Universidade de Aveiro Departamento de Matemática
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Artur Miguel Capêllo Brito da Cruz

Cálculo Quântico Simétrico

Symmetric Quantum Calculus

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Tese de doutoramento apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, Programa Doutoral em Matemática e Aplicações, PDMA 2008-2012, da Universidade de Aveiro e Universidade do Minho, realizada sob a orientação científica do Doutor Delfim Fernando Marado Torres, Professor Associado com Agregação do Departamento de Matemática da Universidade de Aveiro, e co-orientação da Doutora Natália da Costa Martins, Professora Auxiliar do Departamento de Matemática da Universidade de Aveiro.

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Cálculo quântico, cálculo das variações, condições necessárias do tipo Euler-Lagrange, derivada simétrica, integral diamond, desigualdades integrais, escalas temporais.

resumo

Generalizamos o cálculo Hahn variacional para problemas do cálculo das variações que envolvem derivadas de ordem superior. Estudamos o cálculo quântico simétrico, nomeadamente o cálculo quântico alpha,beta-simétrico, q-simétrico e Hahn-simétrico. Introduzimos o cálculo quântico simétrico variacional e deduzimos equações do tipo Euler-Lagrange para o cálculo q-simétrico e Hahn simétrico. Definimos a derivada simétrica em escalas temporais e deduzimos algumas das suas propriedades. Finalmente, introduzimos e estudamos o integral diamond que generaliza o integral diamond-alpha das escalas temporais.

keywords

Quantum calculus, calculus of variations, necessary optimality conditions of Euler-–Lagrange type, symmetric derivative, diamond integral, integral inequalities, time scales.

abstract

We generalize the Hahn variational calculus by studying problems of the calculus of variations with higher-order derivatives. The symmetric quantum calculus is studied, namely the alpha,beta-symmetric, the q-symmetric, and the Hahn symmetric quantum calculus. We introduce the symmetric quantum variational calculus and an Euler-Lagrange type equation for the q-symmetric and Hahn's symmetric quantum calculus is proved. We define a symmetric derivative on time scales and derive some of its properties. Finally, we introduce and study the diamond integral, which is a refined version of the diamond-alpha integral on time scales.

Contents

Introduction

Four years ago I started the Ph.D. Doctoral Programme in Mathematics and Applications offered jointly by University of Aveiro and University of Minho. In the first year we, students, had several one-semester courses in distinct fields of mathematics. Many interesting subjects were discussed in these courses and one day, one of this discussions led my Advisor, Professor Delfim F. M. Torres, to propose me that the subject of my Ph.D. thesis should be symmetric quantum calculus and/or symmetric calculus on time scales. Gladly, I accepted the offer and here we present the result of our work.

In classical calculus, the symmetric derivative is defined by

$$f^s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

The definition of the symmetric derivative emerged from the necessity to extend the notion of differentiability to points that do not have classical derivative and its applications are important in many problems, namely in the study of trigonometric series. Its study began in the mid-19th century and since then, according to Thomson [117], many important figures in analysis contributed for this topic. Thomson also highlights the importance of authors such Charzyński, Denjoy, Kijtchine, Marcinkiewicz, Sierpinski and Zygmund (see, e.g., [42, 45, 75, 93, 113, 124]). Thomson refers that the study of the symmetric properties of real functions is as relevant as ever. In the last decades we have seen many new material produced in this topic (see, e.g., [16, 22, 23, 24, 38, 52, 53, 54, 59, 61, 77, 78, 101]), in spite the difficulty of obtaining new results.

However, the idea of my advisor was not to develop new results for the classical symmetric calculus, but to introduce the symmetric quantum calculus. The fusion of these two subjects, symmetric calculus and quantum calculus, seemed natural to him since he is an active researcher on the time scale theory (some approaches of quantum calculus can be seen as a particular case of time scale calculus) and knew the absence of the notion of the symmetric derivative on time scale calculus.

Quantum from the Latin word "quantus" (in Portuguese "quantos") literally means how much. Usually we associate the term quantum to the minimum amount of any measure or entity. In mathematics, the quantum calculus refers to the "calculus without limits". Usually, the quantum calculus is identified with the q-calculus and the h-calculus and both of them were studied by Kac and Cheung in their book [73].

In 1750 Euler proved the pentagonal number theorem which was the first example of a q-series and, in some sense, he introduced the q-calculus. The q-derivative was (re)introduced by Jackson [70] and for $q \in]0,1[$, the q-derivative of a function $f : \overline{q^{\mathbb{Z}}} \to \mathbb{R}$ is defined by

$$D_q[f](t) := \frac{f(qt) - f(t)}{(q-1)t}, \quad t \neq 0.$$

Today, and according to Ernst [48, 49], the majority of scientists who use q-calculus are physicists, and he cites Jet Wimp [122]:

> "The field has expanded explosively, due to the fact that applications of basic hypergeometric series to the diverse subjects of combinatorics, quantum theory, number theory, statistical mechanics, are constantly being uncovered. The subject of q-hypergeometric series is the frog-prince of mathematics, regal inside, warty outside."

For $h > 0$, the h-derivative of a function $f : h\mathbb{Z} \to \mathbb{R}$ is defined by

$$D_h[f](t) := \frac{f(t+h) - f(t)}{h}$$

and is also known as finite difference operator. Taylor's "Methods Incrementorum" [116] is considered the first reference of the h-calculus or the calculus of finite differences but it is Jacob Stirling [114] who is considered the founder of the h-calculus. In 1755, Leonhard Euler [51] introduced the symbol Δ for differences. Some of the more important works about h-differences are from authors like Boole [29], Markoff [94], Whittaker and Ronbison [121], Nörlund [105], Milne-Thomson [102] or Jordan [72] (we ordered them historically). Note that the finite difference operator is a discretization of the classical derivative and an immediate application of the h-derivative (but also the q-derivative) is in numerical analysis, especially in numerical differential equations, which aim at the numerical solution of ordinary and partial differential equations. For a deeper understanding of quantum calculus and its history we refer the reader to [48, 50, 62, 73, 76].

Another type of quantum calculus is the Hahn's quantum calculus which can be seen as a generalization of both q-calculus and h-calculus. For $q \in]0, 1[$ and $\omega \geqslant 0$, Hahn's difference operator is defined by

$$D_{q,\omega}[f] := \frac{f(qt+\omega) - f(t)}{(q-1)t + \omega}, \quad t \neq \frac{\omega}{1-q},$$

where f is a function defined on a real interval that contains $\omega_0 := \dfrac{\omega}{1-q}$. Although Hahn defined this operator in 1949, only in 2009 Aldwoah [3, 4] constructed its inverse operator. This construction also provided us tools to generalize both q-calculus and the h-calculus to functions defined in real intervals instead of the sets $\overline{q^{\mathbb{Z}}}$ and $h\mathbb{Z}$. We remark here that the term "quantum calculus" is applied in the literature (and in this thesis) in different contexts: is used to denote the h-calculus and q-calculus, but also to denote the Hahn calculus (that is also a "calculus without limits" but in this theory the domain of each function is a real interval instead of the quantum sets $\overline{q^{\mathbb{Z}}}$ or $h\mathbb{Z}$).

In a first step towards the development of the symmetric quantum calculus we decided to study an application of Hahn's quantum calculus. Since both my advisors mainly work in Calculus of Variations and Optimal Control, we decided to study the higher-order Hahn quantum variational calculus. The calculus of variations is concerned with the problem of extremising functionals and in Chapter 3 the reader can find a brief introduction to this subject. For a study of the calculus of variations within the quantum calculus, we suggest the articles [17, 18, 89, 97].

In this work, we have tried to develop the calculus of variations in the context for symmetric quantum calculus. But soon we understood that this is not always possible, since for example for the α, β-symmetric quantum calculus (see Chapter 5) we do not have (yet?) a Fundamental Theorem of Integral Calculus and/or an Integration by Parts formula. For the symmetric q-calculus and Hahn's symmetric quantum calculus, we were able to introduce and develop the respective calculus of variations.

Another subject that was of our interest was the symmetric calculus on time scales. In 1988, Hilger [67] introduced the theory of time scales which is a theory that was created in order to unify and to extend discrete and continuous analysis into a single theory. A time scale is a nonempty closed subset of $\mathbb{R}$ and both q-calculus and h-calculus can be seen as a particular case of time scale calculus. Since then, many works arose and we highlight [1, 2, 27, 28] for the time scale calculus and [7, 19, 20, 25, 26, 55, 56, 57, 68, 69, 86, 87, 90, 91, 92, 95, 96, 98, 100, 118] for the calculus of variations on time scales. We have made progresses in this field and we present our results in Chapter 8. We were not successful to define a symmetric integral in the time scale context, mainly because there is not a symmetric integral for the classical case. However, we were able to generalize the diamond-α integral which, in some sense, is an attempt to define a symmetric integral on time scales.

We divided this thesis in two parts. In the first part we present some preliminaries about time scale calculus, quantum calculus and calculus of variations. The second part is divided in six chapters where we present the original work and our conclusions. Specifically, in Chapter 4 we develop the higher-order Hahn Quantum Variational Calculus proving the Euler–Lagrange equation for the higher-order problem of the calculus of variations within the Hahn quantum calculus. In Chapter 5 we develop a Symmetric Quantum Calculus: in Section 5.3.3 we present some mean value theorems for the symmetric calculus and in Section 5.3.4 we prove Hölder's, Cauchy-Schwarz's and Minkowski's inequalities in the setting of the α, β-symmetric calculus. In Chapter 6 we develop the q-Symmetric Calculus presenting a necessary optimality condition and a sufficient optimality condition for variational problems involving the q-symmetric derivative. In Chapter 7 we generalize for the Hahn Symmetric Quantum Variational Calculus the results obtained in the preceding chapter. We also prove, in Section 7.3.3, that Leitmann's Direct Method can be applied in Hahn's symmetric quantum variational calculus. In Chapter 8 we introduce the symmetric derivative on time scales and prove some properties of this new derivative. In Section 8.3 we introduce the diamond integral and deduce some results for this new integral.

In Chapter 9, we write our conclusions and some possible directions for future work.

Part I

Synthesis

Chapter 1

Time Scale Calculus

The theory of time scales was born in 1988 with the Ph.D. thesis of Stefan Hilger, done under the supervision of Bernd Aulbach [67]. The aim of this theory was to unify various definitions and results from the theories of discrete and continuous dynamical systems, and to extend such theories to more general classes of dynamical systems.

The calculus of time scales is nowadays an area of great interest of many mathematicians; this can be showed by the numerous papers published in this field.

For a general introduction to the theory of time scales we refer the reader to the excellent books [27, 28]. Here we only give those definitions and results needed in this thesis.

As usual, $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ denote, respectively, the set of real, integer and natural numbers.

A nonempty closed subset of $\mathbb{R}$ is called a *time scale* and is denoted by $\mathbb{T}$. Thus $\mathbb{R}, \mathbb{Z}$ and $\mathbb{N}$ are trivial examples of time scales. Other examples of time scales are: $[-2,5] \cup \mathbb{N}$, $h\mathbb{Z} := \{hz : z \in \mathbb{Z}\}$ for some $h > 0$, $\overline{q^{\mathbb{Z}}} := \{q^z : z \in \mathbb{Z}\} \cup 0$ for some $q \neq 1$ and the Cantor set.

We assume that a time scale has the topology inherited from $\mathbb{R}$ with the standard topology.

