1), \underline{Q}_{ik}\}$.

For the BC-SSIRP model, the corresponding variables, q^C_{itvk}, do not provide information of time of the visit. So we can only limit the demand/consumption for the total time horizon.

Similar reasoning can be applied to inequalities (3.11), (3.12), (3.18)-(3.21). For brevity we give the tightening for the flow and multi-commodity formulations in more detail only.

Consider the arc-load flow models FD-SSIRP and FC-SSIRP. In FD-SSIRP, inequalities (3.70) can be replaced by

$$f^D_{itjuvk} \le \min\{C_{vk}, B1\} x^D_{itjuv}, \qquad \forall v \in V, i, j \in N, t, u \in T, k \in K_v, \tag{3.99}$$

where $B1 = \sum_{j \in N | J_{jk} = -1} \max\{R_{jk}(\overline{T} - u + 1), \underline{Q}_{jk}\}$. In FC-SSIRP, inequalities (3.75) can be replaced by

$$f^C_{imjnvk} \le \min\{C_{vk}, B2\} x^C_{imjnv}, \qquad \forall v \in V, (i, m, j, n) \in S^X_v, k \in K_v, \tag{3.100}$$

where $B2 = \max\{\sum_{j \in N | J_{jk} = -1} R_{jk}\overline{T}, \underline{Q}_{jk}\}$.

Now consider the multi-commodity flow models MD-SSIRP and MC-SSIRP. In MD-SSIRP, inequalities (3.79) can be replaced by

$$v^D_{itjuvkpe} \le min\{C_{vk}, \overline{Q}_{pk}, C1\} x^D_{itjuv}, \ \forall v \in V, i, j, p \in N, t, u, e \in T, k \in K_v : J_{pk} = -1, \tag{3.101}$$

where $C1 = \max\{R_{pk}(\overline{T} - u + 1), \underline{Q}_{pk}\}$. In MC-SSIRP, inequalities (3.82) can be replaced by

$$v^C_{imjnvkpl} \le min\{C_{vk}, \overline{Q}_{pk}, C2\} x^C_{imjnv}, \qquad \forall v \in V, (i, m, j, n) \in S^X_v, (p, l) \in S^A_v, \; k \in K_v : J_{pk} = -1, \tag{3.102}$$

where $C2 = \max\{R_{pk}\overline{T}, \underline{Q}_{pk}\}$.

We can see that $B1$ and $C1$ depend on the time period, while $B2$ and $C2$ do not. This is one of the advantages of the discrete models.

3.5 Computational Experiments

In this section we conduct computational experiments to test and compare the discrete time and the continuous time models. All computations were performed using the optimization software Xpress Optimizer Version 20.00.05 with Xpress Mosel Version 3.0.0, on a computer with an Intel Core 2 Duo processor, with CPU 2.2GHz, and with 4GB of RAM.

We use two sets of instances for the SSIRP with constant consumption rates. The first set consists of 12 real instances from a company in Cape Verde including 2 ships, 4 products and 7 ports. The other set consists of 12 instances from an artificial scenario where the consumption rates of the real instances are doubled as well as the number of ships.

First we describe some characteristics of the instances. The typical planning horizon is two weeks. Here we consider instances with $T = 10$ and $T = 15$. The demand for each product during the planning horizon is, in average, 2.5 times the largest ship tank capacity. The tank capacity at the main ports can cover the demand at that port for a week (without regard the safety stocks). For the small islands typically one or two visits are required. The total number of visits for the tested instances ranged between 12 and 15. The ships have in average 6 tanks.

Computational experiments are conducted to compare the models according to their size, running times and integrality gap without any additional tightening. Based on the information obtained, we select some of the models for further testing. The selected models are used in a branch and cut scheme to solve the two sets of instances.

We also tested the influence of the minimum unload values $\underline{Q}_{ik}$ on solution quality and tractability.

3.5.1 Comparison of the Size of the Models

Now we compare the size of the models without any tightening or addition of cuts. Table 3.1 provides the information of the average number of variables and average number of constraints of the three discrete time and continuous time formulations for a time horizon of 10 and 15 periods (days). Additionally, column "Solved'" gives the number of instances solved to optimality using the default options of Xpress optimizer within a time limit of 3 hours.

For the discrete time models we ignore all variables x^D_{itjuv} with $u > t + 3$, and for the continuous time model we established the upper bound of the number of visits to port i, $\overline{\mu}_i = \underline{\mu}_i + 3$.

Table 3.1: Average size of the tested models.

	Model	$\overline{T}$	—V—	Binary Var.	Cont. Var.	Total Var.	Constraints	Solved
Discrete Models	BD-SSIRP	10	2	3636	1185	4821	22854	10
	FD-SSIRP	10	2	3636	7975	11611	13334	9
	MD-SSIRP	10	2	3636	155955	159591	111668	7
	BD-SSIRP	10	4	7392	2533	9925	74772	4
	FD-SSIRP	10	4	7392	14209	21601	27928	10
	MD-SSIRP	10	4	7392	311975	319367	227748	2
	BD-SSIRP	15	2	5706	1775	7481	36044	5
	FD-SSIRP	15	2	5706	12590	18296	20924	9
	MD-SSIRP	15	2	5706	370570	376276	254543	2
	BD-SSIRP	15	4	11592	3783	15375	159917	4
	FD-SSIRP	15	4	11592	22004	33596	49933	7
	MD-SSIRP	15	4	11592	741240	752832	525598	2
Continuous Models	BC-SSIRP	10	2	2356	606	2962	15288	12
	FC-SSIRP	10	2	2356	5376	7732	8668	12
	MC-SSIRP	10	2	2356	36896	39252	46085	12
	BC-SSIRP	10	4	3278	960	4238	22411	4
	FC-SSIRP	10	4	3278	8000	11278	12511	12
	MC-SSIRP	10	4	3278	41908	45186	55006	3
	BC-SSIRP	15	2	2484	623	3107	16153	6
	FC-SSIRP	15	2	2484	5678	8162	9133	11
	MC-SSIRP	15	2	2484	39074	41558	48726	2
	BC-SSIRP	15	4	3926	1065	4991	27214	4
	FC-SSIRP	15	4	3926	9656	13582	15004	8
	MC-SSIRP	15	4	3926	51004	54930	66596	2

We can see that each continuous time model is smaller than the corresponding discrete time model. Table 3.1 also shows that multi-commodity models are too large and most of the larger instances cannot be solved within the time limit of 3 hours.

Next we study the impact of eliminating some arc-load variables in both types of models. For the discrete time models we eliminate all variables x^D_{itjuv} with $u > t + \alpha$, and for the continuous time models we established the upper bound of the number of visits to port i, as $\overline{\mu}_i = \underline{\mu}_i + \alpha$. If α is small we reduce substantially the set of feasible solutions and it is possible that the instance becomes infeasible. On the other hand if α is large the size of the model increases and the running times tend to be very high. In order to illustrate the effects of α on the optimal solution, we tested the set of 12 real instances with 10 and 15 periods. Each instance was solved for α from 1 to 3. The results are given in Table 3.2. The table gives the number of instances that resulted in the true optimal value using models FD-SSIRP and FC-SSIRP.

Table 3.2: Number of instances where the true optimal solution was obtained. All instances were solved to optimality. We considered $|V| = 2$.

	FD-SSIRP		FC-SSIRP	
α	$\overline{T}$=10	$\overline{T}$=15	$\overline{T}$=10	$\overline{T}$=15
1	5	0	2	0
2	11	2	12	11
3	12	12	12	12

For $\alpha = 1$ the optimal value is worse compared to the true optimal value in most instances. This situation is opposite for $\alpha = 2$. For $\alpha = 3$ we obtain the true optimal value for all the tested instances. A more detailed test (not reported here) revealed that in order to keep the quality of the optimal solution while minimizing the number of variables, for continuous time models, different values of α can be chosen for different ports. Small values of α can be assumed for low activity ports while larger values should be assumed for high activity ones. Additionally, Table 3.2 shows that when the length of the planning horizon is increased the value of α should also increase to obtain the optimal solution.

Figure 3.6 shows the average running times of the arc-load flow models FD-SSIRP and FC-SSIRP (which proved to be the fastest models among all the tested models) when α varies from 1 to 5. It is clear that the running time increases rapidly with the increase of α, and the running times of the discrete time model increase faster than the running time of the continuous time model.

3.5.2 Comparison of the Integrality Gaps

Next we present some computational results in order to compare the integrality gap of the various formulations. The results of the set of real instances are reported in Table 3.3. For each formulation we present the average integrality gap, $gap = \frac{\text{Optimal value - Lower Bound}}{\text{Optimal value}} \times 100$ at the root node for several possible settings. Column N means the original formulation without tightening of constraints and without inclusion of cuts; Column TT means with tightening only; Column C means with inclusion of cuts; and Column (TT+C) means with tightening and inclusion of cuts. When cuts are added we indicate the average number of cuts added (Column Ncuts). Notice that the lower bounds obtained without valid inequalities and tightening are very poor, especially for the arc-load formulations, BD-SSIRP and BC-SSIRP. We can observe

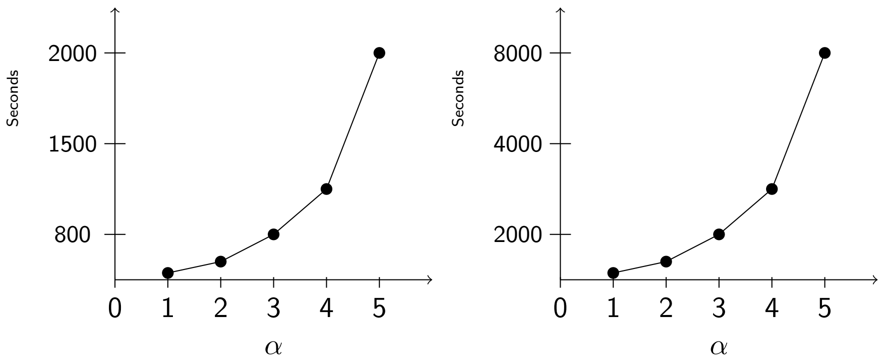

Figure 3.6: Average solution times using the arc-load flow formulations (FC-SSIRP on left and FD-SSIRP on right) on 12 real instances with $\overline{T} = 10$, and $|V| = 2$, when increasing α.

Table 3.3: Average integrality gaps with and without tightening of constraints and inclusion of valid inequalities. We considered $|V| = 2$.

	N		TT		C				TT+C			
Model	$\overline{T} = 10$	$\overline{T} = 15$	$\overline{T} = 10$	$\overline{T} = 15$	$\overline{T} = 10$	Ncuts	$\overline{T} = 15$	Ncuts	$\overline{T} = 10$	Ncuts	$\overline{T} = 15$	Ncuts
BD-SSIRP	57.6	45.3	55.6	43.2	9.2	51	26.1	151.1	7.9	42.3	25.8	108.3
FD-SSIRP	48.9	31.4	47.5	31.1	6.5	60.9	16.1	151.1	3.1	39.1	13.3	94.5
MD-SSIRP	43.3	26	41.3	22.8	6.5	68.3	16.1	151.2	3.1	65.1	13.1	150
BC-SSIRP	57.6	45.3	57.6	45.3	9.2	12.5	26.8	15.3	7.9	12.5	25.8	14.9
FC-SSIRP	48.9	31.4	48.9	31.4	6.5	12.1	16.7	13.8	3.1	10.3	15.1	13.5
MC-SSIRP	43.3	26	41.3	23.6	6.5	15.2	17.4	13.7	3.1	10.3	14.8	13.5

that strengthening the models with the addition of inequalities (3.83), (3.84), (3.90), (3.91) and with the tightening of constraints reduces the integrality gaps considerably. Finally, we observe that the arc-load and the arc-load flow formulations for N and C cases provide essentially the same bounds for both approaches (discrete time and continuous time). With the inclusion of valid inequalities and tightening of constraints the discrete time models provide slightly better gaps than the corresponding continuous time models. This is explained by the fact that, in discrete time models we can provide tighter constraints as explained in Section 3.4.2.

We conduct similar computational experiments for the set of artificial instances with 4 ships, 4 products and 7 ports, where the consumption rate is doubled. Here we report results obtained with the models FD-SSIRP-C and FC-SSIRP only, since the running time was limited to three hours and the multi-commodity formulations are very time consuming.

The results for these two models, including tightening constraints and cuts, are presented in Table 3.4. We give the average initial integrality gap (Gap-I), that is, the average of the integrality gaps at the root node, the average gap provided by Xpress after the three hours limit (Gap-E), and the average running time (Time). We can see that the average initial gap is smaller using FD-SSIRP but the running times are smaller using the continuous model FC-SSIRP.

3.5.3 Impact of Minimum Delivery Quantities

Restrictions on the minimum delivery quantities of each product at each port are considered for the SSIRP with constant consumption rates. In fact, delivering small quantities may result

Table 3.4: Average computational results for FD-SSIRP and FC-SSIRP with $\mid V \mid = 4$.

	FD-SSIRP-C			**FC-SSIRP**		
	Gap-I	Gap-E	Time (sec.)	Gap-I	Gap-E	Time (sec.)
$\overline{T} = 10$	12.9	0	907	13.9	0	476
$\overline{T} = 15$	15.4	5.3	6172	17.8	2.4	5602

in too many port visits. In reality one wants to avoid too many visits to a port due to issues like unpredictable weather conditions and port occupancy. Based on historical data of real instances we conclude that the minimum allowed delivery quantities, $\underline{Q}_{ik}$, are around 40% of the maximum allowed unloading quantities, $\overline{Q}_{ik}$. In order to analyze the real impact of $\underline{Q}_{ik}$, in the objective function value, integrality gap, running time, and the number of branch and bound nodes, we solve the 12 real instances for different values of $\underline{Q}_{ik}$, ranging from 0% to 90% of $\overline{Q}_{ik}$, using the FC-SSIRP model. The results are presented in Figures 3.7 and 3.8 and show that when $\underline{Q}_{ik}$, varies from 0% to 60% the cost increases slowly, but when it is greater than 60% the cost increases significantly. We also observe that time, integrality gap and number of nodes, have small oscillation until 60%, increase significantly between 60% and 80%, and decrease after 80%.

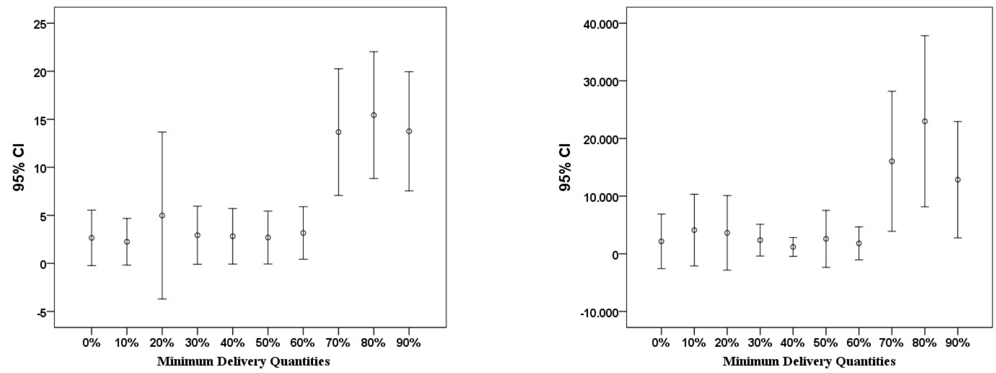

Figure 3.7: Impact of minimum delivery quantities on the integrality gap (left) and number of branch and bound nodes (right).

3.5.4 Comparison of the Running Times and Number of Branch and Bound Nodes

From Section 3.5.1 we see that the multi-commodity formulations are much larger in number of variables and constraints than the arc-load and arc-load flow formulations. However, Section 3.5.2 shows that the reduction in the integrality gap by using the multi-commodity formulations is very small. These two observations lead to the conclusion that the multi-commodity formulations can hardly be competitive compared to the other two formulations. Preliminary results, not reported here, confirm this conclusion. Therefore, in this section we report results for the BD-SSIRP (BC-SSIRP) and FD-SSIRP (FC-SSIRP) models.

A comparison of the running times and number of branch and bound nodes using the BD-SSIRP (BC-SSIRP) and FD-SSIRP (FC-SSIRP) models, for each approach, is shown in Table 3.5. The notation is the same as the one for Table 3.5. For $\overline{T} = 15$, only results with tightening and inclusion of cuts are presented because most of the instances were not solved

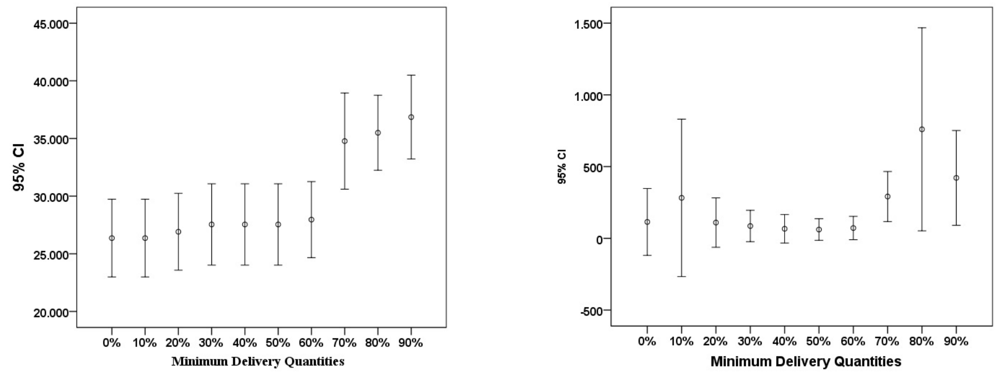

Figure 3.8: Impact of minimum delivery quantities on the solution cost (left) and on the running time (right).

Table 3.5: Average running times and number of branch and bound nodes.