We consider two jump operators: the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$, defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$$

with $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m). A point $t \in \mathbb{T}$ is said to be *right-dense*, *right-scattered*, *left-dense* and *left-scattered* if $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ and $\rho(t) < t$, respectively. A point $t \in \mathbb{T}$ is called *dense* if it is right and left dense. The *backward graininess function* $\nu : \mathbb{T} \to [0, +\infty[$ is defined by

$$\nu(t) := t - \rho(t), \quad t \in \mathbb{T}.$$

The *forward graininess function* $\mu : \mathbb{T} \to [0, +\infty[$ is defined by

$$\mu(t) := \sigma(t) - t, \quad t \in \mathbb{T}.$$

It is clear that when $\mathbb{T} = \mathbb{R}$ one has $\sigma(t) = t = \rho(t)$ and $\nu(t) = \mu(t) = 0$ for any $t \in \mathbb{T}$. When $\mathbb{T} = h\mathbb{Z}$ (for some $h > 0$), then $\sigma(t) = t + h$, $\rho(t) = t - h$ and $\nu(t) = \mu(t) = h$.

In order to introduce the definition of the delta derivative and the nabla derivative we define the sets $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ by

$$\mathbb{T}^\kappa := \begin{cases} \mathbb{T}\backslash\{t_1\} & \text{, if } \mathbb{T} \text{ has a left-scattered maximum } t_1 \\ \\ \mathbb{T} & \text{, otherwise} \end{cases}$$

$$\mathbb{T}_\kappa := \begin{cases} \mathbb{T}\backslash\{t_2\} & \text{, if } \mathbb{T} \text{ has a right-scattered minimum } t_2 \\ \\ \mathbb{T} & \text{, otherwise.} \end{cases}$$

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is delta differentiable at $t \in \mathbb{T}^\kappa$ if there exists a number $f^\Delta(t)$ such that, for all $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \leqslant \varepsilon\,|\sigma(t) - s|\,,$$

for all $s \in U$. We call $f^\Delta(t)$ the *delta derivative* of f at t and we say that f is delta differentiable if f is delta differentiable for all $t \in \mathbb{T}^\kappa$.

We say that a function $f : \mathbb{T} \to \mathbb{R}$ is nabla differentiable at $t \in \mathbb{T}_\kappa$ if there exists a number $f^\nabla(t)$ such that, for all $\varepsilon > 0$, there exists a neighborhood V of t such that

$$\left|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)\right| \leqslant \varepsilon\,|\rho(t) - s|\,,$$

for all $s \in V$. We call $f^\nabla(t)$ the *nabla derivative* of f at t and we say that f is nabla differentiable if f is nabla differentiable for all $t \in \mathbb{T}_\kappa$.

In order to simplify expressions, we denote $f \circ \sigma$ by f^σ and $f \circ \rho$ by f^ρ.

The delta and nabla derivatives verify the following properties.

Theorem 1.0.1 (cf. [27, 28])**.** *Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then we have the following:*

1. *If f is delta differentiable at $t \in \mathbb{T}^\kappa$ or nabla differentiable at $t \in \mathbb{T}_\kappa$, then f is continuous on t;*

2. *If f is continuous at $t \in \mathbb{T}^\kappa$ and t is right-scattered, then f is delta differentiable at t with*
$$f^\Delta(t) = \frac{f^\sigma(t) - f(t)}{\sigma(t) - t};$$

3. *f is delta differentiable at a right-dense point $t \in \mathbb{T}^\kappa$ if the limit*
$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$
exists and in that case
$$f^\Delta(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s};$$

4. *If f is delta differentiable at $t \in \mathbb{T}^\kappa$, then*
$$f^\sigma(t) = f(t) + \mu(t) f^\Delta(t)\,;$$

5. *If f is continuous at $t \in \mathbb{T}_k$ and t is left-scattered, then f is nabla differentiable at t with*
$$f^{\nabla}(t) = \frac{f^{\rho}(t) - f(t)}{\rho(t) - t};$$
6. *f is nabla differentiable at a left-dense point $t \in \mathbb{T}_k$ if the limit*
$$\lim_{s \to t} \frac{f(t) - f(s)}{t - s}$$
exists and in that case
$$f^{\nabla}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s};$$
7. *If f is nabla differentiable at $t \in \mathbb{T}_{\kappa}$, then*
$$f^{\rho}(t) = f(t) - \nu(t) f^{\nabla}(t).$$

Remark 1.0.2. 1. *If $\mathbb{T} = \mathbb{R}$, then the delta and nabla derivative are the usual derivative.*

2. *If $\mathbb{T} = \mathbb{Z}$, then the delta derivative is the* forward difference operator *defined by*
$$f^{\Delta}(t) = f(t+1) - f(t)$$
and the nabla derivative is the backward difference operator *defined by*
$$f^{\nabla}(t) = f(t) - f(t-1).$$
3. *For any time scale $\mathbb{T}$, if f is constant, then $f^{\Delta} = f^{\nabla} \equiv 0$; if $f(t) = kt$ for some constant k, then $f^{\Delta} = f^{\nabla} \equiv k$.*

The delta derivative satisfies the following properties.

Theorem 1.0.3 (cf. [27, 28])**.** *Let $f, g : \mathbb{T} \to \mathbb{R}$ be delta differentiable functions. Let $t \in \mathbb{T}^{\kappa}$ and $\lambda \in \mathbb{R}$. Then*

1. *The function $f + g$ is delta differentiable with*
$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t);$$
2. *The function λf is delta differentiable with*
$$(\lambda f)^{\Delta}(t) = \lambda f^{\Delta}(t);$$
3. *The function fg is delta differentiable with*
$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f^{\sigma}(t) g^{\Delta}(t);$$
4. *The function f/g is delta differentiable with*
$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t) g(t) - f(t) g^{\Delta}(t)}{g(t) g^{\sigma}(t)}$$
provided that $g(t) g^{\sigma}(t) \neq 0$.

The nabla derivative satisfies the following properties.

Theorem 1.0.4 (cf. [27, 28]). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be nabla differentiable functions. Let $t \in \mathbb{T}_\kappa$ and $\lambda \in \mathbb{R}$. Then*

1. *The function $f + g$ is nabla differentiable with*
$$(f+g)^{\nabla}(t) = f^{\nabla}(t) + g^{\nabla}(t);$$
2. *The function λf is nabla differentiable with*
$$(\lambda f)^{\nabla}(t) = \lambda f^{\nabla}(t);$$
3. *The function fg is nabla differentiable with*
$$(fg)^{\nabla}(t) = f^{\nabla}(t)\, g(t) + f^{\rho}(t)\, g^{\nabla}(t);$$
4. *The function f/g is nabla differentiable with*
$$\left(\frac{f}{g}\right)^{\nabla}(t) = \frac{f^{\nabla}(t)\, g(t) - f(t)\, g^{\nabla}(t)}{g(t)\, g^{\rho}(t)}$$
provided that $g(t)\, g^{\rho}(t) \neq 0$.

In this chapter all the intervals are time scales intervals, that is, for $a, b \in \mathbb{T}$, with $a < b$,
$$[a, b]_{\mathbb{T}}$$
denotes the set $\{t \in \mathbb{T} : a \leqslant t \leqslant b\}$; open intervals and half-open intervals are defined accordingly.

Theorem 1.0.5 (Mean value theorem for the delta derivative (cf. [28])). *Let f be a continuous function on $[a, b]_{\mathbb{T}}$ that is delta differentiable on $[a, b[_{\mathbb{T}}$. Then there exist $\xi, \tau \in [a, b[_{\mathbb{T}}$ such that*
$$f^{\Delta}(\tau) \leqslant \frac{f(b) - f(a)}{b - a} \leqslant f^{\Delta}(\xi).$$

Corollary 1.0.6 (Rolle's theorem for the delta derivative (cf. [28])). *Let f be a continuous function on $[a, b]_{\mathbb{T}}$ that is delta differentiable on $[a, b[_{\mathbb{T}}$ and satisfies*
$$f(a) = f(b).$$
Then there exist $\xi, \tau \in [a, b[_{\mathbb{T}}$ such that
$$f^{\Delta}(\tau) \leqslant 0 \leqslant f^{\Delta}(\xi).$$

Another useful result is the following consequence of the Mean Value Theorem.

Corollary 1.0.7 (cf. [28]). *Let f be a continuous function on $[a, b]_{\mathbb{T}}$ that is delta differentiable on $[a, b[_{\mathbb{T}}$. If $f^{\Delta} > 0$, $f^{\Delta} < 0$, $f^{\Delta} \geqslant 0$ or $f^{\Delta} \leqslant 0$, then f is increasing, decreasing, nondecreasing or nonincreasing on $[a, b]_{\mathbb{T}}$, respectively.*

Note that similar mean values Theorems can be enunciate for the nabla derivative.

As usually expected when we generalize some theory, we can lose some nice properties. This situation happens with the chain rule on time scale calculus. The chain rule as we know it in classical calculus is not valid for a general time scale. For a simple example, let $\mathbb{T} = \mathbb{Z}$ and $f, g : \mathbb{T} \to \mathbb{R}$ be such that $f(t) = t^2$ and $g(t) = 3t$. It is simple to verify that $f^{\Delta}(t) = t + \sigma(t) = 2t + 1$ and $g^{\Delta}(t) = 3$, and hence

$$(f \circ g)^{\Delta}(t) = 18t + 9 \neq f^{\Delta}(g(t)) \cdot g^{\Delta}(t) = 18t + 3.$$

However, there are some special chain rules in the context of time scale calculus.

Theorem 1.0.8 (cf. [27]). *Assume that $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. Let $f : \tilde{\mathbb{T}} \to \mathbb{R}$. If $\nu^{\Delta}(t)$ and $f^{\tilde{\Delta}}(\nu(t))$ exist for $t \in \mathbb{T}^{\kappa}$, then*

$$(f \circ \nu)^{\Delta}(t) = \left(f^{\tilde{\Delta}} \circ \nu\right)(t) \cdot \nu^{\Delta}(t)$$

(where $\tilde{\Delta}$ denotes the delta derivative with respect to the time scale $\tilde{\mathbb{T}}$).

For other special chain rules on time scales we refer the reader to [27, 28].

A function $f : \mathbb{T} \to \mathbb{R}$ is called *rd-continuous* if it is continuous at all right-dense points and if its left-sided limits exist and are finite at all left-dense points. We denote the set of all rd-continuous functions on $\mathbb{T}$ by $C_{rd}(\mathbb{T}, \mathbb{R})$ or simply by C_{rd}. Analogously, a function $f : \mathbb{T} \to \mathbb{R}$ is called *ld-continuous* if it is continuous at all left-dense points and if its right-sided limits exist and are finite at all right-dense points. We denote the set of all ld-continuous functions on $\mathbb{T}$ by $C_{ld}(\mathbb{T}, \mathbb{R})$ or simply by C_{ld}. The following results concerning rd-continuity and ld-continuity are useful.

Theorem 1.0.9 (cf. [27]). *Let $\mathbb{T}$ be a time scale and $f : \mathbb{T} \to \mathbb{R}$ a given function.*

1. *If f is continuous, then f is rd-continuous and ld-continuous;*

2. *The forward jump operator, σ, is rd-continuous and the backward jump operator, ρ, is ld-continuous;*

3. *If f is rd-continuous, then f^{σ} is also rd-continuous; if f is ld-continuous, then f^{ρ} is also ld-continuous;*

4. *If $\mathbb{T} = \mathbb{R}$, then f is continuous if, and only if, f is ld-continuous and if, and only if, f is rd-continuous;*

5. *If $\mathbb{T} = h\mathbb{Z}$ (for some $h > 0$), then f is rd-continuous and ld-continuous.*

Delta derivatives of higher-order are defined in the standard way: for $n \in \mathbb{N}$, we define the n^{th}-delta derivative of f to be the function $f^{\Delta^n} : \mathbb{T}^{k^n} \to \mathbb{R}$, defined by $f^{\Delta^n} = \left(f^{\Delta^{n-1}}\right)^{\Delta}$ provided $f^{\Delta^{n-1}}$ is delta differentiable on $\mathbb{T}^{k^n} := \left(\mathbb{T}^{k^{n-1}}\right)^{k}$. Analogously, we can define the *nabla derivatives of higher-order*.

The set of all delta differentiable functions with rd-continuous delta derivatives is denoted by C^1_{rd} or $C^1_{rd}(\mathbb{T}, \mathbb{R})$. In general, for a fixed $n \in \mathbb{N}$, we say that $f \in C^n_{rd}$ if an only if $f^{\Delta} \in C^{n-1}_{rd}$, where $C^0_{rd} = C_{rd}$. Similarly, for a fixed $n \in \mathbb{N}$, we say that $f \in C^n_{ld}$ if an only if $f^{\nabla} \in C^{n-1}_{ld}$, where $C^0_{ld} = C_{ld}$

A function $F : \mathbb{T} \to \mathbb{R}$ is said to be a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$, provided

$$F^{\Delta}(t) = f(t)$$

for all $t \in \mathbb{T}^{\kappa}$. For all $a, b \in \mathbb{T}$, $a < b$, we define the *delta integral* of f from a to b (or on $[a, b]_{\mathbb{T}}$) by

$$\int_a^b f(t)\,\Delta t = F(b) - F(a).$$

Theorem 1.0.10 (cf. [27, 28]). *Every rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ has a delta antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^t f(\tau)\,\Delta\tau \quad \textit{for} \quad t \in \mathbb{T}$$

is a delta antiderivative of f.

The delta integral satisfies the following property:

$$\int_t^{\sigma(t)} f(\tau)\,\Delta\tau = (\sigma(t) - t) f(t).$$

A function $G : \mathbb{T} \to \mathbb{R}$ is said to be a *nabla antiderivative* of $g : \mathbb{T} \to \mathbb{R}$ provided

$$G^{\nabla}(t) = g(t)$$

for all $t \in \mathbb{T}_k$. For all $a, b \in \mathbb{T}$, $a < b$, we define the *nabla integral* of g from a to b (or on $[a, b]_{\mathbb{T}}$) by

$$\int_a^b g(t)\,\nabla t = G(b) - G(a).$$

Theorem 1.0.11 (cf. [27, 28]). *Every ld-continuous function $f : \mathbb{T} \to \mathbb{R}$ has a nabla antiderivative. In particular, if $t_0 \in \mathbb{T}$, then F defined by*

$$F(t) := \int_{t_0}^t f(\tau)\,\nabla\tau \quad \textit{for} \quad t \in \mathbb{T}$$

is a nabla antiderivative of f.

The nabla integral satisfies the following property:

$$\int_{\rho(t)}^{t} g(\tau)\,\nabla\tau = (t - \rho(t))\, g(t).$$

Next, we review the properties of the delta integral.

Theorem 1.0.12 (cf. [27, 28]). *Let $f, g : \mathbb{T} \to \mathbb{R}$ be delta integrable on $[a, b]_{\mathbb{T}}$. Let $c \in [a, b]_{\mathbb{T}}$ and $\lambda \in \mathbb{R}$. Then*

1. $\displaystyle\int_a^a f(t)\,\Delta t = 0$;
2. $\displaystyle\int_a^b f(t)\,\Delta t = \int_a^c f(t)\,\Delta t + \int_c^b f(t)\,\Delta t$;

3. $\displaystyle\int_a^b f(t)\,\Delta t = -\int_b^a f(t)\,\Delta t$;

4. $f+g$ *is delta integrable on* $[a,b]_{\mathbb{T}}$ *and*
$$\int_a^b (f+g)(t)\,\Delta t = \int_a^b f(t)\,\Delta t + \int_a^b g(t)\,\Delta t;$$

5. λf *is delta integrable on* $[a,b]_{\mathbb{T}}$ *and*
$$\int_a^b \lambda f(t)\,\Delta t = \lambda \int_a^b f(t)\,\Delta t;$$

6. fg *is delta integrable on* $[a,b]_{\mathbb{T}}$;

7. *For* $p>0$, $|f|^p$ *is delta integrable on* $[a,b]_{\mathbb{T}}$;

8. *If* f *and* g *are delta differentiable, then*
$$\int_a^b f^{\sigma}(t)\,g^{\Delta}(t)\,\Delta t = f(t)\,g(t)\Big|_a^b - \int_a^b f^{\Delta}(t)\,g(t)\,\Delta t;$$

9. *If* f *and* g *are delta differentiable, then*
$$\int_a^b f(t)\,g^{\Delta}(t)\,\Delta t = f(t)\,g(t)\Big|_a^b - \int_a^b f^{\Delta}(t)\,g^{\sigma}(t)\,\Delta t;$$

10. *If* $f(t)\geqslant 0$ *on* $[a,b]_{\mathbb{T}}$, *then*
$$\int_a^b f(t)\,\Delta t \geqslant 0;$$

11. *If* $|f(t)|\leqslant g(t)$ *on* $[a,b]_{\mathbb{T}}$, *then*
$$\left|\int_a^b f(t)\,\Delta t\right| \leqslant \int_a^b g(t)\,\Delta t.$$

The formulas 8. and 9. in Theorem 1.0.12 are called *integration by parts* formulas. Analogously, the nabla integral satisfies the corresponding properties.

Remark 1.0.13. 1. *If* $\mathbb{T}=\mathbb{R}$, *then*
$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,\nabla t = \int_a^b f(t)\,dt$$
where the last integral is the usual Riemman integral;

2. *If* $\mathbb{T}=h\mathbb{Z}$ *for some* $h>0$, *and* $a,b\in\mathbb{T}$, $a<b$, *then*
$$\int_a^b f(t)\,\Delta t = \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} hf(kh)$$
and
$$\int_a^b f(t)\,\nabla t = \sum_{k=\frac{a}{h}+1}^{\frac{b}{h}} hf(kh);$$

3. *If $a < b$ and $[a, b]_{\mathbb{T}}$ consists of only isolated points, then*

$$\int_a^b f(t)\,\Delta t = \sum_{t\in[a,b[_{\mathbb{T}}} \mu(t)\, f(t)$$

and

$$\int_a^b f(t)\,\nabla t = \sum_{t\in]a,b]_{\mathbb{T}}} \nu(t)\, f(t)\,.$$

The calculus on time scales using the delta derivative and the delta integral is usually known by delta-calculus; the calculus done with the nabla derivative and the nabla integral is known by nabla-calculus. The first developments on the time scale theory was done essentially using the delta-calculus. However, for some applications, in particular to solve problems of the Calculus of Variations and Control Theory in economics [15], is often more convenient to work backwards in time, that is, using the nabla-calculus.

Recently, Caputo provided a duality technique [39] which allows to obtain nabla results on time scales from the delta theory and vice versa (see also [64, 92, 96]).

Another approach to the theory of time scale is the diamond-α calculus, that uses the notion of the diamond-α derivative. To introduce this notion (as defined in [109, 111, 112]) let $t, s \in \mathbb{T}$ and define $\mathbb{T}_k^k := \mathbb{T}_k \cap \mathbb{T}^k$, $\mu_{ts} := \sigma(t) - s$ and $\eta_{ts} := \rho(t) - s$. We say that a function $f : \mathbb{T} \to \mathbb{R}$ is *diamond-α differentiable* on $t \in \mathbb{T}_\kappa^\kappa$ if there exists a number $f^{\diamondsuit_\alpha}(t)$ such that, for all $\varepsilon > 0$, there exists a neighborhood U of t such that, for all $s \in U$,

$$\left|\alpha\left[f^\sigma(t) - f(s)\right]\eta_{ts} + (1-\alpha)\left[f^\rho(t) - f(s)\right]\mu_{ts} - f^{\diamondsuit_\alpha}(t)\,\mu_{ts}\eta_{ts}\right| \leqslant \varepsilon\left|\mu_{ts}\eta_{ts}\right|.$$

A function f is said to be diamond-α differentiable provided $f^{\diamondsuit_\alpha}(t)$ exists for all $t \in \mathbb{T}_\kappa^\kappa$.

Theorem 1.0.14 ([109])**.** *Let $0 \leqslant \alpha \leqslant 1$ and let f be both nabla and delta differentiable on $t \in \mathbb{T}_\kappa^\kappa$. Then f is diamond-α differentiable at t and*

$$f^{\diamondsuit_\alpha}(t) = \alpha f^{\Delta}(t) + (1-\alpha)\, f^{\nabla}(t)\,. \tag{1.0.1}$$

Remark 1.0.15. *If $\alpha = 1$, then the diamond-α derivative reduces to the delta derivative; if $\alpha = 0$, the diamond-α derivative coincides with the nabla derivative.*

Note that equality (1.0.1) is given as the definition of the diamond-α derivative in [112].

Theorem 1.0.16 (cf. [112])**.** *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α differentiable functions. Let $t \in \mathbb{T}_\kappa^\kappa$ and $\lambda \in \mathbb{R}$. Then*

1. *The function $f + g$ is diamond-α differentiable with*

$$(f+g)^{\diamondsuit_\alpha}(t) = f^{\diamondsuit_\alpha}(t) + g^{\diamondsuit_\alpha}(t)\,;$$

2. *The function λf is diamond-α differentiable with*

$$(\lambda f)^{\diamondsuit_\alpha}(t) = \lambda f^{\diamondsuit_\alpha}(t)\,;$$

3. *The function fg is diamond-α differentiable with*

$$(fg)^{\diamondsuit_\alpha}(t) = f^{\diamondsuit_\alpha}(t)\, g(t) + \alpha f^{\sigma}(t)\, g^{\Delta}(t) + (1-\alpha)\, f^{\rho}(t)\, g^{\nabla}(t)\,.$$

Let $a, b \in \mathbb{T}$, $a < b$, $h : \mathbb{T} \to \mathbb{R}$ and $\alpha \in [0,1]$. The *diamond-α integral* of h from a to b (or on $[a,b]_{\mathbb{T}}$) is defined by

$$\int_a^b h(t)\, \diamondsuit_\alpha t = \alpha \int_a^b h(t)\, \Delta t + (1-\alpha) \int_a^b h(t)\, \nabla t,$$

provided that h is delta integrable and nabla integrable on $[a,b]_{\mathbb{T}}$.