	$\overline{T} = 10$								$\overline{T} = 15$	
	N		TT		C		TT+C		TT+C	
	Time	Nodes	Time	Nodes	Time	Nodes	Time	Nodes	Time	Nodes
BD-SSIRP	743	38391	1090	26993	590	26493	412	26942	6305	37236
FD-SSIRP	1614	32249	1347	16035	619	8537	86	916	3773	37213
BC-SSIRP	487	85695	360	85395	112	29453	84	14839	3091	36976
FC-SSIRP	245	26823	78	4320	84	8120	39	3544	2740	36926

within 3 hours for the remaining cases. The tests were performed for the 12 real instances. We observe that tightening constraints and including cuts is essential when solving the instances. The best results where obtained with the improved (with tightened constraints and cuts) FD-SSIRP and FC-SSIRP models. In fact, only this combination allowed us to solve all the tested instances to optimality. We can see that in several cases the number of branch and bound nodes was smaller using the discrete models. This can be justified by the fact that the discrete time model has, on average, slightly better integrality gaps. However, the continuous time model was clearly faster than the discrete one. If we recall that the size of the continuous model is smaller than the size of the discrete one, and the difference on the average integrality gaps is small, we may conclude that this is the expected behavior of the two models, that is, the continuous model should outperform the discrete model, and this difference tends to be larger when $\overline{T}$ increases.

3.6 Conclusions

We present a real short sea inventory routing problem for fuel oil distribution. We provide two types of formulations. A discrete time model for both time varying and constant consumption, and a continuous time model for constant consumption rates. We discuss different extended formulations for both types of formulations, and valid inequalities that allow us to derive tighter formulations.

All the models proposed were compared according to their size, integrality gap and running time using a commercial software. From this comparison we conclude that: i) the extended formulations based on arc-load flow variables with valid inequalities provide the best compromise between integrality gaps and size of model; ii) the discrete time models tend to provide better bounds. However, the running times using the discrete time models are in general worse than the running times using the continuous time model.

From i) and ii) we conclude that, for the constant consumption rate case, the continuous time arc-load flow model with valid inequalities is the best option among all the tested ones to solve small real sized instances. With this formulation we solved instances with up to 15 days to optimality.

Appendix A: glossary of problem and model acronyms

Problem acronyms:

SSIRP:	Short Sea Inventory Routing Problem with constant consumption rates.
SSIRP-V:	Short Sea Inventory Routing Problem with Varying consumption rates.

Model acronyms:

BD-SSIRP-V:	Basic arc-load Discrete time model for the SSIRP-V.
BD-SSIRP:	Basic arc-load Discrete time model for the SSIRP.
BC-SSIRP:	Basic arc-load Continuous time model for the SSIRP.
FD-SSIRP:	Arc-load Flow Discrete time model for the SSIRP.
FC-SSIRP:	Arc-load Flow Continuous time model for the SSIRP.
MD-SSIRP:	Multi-commodity arc-load Discrete time model for the SSIRP.
MC-SSIRP:	Multi-commodity arc-load Continuous time model for the SSIRP.

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Paper III

A. Agra, M. Christiansen, A. Delgado and L. Simonetti:

Hybrid heuristics for a short sea inventory routing problem

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Chapter 4

Hybrid heuristics for a short sea inventory routing problem

Abstract

We consider a fuel oil distribution problem where an oil company is responsible for the routing and scheduling of ships between ports such that the demand for various fuel oil products is satisfied during the planning horizon. The production/ consumption rates are given and assumed to be constant. The objective is to determine distribution policies that minimize the total cost (routing and operating costs), while inventory levels are maintained within their limits. We propose an arc-load flow formulation for the problem which is tightened with valid inequalities. In order to obtain good feasible solutions for planning horizons of several months, we compare different hybridization strategies. Computational results are reported for real small-size instances.

Keywords: Maritime Transportation; Hybrid heuristics; Inventory Routing; Mixed Integer Programming.

4.1 Introduction

Maritime transportation is the major mode of transportation of goods worldwide. The importance of this mode of transportation is obvious for the long distance transportation of cargoes but it is also crucial in local economies where the sea is the natural link between the local developed regions, such as countries formed by archipelagoes. When a company has the responsibility of coordinating the transportation of goods with the inventories at the ports, the underlying planning problem is a maritime inventory routing problem. Such problems are very complex. Usually modest improvements in the supply chain planning can translate into significant cost savings.

In this paper we consider a real maritime Short Sea Inventory Routing Problem (SSIRP) occurring in the archipelago of Cape Verde. An oil company is responsible for the inventory management of different oil products in several tanks located in the main islands. Fuel oil products are imported and delivered to specific islands and stored in large supply storage tanks, so the inventory management does not need to be considered in these tanks. From these islands, fuel oil products are distributed among all the inhabited islands using a small heterogeneous fleet of ships with dedicated tanks. These products are stored in consumption storage tanks with limited capacity. Consumption rates are assumed to be given and constant over a time horizon of several months. Some ports have both supply tanks for some products and consumption tanks of other products.

We have witnessed an increased interest in studying optimization problems within maritime transportation [14, 15, 16] and, in particular, in the last fifteen years, problems combining routing and inventory management [8, 12]. These problems are often called Maritime Inventory Routing Problems (MIRPs). Most of the published MIRP contributions are based on real cases from the industry, see for the single product case [11, 21, 22, 24] and for the multiple products case [7, 13, 28, 30, 33, 35].

This SSIRP is addressed in a companion paper [4] where different mathematical formulations are discussed and compared for the SSIRP considering a shorter time horizon. There, two main approaches to model the problem are considered. One uses a continuous time model where an index indicating the visit number to a particular port is added to most of the variables. This approach was used in [7], [11] and [33] for MIRPs where the production and/or consumption rates are considered given and fixed during the planning horizon. The other approach consists of using a model that combines a discrete and continuous time where the discrete time corresponds to an artificial discretization of the continuous time. Discrete time models have been developed in [2, 22, 23, 24, 28, 30, 34] to overcome the complicating factors with time varying production and consumption rates. In addition, for each approach two new extended formulations are tested in [4].

In [3], the SSIRP for short-term planning is considered. For the short-term plans demand orders are considered, that is, fixed amounts of oil products that must be delivered at each port within a fixed period of time. These orders are determined from the initial stock levels and the consumption rates and lead to a problem with varying demands (corresponding to the demand orders). Several key issues taken into account in the short-term problem are relaxed here or incorporated indirectly in the data. For instance, port operating time windows that are essential in the short-term plan are ignored here. Otherwise, the problems considered originate from the same company in the same region. These problems are solved using the same commercial solver we use here, considering a formulation which is improved by the strengthening of defining inequalities and the inclusion (through separation) of valid inequalities. In [7] a problem similar to the SSIRP is considered with constant consumption rates and dedicated compartments in the ships.

In this paper we develop and compare different hybrid heuristics for the SSIRP. As discussed in [8, 34], most combined maritime routing and inventory management problems described in the literature have particular features and characteristics, and tailor-made methods are developed to solve the problems [12]. These methods are often based on heuristics or decomposition techniques. Recent hybrid heuristics that use MIP solvers as a black-box tool have been proposed. Here we consider and combine three hybrid heuristics: Rolling Horizon (RH), Local Branching (LB) and Feasibility Pump (FP). In RH heuristics the planning horizon is split into smaller sub-horizons. Then, each limited and tractable mixed integer problem is solved to optimality. Within maritime transportation RH heuristics have been used in [25, 28, 32, 33, 34]. Local Branching (LB) was introduced by Fichetti and Lodi [19] to improve feasible solutions. LB heuristics search for local optimal solutions by restricting the number of binary variables that are allowed to change their value in the current solution. Feasibility Pump (FP) was introduced by Fischetti, Glover and Lodi [18] to find initial feasible solutions for MIP problems.

Computational experiments reported in Section 4.6 show that a combined heuristic using the three approaches outperformed the other tested heuristics and, in particular, outperformed the most used approach within MIRPs, the RH heuristic.

To solve each subproblem we consider the arc-load flow (ALF) formulation introduced in

[4], since this was the model with the best performance among all the tested models for this problem with short time horizons. The ALF formulation is improved by a pre-computation of estimates for the number of visits to each port, and with the inclusion of valid inequalities. In particular, we introduce a new family of clique inequalities for MIRPs when continuous time models are used.

The main contributions of this paper, the heuristic strategies and the valid inequalities, can easily be used in other MIRPs.

The remainder of this paper is organized as follows. In Section 4.2, we describe the real problem. The arc-load flow formulation is presented in Section 4.3 and strategies to tighten the formulation are discussed in Section 4.4. In Section 4.5 we describe several hybrid heuristics. The computational experimentations are reported in Section 4.6. Final conclusions are given in Section 4.7.

4.2 Problem description

In Cape Verde, fuel oil products are imported and delivered to specific islands and stored in large supply storage tanks. From these islands, fuel oil products are distributed among all the inhabited islands using a small heterogeneous fleet of ships. The products are stored in consumption storage tanks. Two ports have both supply tanks for some products and consumption tanks of other products, while the remaining ports have only consumption tanks. Not all islands consume all products. The consumptions (which are usually forecasted) are assumed to be constant over the time horizon. It is assumed that each port can receive at most one ship at a time and a minimum interval between the departure of a ship and the arrival of the next one must be considered. Waiting times are allowed.

Each ship has a specified load capacity, fixed speed and cost structure. The cargo hold of each ship is separated into several cargo tanks. The products can not be mixed, so we assume that the ships have dedicated tanks to particular products.

The traveling times between two consecutive ship visits are an estimation based on practical experience. Additionally, we consider set-up times for the coupling and decoupling of pipes, and operating times.

To prevent a ship from delivering small quantities, minimum delivery quantities are considered. The maximum delivered quantity is imposed by the capacity of the consumption storage tank. Safety stocks are considered at consumption tanks. As the capacity of the supply tanks is very large when compared to the total demand over the horizon, we omit the inventory aspects for these tanks.

In each problem instance we are given the initial stock levels at the consumption tanks, initial ship positions (which can be a point at sea) and quantities on board each ship. The inter-island distribution plan consists of designing routes and schedules for the fleet of ships including determining the number of visits to each port and the (un)loading quantity of each product at each port visit. The plan must satisfy the safety stocks of each product at each island and the capacities of the ship tanks. The transportation and operation costs of the distribution plan must be minimized over a finite planning horizon.

4.3 Mathematical model

In [4] a comparison of six different formulations for the SSIRP for a shorter time horizon is given. Three of those formulations consider a time discretization and the other three consider continuous time. For each time option the following formulations are considered: an arc-load formulation, where the model keeps only track of the information of the load on board each ship compartment in each port visit; an arc-load flow formulation, where new variables are used to keep the information about the quantity of each product in each compartment when a ship leaves a port en route to the next one; and a multi-commodity formulation, where the flow on each arc is disaggregated accordingly to its destination. That comparison led to the choice of the continuous time arc-load flow formulation. In this section we present that arc-load flow formulation.

Routing constraints

Let V denote the set of ships. Each ship $v \in V$ must depart from its initial position in the beginning of the planning horizon. The set of ports is denoted by N. For each port we consider an ordering of the visits accordingly to the time of the visit. The ship paths are defined on a network where the nodes are represented by a pair (i, m), where i is the port and m represents the m^{th} visit to port i. Direct ship movements (arcs) from port arrival (i, m) to port arrival (j, n) are represented by (i, m, j, n).

We define S^A as the set of possible port arrivals (i, m), S_v^A as the set of ports that may be visited by ship v, and set S_v^X as the set of all possible movements (i, m, j, n) of ship v.

For the routing we define the following binary variables: x_{imjnv} is 1 if ship v sails from port arrival (i, m) directly to port arrival (j, n), and 0 otherwise; x_{oimv} indicates whether ship v sails directly from its initial position to port arrival (i, m) or not; w_{imv} is 1 if ship v visits port i at arrival (i, m), and 0 otherwise; z_{imv} is equal to 1 if ship v ends its route at port arrival (i, m), and 0 otherwise; z_{ov} is equal to 1 if ship v is not used and 0 otherwise; y_{im} indicates whether a ship is visiting port arrival (i, m) or not.

$$\sum_{(i,m)\in S_v^A} x_{oimv} + z_{ov} = 1, \qquad \forall v \in V, \tag{4.1}$$

$$w_{imv} - \sum_{(j,n)\in S_v^A} x_{jnimv} - x_{oimv} = 0, \qquad \forall v \in V, (i, m) \in S_v^A, \tag{4.2}$$

$$w_{imv} - \sum_{(j,n)\in S_v^A} x_{imjnv} - z_{imv} = 0, \qquad \forall v \in V, (i, m) \in S_v^A, \tag{4.3}$$

$$\sum_{v\in V} w_{imv} = y_{im}, \qquad \forall (i, m) \in S^A, \tag{4.4}$$

$$y_{i(m-1)} - y_{im} \geq 0, \qquad \forall (i, m) \in S^A : m > 1, \tag{4.5}$$

$$x_{oimv}, w_{imv}, z_{imv} \in \{0, 1\}, \qquad \forall v \in V, (i, m) \in S_v^A, \tag{4.6}$$

$$x_{imjnv} \in \{0, 1\}, \qquad \forall v \in V, (i, m, j, n) \in S_v^X, \tag{4.7}$$

$$z_{ov} \in \{0, 1\}, \qquad \forall v \in V, \tag{4.8}$$

$$y_{im} \in \{0, 1\}, \qquad \forall (i, m) \in S^A. \tag{4.9}$$

Equations (4.1) ensure that each ship either departs from its initial position and sails towards another port or the ship is not used. Equations (4.2) and (4.3) are the flow conservation constraints, ensuring that a ship arriving at a port also leaves that port or ends its route. Constraints (4.4) ensure that one ship only visits port (i, m) if y_{im} is equal to one. Constraints (4.5) state that if port i is visited m times, then it must also have been visited $m - 1$ times. Constraints (4.6)-(4.9) define the variables as binary.

Load and unload constraints

Let K represent the set of products and K_v represent the set of products that ship v can transport. Not all ports consume all products. Parameter J_{ik} is 1 if port i is a supplier of product k; -1 if port i is a consumer of product k, and 0 if i is neither a consumer nor a supplier of product k. The quantity of product k on board ship v at the beginning of the planning horizon is given by Q_{vk}, and C_{vk} is the capacity of the compartment of ship v dedicated for product k. The minimum and the maximum discharge quantities of product k at port i are given by $\underline{Q}_{ik}$ and $\overline{Q}_{ik}$, respectively.

In order to model the loading and unloading constraints, we define the following binary variables: o_{imvk} is equal to 1 if product k is loaded onto or unloaded from ship v at port visit (i,m), and 0 otherwise. In addition, we define the following continuous variables: q_{imvk} is the amount of product k loaded onto or unloaded from ship v at port visit (i,m), f_{imjnvk} denotes the amount of product k that ship v transports from port visit (i,m) to port visit (j,n), and f_{oimvk} gives the amount of product k that ship v transports from its initial position to port visit (i,m).

The loading and unloading constraints are given by:

$$f_{oimvk} + \sum_{(j,n)\in S_v^A} f_{jnimvk} + J_{ik}q_{imvk} = \sum_{(j,n)\in S_v^A} f_{imjnvk}, \qquad \forall v \in V, (i,m) \in S_v^A, k \in K_v \tag{4.10}$$

$$f_{oimvk} = Q_{vk}x_{oimv}, \qquad \forall v \in V, (i,m) \in S_v^A, k \in K_v, \tag{4.11}$$

$$f_{imjnvk} \le C_{vk}x_{imjnv}, \qquad \forall\ v \in V, (i,m,j,n) \in S_v^X, k \in K_v, \tag{4.12}$$

$$0 \le q_{imvk} \le C_{vk}o_{imvk}, \qquad \forall v \in V, (i,m) \in S_v^A, k \in K_v : J_{ik} = 1, \tag{4.13}$$

$$\underline{Q}_{ik}o_{imvk} \le q_{imvk} \le \overline{Q}_{ik}o_{imvk}, \qquad \forall v \in V, (i,m) \in S_v^A, k \in K_v : J_{ik} = -1, \tag{4.14}$$

$$\sum_{k\in K_v} o_{imvk} \ge w_{imv}, \qquad \forall v \in V, (i,m) \in S_v^A, \tag{4.15}$$

$$o_{imvk} \le w_{imv}, \qquad \forall v \in V, (i,m) \in S_v^A, k \in K_v, \tag{4.16}$$

$$f_{imjnvk} \ge 0, \qquad \forall v \in V, (i,m,j,n) \in S_v^A, k \in K_v, \tag{4.17}$$

$$f_{oimvk}, q_{imvk} \ge 0, \qquad \forall v \in V, (i,m) \in S_v^A, k \in K_v, \tag{4.18}$$

$$o_{imvk} \in \{0,1\}, \qquad \forall v \in V, (i,m) \in S_v^A, k \in K_v. \tag{4.19}$$

Equations (4.10) are the flow conservation constraints. Equations (4.11) determine the quantity on board when ship v sails from its initial port position to port arrival (i,m). Constraints (4.12) require that the vehicle capacity is obeyed. Constraints (4.13) impose an upper bound on the quantity loaded at a supply port. Constraints (4.14) impose lower and upper limits on the unloaded quantities. Constraints (4.15) ensure that if ship v visits port arrival (i,m), then at least one product must be (un)loaded. Constraints (4.16) ensure that if ship v (un)loads one product at visit (i,m), then w_{imv} must be one. Constraints (4.17)-(4.19) are the non-negativity and integrality constraints.

Time constraints

In order to keep track of the inventory level it is necessary to determine the start and the end times at each port arrival. We define the following parameters: T_{ik}^Q is the time required to load/unload one unit of product k at port i; T_{ik}^S is the set-up time required to operate product k at port i. T_{ijv} is the traveling time between port i and j by ship v; T_{iv}^O indicates the traveling time required by ship v to sail from its initial position to port i; T_i^B is the minimum interval between the departure of one ship and the next arrival at port i. T is the length of the time horizon. Given the start time t_{im} and end time t^E_{im} variables for port arrival (i,m), the time constraints can be written as:

$$t^E_{im} \geq t_{im} + \sum_{v \in V} \sum_{k \in K_v} T^Q_{ik} q_{imvk} + \sum_{v \in V} \sum_{k \in K_v} T^S_{ik} o_{imvk}, \qquad \forall (i,m) \in S^A, \tag{4.20}$$

$$t_{im} - t^E_{i(m-1)} - T^B_i y_{im} \geq 0, \qquad \forall (i,m) \in S^A : m > 1, \tag{4.21}$$

$$t^E_{im} + T_{ijv} - t_{jn} \leq T(1 - x_{imjnv}), \qquad \forall v \in V, (i,m,j,n) \in S^X_v, \tag{4.22}$$

$$\sum_{v \in V} T^O_{iv} x_{oimv} \leq t_{im}, \qquad \forall (i,m) \in S^A, \tag{4.23}$$

$$t_{im}, t^E_{im} \geq 0, \qquad \forall (i,m) \in S^A. \tag{4.24}$$

Constraints (4.20) define the end time of service at port visit (i,m). Constraints (4.21) impose a minimum interval between two consecutive visits at port i. Constraints (4.22) relate the end time of port visit (i,m) to the start time of port visit (j,n) when ship v sails directly from port visit (i,m) to (j,n). Constraints (4.23) ensure that if ship v travels from its initial position directly to port visit (i,m), then the start time is at least the traveling time between the two positions. Constraints (4.24) define the continuous time variables.