Theorem 1.0.17 (cf. [88])**.** *Let $f, g : \mathbb{T} \to \mathbb{R}$ be diamond-α integrable on $[a,b]_{\mathbb{T}}$. Let $c \in [a,b]_{\mathbb{T}}$ and $\lambda \in \mathbb{R}$. Then*

1. $\displaystyle\int_a^a f(t)\, \diamondsuit_\alpha t = 0;$

2. $\displaystyle\int_a^b f(t)\, \diamondsuit_\alpha t = \int_a^c f(t)\, \diamondsuit_\alpha t + \int_c^b f(t)\, \diamondsuit_\alpha t;$

3. $\displaystyle\int_a^b f(t)\, \diamondsuit_\alpha t = -\int_b^a f(t)\, \diamondsuit_\alpha t;$

4. *$f + g$ is diamond-α integrable on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b (f+g)(t)\, \diamondsuit_\alpha t = \int_a^b f(t)\, \diamondsuit_\alpha t + \int_a^b g(t)\, \diamondsuit_\alpha t;$$

5. *λf is diamond-α integrable on $[a,b]_{\mathbb{T}}$ and*

$$\int_a^b \lambda f(t)\, \diamondsuit_\alpha t = \lambda \int_a^b f(t)\, \diamondsuit_\alpha t;$$

6. *fg is diamond-α integrable on $[a,b]_{\mathbb{T}}$;*

7. *For $p > 0$, $|f|^p$ is diamond-α integrable on $[a,b]_{\mathbb{T}}$;*

8. *If $f(t) \geqslant 0$ on $[a,b]_{\mathbb{T}}$, then*

$$\int_a^b f(t)\, \diamondsuit_\alpha t \geqslant 0;$$

9. *If $|f(t)| \leqslant g(t)$ on $[a,b]_{\mathbb{T}}$, then*

$$\left| \int_a^b f(t)\, \diamondsuit_\alpha t \right| \leqslant \int_a^b g(t)\, \diamondsuit_\alpha t.$$

Remark 1.0.18. *In [13], Ammi et al. show that, in general, we do not have*

$$\left(s \to \int_a^s f(\tau)\, \diamondsuit_\alpha \tau \right)^{\diamondsuit_\alpha} (t) = f(t)\,, \quad t \in \mathbb{T}.$$

Hence, we do not have an integral by parts formula for the diamond-α integral and this is a great limitation for the development of the Calculus of Variations for problems involving diamond-α integrals.

Remark 1.0.19. *1. If $\mathbb{T} = \mathbb{R}$, then a bounded function f on $[a, b]$ is diamond-α integrable on $[a, b]$ if, and only if, is Riemann integrable on $[a, b]$, and in this case*

$$\int_a^b f(t) \diamondsuit_\alpha t = \int_a^b f(t)\, dt;$$

2. If $\mathbb{T} = \mathbb{Z}$ and $a, b \in \mathbb{T}$, $a < b$, then $f : \mathbb{T} \to \mathbb{R}$ is diamond-α integrable on $[a, b]_\mathbb{T}$ and

$$\int_a^b f(t) \diamondsuit_\alpha t = \alpha f(a) + (1 - \alpha) f(b) + \sum_{t=a+1}^{b-1} f(t).$$

Chapter 2

Quantum Calculus

Quantum calculus is usually known as "calculus without limits". There are several types of quantum calculus. In this thesis we are concerned with h-calculus, q–calculus and Hahn's calculus. In each type of quantum calculus, we can make a study towards the future, known as the forward quantum calculus, or towards the past, the backward quantum calculus. Moreover, there are different approaches for each type of quantum calculus. Some authors choose the set of study to be a subset of the real numbers containing isolated points, others use subintervals of $\mathbb{R}$.

In this chapter we review some definitions and basic results about the quantum calculus.

2.1 The h-calculus

The h-calculus is also known as the calculus of finite differences and Boole at [29] described it as:

> "The calculus of finite differences may be strictly defined as the science which is occupied about the ratios of the simultaneous increments of quantities mutually dependent. The differential calculus is occupied about the limits of which such ratios approach as the increments are indefinitely diminished."

Many authors contributed to the calculus of finite differences like Boole [29, 30], Milne-Thomson [102], Nörlund [105], just to name a few. In this section we present the h-calculus as Kac and Cheung do in their book [73].

We consider that the set of study is

$$h\mathbb{Z} := \{hz : z \in \mathbb{Z}\}$$

for some $h > 0$.

Definition 2.1.1 (cf. [73])**.** *Let $f : h\mathbb{Z} \to \mathbb{R}$ be a function and let $t \in h\mathbb{Z}$. The* h-derivative *of f or the* forward difference operator *of f at t is given by*

$$\Delta_h [f] (t) := \frac{f (t + h) - f (t)}{h}.$$

Note that, if a function f is differentiable (in the classical sense) at t, then

$$\lim_{h\to 0}\Delta_h [f](t) = f'(t),$$

where f' is usual the derivative.

Example 2.1.2. *Let $f(t) = t^n$. Then*

$$\Delta_h [f](t) = nt^{n-1} + \frac{n(n-1)}{2}t^{n-2}h + \ldots + h^{n-1}.$$

The h-derivative has the following properties.

Theorem 2.1.3 (cf. [73])**.** *Let $f, g : h\mathbb{Z} \to \mathbb{R}$ be functions, $t \in h\mathbb{Z}$ and $\alpha \in \mathbb{R}$. Then*

1. $\Delta_h [f+g](t) = \Delta_h [f](t) + \Delta_h [g](t)$;
2. $\Delta_h [\alpha f](t) = \alpha \Delta_h [f](t)$;
3. $\Delta_h [fg](t) = f(t)\Delta_h [g](t) + \Delta_h [f](t)\, g(t+h)$;
4. $\Delta_h \left[\frac{f}{g}\right](t) = \frac{\Delta_h [f](t)\, g(t) - f(t)\Delta_h [g](t)}{g(t)\, g(t+h)}$.

Definition 2.1.4 (cf. [73])**.** *A function $F : h\mathbb{Z} \to \mathbb{R}$ is said to be an h-*antiderivative *of $f : h\mathbb{Z} \to \mathbb{R}$ provided*

$$\Delta_h [F](t) = f(t),$$

for all $t \in h\mathbb{Z}$.

Definition 2.1.5 (cf. [73])**.** *Let $f : h\mathbb{Z} \to \mathbb{R}$ be a function. If $a, b \in h\mathbb{Z}$, we define the h-*integral *of f from a to b by*

$$\int_a^b f(t)\,\Delta_h t = \begin{cases} h\Big[f(a) + f(a+h) + \ldots + f(b-h)\Big] & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -h\Big[f(b) + f(b+h) + \ldots + f(a-h)\Big] & \text{if} \quad a > b. \end{cases}$$

Theorem 2.1.6 (Fundamental theorem of the h-integral calculus (cf. [73]))**.** *Let $F : h\mathbb{Z} \to \mathbb{R}$ be an h-antiderivative of $f : h\mathbb{Z} \to \mathbb{R}$. If $a, b \in h\mathbb{Z}$, then*

$$\int_a^b f(t)\,\Delta_h t = F(b) - F(a).$$

The h-integral has the following properties.

Theorem 2.1.7 (cf. [73])**.** *Let $f, g : h\mathbb{Z} \to \mathbb{R}$ be functions. Let $a, b, c \in h\mathbb{Z}$ and $\alpha \in \mathbb{R}$. Then*

1. $\int_a^b f(t)\,\Delta_h t = \int_a^c f(t)\,\Delta_h t + \int_c^b f(t)\,\Delta_h t$;

2. $\displaystyle\int_a^b f(t)\,\Delta_h t = -\int_b^a f(t)\,\Delta_h t$;

3. $\displaystyle\int_a^b (f+g)(t)\,\Delta_h t = \int_a^b f(t)\,\Delta_h t + \int_a^b g(t)\,\Delta_h t$;

4. $\displaystyle\int_a^b \alpha f(t)\,\Delta_h t = \alpha\int_a^b f(t)\,\Delta_h t$;

5. $\displaystyle\int_a^b f(t)\,\Delta_h[g](t)\,d_h t = f(t)\,g(t)\Big|_a^b - \int_a^b D_h[f](t)\,g(t+h)\,\Delta_h t$.

We can also define a backward h-calculus where the *backward h-derivative*, or the *backward difference operator*, is defined by the following quotient

$$\nabla_h[f](t) := \frac{f(t) - f(t-h)}{h}.$$

The results of the backward h-calculus are similar to the forward h-calculus.

It is worth noting that the h-calculus (as presented by Kac and Cheung [73]) can be seen as a particular case of the time scale calculus.

2.2 The q-calculus

The q-derivative, like the h-derivatve, is a discretization of the classical derivative and therefore, has immediate applications in numerical analysis. However, and according to Ernst [48], is also a generalization of many subjects, like hypergeometric series, complex analysis and particle physics. The q-difference operator and its inverse operator, the Jackson q-integral, were first defined by Jackson [70, 71] and due to its applications the q-calculus is a popular subject today. In this section we present the q-calculus as Kac and Cheung do in their book [73].

Let $q \in \,]0,1[$ and let I be a real interval containing 0.

Definition 2.2.1 (cf. [73])**.** *Let $f : I \to \mathbb{R}$ be a function and let $t \in I$. The* q-derivative*, or* Jackson's difference operator*, of f at t is given by*

$$D_q[f](t) := \frac{f(qt) - f(t)}{(q-1)\,t}, \ \text{ if } t \neq 0\,,$$

and $D_q[f](0) := f'(0)$, provided f is differentiable at 0*.*

Note that, if a function is f differentiable (in the classical sense) at t, then

$$\lim_{q\to 1} D_q[f](t) = \lim_{q\to 1}\frac{f(qt) - f(t)}{(q-1)\,t} = f'(t)\,,$$

where f' is usual the derivative.

Example 2.2.2. *Let $f(t) = t^n$. Then*

$$D_q[f](t) = \frac{q^n - 1}{q - 1}t^{n-1} = \left(q^{n-1} + \ldots + 1\right)t^{n-1}.$$

The q-derivative has the following properties.

Theorem 2.2.3 (cf. [73]). *Let $f, g : I \to \mathbb{R}$ be functions and $\alpha \in \mathbb{R}$. Then*

1. $D_q[f+g](t) = D_q[f](t) + D_q[g](t)$;
2. $D_q[\alpha f](t) = \alpha D_q[f](t)$;
3. $D_q[fg](t) = f(t) D_q[g](t) + D_q[f](t) g(qt)$;
4. $D_q\left[\frac{f}{g}\right](t) = \frac{D_q[f](t) g(t) - f(t) D_q[g](t)}{g(t) g(qt)}$.

Definition 2.2.4 (cf. [73]). *A function $F : I \to \mathbb{R}$ is said to be a q-*antiderivative *of $f : I \to \mathbb{R}$ provided*

$$D_q[F](x) = f(x),$$

for all $t \in I$.

Definition 2.2.5 (cf. [73]). *Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ the q-*integral*, or* Jackson integral*, of f from a to b is given by*

$$\int_a^b f(t)\, d_q t := \int_0^b f(t)\, d_q t - \int_0^a f(t)\, d_q,$$

where

$$\int_0^x f(t)\, d_q t := (1-q)\, x \sum_{k=0}^{+\infty} q^k f\left(xq^k\right),\ x \in I,$$

*provided that the series converges at $x = a$ and $x = b$. In that case, f is called q-*integrable on *$[a, b]$. We say that f is q-*integrable over *I if it is q-*integrable *over $[a, b]$ for all $a, b \in I$.*

Theorem 2.2.6 (Fundamental theorem of the q-integral calculus (cf. [73])). *Let $F : I \to \mathbb{R}$ be an anti-derivative of $f : I \to \mathbb{R}$ and let F be continuous at 0. For $a, b \in I$ we have*

$$\int_a^b f(t)\, d_q t = F(b) - F(a).$$

The q-integral has the following properties.