Inventory constraints

The inventory constraints are considered for each unloading port. They ensure that the stock levels are within the corresponding bounds and link the stock levels to the (un)loaded quantities.

For each consumption port i, and for each product k, the consumption rate, R_{ik}, the minimum $\underline{S}_{ik}$, the maximum $\overline{S}_{ik}$ and the initial stock S^0_{ik} levels, are given. The parameter $\overline{\mu}_i$ denotes the maximum number of visits at port i.

We define the nonnegative continuous variables s_{imk} and s^E_{imk} indicating the stock levels at the start and at the end of port visit (i,m) for product k, respectively. The inventory constraints are as follows:

$$s_{i1k} = S^0_{ik} - R_{ik} t_{i1}, \qquad \forall i \in N, k \in K : J_{ik} = -1, \tag{4.25}$$

$$s^E_{imk} = s_{imk} + \sum_{v \in V} q_{imvk} - R_{ik}(t^E_{im} - t_{im}), \qquad \forall (i,m) \in S^A, k \in K : J_{ik} = -1, \tag{4.26}$$

$$s_{imk} = s^E_{i(m-1)k} - R_{ik}(t_{im} - t^E_{i(m-1)}), \qquad \forall (i,m) \in S^A : m > 1, k \in K : J_{ik} = -1, \tag{4.27}$$

$$\underline{S}_{ik} \leq s_{imk}, s^E_{imk} \leq \overline{S}_{ik}, \qquad \forall (i,m) \in S^A, k \in K : J_{ik} = -1, \tag{4.28}$$

$$\underline{S}_{ik} \leq s^E_{i\overline{\mu}_i k} - R_{ik}(T - t^E_{i\overline{\mu}_i}) \leq \overline{S}_{ik}, \qquad \forall i \in N, k \in K : J_{ik} = -1. \tag{4.29}$$

Equations (4.25) calculate the stock level of each product at the first visit. Equations (4.26) calculate the stock level of each product when the service ends at port visit (i,m). Equations (4.27) relate the stock level at the start of port visit (i,m) to the stock level at the end of port visit $(i,m-1)$. The upper and lower bounds on the stock levels are ensured by constraints (4.28)-(4.29).

Objective function

The objective is to minimize the total routing costs including traveling, operating and set-up costs. The traveling cost of ship v from port i to port j is denoted by C^T_{ijv}, while C^T_{oiv} represents the traveling cost of ship v from its initial port positions to port i. The set-up cost of product k at port i is denoted by C^O_{ik}. The objective function is as follow:

$$\sum_{v\in V}\sum_{(i,m,j,n)\in S^X_v} C^T_{ijv}x_{imjnv} + \sum_{v\in V}\sum_{(i,m)\in S^A_v} C^T_{oiv}x_{oimv} + \sum_{v\in V}\sum_{(i,m)\in S^A_v}\sum_{k\in K_v} C^O_{ik}o_{imvk}. \quad (4.30)$$

The formulation defined by (4.1)-(4.30) is denoted by F-SSIRP, and the feasible set will be denoted by X.

4.4 Tightening the formulation

Tightening the formulation provided in the previous section is essential to speed up the solution approaches (Branch and Bound and hybrid heuristics), and to provide tighter bounds that will be used in Section 4.6 to evaluate the quality of the tested heuristics. The tightening is done by including new inequalities. Many families of inequalities were tested. Here we present only the ones that provided best results from a preliminary study.

4.4.1 Tightening time constraints

Time constraints (4.22) linking the time variables with the routing variables are very weak, see Desrosiers, Dumas, Solomon, and Soumis (1995). Parameter T works as a big M constant. An approach to tighten such constraints is to establish time windows to the time events.

$$A_{im} \le t_{im} \le B_{im}, \qquad \forall (i,m) \in S^A, \quad (4.31)$$

$$A^E_{im} \le t^E_{im} \le B^E_{im}, \qquad \forall (i,m) \in S^A. \quad (4.32)$$

Then, constraints (4.22) can be replaced by the stronger inequalities

$$t^E_{im} - t_{jn} + (B^E_{im} + T_{ijv} - A_{jn})x_{imjnv} \le B^E_{im} - A_{jn}.$$

These inequalities can be further strengthened as follows (see Proposition 1 in [5]):

$$t^E_{im} - t_{jn} + \sum_{v\in V|(i,m,j,n)\in S^X_v} max\{0, B^E_{im} + T_{ijv} - A_{jn}\}x_{imjnv} \le B^E_{im} - A_{jn}, \forall (i,m),(j,n) \in S^A. \quad (4.33)$$

One can take $A_{im} = A^E_{im} = 0$ and $B_{im} = B^E_{im} = T$. However, by reducing the widths of the time windows we strengthen inequalities (4.33). In this SSIRP we are dealing with multiple ships, multiple products, and all supply ports also act as demand ports of other products. Because of this characteristics it is hard to derive tight time windows.

For simplicity, we provide only those time windows formulas that proved to be most effective for our case. Other rules can be derived adapting the ones given in [10] for the single item case. Since inventory aspects are only relevant for consumption tanks, and since all the loading ports of certain products are also consumption ports of other products, time windows are established based on the unloading products only.

The start of time windows are computed as follows:

$$A_{im} = min_{v\in V}\{T^O_{iv}\} + (m-1) * \left(T^B_i + min_{k\in K|J_{ik}=-1}\left\{T^Q_{ik}\underline{Q}_{ik} + T^S_{ik}\right\}\right),$$

$$A^E_{im} = min_{v\in V}\{T^O_{iv}\} + (m-1) * T^B_i + m * min_{k\in K|J_{ik}=-1}\left\{T^Q_{ik}\underline{Q}_{ik} + T^S_{ik}\right\},$$

and the end of time windows are computed as follows:

$$B_{im} = min\left\{T, min_{k\in K|J_{ik}=-1}\left\{\left(S^0_{ik} + (m-1)*\overline{S}_{ik} - \underline{S}_{ik}\right)/R_{ik} - T^S_{ik}\right\}\right\},$$
$$B^E_{im} = min\left\{T, min_{k\in K|J_{ik}=-1}\left\{\left(S^0_{ik} + m*\overline{S}_{ik} - \underline{S}_{ik}\right)/R_{ik} - T^S_{ik}\right\} - T^B_i\right\}.$$

The end of time windows can be further strengthened. Let $\underline{\mu_i}$ denote a lower bound on the number of visits to port i, $i \in N$ (see in Section 4.4.2 how to compute these parameters). If $m \leq \underline{\mu_i}$, then T in the B_{im} formula given above can be replaced by

$$T - (\underline{\mu_i} - m) * T^B_i - (\underline{\mu_i} - m + 1) * min_{k\in K|J_{ik}=-1}\left\{T^Q_{ik}\underline{Q}_{ik} + T^S_{ik}\right\},$$

and, if $m < \underline{\mu_i}$, then T in the B^E_{im} formula can be replaced by

$$T - (\underline{\mu_i} - m) * \left\{T^B_i + min_{k\in K|J_{ik}=-1}\left\{\underline{Q}_{ik}T^Q_{ik} + T^S_{ik}\right\}\right\}.$$

4.4.2 Lower bounds on the number of visits

A common approach to tighten formulations for routing problems is to include constraints imposing a minimum number of visits to each node. The impact on the reduction of the integrality gap is usually high. Equations

$$y_{i\underline{\mu_i}} = 1, \forall i \in N \tag{4.34}$$

can be added to each model. These parameters $\underline{\mu_i}$ can be computed from the inventory information and traveling times. However, since the traveling times between islands are small, the number of visits is better estimated through the inventory information and storage capacities (at ships and ports).

For each port $i \in N$ where product k is unloaded, $J_{ik} = -1$, let

$$D^N_{ik} = max\{T \times R_{ik} - S^0_{ik} + \underline{S}_{ik}, \quad \underline{Q}_{ik}\}$$

denote the net consumption over the time horizon. The minimum number of visits to port i for unloading product k is given by

$$\underline{\lambda}_{ik} = \left\lceil \frac{D^N_{ik}}{\overline{Q}_{ik}} \right\rceil.$$

In the real problem, each product has a single origin. As inventory management at supply tanks is disregarded, the minimum number of visits to load a product can be estimated using the total consumption supplied by that origin. The consumption of that product must be satisfied either from that port or from the quantity in the ship tanks at the beginning of the planning horizon.

For each product $k \in K$, loaded at port $i \in N$ ($J_{ik} = 1$) let

$$D^N_{ik} = \sum_{j\in N|J_{jk}=-1} (T \times R_{jk} - S^0_{jk} + \underline{S}_{jk}),$$

denote the net consumption of this product over the time horizon. The minimum number of loadings of product k at port i is given by

$$\underline{\lambda}_{ik} = \left\lceil \frac{D^N_{ik} - \sum_{v\in V} Q_{vk}}{max\{C_{vk} : v \in V\}} \right\rceil.$$

A lower bound on the total number of visits to port $i \in N$ can be given by the following equation:

$$\underline{\mu}_i = max\{\underline{\lambda}_{ik} : k \in K\}. \tag{4.35}$$

Better bounds can be obtained by solving subproblems for each port. A subproblem is solved for the consumption products at the port and, if the port is also a supplier of other products, another subproblem is solved for the supply products.

Although the subproblems are NP-hard, they can be solved very quickly using a commercial software.

First we state the subproblem for consumption products. All the routing constraints are ignored in the subproblems. For these subproblems associated to each port the inventory and time constraints are the same as for the original model. The ship capacity for each product is overestimated by the maximum of the ship capacities for that product.

Let $\overline{C}_k = max\{C_{vk} : v \in V, k \in K_v\}$. For each port i let $M_i = \{1, 2, \cdots, \overline{\mu}_i\}$. The subproblem is defined as follows:

$$NV^D(i) : min \sum_{m \in M_i} y_{im} \tag{4.36}$$

$$s.t.$$

$$\overline{q}_{imk} \leq \overline{C}_k \overline{o}_{imk}, \quad \forall m \in M_i, k \in K, J_{ik} = -1 \tag{4.37}$$

$$\underline{Q}_{ik} \overline{o}_{imk} \leq \overline{q}_{imk} \leq \overline{Q}_{ik} \overline{o}_{imk}, \quad \forall m \in M_i, k \in K : J_{ik} = -1, \tag{4.38}$$

$$\overline{o}_{imk} \leq y_{im}, \quad \forall m \in M_i, k \in K : J_{ik} = -1, \tag{4.39}$$

$$\text{Constraints } (4.25) - (4.29) \text{ for node } i$$

$$\text{Constraints } (4.20), (4.21), (4.24) \text{ for node } i$$

$$y_{im} \in \{0, 1\}, \quad \forall m \in M_i, \tag{4.40}$$

$$\overline{o}_{imk} \in \{0, 1\}, \quad \forall m \in M_i, k \in K : J_{ik} = -1, \tag{4.41}$$

$$\overline{q}_{imk} \geq 0, \quad \forall m \in M_i, k \in K : J_{ik} = -1, \tag{4.42}$$

where $\overline{o}_{imk} = \sum_{v \in V} o_{imkv}$, $\overline{q}_{imk} = \sum_{v \in V} q_{imkv}$.

The objective function (4.36) minimizes the number of visits at port i. Constraints (4.37) - (4.39) have a similar meaning as constraints (4.13), (4.14), (4.16), only now the ship is ignored and an overestimation of the ship capacities is used.

If port i is also a supplier, for we define the following subproblem, $NV^S(i)$, where only the ship tank capacities are considered.

$$min\{\sum_{v \in V} u_{iv} : \sum_{v \in V} C_{vk} u_{iv} \geq \sum_{j \in N : J_{jk} = -1} D^N_{jk} - \sum_{v \in V} Q_{vk}, \forall k \in K : J_{ik} = 1, u_{iv} \in \mathbb{Z}_+, v \in V\},$$

where u_{iv} indicates the number of visits of ship v to port i.

If port i is simultaneously a consumption and a supply port, the minimum number of visits is the maximum between $NV^D(i)$ and $NV^S(i)$. These two subproblems will be called port subproblems.

4.4.3 Integer knapsack inequalities

Inequalities from knapsack relaxations have previously been used for MIRPs, see for instance [24, 27, 34].

Let $D_k(S)$ denote the total demand of product k, from ports in S during the planning horizon, where $S \subseteq N$ and $J_{ik} = -1$ for all $i \in S$. Hence, $D_k(S) = \sum_{i \in S} T \times R_{ik}$. Let $ND_k(S)$ denote the amount of demand $D_k(S)$ that must be transported from ports in $N \setminus S$. That is, $ND_k(S) = D_k(S) - \sum_{v \in V} Q_{vk} - \sum_{i \in S}(S^0_{ik} - \underline{S}_{ik})$. Then, the following integer set is a relaxation of X :

$$RX = \left\{ \chi \in \mathbb{Z}_+^{|V|} : \sum_{v \in V} C_{vk} \chi_v \geq ND_k(S) \right\}.$$

where

$$\chi_v = \sum_{(i,m) \in S_v^A | i \in N \setminus S} \; \sum_{(j,n) \in S_v^A | j \in S} x_{imjnv},$$

denotes the number of times ship v visits a port in S coming from a port not in S during the planning horizon T.

Valid inequalities for RX are valid for X. A particular case of these inequalities is the following Gomory cut

$$\sum_{v \in V} \; \sum_{(i,m) \in S_v^A | i \in N \setminus S} \; \sum_{(j,n) \in S_v^A | j \in S} \left\lceil \frac{C_{vk}}{Q} \right\rceil x_{imjnv} \geq \left\lceil \frac{ND_k(S)}{Q} \right\rceil, \tag{4.43}$$

where Q can be any positive number. We take $Q = \overline{C}_k$.

However, when $|\ V\ | = 2$ the convex hull of RX can be completely described in polynomial time, see [6]. When $|\ V\ | > 2$ facet defining inequalities for restrictions of RX to two variables χ_v can be lifted using the lifting function ω_3 presented in [6]. This approach was used in [3]. Here we provide an example.

Example 4.4.1. *Let $N = \{1, 2, \cdots, 7\}$, $V = \{1, 2, 3, 4\}$, $K = \{1, 2, 3, 4\}$. Fix port $i = 6$, and consider the capacities of the compartments dedicated to product $k = 1$: $C_{11} = 900$, $C_{21} = 600$, $C_{31} = 920$, and $C_{41} = 700$. Suppose that for $i = 6$ and $k = 1$ with $J_{61} = -1$, we have $ND_{61} = 3675$. The following relaxation is derived*

$$RX = \{\chi_v \in \mathbb{Z}_+ : 900\chi_1 + 600\chi_2 + 920\chi_3 + 700\chi_4 \geq 3675\}.$$

Inequality $3\chi_1 + 2\chi_2 \geq 13$ is a facet-defining inequality for RX restricted to $\chi_3 = \chi_4 = 0$. The lifting function associated with this inequality is:

$$\begin{aligned} \varphi(z) \quad &= \max \quad 13 - 3\chi_1 - 2\chi_2 \\ s.\ t. \quad & \qquad\qquad 900\chi_1 + 600\chi_2 \geq 3675 - z, \\ & \qquad\qquad \chi_1, \chi_2 \in \mathbb{Z}_+. \end{aligned}$$

In order to lift simultaneously the coefficients of χ_3 and χ_4, the lifting function $\varphi(z)$ can be overestimated by the subadditive lifting function ω_3 described in [6]. Both functions are depicted in Figure 4.1. Then the lifted inequality $3\chi_1 + 2\chi_2 + \omega_3(920)\chi_3 + \omega_3(700)\chi_4 \geq 13 \Leftrightarrow 3\chi_1 + 2\chi_2 + 3.26667\chi_3 + 3\chi_4 \geq 13$ is valid for RX.

Notice that if only three variables are considered then one can use $\varphi(z)$ instead of ω_3 which gives a better coefficient for χ_3 since $\varphi(920) = 3$.

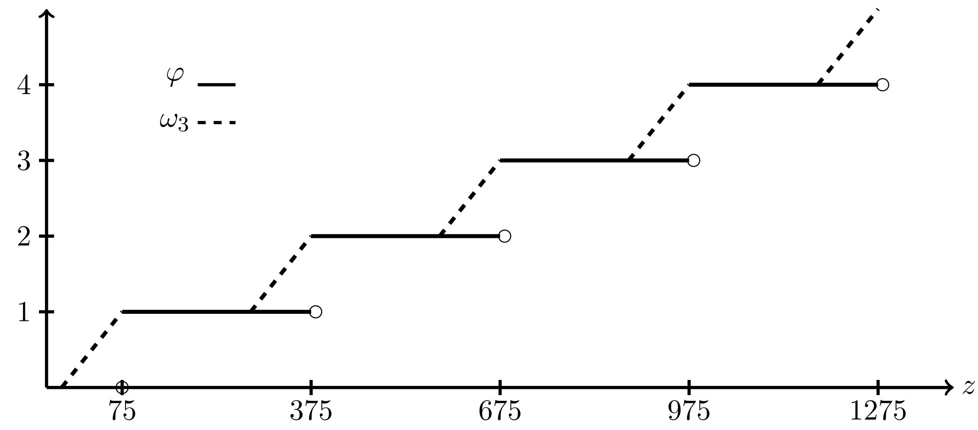

Figure 4.1: Lifting function φ and subadditive function ω_3.

Similar knapsack inequalities can be derived for loading ports and for relaxations using the operating variables o_{imvk} instead of the traveling variables. For brevity we omit those inequalities.

4.4.4 Clique inequalities

The name *clique inequalities* has been used for different families of valid inequalities for vehicle routing problems. Here we introduce a family of clique inequalities which can be regarded as a generalization of the subtour elimination constraints (SEC):

$$x_{imjnv} + x_{jnimv} \leq 1$$

Although subtour elimination constraints including more than two variables can be useful to improve the integrality gap, our experience showed that good computational results can be obtained using SEC including only two variables. These inequalities can be regarded a particular case of clique inequalities on a given conflict graph. Consider the conflict graph $G = (\mathcal{N}, E)$, where each node in $\mathcal{N}$, denoted by (i, m, j, n, v), corresponds to a variable x_{imjnv}, and there is an edge in E between two nodes if the corresponding variables cannot be set simultaneously to one (the two nodes are in conflict).

Definition 4.4.2. *Let $G = (\mathcal{N}, E)$ be a conflict graph. Then we define the following pairs of incompatible variables:*

(i) x_{imjnv} *and* x_{jnimv}, $\forall v \in V, (i, m, j, n) \in S_{X_v}$.

(ii) x_{imjnv_1} *and* x_{imlwv_2}, $\forall v_1, v_2 \in V, (i, m, j, n) \in S_{X_{v_1}}, (i, m, l, w) \in S_{X_{v_2}}$.

(iii) x_{lwjnv_1} *and* x_{imjnv_2}, $\forall v_1, v_2 \in V, (l, w, j, n) \in S_{X_{v_1}}, (i, m, j, n) \in S_{X_{v_2}}$.

(iv) x_{lwjnv_1} *and* x_{jnimv_2}, $\forall v_1, v_2 \in V : v_1 \neq v_2, (l, w, j, n) \in S_{X_{v_1}}, (j, n, i, m) \in S_{X_{v_2}}$.

As consequence of the above discussion we have the following result:

Proposition 4.4.1. *If $C \subset \mathcal{N}$ is a clique in the conflict graph G, then the inequality*

$$\sum_{(i,m,j,n,v) \in C} x_{imjnv} \leq 1 \tag{4.44}$$

is valid for X.

Remark 4.4.3. *An inequality based on a pair of incompatible inequalities of type (i) is a SEC.*

In order to separate clique inequalities we need to consider weights on the nodes. The weight of node (i, m, j, n, v) is given by the value of the variable x_{imjnv} in the linear solution. Finding the most violated clique inequality implies to solve the maximum weight clique problem, which is known to be strongly NP-hard. Here we use a simple greedy separation heuristic. First, find the maximum weight clique with two nodes and update C accordingly. Then augment set C in a greedy fashion. In each iteration add to C the maximum weight node that forms a clique with the nodes in C, that is, $C \leftarrow C \cup \{v^*\}$ where

$$v^* = argmax\{w_v : \forall u \in C, \{u, v\} \in E\}.$$

and w_v is the weight of node v. The process stops when a maximal clique is found. If the resulting clique inequality (4.44) is violated then it is added as a cut, otherwise no new inequality is added.

Figure 4.2 shows an example of a linear relaxation solution and the respective conflict graph. Starting with the maximum weight clique with two nodes

$$C = \{(1, 1, 2, 1, 2), (1, 1, 2, 2, 2)\}.$$

C is further expanded. First with $(2, 2, 1, 1, 2)$ and then with $(3, 1, 1, 1, 1)$.
Hence, $C = \{(1, 1, 2, 1, 2), (1, 1, 2, 2, 2), (2, 2, 1, 1, 2), (3, 1, 1, 1, 1)\}$. The (violated) maximal clique inequality is

$$x_{11212} + x_{11222} + x_{31111} + x_{22112} \leq 1$$

.

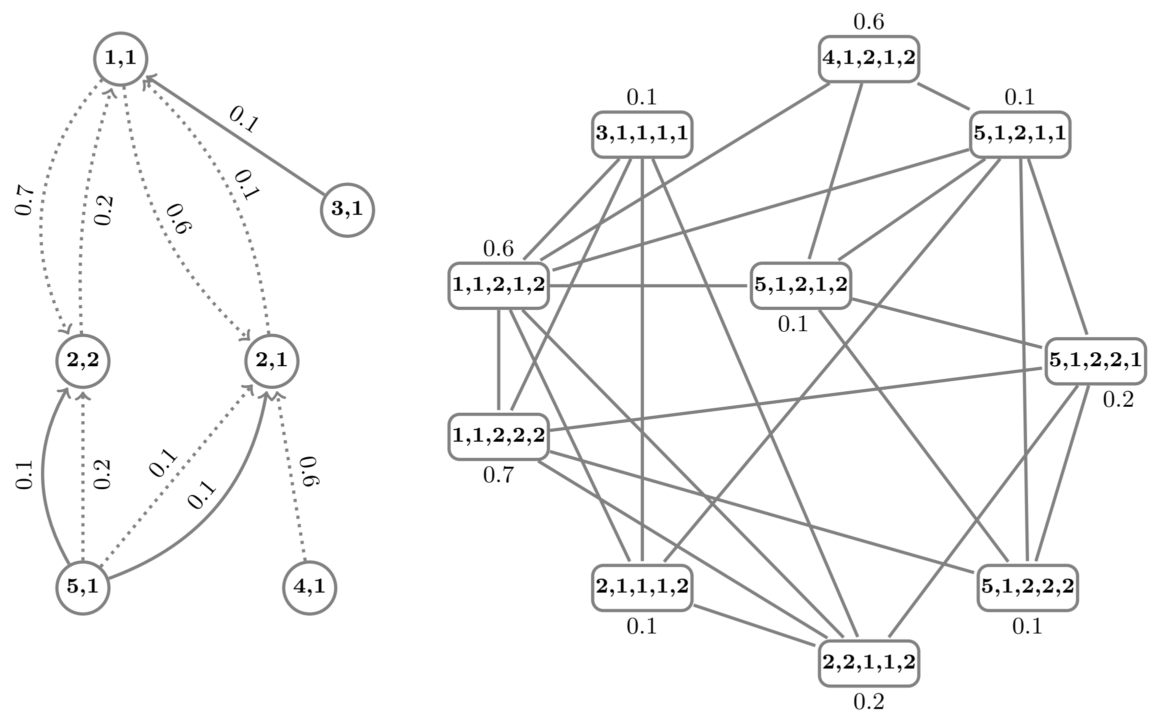

Figure 4.2: Example of a partial linear relaxation on the left. The two types of arcs represent different ships. The corresponding conflict graph is given on the right.

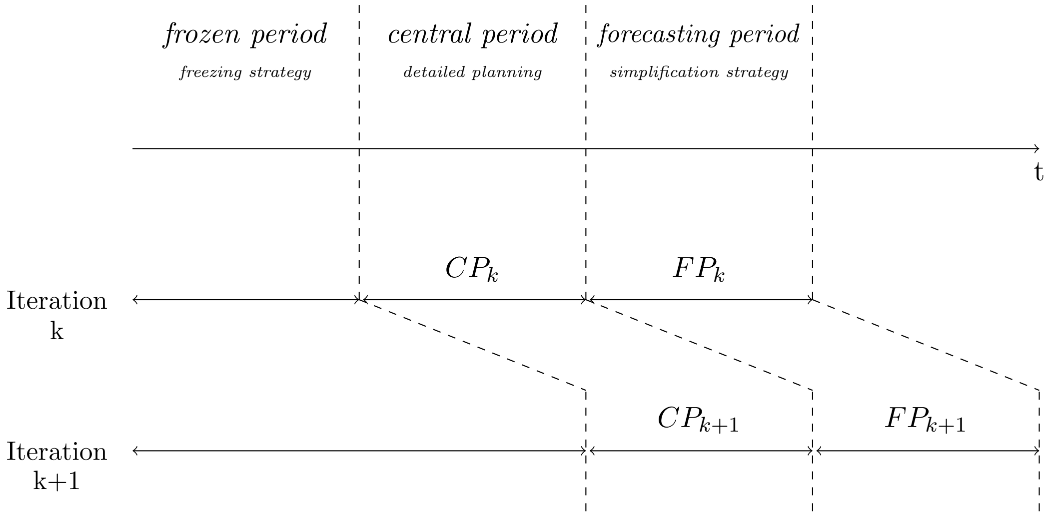

Figure 4.3: The rolling horizon heuristic

4.5 Hybrid heuristics

The formulation F-SSIRP tightened with the strategies discussed in the previous section can hardly be used to solve real instances using a generic MIP solver. However, recent hybrid heuristics have been proposed that use MIP solvers as a black-box tool. Here we consider and combine three such heuristic procedures: rolling horizon, local branching and feasibility pump.

4.5.1 Rolling Horizon heuristic

When considering a planning horizon of several months, the tested instances become too large to be handled by commercial software. To provide feasible solutions we have developed a Rolling Horizon (RH) heuristic. The main idea of the RH heuristic is to split the planning horizon into smaller sub-horizons, and then repeatedly solve limited and tractable mixed integer problem for the shorter sub-horizons. In transportation problems, RH heuristics have been used in several related works [9, 28, 31, 32].

In each iteration k of the RH heuristic (except the first and last one), the sub-horizon considered is divided into three parts: (i) a frozen part where binary variables are fixed; (ii) a central part (CP_k) where no restriction or relaxation is considered, and (iii) a forecasting period (FP_k) where binary variables are relaxed. The central period in iteration k becomes a frozen period in iteration $k+1$, and the forecasting period from iteration k becomes the central period in iteration $k+1$, see Figure 4.3. The process is repeated until the whole planning horizon is covered. In each iteration the limited mixed integer problem is solved. When moving from iteration k to iteration $k+1$ we (a) fix the values of the binary variables, (b) update the initial stock level of each product at each port, (c) calculate the quantity of each product on board each ship, and (d) update, for each ship, the initial position and the travel time and cost from that position to every port, see Algorithm 1. Based on preliminary tests we set $CP_k = FP_k = 5$ days.

Algorithm 1 Rolling Horizon heuristic

k $\leftarrow$ 1
U $\leftarrow$ number of iterations to cover the planning horizon $[1, \cdots, T]$
while $k \leq U$ **do**
  Relax binary variables in forecasting period FP_k
  Solve a limited mixed integer problem defined by CP_k and FP_k
  Freeze the variables $x_{imjnv}, x_{oimv}, o_{imvk}, w_{imv}, z_{imv}$ and y_{im} in CP_k
  if $k < U$ **then**
    Update the initial stock level of product k at port i
    Calculate the quantity of each product on board each ship v
    Update, for each ship v, the initial position and the travel time and cost from that position to every port i
  end if
  $k \leftarrow k + 1$
end while

4.5.2 Local Branching heuristic

Local Branching (LB) was introduced in [19] to improve a given feasible solution. The LB heuristic searches for a local optimum by restricting the number of variables that can change their value in the current feasible solution.

More formally, consider a feasible set of the form $\{(u, v) \in \{0, 1\}^n \times \mathbb{R}^m \cap P\}$ where P is a polyhedron. Given a feasible solution $(\overline{u}, \overline{v})$, let $\overline{S} = \{j \in \{1, \cdots, n\} : \overline{u}_j = 1\}$ denote the set of indices of the binary variables that are set to 1. The extra constraint

$$\sum_{j \in \overline{S}} (1 - u_j) \leq \Delta, \tag{4.45}$$

is considered, where Δ is a given positive integer parameter, indicating the number of variables $u_j, j \in \overline{S}$ that are allowed to flip from one to zero.

Many strategies were tested to combine the two heuristic approaches RH and LB. Here we present only three such strategies. In the RH, the problem is decomposed into subproblems. In each iteration the subproblem is solved to optimality. For the combined heuristics we used the same decomposition as for the RH. For all three combined strategies, for each subproblem, a constraint (4.45) with $\Delta = 0$ is added on the variables of the frozen period. Doing so, we allow the continuous variables to change their value within the frozen period. The strategies differ in the solution approach for each subproblem, and on whether they perform a local search in the neighborhood of the final solution or not.

LB1: For each subproblem, the solver is interrupted when the first feasible solution is reached.

LB2: Solve each subproblem twice. First the solver is run until either an integrality gap ($gap = 100 \times (UB - LR)/LR$ where UB is the best known upper bound and LR is the best known lower bound) less than or equal to 10% is achieved or a maximum time limit is reached. Then a constraint (4.45) with $\Delta = 2$ is added over the variables in the central period, and the subproblem is solved again until a gap of 5% is reached or the time limit is attained.

LB3: Obtain a feasible solution with LB2. For a t, $0 < t < T$, impose a constraint (4.45) with $\Delta = 0$ for the period $[0, T - t]$, and a constraint (4.45) with $\Delta = 6$ for the period $[T - t, T]$. Solve the new problem. Using the new solution impose new constraints on periods $[0, T - 2t]$, with $\Delta = 0$, and $[T - 2t, T]$, with $\Delta = 6$, and solve the problem again. This procedure is repeated until at least one of the following stopping criterion is reached: (i) time limit; (ii) maximum number of iterations without improvement and (iii) a maximum number of iterations. This algorithm is detailed in Algorithm 2. In our experiments we used $t = 5$ days, and a maximum number of 5 iterations.

Algorithm 2 LB3 heuristic

// first part (obtain a feasible solution for a planning horizon T, $\overline{x}_{imjnv}$)
T $\leftarrow$ length of the planning horizon
$T_1 \leftarrow$ length of the sub-horizon
Solve the problem for a time horizon of $T_1 = 2t$ periods
Save the feasible solution, $\overline{x}_{imjnv}$, and compute $\overline{S}$
$T_1 \leftarrow T_1 + t$
$\Delta_1 \leftarrow 0$
$\Delta_2 \leftarrow 6$
Bin $\leftarrow 0$
while $T_1 \leq T$ **do**
  Using the port subproblem $NV^D(i)$, determine the minimum number of visits at each port i for time horizon $[0, T_1]$
  Add constraints $\sum_{j \in \overline{S}}(1 - \overline{x}_j) \leq \Delta_1$ for time horizon $[0; T_1 - 3t]$
  if $Bin = 0$ **then**
    Solve the problem until gap $\leq 10\%$ or time limit is reached
    Bin $\leftarrow 1$
  else
    Add constraints $\sum_{j \in \overline{S}}(1 - \overline{x}_j) \leq \Delta_2$ for time horizon $[T_1 - 3t; T_1]$
    Solve the problem until gap $\leq 5\%$ or time limit is reached
    Bin $\leftarrow 0$
    $T_1 \leftarrow T_1 + t$
    Remove all added constraints and update the model
  end if
  Update the solution, $\overline{x}_{imjnv}$ and $\overline{S}$
end while
// second part (improve the feasible solution , $\overline{x}_{imjnv}$)
number of iterations $\leftarrow 1$
while *number of iterations* $\leq$ *max number of iterations* and *solution improves* **do**
  Reduce the fixed period of variables with t days: $T_1 \leftarrow T_1 - t$
  Add constraints $\sum_{j \in \overline{S}}(1 - \overline{x}_j) \leq \Delta_2$
  Update the solution, $\overline{x}_{imjnv}$ and $\overline{S}$
  number of iterations $\leftarrow$ *number of iterations*+1
end while

4.5.3 Feasibility Pump heuristic

Feasibility Pump (FP) was introduced by Fischetti, Glover and Lodi [18] as a heuristic scheme to find a feasible solution for a given mixed integer program. Such a procedure can be useful for those problems where finding an initial solution can be an hard task. FP is a rounding scheme that generates a sequence of fractional solutions from the linear relaxation which are rounded. The heuristic stops when a feasible solution is found or other stopping criteria is reached.

Here we use FP to speed-up the finding of an initial feasible solution. Although we followed the underlying ideas of FP, it was necessary to adjust this heuristic scheme to our MIRP. We focus on the problem at hand and not on the general FP scheme.

In this section, and for simplicity, we denote the points in the space of variables of F-SSIRP by x. First the linear relaxation of F-SSIRP is solved and a linear solution x^* is obtained. Then the binary variables with fractional values are rounded, and a solution $\overline{x}$ is obtained. If $\overline{x}$ is feasible ($\overline{x} \in X$) we stop. Otherwise, a new fractional solution is derived by finding the linear solution in the linear relaxation of X that minimizes a distance function to $\overline{x}$. The process is repeated until a feasible solution is found or a predefined maximal number of iterations is reached. If the rounding procedure stops without a feasible solution, then we run the solver.

Next we address the main steps of the FP algorithm in more detail.

Rounding scheme

For the rounding scheme we first consider the routing variables, x_{imjnv}. We set $\overline{x}_{imjnv} = 1$ whenever $x_{imjnv} > 0.5$ and $\overline{x}_{imjnv} = 0$ whenever $x_{imjnv} < \epsilon$, for small ϵ. Using the routing flow conservation constraints we fix the value of the remaining routing variables. Then the remaining binary variables x_{oimv}, w_{imv}, z_{imv}, y_{im} o_{im} are trivially fixed. This guided rounding scheme provided better results than rounding all binary variables simultaneously or rounding all the routing variables simultaneously first. Sophisticated rounding schemes are discussed in [20]. In our experiments we use $\epsilon = 0.1$.

The distance function

Given a 0-1 MIP solution obtained by rounding $\overline{x}$ we define the following distance function

$$\begin{aligned}
\phi(x_{imjnv}, \overline{x}_{imjnv}) &= \sum_{v \in V} \sum_{(i,m,j,n) \in S_v^X} |x_{imjnv} - \overline{x}_{imjnv}| \\
&= \sum_{v \in V} \sum_{(i,m,j,n) \in S_v^X | \overline{x}_{imjnv}=1} (1 - x_{imjnv}) \\
&\quad + \sum_{v \in V} \sum_{(i,m,j,n) \in S_v^X | \overline{x}_{imjnv}=0} x_{imjnv}
\end{aligned} \tag{4.46}$$

If $\phi(x_{imjnv}, \overline{x}_{imjnv}) = 0$, then a feasible solution can be derived. Otherwise a new linear solution x^* is obtained by solving the problem:

$$min\{\phi(x_{imjnv}, \overline{x}_{imjnv}) : x \in X_L\}$$

where X_L denotes the linear relaxation of the feasible set X of $F - SSIRP$.

Random perturbation

During the execution of the procedure two problems may arise: (i) the algorithm can be caught in a cycle, i.e., the same sequence is visited after consecutively and (ii) the convergence to a feasible solution is very slow.