Theorem 2.2.7 (cf. [73]). *Let $f, g : I \to \mathbb{R}$ be functions. Let $a, b, c \in I$ and $\alpha \in \mathbb{R}$. Then*

1. $\int_a^b f(t)\, d_q t = \int_a^c f(t)\, d_q t + \int_c^b f(t)\, d_q t$;
2. $\int_a^b f(t)\, d_q t = -\int_b^a f(t)\, d_q t$;
3. $\int_a^b (f+g)(t)\, d_q t = \int_a^b f(t)\, d_q t + \int_a^b g(t)\, d_q t$;
4. $\int_a^b \alpha f(t)\, d_q t = \alpha \int_a^b f(t)\, d_q t$;

5. $\displaystyle\int_a^b f(t)\, D_q[g](t)\, d_h t = f(t)\, g(t)\Big|_a^b - \int_a^b D_q[f](t)\, g(qt)\, d_h t.$

The *backward q-derivative* of $f : I \to \mathbb{R}$ at $t \neq 0$ is defined by the quotient

$$\frac{f(t) - f\left(q^{-1}t\right)}{\left(1-q^{-1}\right)t}$$

and from it one can obtain the backward q-quantum calculus. Note that the q-calculus as defined in this section is not a particular case of the time scale calculus. Namely, the q-integral does not coincide with the delta integral for $\mathbb{T} = \overline{q^{\mathbb{Z}}}$.

2.3 The Hahn calculus

Hahn, in [66], introduced his difference operator as a tool for constructing families of orthogonal polynomials. Hahn's quantum difference operator unifies (in the limit) Jackson's q-difference operator and the forward difference operator. Recently, Aldwoah [3] defined a proper inverse operator of Hahn's difference operator, and the associated integral calculus was developed in [3, 4, 14]. In this section we present the Hahn calculus as Aldwoah did in his Ph.D. thesis [3].

Let $q \in]0,1[$ and $\omega \geqslant 0$. We introduce the real number

$$\omega_0 := \frac{\omega}{1-q}.$$

Let I be a real interval containing ω_0. For a function f defined on I, the *Hahn difference operator* of f is given by

$$D_{q,\omega}[f](t) := \frac{f(qt+\omega) - f(t)}{(q-1)\,t + \omega}\,,\ \text{if } t \neq \omega_0\,,$$

and $D_{q,\omega}[f](\omega_0) := f'(\omega_0)$, provided f is differentiable at ω_0. We usually call $D_{q,\omega}[f]$ the *q,ω-derivative of f*, and f is said to be *q,ω-differentiable on I* if $D_{q,\omega}[f](\omega_0)$ exists.

Remark 2.3.1. *The $D_{q,\omega}$ operator generalizes (in the limit) the well known forward difference and the Jackson difference operators [48, 73]. Indeed, when $q \to 1$ we obtain the* forward difference operator

$$\Delta_\omega[f](t) := \frac{f(t+\omega) - f(t)}{\omega};$$

when $\omega = 0$ we obtain the Jackson difference operator

$$D_q[f](t) := \frac{f(qt) - f(t)}{(q-1)\,t}\,,\ \textit{if } t \neq 0\,,$$

and $D_q[f](0) = f'(0)$ provided $f'(0)$ exists. Notice also that, under appropriate conditions,

$$\lim_{q\to 1} D_{q,0}[f](t) = f'(t)\,.$$

Hahn's difference operator has the following properties.

Theorem 2.3.2 ([3, 4, 14]). *Let f and g be q,ω-differentiable on I and $t \in I$. One has,*

1. $D_{q,\omega}[f](t) \equiv 0$ *on I if, and only if, f is constant;*
2. $D_{q,\omega}\left[f+g\right](t) = D_{q,\omega}\left[f\right](t) + D_{q,\omega}\left[g\right](t)$*;*
3. $D_{q,\omega}\left[fg\right](t) = D_{q,\omega}\left[f\right](t)\, g(t) + f(qt+\omega)\, D_{q,\omega}\left[g\right](t)$*;*
4. $D_{q,\omega}\left[\frac{f}{g}\right](t) = \dfrac{D_{q,\omega}\left[f\right](t)\, g(t) - f(t)\, D_{q,\omega}\left[g\right](t)}{g(t)\, g(qt+\omega)}$ *if $g(t)\, g(qt+\omega) \neq 0$;*
5. $f(qt+\omega) = f(t) + (t(q-1)+\omega)\, D_{q,\omega}\left[f\right](t)$.

For $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ define $[k]_q := \dfrac{1-q^k}{1-q}$ and let $\sigma(t) := qt+\omega$, $t \in I$. Note that σ is a contraction, $\sigma(I) \subseteq I$, $\sigma(t) < t$ for $t > \omega_0$, $\sigma(t) > t$ for $t < \omega_0$, and $\sigma(\omega_0) = \omega_0$. The following technical result is used several times in this thesis.

Lemma 2.3.3 ([3, 14]). *Let $k \in \mathbb{N}$ and $t \in I$. Then,*

1. $\sigma^k(t) = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{k\text{-times}}(t) = q^k t + \omega\, [k]_q$*;*
2. $\left(\sigma^k(t)\right)^{-1} = \sigma^{-k}(t) = \dfrac{t - \omega\,[k]_q}{q^k}$.

Following [3, 4, 14] we define the notion of q,ω-*integral* (also known as the *Jackson–Nörlund integral*) as follows.

Definition 2.3.4. *Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ the q,ω*-integral of *f from a to b is given by*

$$\int_a^b f(t)\, d_{q,\omega}t := \int_{\omega_0}^b f(t)\, d_{q,\omega}t - \int_{\omega_0}^a f(t)\, d_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)\, d_{q,\omega}t := (x(1-q)-\omega) \sum_{k=0}^{+\infty} q^k f\left(q^k x + \omega\,[k]_q\right),\ x \in I\,,$$

provided that the series converges at $x = a$ and $x = b$. In that case, f is called q,ω-integrable on *$[a, b]$. We say that f is q,ω*-integrable over *I if it is q,ω*-integrable *over $[a, b]$ for all $a, b \in I$.*

Remark 2.3.5. *The q,ω*-integral *generalizes (in the limit) the Jackson q-integral and the Nörlund sum (cf. [73]). When $\omega = 0$, we obtain the* Jackson q-integral

$$\int_a^b f(t)\, d_qt := \int_0^b f(t)\, d_qt - \int_0^a f(t)\, d_qt,$$

where

$$\int_0^x f(t)\, d_qt := x(1-q) \sum_{k=0}^{+\infty} q^k f\left(xq^k\right)$$

provided that the series converge at $x = a$ and $x = b$. When $q \to 1$, we obtain the Nörlund sum

$$\int_a^b f(t)\,\Delta_\omega t := \int_{+\infty}^b f(t)\,\Delta_\omega t - \int_{+\infty}^a f(t)\,\Delta_\omega t,$$

where

$$\int_{+\infty}^x f(t)\,\Delta_\omega t := -\omega \sum_{k=0}^{+\infty} f(x + k\omega)$$

provided that the series converge at $x = a$ and $x = b$. Note that the Nörlund sum is, in some sense, a generalization of the h-integral for the case where a and b are any real numbers (not necessarily $a, b \in h\mathbb{Z}$ like in Definition 2.1.5).

Proposition 2.3.6 (cf. [3, 4, 14])**.** *If $f : I \to \mathbb{R}$ is continuous at ω_0, then f is q,ω*-integrable over *I.*

Theorem 2.3.7 (Fundamental theorem of Hahn's calculus [3, 14])**.** *Assume that $f : I \to \mathbb{R}$ is continuous at ω_0 and, for each $x \in I$, define*

$$F(x) := \int_{\omega_0}^x f(t)\, d_{q,\omega} t.$$

Then F is continuous at ω_0. Furthermore, $D_{q,\omega}[F](x)$ exists for every $x \in I$ with

$$D_{q,\omega}[F](x) = f(x).$$

Conversely,

$$\int_a^b D_{q,\omega}[f](t)\, d_{q,\omega} t = f(b) - f(a)$$

for all $a, b \in I$.

The q,ω-*integral* has the following properties.

Theorem 2.3.8 ([3, 4, 14])**.** *Let $f, g : I \to \mathbb{R}$ be q,ω*-integrable on *I, $a, b, c \in I$ and $k \in \mathbb{R}$. Then,*

1. $\displaystyle\int_a^a f(t)\, d_{q,\omega} t = 0;$
2. $\displaystyle\int_a^b k f(t)\, d_{q,\omega} t = k \int_a^b f(t)\, d_{q,\omega} t;$
3. $\displaystyle\int_a^b f(t)\, d_{q,\omega} t = -\int_b^a f(t)\, d_{q,\omega} t;$
4. $\displaystyle\int_a^b f(t)\, d_{q,\omega} t = \int_a^c f(t)\, d_{q,\omega} t + \int_c^b f(t)\, d_{q,\omega} t;$
5. $\displaystyle\int_a^b (f(t) + g(t))\, d_{q,\omega} t = \int_a^b f(t)\, d_{q,\omega} t + \int_a^b g(t)\, d_{q,\omega} t;$
6. *every Riemann integrable function f on I is q,ω*-integrable on *I;*

7. if $f, g : I \to \mathbb{R}$ are q,ω-differentiable and $a, b \in I$, then

$$\int_a^b f(t)\, D_{q,\omega}[g](t)\, d_{q,\omega}t = f(t)\, g(t) \Big|_a^b - \int_a^b D_{q,\omega}[f](t)\, g(qt+\omega)\, d_{q,\omega}t.$$

Property 7 of Theorem 2.3.8 is known as *q,ω-integration by parts*. Note that

$$\int_{\sigma(t)}^t f(\tau)\, d_{q,\omega}\tau = (t(1-q)-\omega)\, f(t).$$

Lemma 2.3.9 (cf. [33]). *Let $b \in I$ and f be q,ω-*integrable *over I. Suppose that*

$$f(t) \geqslant 0, \quad \forall t \in \left\{ q^n b + \omega\, [n]_q : n \in \mathbb{N}_0 \right\}.$$

1. If $\omega_0 \leqslant b$, then

$$\int_{\omega_0}^b f(t) d_{q,\omega}t \geqslant 0.$$

2. If $\omega_0 > b$, then

$$\int_b^{\omega_0} f(t) d_{q,\omega}t \geqslant 0.$$

Remark 2.3.10. *In general it is not true that*

$$\left| \int_a^b f(t)\, d_{q,\omega}t \right| \leqslant \int_a^b |f(t)|\, d_{q,\omega}t, \quad a, b \in I.$$

For a counterexample, see [3, 14]. This illustrates well the difference with other non-quantum integrals, e.g., the time scale integrals [88, 104].

Similarly to the previous sections, one can define the *backward Hahn's derivative* for $t \neq \omega_0$ by

$$\frac{f(t) - f\left(q^{-1}(t-\omega)\right)}{(1-q^{-1})\, t + q^{-1}\omega}.$$

To conclude this section, we note that the Hahn Calculus is not a particular case of the time scale calculus. We stress that the basic difference between the Hahn calculus and the time scale calculus is that in the Hahn calculus we deal with functions defined in real intervals and the derivative is a "discrete derivative", while in the time scale calculus, this kind of derivative is only defined for discrete time scales.