Both problems (i) and (ii) are solved by performing a restart, that is, a new 0-1 MIP solution is derived by performing a random perturbation step. This step is similar to the one given in [1] and it is applied to the routing variables on the rounding scheme, that is, $\overline{x}_{imjnv} = \lfloor x^*_{imjnv} + \rho(z) \rfloor$ where $z \in [0, 1]$ is a uniform random variable and $\rho(z) = 2z(1 - z)$ if $z \leq 0.5$ and $\rho(z) = 1 - 2z(1 - z)$ if $z > 0.5$.

To measure the convergence speed we compute the difference between the value of the distance function in two consecutive solutions. When this difference is very small (smaller than a given δ) we perform the random perturbation.

Algorithm 3 describes the FP heuristic. In the computational results we set $\delta = 0.1$ and a maximum number of 50 iterations.

Algorithm 3 Feasibility Pump heuristic

Relax binary variables
Solve LP-relaxation of F-SSIRP. Let x^* denote its optimal solution
Obtain $\overline{x}$ by rounding x^*
number of iterations $\leftarrow 1$
while *number of iterations* $\leq$ *max number of iterations* and $\phi(x_{imjnv}, \overline{x}_{imjnv}) > 0$ **do**
  Solve the LP: $x^* \leftarrow argmin\{\phi(x_{imjnv}, \overline{x}_{imjnv}) : x \in X_L\}$
  Obtain $\overline{x}$ by rounding x^*
  if $\phi(\overline{x}_{imjnv}, x^*_{imjnv}) < \delta$ **then**
    Apply the random perturbation step
  end if
  number of iterations $\leftarrow$ *number of iterations*$+1$
end while

4.6 Computational experimentation

In this section we report the computational results when testing different hybrid heuristic approaches.

All computations were performed using the optimization software Xpress Optimizer Version 20.00.05 with Xpress Mosel Version 3.0.0, on a computer with processor Intel Core 2 Duo 2.2GHz and with 4GB of RAM.

We tested 12 real instances from a company in Cape Verde with 2 different ships, 7 ports and 4 products.

First we report a summary of results that testify the model choices. These tests were run for periods of 15 days. Then we report the results from the tests conducted to compare several hybrid strategies for periods of 2 and 6 months.

4.6.1 Model tuning

First we consider the use of port subproblems to estimate the minimum number of port visits. Figure 4.4, on the left, shows the minimum number of visits calculated using the formula (4.35), calculated using the subproblems, and the number of visits in the optimal solution for the 12 instances tested. On the right, the figure depicts the integrality gap (GAP), given by $GAP = 100 \times (OPT - LR)/OPT$ where OPT is the optimal value, obtained using the Xpress optimizer, and LR is the value of the linear relaxation. We consider the cases: "initial" when no minimum number of visits is imposed, "formula" when the minimum is obtained using (4.35), "subproblem" when the minimum is obtained using port subproblems and "exact" when we consider the minimum equal to the number of visits in the optimal solution.

In average, the initial integrality gap is 26.7%, drops to 24.1% using equations (4.35), and drops to 17.7% using subproblems. If the exact value in the optimal solution is used, the average gap is 13.2%.

Table 4.1 summarizes the integrality gaps when model F-SSIRP is used. TT means that the time constraints were tightened, SP means that the minimum number of visits was estimated

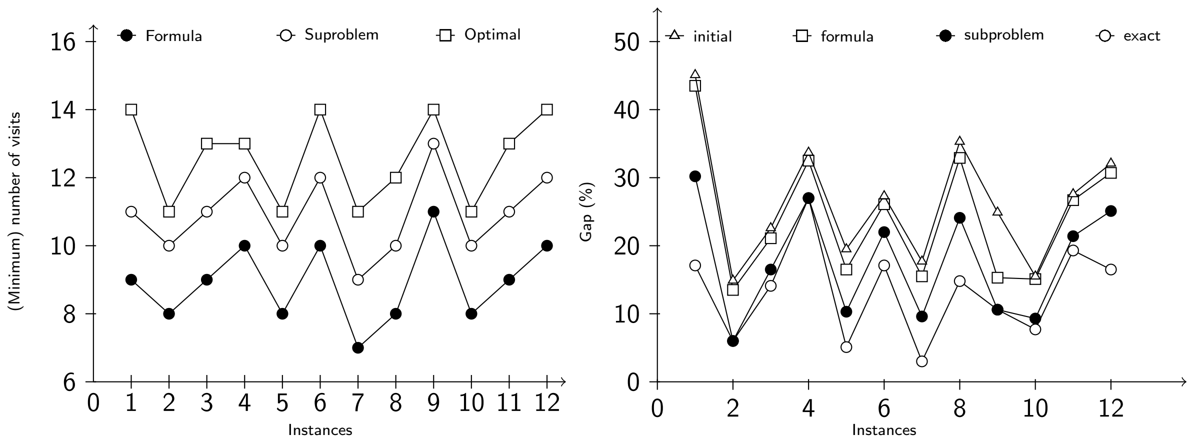

Figure 4.4: Estimation of the minimum number of visits (on the left) and its impact on the integrality gap (on the right) .

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using the port subproblem. IK indicates that the Integer Knapsack inequalities are added, and C means that the clique inequalities are added.

Table 4.1: Evolution of the average integrality gap with model tightening.

F-SSIRP + TT	F-SSIRP + TT + SP	F-SSIRP + TT + SP + IK	F-SSIRP+TT+SP+IK+C
26.7	17.7	10.9	10.9

In Table 4.2 we present the average solutions times, the number of B&B nodes, and the number of cuts added in each case. We can see that although the clique inequalities do not improve the integrality gap significantly, they are important with regard to the reduction in number of B&B nodes and running time.

4.6.2 Hybrid heuristics

In this section we report experiments carried out for comparing the hybrid heuristics in terms of running time, integrality gap and number of B&B nodes over two planning horizons: 2 and 6 months. Since the optimal solutions could not be obtained for these time horizons, the integrality gap (GAP) is computed as $GAP = 100 \times (UB - LR)/LR$ where UB is the value obtained by the heuristic and LR is the value of the linear relaxation. The value LR is obtained using the port subproblems to estimate the number of visits, and including IK and C inequalities. These model strengthening techniques are used whenever the optimization of the model F-SSIRP occurs as a subproblem embedded in a hybrid heuristic. The valid inequalities are added only at the root node.

For a time horizon of 2 months, Table 4.3 shows the performance of the RH heuristic, LB1 and LB1 combined with FP. It reports the time in seconds, the number of B&B and the integrality gap for each heuristic. The performance of LB2 and LB2 combined with FP is given in Table 4.4, and the performance of LB3 and LB3 combined with FP is given in Table 4.5.

Table 4.2: Comparison of time (in seconds), and B&B nodes using valid inequalities.

	F-SSIRP+TT		F-SSIRP+TT+SP+IK			F-SSIRP+TT+SP+IK+C		
Inst.	Time	Nodes	Time	Nodes	Cuts	Time	Node	Cuts
1	288	23788	38	1017	12	36	1015	16
2	11	19	25	1491	5	9	7	6
3	31	1377	51	3451	9	55	5678	16
4	63	3970	26	919	9	17	575	10
5	19	2777	15	2307	7	16	533	11
6	69	6188	23	2433	9	23	2433	9
7	15	754	8	379	5	6	327	6
8	20	8785	18	2917	10	10	622	11
9	40	8071	23	1423	7	24	603	9
10	40	1551	23	3535	9	9	3	13
11	58	16729	111	5383	9	73	2509	11
12	71	9299	41	8003	8	41	8003	8
Average	60.4	6942.3	33.5	2771.5	8.3	26.6	1859.0	10.5

Table 4.3: Computational results using RH, LB1 and LB1+FP for $T = 2$ months.

	RH			**LB1**			**LB1+FP**		
Inst.	**Time**	**Nodes**	**Gap**	**Time**	**Nodes**	**Gap**	**Time**	**Nodes**	**Gap**
1	1409	141380	37,1	45	1631	24,8	62	1753	27,7
2	951	148330	26,0	31	692	18,1	88	3229	31,2
3	1421	119833	12,4	365	30027	30,2	401	12420	16,8
4	4908	349909	41,1	51	2118	22,0	110	1700	28,2
5	649	105135	33,5	81	2829	30,8	126	2744	36,2
6	711	106265	33,0	598	53813	38,3	405	29366	30,9
7	362	47432	29,5	384	24356	28,2	321	22785	18,7
8	1285	160392	28,0	225	17487	29,1	256	16439	23,4
9	1107	122907	31,5	684	60289	33,6	322	13265	22,1
10	865	105245	25,8	97	3706	27,0	108	11027	27,1
11	985	143251	28,5	97	3706	28,1	64	2023	26,9
12	1106	167755	30,2	3	13	24,3	74	2838	32,9
Av.	**1313,3**	**143152,8**	**29,7**	**221,8**	**16722,3**	**27,9**	**194,8**	**9965,8**	**26,8**

Table 4.4: Computational results for LB2 and LB2+FP for $T = 2$ months.

	LB2			**LB2+FP**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	277	19887	23,2	106	4014	16,1
2	104	7982	11,8	72	3859	12,4
3	817	54236	21,8	780	48717	20,7
4	155	10214	22,6	192	12692	18,6
5	552	31737	15,2	252	10013	17,8
6	1755	122197	20,4	940	78983	20,4
7	1066	79101	21,3	481	26912	16,2
8	734	63262	20,0	672	28244	25,4
9	846	54919	16,7	1083	41811	21,7
10	1047	52706	17,5	397	7660	14,1
11	285	10004	20,6	423	11650	18,4
12	744	27989	11,2	456	12493	14,7
Average	**698,5**	**44519,5**	**18,5**	**487,8**	**23920,7**	**18,1**

Table 4.5: Computational results for LB3 and LB3 + FP for $T = 2$ months.

	LB3			**LB3+FP**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	301	20561	20,5	107	4014	12,9
2	105	7982	8,6	144	7718	12,4
3	951	64918	18,8	781	48717	18,1
4	185	15624	18,2	384	25384	18,6
5	573	33366	11,9	504	20026	17,8
6	2018	131345	20,4	1211	86043	20,4
7	1079	79303	18,5	485	26943	12,9
8	760	64206	17,0	686	28353	17,0
9	850	54919	13,7	1088	41811	18,7
10	1050	52706	14,5	399	7660	11,0
11	312	10770	17,9	425	11650	15,7
12	753	28264	7,8	461	12494	11,5
Average	**744,8**	**46997,0**	**15,7**	**556,3**	**26734,4**	**15,6**

We can see that LB heuristics combined with FP are, in average, faster than the LB heuristics which are in turn faster than the RH heuristic. The use of FP is more relevant on those harder instances, where the solver is not able to find good initial feasible solutions quickly. As expected, LB1 is faster than LB2, and LB2 is faster than LB3. However, the quality of the solutions obtained varies in the opposite direction. The most sophisticated heuristic, LB3 combined with FP, provides solutions with an integrality gap which is, in average, half of the integrality gap of the usual RH heuristic. The running time is almost a third of the running time of the RH heuristic.

Tables 4.6 and 4.7 give the computational results for 6 months for heuristics RH, LB1, and LB2 and LB3 combined FP. The behavior of these algorithms is similar to the case of 2 months. Only the gaps are higher. However, as this gap is computed by use of the linear relaxation value we do not know whether this increase results from a deterioration of the upper bound, the lower bound, or both.

Table 4.6: Computational results for RH and LB1 for $T = 6$ months.

	RH			**LB1+FP**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	3324	107998	42,6	2816	25114	24,3
2	10258	207125	44,8	1937	23517	28,7
3	3451	62775	45,6	2872	57014	26,1
4	4631	115802	41,6	1040	14311	26,5
5	6149	103324	47,7	3689	48353	32,8
6	10288	139427	42,5	3977	77989	31,5
7	7219	105059	42,4	1468	35739	27,8
8	3776	166414	46,2	1213	34326	32,5
9	4196	209323	47,2	7792	102636	29,7
10	2658	113510	45,1	4854	39172	30,5
11	13244	208361	44,8	569	12772	27,9
12	2079	93102	45,1	3042	35513	29,4
Average	**5939,4**	**136018,3**	**44,6**	**2939,1**	**42204,7**	**29,0**

Table 4.7: Computational results for LB2 and LB3 for $T = 6$ months

	LB2+FP			**LB3+FP**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	4404	166993	23,1	4551	167148	21,1
2	1260	78999	20,7	1300	79060	18,6
3	2469	83566	23,8	2507	83647	22,0
4	1736	83330	20,3	1819	83457	18,2
5	2917	99785	28,2	3142	100031	26,6
6	3109	114450	28,7	3125	114455	27,1
7	2899	102661	31,9	3004	102776	30,4
8	2349	113899	28,7	2480	114137	27,1
9	3894	142451	21,1	4109	142606	19,2
10	1392	53626	20,7	1598	53742	18,7
11	2308	110136	24,4	2454	110286	22,6
12	1607	67245	24,5	1881	67355	22,8
Average	**2528,7**	**101428,4**	**24,7**	**2664,1**	**101558,5**	**22,9**

To test the heuristic approaches that performed best on the larger instances, we created two artificial future scenarios where the demands as well as the number of ships are increased. One scenario with three ships and demands that are 1.5 times the current demands, and another scenario with four ships and double demands. Each scenario is identified by the number of ships ($\mid V \mid= 3$ and $\mid V \mid= 4$). We opted not to reduce the length of each sub horizon. All the tested heuristics run within a reasonable computational time effort for 2 months. For 6 months, RH, LB2 and LB3 heuristics were too time consuming.

In Table 4.8 we give the computational results. For $\mid V \mid= 3$ we used a variant of LB2, where only the first run (until a gap of 10%) is performed, combined with FP. For $\mid V \mid= 4$ we used LB1 combined with FP. We could not solve most of the linear relaxations within 1 day time limit. To compute the lower bound we computed the linear relaxation of the model obtained from F-SSIRP by removing all time and inventory constraints, and with the additional cuts discussed in Section 4.4. Additionally we imposed, for each port i and each product k such that $J_{ik} = -1$, the constraint $\sum_{v\in V}\sum_{m=1}^{\mu_i} q_{imvk} \geq T \times R_{ik} + \underline{S}_{ik} - S^0_{ik}$.

Table 4.8: Computational results for larger instances with 3 and 4 ships

	—V—=3			—V—=4		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	988	10154	27,0	5218	45921	31,0
2	1096	20695	29,7	5017	44186	35,1
3	924	30403	29,8	4633	51406	24,5
4	2120	34692	30,3	6804	47798	28,2
5	2120	49307	32,7	5706	49415	35,9
6	2199	25836	36,9	10988	55062	40,8
7	1158	32612	33,7	3338	48450	31,2
8	2340	62303	33,3	4173	54671	30,7
9	1486	51884	29,9	6813	52666	35,2
10	1857	51934	35,0	9958	47864	34,3
11	2275	25875	31,1	4581	49583	36,6
12	2628	30691	31,1	5064	47717	31,2
Average	**1765,9**	**35532,2**	**31,7**	**6024,4**	**49561,6**	**32,9**

4.7 Conclusions

We have presented a mathematical model for the short sea inventory routing problem. This model is tightened with valid inequalities and an estimation of the minimum number of visits to each port by solving some port subproblems. In particular we introduced new clique inequalities that can be used to tighten continuous time maritime inventory routing models.

Given the long time horizons, we propose and compare different strategies of combining three well-known heuristics that use the mathematical model as a black-box. The Rolling Horizon heuristic is used to decompose the original problem into smaller and more tractable problems, the Feasibility Pump heuristic is used to find initial solutions for MIP problems, and the Local Branching heuristic is used to improve feasible solutions.

The best strategy tested combines all the three heuristics, and allowed us to obtain solutions whose integrality gap is in average half of the integrality gap obtained using the rolling horizon heuristic alone. We provided computational results for time horizons up to 6 months.

In order to evaluate the quality of the solutions obtained by the hybrid procedures, an important future direction of research is to investigate approaches to derive tight lower bounds, specially for long time horizons where the size of the linear relaxation model is quite large.

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Paper IV

A. Agra, M. Christiansen, A. Delgado and L. M. Hvattum:

A Maritime Inventory Routing Problem with Stochastic Sailing and Port Times

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Chapter 5

A Maritime Inventory Routing Problem with Stochastic Sailing and Port Times

Abstract

We consider a stochastic short sea shipping problem where a company is responsible for both the distribution of oil products between islands and the inventory management of those products at unloading ports. Ship routing and scheduling is associated to uncertainty in weather conditions and unpredictable waiting times in ports, and in this work, both sailing times and port times are considered to be stochastic parameters.

A two-stage stochastic programming model with recourse is presented where the first-stage consists of routing, loading and unloading decisions, and the second stage consists of scheduling decisions. The model is solved using a decomposition approach similar to an L-shaped algorithm where optimality cuts are added dynamically, and this solution process is embedded within the sample average approximation method. A computational study based on ten real-world instances is presented.

Keywords: Stochastic programming; Maritime transportation; Uncertainty; L-shaped method; Sample average approximation; Travel time; Service time.

5.1 Introduction

Maritime transportation is characterized by high levels of uncertainty. In practice, operational plans are often adjusted due to factors such as changing weather conditions, ports congestions, or mechanical problems at port. A plan that minimizes the transportation and port costs based on expected sailing and port times may not necessarily be good, as it does not account for consequences resulting from delays. Hence, in most practical situations it will be beneficial to consider the possibility of delays when trying to minimize costs.

In this paper we consider a maritime inventory routing problem occurring at the archipelago of Cape Verde. A deterministic variant of this problem was solved to optimality in [1] for short time horizons. Heuristics for the same problem with time horizons up to 6 months were developed in [2]. The deterministic methods assume known and fixed sailing times, but the planner needs to face the uncertainty associated with the ships sailing between ports. This may somehow be circumvented by the inclusion of safety stocks at inventories or by artificially increasing the sailing times to compensate for delays. However, in this paper we consider explicitly uncertainty in both sailing times between ports and waiting times at ports, over a short time horizon. Bad weather can lead to both longer sailing and port times. The ports are used by several independent shipping companies, and limited coordination between the various operators can result in heavy port congestion. This may come from limited capacities in the inner port area, at berths, and of pipes and other important equipment for performing the (un-)loading operations. In addition, delays may occur due to mechanical problems at port. By taking this into account, good distribution plans can be found that explicitly takes into account the real possibility of violating inventory limits at production or consumption ports.

This paper describes a stochastic programming model with recourse where the routes and the quantities to load and unload must be fixed *a priori*, that is, before actual values of the uncertain parameters are revealed, while the schedule of the loading and unloading operations can be adjusted according to the observed sailing and port times.

The solution method combines the use of the sample average approximation method with a decomposition procedure resembling an L-shaped method [5, 11]. For a given set of scenarios, the corresponding two-stage model is solved to obtain a candidate solution. This is repeated for several different sets of scenarios to obtain several candidate solutions. To choose the best solution, these candidate solutions are evaluated for a larger and independent set of scenarios. To solve the two-stage model for a given set of scenarios, the problem is decomposed into a master problem and one subproblem for each scenario, where the second-stage decisions are considered in the subproblems. We show that feasibility is always guaranteed for the solution obtained in the first stage. Then we show how to derive optimality cuts from the subproblems that are added dynamically to the master problem.

The remainder of this paper is organized as follows. In Section 5.2 we describe the real problem and review some relevant literature. Then, in Section 5.3 we present a scenario based mathematical formulation for the problem. The solution approach based on decomposing the problem is discussed in Section 5.4. In Section 5.5 we describe how the stochastic sailing and port times have been modeled, and how scenarios have been generated. Section 5.6 contains computational results for ten real-world instances, and in Section 5.7 we present the main conclusions of this work.

5.2 Problem description and literature review

In Cape Verde, fuel oil products are imported and delivered to specific islands and stored in large supply storage tanks. From these islands, fuel oil products are distributed among all inhabited islands using a small heterogeneous fleet of ships. Products are stored in separate consumption storage tanks with limited capacity. Some ports have both supply tanks for some products and consumption tanks for other products. As the capacities of the supply tanks are very large compared to the total consumption over the planning horizon, the inventory aspects for these tanks can be ignored. Not all islands consume all products. Consumption rates are assumed to be constant over the time horizon. Each port can receive at most one ship at a time, and in some ports there exists a minimum time interval between the departure of one ship and the arrival of the next ship.

Each ship has a specified capacity, fixed speed, and cost structure. The cargo hold of each ship is separated into several cargo tanks. The products cannot be mixed, so we assume that the ships have dedicated tanks for the particular products. The ships are either sailing, waiting outside a port or operating. Here, operating is the common term for loading and unloading.

At port, we consider set-up times for the coupling and decoupling of pipes and operation times which depend on the amount loaded or unloaded. Minimum and maximum unloading quantities can be derived. The maximum unloading quantity is imposed by the inventory capacity at the consumption port and by the ship cargo tank capacity.

The driving force in the problem is the need for fuel oil products in the consumption storage tanks. If the demand is not satisfied, the backlogged demand will be penalized by a cost.

The traveling times depend upon the weather conditions and are considered stochastic. The uncertain time parameter at port is manly related to the time from arrival to start of operation. Hence, a specified waiting time before start of service is defined as stochastic, while the operation times are deterministic.

The inter-island distribution plan consists of routes and schedules for the fleet of ships, and describes the number of visits to each port and the quantity of each product to be loaded or unloaded at each port visit. This plan must satisfy the capacities of the ships and consumption inventories while minimizing the sailing and port costs as well as the expected penalty costs of backlogged demand. There is great flexibility in the route pattern of a ship, such that a ship may visit several loading ports as well as unloading ports in succession and the quantities loaded or unloaded are variable as well as the number of visits at each port. The problem described here will be referred to as a *stochastic maritime inventory routing problem* (SMIRP), and a scenario based stochastic programming model for the problem is given in Section 5.3.

The amount of literature on maritime transportation optimization has increased steadily over the last decades, as evidenced through the recent survey in [7]. Despite being a transportation mode that is heavily influenced by uncertainty, most of the literature on maritime routing and scheduling involves solving static and deterministic problem variants. However, some contributions exist, and we describe some that are considering problems close to the stochastic maritime inventory routing problem of this paper.

An inventory routing problem with uncertain demands and sailing times was solved heuristically by Cheng and Duran [6]. Rakke et al. [17] and Sherali and Al-Yakoob [18, 19] introduce penalty functions for deviating from the customer contracts and the inventory limits, respectively. Christiansen and Nygreen [8] introduce soft inventory levels to handle uncertainties in sailing and port times, and these levels are transformed into soft time windows.

Agra et al. [3] solved a full-load ship routing and scheduling problem with uncertain travel times using robust optimization. Weather conditions affect both sailing speeds and the loading and unloading operations for supply vessels servicing offshore installations, and various heuristic strategies to achieve robust weekly voyages and schedules were analyzed by Halvorsen-Weare and Fagerholt [9]. Heuristic strategies for obtaining robust solutions with uncertain sailing times was also discussed by Halvorsen-Weare et al. [10] for the delivery of liquefied natural gas. None of the aforementioned research has used stochastic programming to model uncertain sailing and port times.

A stochastic model for a particular version of the *vehicle routing problem* (VRP) with stochastic travel times was presented by Lambert et al. [14], and a heuristic solution method was proposed. Considering a VRP with stochastic travel times and service times, Laporte et al. [15] presented a chance constrained formulation as well as two recourse formulations. The recourse problem was solved to optimality for up to 20 nodes and 5 scenarios using an integer L-shaped method. The VRP with stochastic travel and service times was also studied by Kenyon and Morton [13], considering stochastic programming models that minimized the expected completion time or maximized the probability of completing the routes within a given deadline. An integer L-shaped algorithm was used by Teng et al. [20] to solve a time-constrained *traveling salesman problem* with stochastic travel and service times with up to 35 nodes. Although these papers present stochastic programming models for routing problems with uncertain travel times and service times, they do not consider heterogeneous fleets, a variable number of visits, nor inventory constraints.

5.3 Mathematical Model

In this section we introduce a two-stage stochastic programming model with recourse for the SMIRP problem. The routes and the quantities to load and unload are determined in the first stage. However, the schedule of the loading and unloading operations can be adjusted in the second stage. Thus, also the inventory level variables are allowed to change according to the realization of the stochastic parameters. In the following we first describe the variables and constraints related to the first stage (Section 5.3.1), and then the variables and constraints related to the second stage (Section 5.3.2).

5.3.1 First stage

First we model the routing and the loading and unloading constraints.

Routing constraints:

Let V denote the set of ships. Each ship $v \in V$ must depart from its initial position in the beginning of the planning horizon. For each port we consider an ordering of the visits accordingly to the time of visit. The ship paths are defined on a network where the nodes are represented by a pair (i, m), where i is the port and m is the m^{th} visit to port i. A direct ship movement (arc) from port arrival (i, m) to port arrival (j, n) is represented by (i, m, j, n).

We define S^A as the set of possible port arrivals (i, m), S_v^A as the set of port arrivals that may be visited by ship v, S^X as the set of all possible ship movements (i, m, j, n), and set S_v^X as the set of all possible movements of ship v. The set of ships that can visit port i is denoted V_i.

For the routing we define the following binary variables: x_{imjnv} that is 1 if ship v sails from port arrival (i, m) directly to port arrival (j, n), and 0 otherwise; x_{imv}^O that indicates whether ship v sails directly from its initial position to port arrival (i, m) or not; w_{imv} is 1 if ship v visits port i at arrival (i, m), and 0 otherwise; z_{imv} is equal to 1 if ship v ends its route at port arrival (i, m), and 0 otherwise; z_v^O is equal to 1 if ship v is not used and 0 otherwise; y_{im} indicates whether a ship is visiting port arrival (i, m) or not.

$$\sum_{(i,m)\in S_v^A} x_{imv}^O + z_v^O = 1, \qquad v \in V, \tag{5.1}$$

$$w_{imv} - \sum_{(j,n)\in S_v^A} x_{jnimv} - x_{imv}^O = 0, \qquad v \in V, (i, m) \in S_v^A, \tag{5.2}$$

$$w_{imv} - \sum_{(j,n)\in S_v^A} x_{imjnv} - z_{imv} = 0, \qquad v \in V, (i, m) \in S_v^A, \tag{5.3}$$

$$\sum_{v\in V_i} w_{imv} = y_{im}, \qquad (i, m) \in S^A, \tag{5.4}$$

$$y_{i(m-1)} - y_{im} \geq 0, \qquad (i, m) \in S^A : m > 1, \tag{5.5}$$

$$x_{imv}^O, w_{imv}, z_{imv} \in \{0, 1\}, \qquad v \in V, (i, m) \in S_v^A, \tag{5.6}$$

$$x_{imjnv} \in \{0, 1\}, \qquad v \in V, (i, m, j, n) \in S_v^X, \tag{5.7}$$

$$z_v^O \in \{0, 1\}, \qquad v \in V, \tag{5.8}$$

$$y_{im} \in \{0, 1\}, \qquad (i, m) \in S^A. \tag{5.9}$$

Equations (5.1) ensure that each ship either departs from its initial position and sails towards another port or the ship is not used. Equations (5.2) and (5.3) are the arc flow conservation constraints, ensuring that a ship arriving at a port also leaves that port or ends its route. Constraints (5.4) ensure that one ship only visits port (i, m) if y_{im} is equal to one. Constraints (5.5) state that if port i is visited m times, then it must also have been visited $m-1$ times. Constraints (5.6) - (5.9) define the variables as binary.

Loading and unloading constraints

Let K represent the set of products and K_v represent the set of products that ship v can transport. Not all ports consume all products. Parameter J_{ik} is 1 if port i is a supplier of product k; -1 if port i is a consumer of product k, and 0 if i is neither a consumer nor a supplier of product k. The quantity of product k on board of ship v at the beginning of the planning horizon is given by Q^O_{vk} and C_{vk} is the capacity of the compartment of ship v dedicated for product k. The minimum and the maximum discharge quantities of product k at port i are given by $\underline{Q}_{ik}$ and $\overline{Q}_{ik}$, respectively. Parameter T is the length of the time horizon.

To model the loading and unloading constraints, we define the following binary variables: o_{imvk} is equal to 1 if product k is loaded onto or unloaded from ship v at port visit (i, m), and 0 otherwise. In addition, we define the following continuous variables: q_{imvk} is the amount of product k loaded onto or unloaded from ship v at port visit (i, m); f_{imjnvk} denotes the amount of product k that ship v transports from port visit (i, m) to port visit (j, n), and f^O_{imvk} gives the amount of product k that ship v transports from its initial position to port visit (i, m).

The loading and unloading constraints are given by:

$$f^O_{jnvk} + \sum_{(i,m)\in S^A_v} f_{imjnvk} + J_{jk} q_{jnvk} = \sum_{(i,m)\in S^A_v} f_{jnimvk}, \qquad v \in V, (j, n) \in S^A_v, k \in K_v, \tag{5.10}$$

$$f^O_{imvk} = Q^O_{vk} x^O_{imv}, \qquad v \in V, (i, m) \in S^A_v, k \in K_v, \tag{5.11}$$

$$f_{imjnvk} \le C_{vk} x_{imjnv}, \qquad v \in V, (i, m, j, n) \in S^X_v, k \in K_v, \tag{5.12}$$

$$0 \le q_{imvk} \le C_{vk} o_{imvk}, \qquad v \in V, (i, m) \in S^A_v, k \in K_v : J_{ik} = 1, \tag{5.13}$$

$$\underline{Q}_{ik} o_{imvk} \le q_{imvk} \le \overline{Q}_{ik} o_{imvk}, \qquad v \in V, (i, m) \in S^A_v, k \in K_v : J_{ik} = -1, \tag{5.14}$$

$$\sum_{k\in K_v} o_{imvk} \ge w_{imv}, \qquad v \in V, (i, m) \in S^A_v, \tag{5.15}$$

$$\sum_{(i,m)\in S_{A_v}} \sum_{v\in V} \sum_{k\in K_v : J_{ik}=-1} q_{imvk} \ge \sum_{i\in N} \sum_{k\in K : J_{ik}=-1} R_{ik} T, \tag{5.16}$$

$$o_{imvk} \le w_{imv}, \qquad v \in V, (i, m) \in S^A_v, k \in K_v, \tag{5.17}$$

$$f_{imjnvk} \ge 0, \qquad v \in V, (i, m, j, n) \in S^A_v, k \in K_v, \tag{5.18}$$

$$f^O_{imvk}, q_{imvk} \ge 0, \qquad v \in V, (i, m) \in S^A_v, k \in K_v, \tag{5.19}$$

$$o_{imvk} \in \{0, 1\}, \qquad v \in V, (i, m) \in S^A_v, k \in K_v. \tag{5.20}$$

Equations (5.10) are the load flow conservation constraints. Equations (5.11) determine the quantity on board when ship v sails from its initial port position to port arrival (i, m). Constraints (5.12) guarantee that the ships' tank capacities are not exceeded. Constraints (5.13) impose an upper bound on the quantity loaded at the supply ports. Constraints (5.14) impose lower and upper limits on the unloaded quantities. Constraints (5.15) ensure that if ship v visits port arrival (i, m), then at least one product must be (un)loaded. Constraints (5.16) ensure that the sum of delivered goods should not be less than the sum of the consumption over the entire horizon T. Constraints (5.17) ensure that if ship v (un)loads one product at visit (i, m), then w_{imv} must be one. Constraints (5.18)-(5.20) are the non-negativity and integrality requirements.

5.3.2 Second stage

Now we present the second stage model where the variables can be adjusted to the scenario. The set of scenarios Ω will be indexed by c.

Time constraints

To keep track of the inventory level it is necessary to determine the start and the end times at each port arrival. We define the following parameters: T^Q_{ik} is the time required to load/unload one unit of product k at port i; T^S_{ik} is the set up time required to operate product k at port i. T_{ijvc} is the sailing time between port i and j by ship v for scenario c; T^O_{ivc} indicates the sailing time required by ship v to travel from its initial port position to port i for scenario c; T^B_i is the minimum interval between the departure of one ship and the next arrival at port i; T^W_{imc} is the waiting time at port arrival (i,m) for scenario c. The parameter μ_i denotes the maximum number of visits at port i. For each scenario c we define the start time t_{imc} and the end time t^E_{imc} variables for port arrival (i,m). Variables t^+_{ic} give the remaining time from the end of the last visit at port i until time T for scenario c, when this visit occurs before time T.

Assuming that a ship travels from (i,m) to (j,n) under scenario c and loads product k using vessel v, Figure 5.1 shows the parameters involved when calculating the time variables for node (j,n).

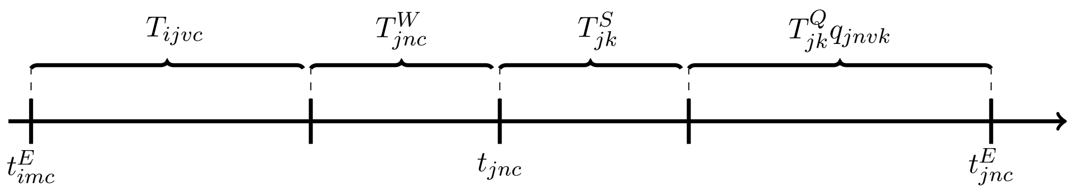

Figure 5.1: Illustration of the parameters involved when calculating start and end times for node (j,n).

The set of time constraints is as follow:

$$t^E_{imc} \geq t_{imc} + \sum_{v\in V}\sum_{k\in K_v} T^S_{ik}o_{imvk} + \sum_{v\in V}\sum_{k\in K_v} T^Q_{ik}q_{imvk}, \qquad (i,m)\in S^A, c\in\Omega, \tag{5.21}$$

$$t_{imc} - t^E_{i(m-1)c} - T^B_i y_{im} \geq 0, \qquad (i,m)\in S^A : m>1, c\in\Omega, \tag{5.22}$$

$$t^E_{imc} + \sum_{v\in V_i\cap V_j} T_{ijvc}x_{imjnv} + T^W_{jnc} - t_{jnc} \leq M(1-\sum_{v\in V_i\cap V_j} x_{imjnv}), (i,m,j,n)\in S^X, c\in\Omega, \tag{5.23}$$

$$\sum_{v\in V} T^O_{ivc}x^O_{imv} + T^W_{imc} \leq t_{imc}, \qquad (i,m)\in S^A, c\in\Omega, \tag{5.24}$$

$$t^+_{ic} \geq T - t^E_{i\mu_i c}, \qquad i\in N, c\in\Omega, \tag{5.25}$$

$$t_{imc}, t^E_{imc} \geq 0, \qquad (i,m)\in S^A, c\in\Omega, \tag{5.26}$$

$$t^+_{ic} \geq 0, \qquad i\in N, c\in\Omega. \tag{5.27}$$

Constraints (5.21) define the end time of service at port visit (i,m). Constraints (5.22) impose a minimum interval between two consecutive visits at port i. Constraints (5.23) relate the end time of port visit (i,m) to the start time of port visit (j,n) when ship v sails directly from port (i,m) to (j,n). The big-M constant, denoted by M was set to $2T$, since the start time of a visit can occur after time T. These constraints are a stronger version of the usual family of constraints $t^E_{imc} + T_{ijvc} + T^W_{jnc} - t_{jnc} \leq M(1 - x_{imjnv})$ defined for each $v \in V$. Constraints (5.24) ensure that if ship v travels from its initial position directly to port visit (i, m), then the start time is at least the sailing time between the two positions plus the waiting time at port visit (i, m). Constraints (5.25) together with (5.27) determine the time gap between the last visit to port i and time T. The continuous time variables are declared as non-negative in (5.26) and (5.27).

Inventory constraints

The inventory constraints are considered for each unloading port i $(J_{ik} = -1)$. They ensure that the inventory levels are kept within the corresponding bounds, and link the inventory levels to the unloaded quantities.

For each consumption port i, and for each product k, the demand rate, R_{ik}, the minimum $\underline{S}_{ik}$, the maximum $\overline{S}_{ik}$, and the initial S^O_{ik} inventory levels are given.