Chapter 3

Calculus of Variations

In 1696, Johann Bernoulli challenge the mathematical community (and in particular his brother James) to solve the following problem: in a vertical plane, consider two fixed points A and B, with A higher than B. Determine the curve from A to B along which a particle (initially at rest) slides in minimum time, with the only external force acting on the particle being gravity. This curve is called the brachistochrone (from Greek, brachistos - the shortest, chronos - time). This problem was solved by both Bernoulli brothers, but also by Newton, L'Hôpital, Leibniz and later by Euler and Lagrange. This problem is considered the first problem in calculus of variations and the solution provided by Euler was so resourceful that gave us the principles to solve other variational problems, such as:

-Find among all the closed curves with the same length, the curve enclosing the greatest area (this problem is called the isoperimetric problem);

-Find the shortest path (i.e., geodesic) between two given points on a surface;

-Find the curve between two given points in the plane that yields a surface of revolution of minimum area when revolved around a given axis. This curve is called the catenary and the surface is called a catenoid.

In the eighteenth century mathematicians like the Bernoulli brothers, Leibniz, Euler, Lagrange and Legendre all worked in theory of calculus of variations. In the nineteenth century names like Jacobi and Weierstrass contributed to the subject and in the twentieth century Hilbert, Noether, Tonelli, Lebesgue, just to name a few, made advances to the theory. For a deeper understanding of the history of the calculus of variations we refer the reader to Goldstine [65].

Historically, there is an intricate relation between calculus of variations and mechanics. In fact, many physical principles may be formulated in terms of variational problems, via Hamilton's principle of least action. Applications can be found in general relativity theory, quantum field theory and particle physics. Even recently, calculus of variations provided tools to discover the solution to the three-body problem [103]. However, applications of calculus of variations can be found in other areas such as economics [15] and optimal control theory [108].

The calculus of variations is concerned with the problem of extremizing functionals, and so, we can consider it as a branch of optimization.

Next, we review the basics of the theory of the calculus of variations.

Let $L(x, y, z)$ be a function with continuous first and second partial derivatives with respect to all its arguments. The simplest variational problem is defined by

$$\begin{cases} \mathcal{L}[y] = \displaystyle\int_a^b L\left(t, y(t), y'(t)\right) dt \quad \rightarrow \quad \text{extremize} \\ y \in C^1([a,b], \mathbb{R}) \\ y(a) = \alpha \\ y(b) = \beta \end{cases} \tag{3.0.1}$$

where $a, b, \alpha, \beta \in \mathbb{R}$ and $a < b$. By extremize we mean maximize or minimize.

We say that y is an admissible function for problem (3.0.1) if $y \in C^1([a,b], \mathbb{R})$ and y satisfies the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$. We say that y_* is a local minimizer (resp. local maximizer) for problem (3.0.1) if y_* is an admissible function an there exists $\delta > 0$ such that

$$\mathcal{L}[y_*] \leqslant \mathcal{L}[y] \quad (\text{resp. } \mathcal{L}[y_*] \geqslant \mathcal{L}[y])$$

for all admissible y with $\|y_* - y\| < \delta$ and where

$$\|y\| = \max_{t \in [a,b]} \left\{ |y(t)| + \left|y'(t)\right| \right\}.$$

A necessary condition for a function to be an extremizer for problem (3.0.1) is given by the Euler–Lagrange equation.

In what follows, $\frac{\partial L}{\partial y}$ denotes the partial derivative of L with respect to its $2nd$ argument and $\frac{\partial L}{\partial y'}$ denotes the partial derivative of L with respect to its $3rd$ argument.

Theorem 3.0.1 (Euler–Lagrange equation (cf. [120]))**.** *Under the hypotheses of the problem* (3.0.1)*, if an admissible function y_* is a local extremizer for problem* (3.0.1)*, then y_* satisfies the Euler–Lagrange equation*

$$\frac{\partial L}{\partial y}\left(t, y(t), y'(t)\right) = \frac{d}{dt}\left[\tau \rightarrow \frac{\partial L}{\partial y'}\left(\tau, y(\tau), y'(\tau)\right)\right](t) \tag{3.0.2}$$

for all $t \in [a,b]$.

Example 3.0.2. *The problem of discovering the curve y which has a minimum arc length between the points* $(0,0)$ *and* $(1,1)$ *can be enunciate by*

$$\begin{cases} \mathcal{L}[y] = \displaystyle\int_0^1 \sqrt{1 + [y'(t)]^2} dt \quad \rightarrow \quad \textit{minimize} \\ y \in C^1([0,1], \mathbb{R}) \\ y(0) = 0 \\ y(1) = 1. \end{cases}$$

Clearly, our knowledge of ordinary geometry suggest that the curve which minimizes the arc length is the straight line connecting $(0,0)$ *to* $(1,1)$*. Using the tools of the calculus of variations we prove, in a rigorous way, that our geometric intuition is indeed correct.*

The Euler-Lagrange equation associated to $\mathcal{L}$ is

$$\frac{d}{dt}\left(\frac{y'(t)}{\sqrt{1+[y'(t)]^2}}\right)=0 \Leftrightarrow \frac{y''(t)}{\left(1+[y'(t)]^2\right)^{\frac{3}{2}}}=0$$

for all $t \in [0,1]$. Obviously, the solution of this second-order equation is $y(t) = At + B$ for some $A, B \in \mathbb{R}$. Using the boundary conditions $y(0) = 0$ and $y(1) = 1$ we conclude that $y(t) = t$, $t \in [0,1]$, is the only candidate to be a minimizer. To prove that $y(t) = t$ is indeed the solution, we use a sufficient condition (see Example 3.0.10).

Historically, variational problems were often formulated with some kind of constraint. Dido, the queen founder of Carthage, when arrived on the coast of Tunisia, asked for a piece of land. Her request was satisfied provided that the land could be encompassed by an oxhide. She cut the oxhide into very narrow strips and joined them together into a long thin strip and used it to encircle the land. This land became Carthage. Dido's problem, the *isoperimetric* (same perimeter) *problem* is to find a curve y that satisfies the variational problem

$$\begin{cases}\mathcal{L}[y]=\int_a^b L\left(t,y(t),y'(t)\right)dt \quad \rightarrow \quad \text{extremize}\\[1ex] y\in C^1([a,b],\mathbb{R})\\[1ex] y(a)=\alpha\\[1ex] y(b)=\beta\end{cases} \tag{3.0.3}$$

but subject to the constraint

$$\gamma=\int_a^b G\left(t,y(t),y'(t)\right)dt, \tag{3.0.4}$$

where G is a given function satisfying the same condition that the Lagrangian L and γ is a real fixed value.

A necessary condition for a function to be an extremizer for the isoperimetric problem (3.0.3)–(3.0.4) is given by the following result.

Theorem 3.0.3 (cf. [120])**.** *Suppose that $\mathcal{L}$ has an extremum at $y \in C^1([a,b],\mathbb{R})$ subject to the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ and the isoperimetric constraint* (3.0.4)*. Then there exist two real numbers λ_0 and λ_1, not both zero, such that*

$$\frac{d}{dt}\frac{\partial K}{\partial y'}=\frac{\partial K}{\partial y},$$

where

$$K=\lambda_0 L-\lambda_1 G.$$

Another necessary condition for problem (3.0.1) is given by the Legendre condition.

Theorem 3.0.4 (Legendre's condition (cf. [120]))**.** *Let $\mathcal{L}$ be the functional of problem* (3.0.1) *and suppose that $\mathcal{L}$ has a local minimum for the curve y. Then,*

$$\frac{\partial}{\partial y'}\left[\tau \to \frac{\partial L}{\partial y'}\left(\tau, y\left(\tau\right), y'\left(\tau\right)\right)\right](t) \geqslant 0$$

for all $t \in [a,b]$.

The first-order variational problem (3.0.1) can be extended to functionals involving higher-order derivatives. We formulate the r-order variational problem as follows

$$\begin{cases} \mathcal{L}\left[y\right] = \displaystyle\int_a^b L\left(t, y\left(t\right), y'\left(t\right), \ldots, y^{(r)}\left(t\right)\right) dt \quad \to \quad \text{extremize} \\ y \in C^r\left(\left[a,b\right], \mathbb{R}\right) \\ y\left(a\right) = \alpha_0, \quad y\left(b\right) = \beta_0 \\ \vdots \\ y^{(r-1)}\left(a\right) = \alpha_{r-1}, \quad y^{(r-1)}\left(b\right) = \beta_{r-1} \end{cases} \tag{3.0.5}$$

where $r \in \mathbb{N}$, $a, b, \in \mathbb{R}$ with $a < b$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 0, \ldots, r-1$, are given. Also, we must assume that the Lagrangian function L has continuous partial derivatives up to the order $r+1$ with respect to all its arguments. Clearly, in the problem of order r we need to restrict the space of functions and we say that y is an admissible function if $y \in C^r\left(\left[a,b\right], \mathbb{R}\right)$ and satisfies all the boundary conditions of problem (3.0.5). Like the first-order case, we can obtain a necessary condition for a function to be an extremizer.

Theorem 3.0.5 (Higher-order Euler–Lagrange equation (cf. [120]))**.** *Under the hypotheses of problem* (3.0.5)*, if an admissible function y_* is a local extremizer for problem* (3.0.1)*, then y_* satisfies the Euler–Lagrange equation*

$$\sum_{i=0}^{r} \left(-1\right)^i \frac{d^i}{dt^i}\left[\tau \to \partial_{i+2} L\left(\tau, y\left(\tau\right), y'\left(\tau\right), \ldots, y^{(r)}(\tau)\right)\right](t) = 0$$

for all $t \in [a,b]$ and where $\partial_i L$ denotes the partial derivative of L with respect to its ith argument.

Let us recall the definition of a convex set and a convex function.

Definition 3.0.6. *Let $\Omega \subseteq \mathbb{R}^2$. The set Ω is said to be convex if*

$$\left(1-t\right) z_1 + t z_2 \in \Omega$$

for all $t \in [0,1]$ and $z_1, z_2 \in \Omega$.

Definition 3.0.7. *Let $\Omega \subseteq \mathbb{R}^2$ be a convex set. A function $f : \Omega \to \mathbb{R}$ is said to be convex if*

$$f\left[\left(1-t\right) z_1 + t z_2\right] \leqslant \left(1-t\right) f(z_1) + t f(z_2)$$

for all $z_1, z_2 \in \Omega$ and $t \in [0,1]$.

A sufficient condition for a minimum for problem (3.0.3) is now presented.

Theorem 3.0.8 (cf. [120]). *Let $L : D \subseteq \mathbb{R}^3 \to \mathbb{R}$ be a function. Suppose that, for each $t \in [a,b]$, the set*

$$\Omega_t := \left\{\left(y, y'\right) \in \mathbb{R}^2 : \left(t, y, y'\right) \in D\right\}$$

is convex and that L is a convex function of the variables $(y, y') \in \Omega_t$. If y_* satisfies the Euler–Lagrange equation (3.0.2), then y_* is a minimizer to problem (3.0.3).

Note that when a function is smooth or has continuous first- and second-order partial derivatives we can apply the following result which is more practical to prove that a function is convex.