We define the nonnegative continuous variables s_{imkc} and s^E_{imkc} indicating the inventory levels at the start and at the end of port visit (i, m) for scenario c, respectively; r_{imkc} and r^E_{imkc} indicate the backlog of product k at the start and at the end of port visit (i, m) for scenario c, respectively. The inventory constraints are as follow:

$$s_{i1kc} = S^O_{ik} - R_{ik}t_{i1c} + r_{i1kc}, \qquad i \in N, k \in K : J_{ik} = -1, c \in \Omega, \tag{5.28}$$

$$s^E_{imkc} + r_{imkc} = s_{imkc} + r^E_{imkc} + \sum_{v \in V} q_{imvk} - R_{ik}(t^E_{imc} - t_{imc}), \qquad (i, m) \in S^A, \quad k \in K : J_{ik} = -1, c \in \Omega, \tag{5.29}$$

$$s_{imkc} + r^E_{i(m-1)kc} = s^E_{i(m-1)kc} + r_{imkc} - R_{ik}(t_{imc} - t^E_{i(m-1)c}), \qquad (i, m) \in S^A : m > 1, \quad k \in K : J_{ik} = -1, c \in \Omega, \tag{5.30}$$

$$s_{imkc}, s^E_{imkc} \leq \overline{S}_{ik}, \qquad (i, m) \in S^A, k \in K : J_{ik} = -1, c \in \Omega, \tag{5.31}$$

$$s^E_{i\mu_i kc} - R_{ik}t^+_{ic} \geq \underline{S}_{ik}, \qquad i \in N, k \in K : J_{ik} = -1, c \in \Omega, \tag{5.32}$$

$$s_{imkc}, s^E_{imkc}, r_{imkc}, r^E_{imkc} \geq 0, \qquad (i, m) \in S^A, k \in K : J_{ik} = -1, c \in \Omega. \tag{5.33}$$

Equations (5.28) calculate the inventory level of each product at the first visit. Equations (5.29) calculate the inventory level of each product when the service ends at port visit (i, m). Equations (5.30) relate the inventory level at the start of port visit (i, m) to the inventory level at the end of port visit $(i, m - 1)$. Constraints (5.31) ensure that the capacities of the depots are not exceeded. Constraints (5.32) impose a lower bound on the inventory level at time T, or at the end of the last visit, for each product. When the last visit occurs before T, the inventory level at the last visit needs to be reduced by the consumption until time T. The quantity below this lower bound is penalized as backlogged demand. Finally, non-negativity requirements (5.33) are imposed on the inventory and backlog variables.

5.3.3 Objective function

The objective is to minimize the sailing, setup and operating costs plus the penalty for backlogged demand. The sailing cost of ship v from port i to port j is denoted by C^T_{ijv}, while C^{TO}_{oiv} represents the sailing cost of ship v from its initial port position to port i. The operating cost of product k at port i is denoted by C^O_{ik}. The penalty cost for backlogging of product k at port i is denoted C^P_{ik}. The objective function is as follow:

$$z = \min \sum_{v \in V} \sum_{(i,m,j,n) \in S^X_v} C^T_{ijv} x_{imjnv} + \sum_{v \in V} \sum_{(i,m) \in S^A_v} C^{TO}_{oiv} x^O_{imv} +$$
$$\sum_{v \in V} \sum_{(i,m) \in S^A_v} \sum_{k \in K_v} C^O_{ik} o_{imvk} + \sum_{c \in \Omega} \frac{1}{|\Omega|} \left(\sum_{(i,m) \in S^A} \sum_{k \in K_v} C^P_{ik} (r_{imkc} + r^E_{imkc}) \right). \quad (5.34)$$

We penalize backlogged demand for each port visit. For the last visit, or time T if the last visit is earlier, we penalize the difference between the lower bound of the inventory and the actual inventory level in addition to any backlog.

5.4 Solution approach

Since the complete model is too large to be solved efficiently, it is decomposed into a master problem and one subproblem for each scenario, following the idea of the L-shaped algorithm [5]. Let the problem (5.1) - (5.34) be re-written as:

$$z = \min C(X) + \sum_{c \in \Omega} \frac{1}{|\Omega|} H(X, c)$$
$$s.t. \ (5.1) - (5.20)$$

where

$$C(X) = \sum_{v \in V} \sum_{(i,m,j,n) \in S^X_v} C^T_{ijv} x_{imjnv} + \sum_{v \in V} \sum_{(i,m) \in S^A_v} C^{TO}_{oiv} x^O_{imv}$$
$$+ \sum_{v \in V} \sum_{(i,m) \in S^A_v} \sum_{k \in K_v} C^O_{ik} o_{imvk}$$

and

$$H(X, c) = \min \sum_{(i,m) \in S^A} \sum_{k \in K : J_{ik} = -1} C^P_{ik} (r_{imkc} + r^E_{imkc})$$
$$s.t. \ (5.21) - (5.33), \text{with } \Omega = \{c\}.$$

The master problem consists of the first stage, but with iteratively added variables and constraints to reflect the recourse costs. The subproblems consider fixed first stage decisions, and are solved for each scenario to supply optimality cuts to the master problem.

The problem (5.1) - (5.34) has relatively complete recourse, since feasibility in the second stage is guaranteed if the inventory levels do not exceed the capacities of the inventories. Hence, for each feasible solution to the first stage, the second stage has always a feasible solution (it suffices to delay the unloading when necessary). The details for solving the problem are given in Section 5.4.2. To solve problems with a large number of scenarios, the sample average approximation method is used as described in Section 5.4.1.

5.4.1 Sample average approximation method

To solve the SMIRP with many scenarios, we apply the sample average approximation method [21]. First we consider M separate sets of scenarios. Each set of scenarios, $i \in \{1, \dots M\}$ contains a small number of m scenarios, $\{c^{i1}, \dots, c^{im}\}$. The model (5.1) - (5.34) is solved for each set of scenarios i using a decomposition approach. Let X^i denote the obtained first stage solution. The M candidate solutions $X^1, \ldots, X^M$, are then compared using a different, and much larger, set of n scenarios $\{\hat{c}^1, \ldots, \hat{c}^n\}$. The best solution is given by $X^* = argmin\{z_n(X^i) : i \in \{1, \ldots, M\}\}$ where $z_n(X) = C(X) + \frac{1}{n}\sum_{j=1}^{n} H(X, \hat{c}^j)$.

With the first stage solutions $X^1, \ldots, X^M$ being obtained, the optimal values are denoted $z_m^i = z_m(X^i) = C(X^i) + \frac{1}{m}\sum_{j=1}^{m} H(X^i, c^{ij})$. The average value over all sets of scenarios, $\bar{z}_m = \frac{1}{M}\sum_{i=1}^{M} z_m^i$ is a statistical estimate for a lower bound on the optimal value of the true problem.

For the larger set of n scenarios, which can be regarded as a benchmark scenario set representing the true distribution (see [12]), the cost $z_n(X^i)$ of each solution $X^i, i \in \{1, \ldots, M\}$ is computed as well as $X^* = argmin\{z_n(X^i) : i \in \{1, \ldots, M\}\}$. The best value, $z_n(X^*)$, is a statistical estimate for an upper bound on the optimal value. The estimated optimality gap (GAP) is given by $GAP = z_n(X^*) - \bar{z}_m$.

When employing a scenario generation method it is desirable that no matter which set of scenarios is used, by solving the two-stage model, one obtains approximately the same value for the optimal solution. This is named as stability requirement conditions in [12]. Here we evaluate stability (following [21]) through the computations of the variances:

$$\hat{\sigma}^2_{z_n(X^*)} = \frac{1}{(n-1)n}\sum_{j=1}^{n}\left(C(X^*) + H(X^*, \hat{c}^j) - z_n(X^*)\right)^2, \tag{5.35}$$

$$\hat{\sigma}^2_{\bar{z}_m} = \frac{1}{(M-1)M}\sum_{i=1}^{M}(z_m^i - \bar{z}_m)^2, \tag{5.36}$$

where $\hat{\sigma}^2_{\bar{z}_m}$ is the variance between samples and $\hat{\sigma}^2_{z_n(X^*)}$ is the variance within the larger sample. The estimated variance of the estimated optimality gap is

$$\hat{\sigma}^2_G = \hat{\sigma}^2_{z_n(X^*)} + \hat{\sigma}^2_{\bar{z}_m}.$$

5.4.2 Optimization process

To solve the model (5.1) - (5.34) for a set of scenarios Ω, we first solve to optimality a master problem including only one scenario. Since a feasible solution to the first stage can be completed with a feasible solution to the second stage for each scenario, the resulting values for the first stage decision variables are feasible for the complete problem with all scenarios. However, we need to check whether the solution is optimal for the complete model. To do that we check, for each scenario, whether there is backlogged demand when the deliveries are made as early as possible. If such a scenario with backlogged demand is found, we add to the master problem additional variables and constraints (which are implied by the time constraints and inventory constraints) enforcing the backlogged demand to be counted in the objective function. Then the revised master problem is solved again, and the process is repeated until all the optimality constraints are satisfied. Hence, as in the L-shaped method, the master problem initially disregards the recourse cost, and an improved estimation of the recourse cost is gradually added to the master problem by solving optimality subproblems and adding corresponding cuts. The algorithm may also be terminated if the additional recourse cost added in an iteration is less than a given small amount ϵ. A formal description of this process is given below.

Algorithm 4 Optimization procedure for an input set of scenarios Ω.

1: Choose a scenario $c \in \Omega$
2: Solve the master problem with one scenario, c
3: **while** There are new violated optimality cuts and a change in the objective function greater than ϵ **do**
4: Add all the violated optimality cuts
5: Solve again the master problem with the new cuts
6: **end while**

Next we explain how separation of constraints imposing backlog for each scenario (optimality cuts) is done in each iteration.

The backlog variables are bounded as follows:

$$r_{imkc} \geq R_{ik}t_{imc} - S^O_{ik} - \sum_{n \leq m-1} \sum_{v \in V} q_{invk}, (i,m) \in S^A, k \in K : J_{ik} = -1, c \in \Omega, \quad (5.37)$$

$$r^E_{imkc} \geq R_{ik}t^E_{imc} - S^O_{ik} - \sum_{n \leq m} \sum_{v \in V} q_{invk}, (i,m) \in S^A, m < \mu_i, k \in K : J_{ik} = -1, c \in \Omega, \quad (5.38)$$

$$r^E_{i\mu_i kc} \geq R_{ik}t^E_{i\mu_i c} + R_{ik}t^+_{ic} + \underline{S}_{ik} - S^O_{ik} - \sum_{n \leq \mu_i} \sum_{v \in V} q_{invk}, i \in N, k \in K : J_{ik} = -1, c \in \Omega. \quad (5.39)$$

Constraints (5.37) - (5.38) are implied by (5.28) - (5.30) (adding alternately (5.29) and (5.30) from (i,m) to $(i,1)$ and then (5.28)), and from the non-negativity requirements on the inventory variables (5.33). Constraints (5.39) are implied by (5.28) - (5.30) and by (5.32).

The minimum backlog occurs when the time variables t_{imc} and t^E_{imc} are set to the earliest feasible times. Once these variables are defined, separation over (5.37) - (5.38) is trivial since the right hand side is fixed. So we focus now on finding tight bounds for the time variables. First observe that the starting and ending times of each operation are established either from the (maximum) inventory levels (inventory constraints) or from the duration of the several operations the ships perform (time constraints). In the first case we need to ensure that the inventory capacity is not exceeded. Hence we have:

$$t_{imc} \geq \frac{S^O_{ik} + \sum_{n \leq m-1} \sum_{v \in V} q_{invk} - \overline{S}_{ik}}{R_{ik}}, \quad (i,m) \in S^A, k \in K : J_{ik} = -1, c \in \Omega, \quad (5.40)$$

$$t^E_{imc} \geq \frac{S^O_{ik} + \sum_{n \leq m} \sum_{v \in V} q_{invk} - \overline{S}_{ik}}{R_{ik}}, \quad (i,m) \in S^A, k \in K : J_{ik} = -1, c \in \Omega. \quad (5.41)$$

Constraints (5.40) and (5.41) follow from (5.28) - (5.32). For a given feasible solution for the first stage, the right hand sides of (5.40) and (5.41) are constant.

For the second case, the time variables are determined from the time constraints (5.21) - (5.27). For a feasible solution of the first stage, and for each scenario, most of the constraints (5.21) - (5.27) are not tight and many variables do not need to be considered. We can see that the t^E_{imc}-variables are bounded by (5.21) while the t_{imc}-variables are bounded by (5.22) (from the end time of the visit to the same port) and by (5.23) (from the last ship operation). These cases can be represented in a network $\mathcal{N} = (P, A, W)$, where P is the set of nodes, A is the set of arcs and W is the set of weights. The set of nodes P is given by the origin of each ship, represented by O_v, a node (i,m) representing the starting time of each port visit and a node

$(\overline{i,m})$ representing the end time of the visit. Each arc in A corresponds to a routing variable set to one. That is, there is an arc from node O_v to node (i,m) if $x^O_{imv} = 1$, and there is an arc from node $(\overline{i,m})$ to node (j,n) if $x_{imjnv} = 1$ for some v. The arcs have weights $T^O_{ivc} + T^W_{imc}$ and $T_{ijvc} + T^W_{jnc}$, respectively. There is an arc from node (i,m) to node $(\overline{i,m})$ with weight $\sum_{v\in V}\sum_{k\in K_v} T^S_{ik} o_{imvk} + \sum_{v\in V}\sum_{k\in K_v} T^Q_{ik} q_{imvk}$, and there is an arc from node $(\overline{i,m})$ to node $(i,m+1)$ with weight T^B_i. Finally, we consider an arc from O_v to each node visited by ship v. The weight from O_v to (i,m) is given by the right hand side of (5.40) and the weight from O_v to $\overline{(i,m)}$ is given by the right hand side of (5.41).

The weight of each path from one origin to a node gives a lower bound for the time variable corresponding to that node. Hence the earliest time associated to a node corresponds to the weight of the longest path from one origin to that node (one can always establish an artificial origin which is linked to all ship origins O_v and with null weight). Since the graph is acyclic, finding the longest path to each node can be done in polynomial time. However, for this particular graph, it is easy to derive a linear labeling correcting algorithm.

The time variables can then be restricted using these paths or sub-paths. For each (sub)path $\Pi^{(i,m)}_{(j,n)}$, from visit (j,n) to visit (i,m) of a ship v, we define the set of nodes (port visits) as $\mathcal{N}(\Pi^{(i,m)}_{(j,n)})$ and the set of arcs as $\mathcal{A}\left(\Pi^{(i,m)}_{(j,n)}\right)$. Let (i_v, m_v) denote the first visit of ship v after leaving the origin.

If the earliest time for a visit $(i,m) \in S^A$ is determined only by the schedule of operations for a given ship $v \in V$, then t_{imc} and t^E_{imc} are restricted as follows:

$$
\begin{aligned}
t_{imc} \geq & \sum_{(\ell,u)\in\mathcal{N}(\Pi^{(i,m)}_{(i_v,m_v)})} T^W_{\ell uc} + \sum_{(\ell,u)\in\mathcal{N}(\Pi^{(i,m)}_{(i_v,m_v)})\setminus\{(i,m)\}} \sum_{k\in K} \left(T^S_{\ell k} o_{\ell uvk} + T^Q_{\ell k} q_{\ell uvk}\right) \\
& + T^O_{i_v vc} + \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi^{(i,m)}_{(i_v,m_v)})} T_{\ell tvc} \\
& - T\left(1 + \mid \mathcal{A}(\Pi^{(i,m)}_{(i_v,m_v)}) \mid - x^O_{i_v m_v v} - \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi^{(i,m)}_{(i_v,m_v)})} x_{\ell utwv}\right),
\end{aligned} \tag{5.42}
$$

$$
\begin{aligned}
t^E_{imc} \geq & \sum_{(\ell,u)\in\mathcal{N}(\Pi^{(i,m)}_{(i_v,m_v)})} T^W_{\ell uc} + \sum_{(\ell,u)\in\mathcal{N}(\Pi^{(i,m)}_{(i_v,m_v)})} \sum_{k\in K} \left(T^S_{\ell k} o_{\ell uvk} + T^Q_{\ell k} q_{\ell uvk}\right) \\
& + T^O_{i_v vc} + \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi^{(i,m)}_{(i_v,m_v)})} T_{\ell tvc} \\
& - T\left(1 + \mid \mathcal{A}(\Pi^{(i,m)}_{(i_v,m_v)}) \mid - x^O_{i_v m_v v} - \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi^{(i,m)}_{(i_v,m_v)})} x_{\ell utwv}\right).
\end{aligned} \tag{5.43}
$$

Validity of (5.42) and (5.43) is implied by (5.21) - (5.23). In Appendix we provide a list of the remaining inequalities defined for each possible subpath.

The overall separation procedure for each iteration works as follow:

Algorithm 5 Separation procedure

Construct the network $\mathcal{N} = (P, A, W)$
Determine the longest path from the origin to each node
Associate the corresponding time variables to each node
Set the backlog variables to the minimum value using (5.37)-(5.39)
for each node **do**
  if the corresponding backlog variable has value strictly greater than its value in the current solution **then**
    add the inequality (5.37)-(5.39) determining its value
    add the time constraints (5.40)-(5.43) corresponding to the weight of the longest path
    (Use subpaths of each ship that are not contained in other subpaths of the same ship in the critical path)
  end if
end for

Example 5.4.1. *Consider an instance with 2 ships, v_1 and v_2, and 3 ports and assume that there is only one scenario. Hence we omit the corresponding scenario index from all variables and parameters. Let the paths resulting from the first stage solution be $x^O_{11v_1} = x_{1132v_1} = 1$ and $x^O_{21v_2} = x_{2131v_2} = x_{3112v_2} = 1$. Assume the weights of the arcs are those given in Figure 5.2. For instance, $T^B_1 = T^B_3 = 0.5$, $\sum_{k\in K}\left(T^S_{1k} o_{11v_1k} + T^Q_{1k} q_{11v_1k}\right) = 1$ and $T_{13v_1} = 6$.*

For simplicity we omit arcs with weights resulting from (5.40) and (5.41).