Theorem 3.0.9 (cf. [120]). *Let $\Omega \subseteq \mathbb{R}^2$ be a convex set and let $f : \Omega \to \mathbb{R}$ be a function with continuous first- and second-order partial derivatives. The function f is convex if, and only if, for each $(y, w) \in \Omega$:*

$$\begin{aligned} \frac{\partial^2 f}{\partial y^2}(y,w) &\geqslant 0 \\ \frac{\partial^2 f}{\partial w^2}(y,w) &\geqslant 0 \\ \frac{\partial^2 f}{\partial y^2}(y,w)\frac{\partial^2 f}{\partial w^2}(y,w) - \frac{\partial^2 f}{\partial w \partial y}(y,w) &\geqslant 0. \end{aligned}$$

Example 3.0.10. *In Example 3.0.2, we have proven that the function $y(t) = t$ is a candidate to the problem of finding the curve with the shortest arc length connecting the points $(0,0)$ and $(1,1)$ of the plane. In this example $\Omega_t = \mathbb{R}^2$ is a convex set and since for all $(y, y') \in \Omega_t$ we have*

$$\frac{\partial^2 L}{\partial y^2} = \frac{\partial^2 L}{\partial y' \partial y} = 0$$

and

$$\frac{\partial^2 L}{\partial y'^2} = \frac{1}{\left[\sqrt{1 + \left[y'(t)\right]^2}\right]^3} > 0,$$

hence L is convex. By Theorem 3.0.8, we can conclude that the line segment $y(t) = t$ is the solution to the problem.

For further information about the classical calculus of variations we suggest the reader the following books [63, 120].

The study of the calculus of variations in the context of the time scale theory is very recent (2004) and was initiated with the well known paper of Martin Bohner [25]. One of the main results of [25] is the Euler–Lagrange equation for the first-order variational problem involving the delta derivative:

$$\begin{cases} \mathcal{L}[y] = \displaystyle\int_a^b L\left(t, y^\sigma(t), y^\Delta(t)\right)\Delta t \quad \rightarrow \quad \text{extremize} \\ y(a) = \alpha \\ y(b) = \beta \end{cases} \tag{3.0.6}$$

where $a, b \in \mathbb{T}$, $a < b$; $\alpha, \beta \in \mathbb{R}^n$ with $n \in \mathbb{N}$, and $L : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$ is a C^2 function.

Theorem 3.0.11 (Euler–Lagrange equation on time scales [25])**.** *If $y_* \in C^1_{rd}$ is a local extremum of* (3.0.6)*, then the Euler–Lagrange equation*

$$(\partial_3 L)^\Delta \left(t, y_*^\sigma(t), y_*^\Delta(t)\right) = \partial_2 L \left(t, y_*^\sigma(t), y_*^\Delta(t)\right)$$

holds for all $t \in [a, b]^k$.

Since the pioneer work of Martin Bohner [25], many researchers developed the calculus of variations on time scales in several different directions, namely: with non-fixed boundary conditions [68, 69], with two independent variables [26], with higher-order delta derivatives [56, 55], with higher-order nabla derivatives [95], with isoperimetric constraints [7, 57, 91], infinite horizon variational problems [86, 100], with a Lagrangian depending on a delta indefinite integral that depends on the unknown function [98], and with an invariant group of parameter-transformations [19, 20, 96].

To end this chapter we refer the reader to the survey paper [118], that gives an excellent overview on a more general approach of the calculus of variations on time scales that allows to obtain as particular cases the delta and nabla calculus of variations.

Part II

Original Work

Chapter 4

Higher-order Hahn's Quantum Variational Calculus

In this chapter we present our achievements made in the calculus of variations within the Hahn's quantum calculus. In Section 4.2.1 we prove a higher-order fundamental lemma of the calculus of variations with Hahn's operator (Lemma 4.2.8). In Section 4.2.2 we deduce a higher-order Euler–Lagrange equation for Hahn's variational calculus (Theorem 4.2.12). Finally, we provide in Section 4.2.3 a simple example of a quantum optimization problem where our Theorem 4.2.12 leads to the global minimizer, which is not a continuous function.

4.1 Introduction

Many physical phenomena are described by equations involving nondifferentiable functions, e.g., generic trajectories of quantum mechanics [58]. Several different approaches to deal with nondifferentiable functions are followed in the literature of variational calculus, including the time scale approach, which typically deal with delta or nabla differentiable functions [56, 87, 95], the fractional approach, allowing to consider functions that have no first-order derivative but have fractional derivatives of all orders less than one [6, 47, 60], and the quantum approach, which is particularly useful to model physical and economical systems [17, 43, 85, 89, 97].

Roughly speaking, a quantum calculus substitutes the classical derivative by a difference operator, which allows one to deal with sets of nondifferentiable functions. Several dialects of quantum calculus are available [50, 73]. For motivation to study a nondifferentiable quantum variational calculus we refer the reader to [8, 17, 43].

In 1949 Hahn introduced the difference operator $D_{q,\omega}$ defined by

$$D_{q,\omega}\left[f\right](t) := \frac{f\left(qt+\omega\right)-f\left(t\right)}{\left(q-1\right)t+\omega},$$

where f is a real function, $q \in]0,1[$ and $\omega \geqslant 0$ are real fixed numbers [66]. Hahn's difference operator has been applied successfully in the construction of families of orthogonal polynomials as well as in approximation problems [12, 46, 107]. However, during 60 years, the construction of the proper inverse of Hahn's difference operator remained an open question. Eventually, the problem was solved in 2009 by Aldwoah [3] (see also [4, 14]). Here we introduce the higher-order Hahn's quantum variational calculus, proving Hahn's quantum analog of the higher-order Euler–Lagrange equation. As particular cases we obtain the q-calculus Euler–Lagrange equation [17] and the h-calculus Euler–Lagrange equation [21, 74].

Variational functionals that depend on higher-order derivatives arise in a natural way in applications of engineering, physics, and economics. Let us consider, for example, the equilibrium of an elastic bending beam. Let us denote by $y(x)$ the deflection of the point x of the beam, $E(x)$ the elastic stiffness of the material, that can vary with x, and $\xi(x)$ the load that bends the beam. One may assume that, due to some constraints of physical nature, the dynamics does not depend on the usual derivative $y'(x)$ but on some quantum derivative $D_{q,\omega}[y](x)$. In this condition, the equilibrium of the beam correspond to the solution of the following higher-order Hahn's quantum variational problem:

$$\int_0^L \left[\frac{1}{2}\left(E(x)D^2_{q,\omega}[y](x)\right)^2 - \xi(x)y\left(q^2x+q\omega+\omega\right)\right]dx \longrightarrow \min. \tag{4.1.1}$$

Note that we recover the classical problem of the equilibrium of the elastic bending beam when $(\omega, q) \to]0, 1[$. Problem (4.1.1) is a particular case of problem (4.2.2) investigated in Section 4.2. Our higher-order Hahn's quantum Euler–Lagrange equation (Theorem 4.2.12) gives the main tool to solve such problems.

4.2 Main results

Let I be a real interval containing $\omega_0 := \dfrac{\omega}{1-q}$. We define the q,ω-derivatives of higher-order in the usual way: the rth q,ω-derivative ($r \in \mathbb{N}$) of $f : I \to \mathbb{R}$ is the function $D^r_{q,\omega}[f] : I \to \mathbb{R}$ given by $D^r_{q,\omega}[f] := D_{q,\omega}[D^{r-1}_{q,\omega}[f]]$, provided $D^{r-1}_{q,\omega}[f]$ is q,ω-differentiable on I and where $D^0_{q,\omega}[f] := f$.

For $s \in I$ we define

$$[s]_{q,\omega} := \left\{q^n s + \omega\,[n]_q : n \in \mathbb{N}_0\right\} \cup \{\omega_0\}. \tag{4.2.1}$$

Let $a, b \in I$ and $a < b$. We define the q,ω-interval from a to b by

$$[a,b]_{q,\omega} := \left\{q^n a + \omega\,[n]_q : n \in \mathbb{N}_0\right\} \cup \left\{q^n b + \omega\,[n]_q : n \in \mathbb{N}_0\right\} \cup \{\omega_0\},$$

i.e., $[a,b]_{q,\omega} = [a]_{q,\omega} \cup [b]_{q,\omega}$, where $[a]_{q,\omega}$ and $[b]_{q,\omega}$ are given by (4.2.1).

We introduce the linear space $\mathcal{Y}^r = \mathcal{Y}^r([a,b], \mathbb{R})$ by

$$\mathcal{Y}^r := \left\{y : I \to \mathbb{R} \mid D^i_{q,\omega}[y], i = 0, \ldots, r, \text{ are bounded on } [a,b]_{q,\omega} \text{ and continuous at } \omega_0\right\}$$

endowed with the norm $\|y\|_{r,\infty} := \sum_{i=0}^r \left\|D^i_{q,\omega}[y]\right\|_\infty$, where $\|y\|_\infty := \sup_{t\in[a,b]_{q,\omega}} |y(t)|$. The following notations are in order: $\sigma(t) := qt + \omega$, $y^\sigma(t) = y^{\sigma^1}(t) = (y \circ \sigma)(t) = y(qt + \omega)$, and $y^{\sigma^k} = y^{\sigma^{k-1}} \circ \sigma$, $k = 2, 3, \ldots$ Our main goal is to establish necessary optimality conditions for the higher-order q,ω-variational problem

$$\begin{cases} \mathcal{L}[y] = \displaystyle\int_a^b L\left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t), \ldots, D_{q,\omega}^r[y](t)\right) d_{q,\omega}t \longrightarrow \text{extremize} \\ y \in \mathcal{Y}^r([a,b],\mathbb{R}) \\ y(a) = \alpha_0\,, \quad y(b) = \beta_0\,, \\ \vdots \\ D_{q,\omega}^{r-1}[y](a) = \alpha_{r-1}\,, \quad D_{q,\omega}^{r-1}[y](b) = \beta_{r-1}\,, \end{cases} \tag{4.2.2}$$

where $r \in \mathbb{N}$ and $\alpha_i, \beta_i \in \mathbb{R}$, $i = 0, \ldots, r-1$, are given. By extremize we mean maximize or minimize.

Definition 4.2.1. *We say that y is an admissible function for* (4.2.2) *if $y \in \mathcal{Y}^r([a,b],\mathbb{R})$ and y satisfies the boundary conditions $D_{q,\omega}^i[y](a) = \alpha_i$ and $D_{q,\omega}^i[y](b) = \beta_i$ of problem* (4.2.2)*, $i = 0, \ldots, r-1$.*

The Lagrangian L is assumed to satisfy the following hypotheses:

(H1) $(u_0, \ldots, u_r) \to L(t, u_0, \ldots, u_r)$ is a $C^1(\mathbb{R}^{r+1}, \mathbb{R})$ function for any $t \in [a,b]$;

(H2) $t \to L(t, y(t), D_{q,\omega}[y](t), \ldots, D_{q,\omega}^r[y](t))$ is continuous at ω_0 for any admissible y;

(H3) functions $t \to \partial_{i+2}L(t, y(t), D_{q,\omega}[y](t), \cdots, D_{q,\omega}^r[y](t))$, $i = 0, 1, \cdots, r$, belong to $\mathcal{Y}^1([a,b],\mathbb{R})$ for all admissible y, where $\partial_i L$ denotes the partial derivative of L with respect to its ith argument.

Definition 4.2.2. *We say that y_* is a local minimizer (resp. local maximizer) for problem* (4.2.2) *if y_* is an admissible function and there exists $\delta > 0$ such that*

$$\mathcal{L}[y_*] \leqslant \mathcal{L}[y] \quad (\textit{resp. } \mathcal{L}[y_*] \geqslant \mathcal{L}[y]\,)$$

for all admissible y with $\|y_* - y\|_{r,\infty} < \delta$.