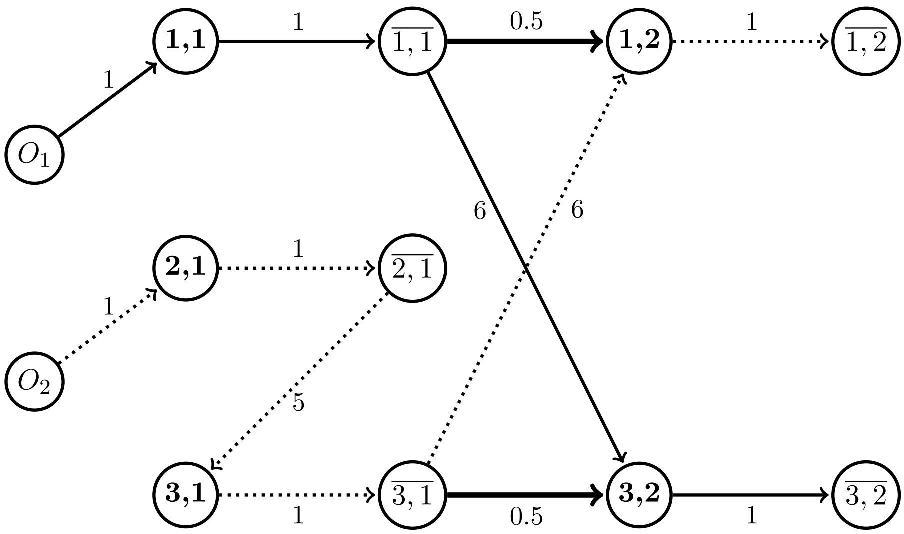

Figure 5.2: Example of a graph G for a set of three ports and two ships.

We can see that $t_{11} = 1, t^E_{11} = 2, t_{21} = 1, t^E_{21} = 2, t_{31} = 7, t^E_{31} = 8, t_{12} = 14, t^E_{12} = 15, t_{32} = 8.5, t^E_{32} = 9.5$.

Suppose there is backlog at nodes $(1,2)$ *and* $(3,2)$*. In addition to the inequalities* (5.37) *for* $(1,2)$ *and* $(3,2)$ *defining the lower bound on the backlog, the following inequalities, corresponding* *to the critical paths to nodes* $(1,2)$ *and* $(3,2)$ *are added to limit the time variables:*

$$\begin{aligned}
t_{31}^{E} &\geq \sum_{k\in K}\left(T_{2k}^{S}o_{21v_2k}+T_{2k}^{Q}q_{21v_2k}\right)+\sum_{k\in K}\left(T_{3k}^{S}o_{31v_2k}+T_{3k}^{Q}q_{31v_2k}\right)\\
&\quad +T_{2v_2}^{O}+T_{23v_2}-T(2-x_{021v_2}-x_{2131v_2}),\\
t_{12} &\geq \sum_{k\in K}\left(T_{2k}^{S}o_{21v_2k}+T_{2k}^{Q}q_{21v_2k}\right)+\sum_{k\in K}\left(T_{3k}^{S}o_{31v_2k}+T_{3k}^{Q}q_{31v_2k}\right)\\
&\quad +T_{2v_2}^{O}+T_{23v_2}+T_{31v_2}-T(3-x_{021v_2}-x_{2131v_2}-x_{3112v_2}),\\
t_{32} &\geq t_{31}^{E}+T_{3}^{B}.
\end{aligned}$$

5.5 Stochastic times and sample scenarios generation

In the SMIRP problem, the sailing and waiting times at ports are assumed to be random, following known probability distributions. We now describe the distributions used and how scenarios are generated for the stochastic programming model.

For the sailing times we assume that there are three potential events that affect all the sailing times simultaneously. These correspond to “good weather”, “moderate weather” and “bad weather”. For good weather, the sailing times are obtained directly from the sailing distance and the ship speed. For moderate weather, the sailing times are 1.5 times the corresponding sailing times in good weather, and for bad weather the sailing times are 2.0 times those in good weather. From the historical data for the season we are considering, a probability is associated to each event.

Contrary to the sailing times, where the weather usually affects all the islands simultaneously, waiting times due to port occupancy depend only on the port. For each visit to each port, we assume that the random variable indicating whether the port is occupied or not follows a Bernoulli distribution with parameter $p \in [0,1]$ (p is the probability of the port being occupied). If the port is occupied then the random variable W indicating the waiting time is given by a truncated exponential distribution

$$F(w)=\begin{cases} 0, & w<0,\\ (1-e^{-\lambda w})/A, & 0\leq w\leq M,\\ 1, & w>M,\end{cases}$$

where $A=1-e^{-M\lambda}$, and λ is such that the expected value of waiting time is $1/\lambda-\frac{Me^{-M\lambda}}{A}$, and M represents the maximum waiting time. Parameters p, λ, and M are obtained from historical data.

The weather events are trivially generated using the given probabilities. For each visit to each port the waiting times are randomly generated as follows: let $p \in [0,1]$ be the probability of the port being occupied. Generate an uniform random variable $U_1 \in [0,1]$. If $U_1 > p$ we assume that the port is not occupied. Otherwise we randomly generate a waiting time from the truncated exponential distribution using the inverse transformation method. The waiting time is given by

$$W=\begin{cases} \frac{ln(1-AU_2)}{-\lambda}, & \text{if } U_1\leq p;\\ 0, & \text{if } U_1>p.\end{cases}$$

where U_2 is an uniform random variable, $U_2 \in [0,1]$.

To generate the set of scenarios, Ω, we first fix the number of scenarios $n = \mid \Omega \mid$ *a priori*. Then each scenario is generated separately, first by generating the sailing times at random and then by generating a random waiting time for each port visit.

5.6 Computational results

In this section we report the results from the computational experimentation conducted to test the stochastic model. All computations were performed using the optimization software Xpress Optimizer Version 20.00.05 with Xpress Mosel Version 3.0.0, on a computer with processor Intel Core 2 Duo 2.2GHz and with 4GB of RAM. In Algorithm 4 we use $\epsilon = 0.01$. Ten real-world instances are used in the testing, considering two different ships, seven ports, four products, and a time horizon of eight days. The instances differ on the initial inventory levels.

First we test the effectiveness of the decomposition method, through a comparison by solving the full stochastic programming model directly using commercial software. Then, we test the sample average approximation method using the decomposition method. Finally we compute estimations of the Value of the Stochastic Solution and the Expected Value of Perfect Information.

5.6.1 Effectiveness of decomposed model

To test the effectiveness of the decomposed model we compared its performance with the use of Xpress Optimizer to directly solve the stochastic programming model with 10 scenarios. The results are reported in Table 5.1. The column "Opt" gives the optimal values, the columns "Nodes" indicate the number of branch and bound nodes, the columns "Seconds" report the running time in seconds to solve the instance. For the decomposed model we report additionally the number of cuts added in the column "Ncuts" and the number of iterations in the column "Iterations", that is, the number of times we solve the separation problem to add backlog and time constraints.

Table 5.1: Effectiveness of the L-shaped method

		full model		**decomposed model**			
Instance	Opt	Nodes	Seconds	Nodes	Seconds	Ncuts	Iterations
1	16210	5888	1498	3503	390	20	3
2	17610	20292	5397	16436	1061	24	4
3	18500	8495	2111	3863	434	65	3
4	17248.6	9253	1644	5377	526	78	4
5	15410	8177	2284	5356	384	18	3
6	18576.8	42774	7799	7247	854	32	4
7	15362.3	20720	4546	6658	603	45	4
8	17008	27740	5564	7558	740	28	4
9	13330	1911	362	1462	146	16	3
10	14550	46407	9200	3821	351	25	4
Average	16380.57	19165.7	4040.5	6128.1	548.9	35.1	3.6

As expected, the running times of the decomposition method are much lower than the running times obtained by solving the complete model. Additionally, we can see that the number of times the separation problem is called is at most 4 and few cuts are added.

5.6.2 Testing different sizes of sets of scenarios

Next we follow the solution approach described in Section 5.4, see [21]. Each instance is solved for M independent sets of scenarios, each set i containing m scenarios.

We conducted tests for $m = 10$ and $m = 50$. In all cases we consider $M = 10$ and the solutions are evaluated using a bigger set of $n = 1000$ scenarios. For each value of m we give two tables (Tables 5.2 and 5.3 for $m = 10$ and Tables 5.4 and 5.5 for $m = 50$). In the first table we present, for each instance, $z_n(X^*)$, $\bar{z}_m$, GAP, $\hat{\sigma}_{z_n(X^*)}$, $\hat{\sigma}_{\bar{z}_m}$, $\hat{\sigma}_G$. In the second table we give, for each instance, the running time to solve the M problems (one problem for each set of scenarios of size m) using the decomposition method, "Seconds M", and the time to compute $z_n(X^k), k \in \{1, \ldots, M\}$ "Seconds n", the average number of iterations, "Iterations", to solve the M problems, that is, the average number of times we solve the separation problem, and the average number of cuts (5.37)-(5.39) added, "Cuts".

Table 5.2: Bounds and variances for $m = 10$.

Instance	$z_n(X^*)$	$\bar{z}_m$	**GAP**	$\hat{\sigma}_{z_n(X^*)}$	$\hat{\sigma}_{\bar{z}_m}$	$\hat{\sigma}_G$
1	16956.8	16358	598.8	24.1	76.7	80.4
2	19080.3	18516.9	563.4	47.2	173.8	180.1
3	21150.9	19660.2	1490.7	95.6	265.1	281.8
4	19613.9	18750.8	863.1	198.4	293.1	353.9
5	18813.2	16658.5	2154.7	72.3	194.2	207.3
6	21182.1	19743.3	1438.8	104.7	210.7	235.3
7	16694.8	16509.5	185.3	76.6	196.8	211.1
8	19325.2	18664.2	661	71.0	227.8	238.6
9	14335.6	14139	196.6	115.2	238.1	264.5
10	17636.8	16721.6	915.2	127.0	324.5	348.5
Average	18479.0	17572.2	906.8	93.2	220.1	240.1

Table 5.3: Times, average number of iterations and average number of cuts for $m = 10$.

Instance	**Seconds M**	**Seconds n**	**Iterations**	**Cuts**
1	77	35	3	16.4
2	180.5	38.5	4	16.4
3	95.57	45.7	3	16.4
4	104.9	42.5	3.4	18.4
5	118.3	41.2	3	15.2
6	181.7	50.9	3	14.4
7	133.5	29.3	3	13.9
8	117.7	54.3	3.2	19.6
9	37.1	42.9	3	20.2
10	123.1	40.7	3	16.3
Average	116.9	42.1	3.2	16.7

We can see that increasing m, the cost of the selected solution decreases in average by 2.8%. Also, the standard deviation $\hat{\sigma}_{\bar{z}_m}$ and the GAP have a little reduction. The price to pay for the improvement of the solution and reduction of variability is an increase in the average running times. The running time is, on average, approximately 2 minutes for $m = 10$, and increases to 10 minutes for $m = 50$.

5.6.3 Importance of a stochastic approach

To evaluate the importance of the stochastic approach we compute estimations of the Value of the Stochastic Solution (VSS) and the Expected Value of Perfect Information (EVPI). The

Table 5.4: Bounds and variances for $m = 50$.

Instance	$z_n(X^*)$	$\bar{z}_m$	**GAP**	$\hat{\sigma}_{z_n(X^*)}$	$\hat{\sigma}_{\bar{z}_m}$	$\hat{\sigma}_G$
1	16498.1	16352.4	145.7	11.5	44.2	45.7
2	19080.3	18375.8	704.5	47.2	219.3	224.3
3	20839.2	19542.5	1296.7	97.6	142.0	172.3
4	19401.7	18027.2	1374.5	198.4	241.2	312.3
5	18490.2	16160	2330.2	66.2	119.3	136.4
6	20177.1	19651.1	526	104.5	96.4	142.2
7	16358.8	16204.4	154.4	76.6	98.5	124.8
8	18148.3	18015.3	133	71.0	198.2	210.6
9	13877.7	13876.4	1.3	115.2	116.5	163.8
10	16875	16336.1	538.9	127.0	286.4	313.3
Average	17974.6	17254.1	720.5	91.5	156.2	184.5

Table 5.5: Times, average number of iterations and average number of cuts for $m = 50$.

Instance	**Seconds M**	**Seconds n**	**Iterations**	**Cuts**
1	426.5	51	3.3	18.8
2	838.2	64.8	3	16.5
3	658.3	59.4	3.1	18.9
4	682.8	57.8	3	20.1
5	627.2	63.9	3	16.8
6	751.9	50.2	4.2	16.8
7	573.3	76.1	3	16.3
8	791	49.5	3	19.8
9	174.5	51.5	4	18.8
10	524.3	56	3	17.8
Average	604.8	58.0	3.3	18.1

results are given in Table 5.6. To compute the VSS we solve the model with one scenario, where the stochastic parameters are set to their expected values. We used the sample average values (considering the larger sample), which are very similar to the theoretical expected values. Solving this deterministic model we obtain the well known expected value solution. The cost of this solution is given in column "EVS". In column $z_n(X^*)$ we give the corresponding value for $m = 50$, and in column "VSS" we give an estimation of the Value of the Stochastic Solution which is the difference between EVS and $z_n(X^*)$. In column "PI" we give the average value of the $n = 1000$ deterministic models, one for each scenario, and in column "EVPI" we give an estimation of the Expected Value of Perfect Information which is the difference $z_n(X^*) - PI$.

We can see, from Table 5.6, the gains for using stochastic programming instead of the deterministic model based on expected values are in general very high. In average, the expected value of the best solution is only 9% above the Expected Value of Perfect Information.

5.7 Conclusions

We presented a two-stage stochastic programming model with recourse for a maritime inventory routing problem where sailing times and port times are random. The model has the property that, for each scenario, a feasible solution to the first stage can always be completed with a feasible solution to the second stage. We proposed a decomposition method where, for a given first stage solution, optimality is checked for the complete model through an efficient separation method.

Ten instances based on real data are solved using the sample approximation method. Com-

Table 5.6: Estimating the VSS and EVPI

Instance	EVS	$z_n(X^*)$	VSS	PI	EVPI
1	42049.2	16498.1	25551.1	16210.1	288.0
2	33020.1	19080.3	13939.8	17620.2	1460.1
3	39871.2	20839.2	19032	18572.6	2266.6
4	49582.5	19401.7	30180.8	17285.9	2115.8
5	58053.6	18490.2	39563.4	15461.7	3028.5
6	41503.3	20177.1	21326.2	18942.0	1235.1
7	32256.6	16204.4	16052.2	15497.8	706.6
8	64144	18148.3	45995.7	17023.8	1124.5
9	41125.7	13877.7	27248	13354.0	523.7
10	25623.9	16875	8748.9	15066.9	1808.1
Average	42723.01	17959.2	24763.8	16503.5	1455.7

putational tests have shown the effectiveness of the decomposition method, and the importance in the use of stochastic programming instead of a deterministic approach.

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Appendix

The following inequalities, for each $(i,m) \in S^A$ and $v \in V$, are implied by (5.21) - (5.23):

$$\begin{aligned} t_{imc} \;\geq\; & t_{jnc} + \sum_{(\ell,u)\in\mathcal{N}(\Pi_{(j,n)}^{(i,m)})\setminus\{(j,n)\}} T_{\ell uc}^{W} + \sum_{(\ell,u)\in\mathcal{N}(\Pi_{(j,n)}^{(i,m)})\setminus\{(i,m)\}} \sum_{k\in K} \left(T_{\ell k}^{S} o_{\ell uvk} + T_{\ell k}^{Q} q_{\ell uvk}\right) \\ & + \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi_{(j,n)}^{(i,m)})} T_{\ell tvc} - T\left(\mid \mathcal{A}(\Pi_{(j,n)}^{(i,m)}) \mid - \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi_{(j,n)}^{(i,m)})} x_{\ell utwv}\right), \end{aligned} \tag{5.44}$$

$$\begin{aligned} t_{imc} \;\geq\; & t_{jnc}^{E} + \sum_{(\ell,u)\in\mathcal{N}(\Pi_{(j,n)}^{(i,m)})\setminus\{(j,n)\}} T_{\ell uc}^{W} + \sum_{(\ell,u)\in\mathcal{N}(\Pi_{(j,n)}^{(i,m)})\setminus\{(j,n),(i,m)\}} \sum_{k\in K} \left(T_{\ell k}^{S} o_{\ell uvk} + T_{\ell k}^{Q} q_{\ell uvk}\right) \\ & + \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi_{(j,n)}^{(i,m)})} T_{\ell tvc} - T\left(\mid \mathcal{A}(\Pi_{(j,n)}^{(i,m)}) \mid - \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi_{(j,n)}^{(i,m)})} x_{\ell utwv}\right), \end{aligned} \tag{5.45}$$

$$
\begin{aligned}
t^E_{imc} \;\geq\; & t_{jnc} + \sum_{(\ell,u)\in\mathcal{N}(\Pi^{(i,m)}_{(j,n)})\setminus\{(j,n)\}} T^W_{\ell uc} + \sum_{(\ell,u)\in\mathcal{N}(\Pi^{(i,m)}_{(j,n)})} \sum_{k\in K} \left(T^S_{\ell k} o_{\ell uvk} + T^Q_{\ell k} q_{\ell uvk}\right) \\
& + \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi^{(i,m)}_{(j,n)})} T_{\ell tvc} - T\left(\mid \mathcal{A}(\Pi^{(i,m)}_{(j,n)}) \mid - \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi^{(i,m)}_{(j,n)})} x_{\ell utwv} \right), \qquad (5.46)
\end{aligned}
$$

$$
\begin{aligned}
t^E_{imc} \;\geq\; & t^E_{jnc} + \sum_{(\ell,u)\in\mathcal{N}(\Pi^{(i,m)}_{(j,n)})\setminus\{(j,n)\}} T^W_{\ell uc} + \sum_{(\ell,u)\in\mathcal{N}(\Pi^{(i,m)}_{(j,n)})\setminus\{(j,n)\}} \sum_{k\in K} \left(T^S_{\ell k} o_{\ell uvk} + T^Q_{\ell k} q_{\ell uvk}\right) \\
& + \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi^{(i,m)}_{(j,n)})} T_{\ell tvc} - T\left(\mid \mathcal{A}(\Pi^{(i,m)}_{(j,n)}) \mid - \sum_{(\ell,u,t,w)\in\mathcal{A}(\Pi^{(i,m)}_{(j,n)})} x_{\ell utw} \right). \qquad (5.47)
\end{aligned}
$$