Definition 4.2.3. *We say that $\eta \in \mathcal{Y}^r([a,b],\mathbb{R})$ is a* variation *if $\eta(a) = \eta(b) = 0$, $\ldots$, $D_{q,\omega}^{r-1}[\eta](a) = D_{q,\omega}^{r-1}[\eta](b) = 0$.*

4.2.1 Higher-order fundamental lemma of Hahn's variational calculus

The chain rule, as known from classical calculus, does not hold in Hahn's quantum context (see a counterexample in [3, 14]). However, we can prove the following.

Lemma 4.2.4. *If f is q,ω-differentiable on I, then the following equality holds:*

$$D_{q,\omega}[f^\sigma](t) = q\,(D_{q,\omega}[f])^\sigma(t)\,, \quad t \in I.$$

Proof. For $t \neq \omega_0$ we have

$$(D_{q,\omega}[f])^\sigma(t) = \frac{f(q(qt+\omega)+\omega) - f(qt+\omega)}{(q-1)(qt+\omega)+\omega} = \frac{f(q(qt+\omega)+\omega) - f(qt+\omega)}{q((q-1)t+\omega)}$$

and

$$D_{q,\omega}\left[f^{\sigma}\right](t)=\frac{f^{\sigma}\left(qt+\omega\right)-f^{\sigma}\left(t\right)}{\left(q-1\right)t+\omega}=\frac{f\left(q\left(qt+\omega\right)+\omega\right)-f\left(qt+\omega\right)}{\left(q-1\right)t+\omega}.$$

Therefore, $D_{q,\omega}\left[f^{\sigma}\right](t)=q\left(D_{q,\omega}\left[f\right]\right)^{\sigma}(t)$. If $t=\omega_0$, then $\sigma\left(\omega_0\right)=\omega_0$. Thus,

$$\left(D_{q,\omega}\left[f\right]\right)^{\sigma}\left(\omega_0\right)=\left(D_{q,\omega}\left[f\right]\right)\left(\sigma\left(\omega_0\right)\right)=\left(D_{q,\omega}\left[f\right]\right)\left(\omega_0\right)=f'\left(\omega_0\right)$$

and $D_{q,\omega}\left[f^{\sigma}\right]\left(\omega_0\right)=\left(f^{\sigma}\right)'\left(\omega_0\right)=f'\left(\sigma\left(\omega_0\right)\right)\sigma'\left(\omega_0\right)=qf'\left(\omega_0\right)$. □

Lemma 4.2.5. *If $\eta\in\mathcal{Y}^{r}\left(\left[a,b\right],\mathbb{R}\right)$ is such that $D^{i}_{q,\omega}\left[\eta\right](a)=0$ (resp. $D^{i}_{q,\omega}\left[\eta\right](b)=0$) for all $i\in\{0,1,\ldots,r\}$, then $D^{i-1}_{q,\omega}\left[\eta^{\sigma}\right](a)=0$ (resp. $D^{i-1}_{q,\omega}\left[\eta^{\sigma}\right](b)=0$) for all $i\in\{1,\ldots,r\}$.*

Proof. If $a=\omega_0$ the result is trivial (because $\sigma\left(\omega_0\right)=\omega_0$). Suppose now that $a\neq\omega_0$ and fix $i\in\{1,\ldots,r\}$. Note that

$$D^{i}_{q,\omega}\left[\eta\right](a)=\frac{\left(D^{i-1}_{q,\omega}\left[\eta\right]\right)^{\sigma}(a)-D^{i-1}_{q,\omega}\left[\eta\right](a)}{\left(q-1\right)a+\omega}.$$

Since, by hypothesis, $D^{i}_{q,\omega}\left[\eta\right](a)=0$ and $D^{i-1}_{q,\omega}\left[\eta\right](a)=0$, then $\left(D^{i-1}_{q,\omega}\left[\eta\right]\right)^{\sigma}(a)=0$. Lemma 4.2.4 shows that

$$\left(D^{i-1}_{q,\omega}\left[\eta\right]\right)^{\sigma}(a)=\left(\frac{1}{q}\right)^{i-1}D^{i-1}_{q,\omega}\left[\eta^{\sigma}\right](a).$$

We conclude that $D^{i-1}_{q,\omega}\left[\eta^{\sigma}\right](a)=0$. The case $t=b$ is proved in the same way. □

Lemma 4.2.6. *Suppose that $f\in\mathcal{Y}^{1}\left(\left[a,b\right],\mathbb{R}\right)$. One has*

$$\int_a^b f(t)\,D_{q,\omega}\left[\eta\right](t)\,d_{q,\omega}t=0$$

for all functions $\eta\in\mathcal{Y}^{1}\left(\left[a,b\right],\mathbb{R}\right)$ such that $\eta(a)=\eta(b)=0$ if, and only if, $f(t)=c$, $c\in\mathbb{R}$, for all $t\in\left[a,b\right]_{q,\omega}$.

Proof. The implication "⇐" is obvious. We prove "⇒". We begin by noting that

$$\underbrace{\int_a^b f(t)\,D_{q,\omega}\left[\eta\right](t)\,d_{q,\omega}t}_{=0}=\underbrace{f(t)\,\eta(t)\Big|_a^b}_{=0}-\int_a^b D_{q,\omega}\left[f\right](t)\,\eta^{\sigma}(t)\,d_{q,\omega}t.$$

Hence,

$$\int_a^b D_{q,\omega}\left[f\right](t)\,\eta\left(qt+\omega\right)d_{q,\omega}t=0$$

for any $\eta\in\mathcal{Y}^{1}\left(\left[a,b\right],\mathbb{R}\right)$ such that $\eta(a)=\eta(b)=0$. We need to prove that, for some $c\in\mathbb{R}$, $f(t)=c$ for all $t\in\left[a,b\right]_{q,\omega}$, that is, $D_{q,\omega}\left[f\right](t)=0$ for all $t\in\left[a,b\right]_{q,\omega}$. Suppose, by contradiction, that there exists $p\in\left[a,b\right]_{q,\omega}$ such that $D_{q,\omega}\left[f\right](p)\neq0$.
(1) If $p\neq\omega_0$, then $p=q^{k}a+\omega\left[k\right]_q$ or $p=q^{k}b+\omega\left[k\right]_q$ for some $k\in\mathbb{N}_0$. Observe that $a\left(1-q\right)-\omega$ and $b\left(1-q\right)-\omega$ cannot vanish simultaneously.

(a) Suppose that $a(1-q)-\omega \neq 0$ and $b(1-q)-\omega \neq 0$. In this case we can assume, without loss of generality, that $p = q^k a + \omega [k]_q$ and we can define

$$\eta(t) = \begin{cases} D_{q,\omega}[f]\left(q^k a + \omega [k]_q\right) & \text{if } t = q^{k+1}a + \omega [k+1]_q \\ \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$\int_a^b D_{q,\omega}[f](t) \cdot \eta(qt+\omega)\, d_{q,\omega}t$$
$$= -\left(a(1-q)-\omega\right) q^k D_{q,\omega}[f]\left(q^k a + \omega [k]_q\right) \cdot D_{q,\omega}[f]\left(q^k a + \omega [k]_q\right) \neq 0,$$

which is a contradiction.
(b) If $a(1-q)-\omega \neq 0$ and $b(1-q)-\omega = 0$, then $b = \omega_0$. Since $q^k \omega_0 + \omega [k]_q = \omega_0$ for all $k \in \mathbb{N}_0$, then $p \neq q^k b + \omega [k]_q$ $\forall k \in \mathbb{N}_0$ and, therefore,

$$p = q^k a + \omega [k]_{q,\omega} \text{ for some } k \in \mathbb{N}_0.$$

Repeating the proof of (a) we obtain again a contradiction.
(c) If $a(1-q)-\omega = 0$ and $b(1-q)-\omega \neq 0$ then the proof is similar to (b).

(2) If $p = \omega_0$ then, without loss of generality, we can assume $D_{q,\omega}[f](\omega_0) > 0$. Since

$$\lim_{n\to+\infty}\left(q^n a + \omega [k]_q\right) = \lim_{n\to+\infty}\left(q^n b + \omega [k]_q\right) = \omega_0$$

(see [3]) and $D_{q,\omega}[f]$ is continuous at ω_0, then

$$\lim_{n\to+\infty} D_{q,\omega}[f]\left(q^n a + \omega [k]_q\right) = \lim_{n\to+\infty} D_{q,\omega}[f]\left(q^n b + \omega [k]_q\right) = D_{q,\omega}[f](\omega_0) > 0.$$

Thus, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$ one has $D_{q,\omega}[f]\left(q^n a + \omega [k]_q\right) > 0$ and $D_{q,\omega}[f]\left(q^n b + \omega [k]_q\right) > 0$.
(a) If $\omega_0 \neq a$ and $\omega_0 \neq b$, then we can define

$$\eta(t) = \begin{cases} D_{q,\omega}[f]\left(q^N b + \omega [N]_q\right) & \text{if} \quad t = q^{N+1}a + \omega [N+1]_q \\ \\ D_{q,\omega}[f]\left(q^N a + \omega [N]_q\right) & \text{if} \quad t = q^{N+1}b + \omega [N+1]_q \\ \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\int_a^b D_{q,\omega}[f](t)\, \eta(qt+\omega)\, d_{q,\omega}t$$
$$= (b-a)(1-q)\, q^N D_{q,\omega}[f]\left(q^N b + \omega [N]_q\right) \cdot D_{q,\omega}[f]\left(q^N a + \omega [N]_q\right) \neq 0,$$

which is a contradiction.
(b) If $\omega_0 = b$, then we define

$$\eta(t) = \begin{cases} D_{q,\omega}[f](\omega_0) & \text{if} \quad t = q^{N+1}a + \omega [N+1]_q \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\begin{aligned} &\int_a^b D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t \\ &\quad = -\int_{\omega_0}^a D_{q,\omega}[f](t)\,\eta(qt+\omega)\,d_{q,\omega}t \\ &\quad = -(a(1-q)-\omega)\,q^N D_{q,\omega}[f]\left(q^N a + \omega [k]_q\right) \cdot D_{q,\omega}[f](\omega_0) \neq 0, \end{aligned}$$

which is a contradiction.
(c) When $\omega_0 = a$, the proof is similar to (b). □

Lemma 4.2.7 (Fundamental lemma of Hahn's variational calculus)**.** *Let $f, g \in \mathcal{Y}^1([a,b],\mathbb{R})$. If*

$$\int_a^b \left(f(t)\,\eta^\sigma(t) + g(t)\,D_{q,\omega}[\eta](t)\right) d_{q,\omega}t = 0$$

for all $\eta \in \mathcal{Y}^1([a,b],\mathbb{R})$ such that $\eta(a) = \eta(b) = 0$, then

$$D_{q,\omega}[g](t) = f(t) \quad \forall t \in [a,b]_{q,\omega}.$$

Proof. Define the function A by $A(t) := \int_{\omega_0}^t f(\tau)\,d_{q,\omega}\tau$. Then, $D_{q,\omega}[A](t) = f(t)$ for all $t \in [a,b]$ and

$$\begin{aligned} \int_a^b A(t)\,D_{q,\omega}[\eta](t)\,d_{q,\omega}t &= A(t)\,\eta(t)\Big|_a^b - \int_a^b D_{q,\omega}[A](t)\,\eta^\sigma(t)\,d_{q,\omega}t \\ &= -\int_a^b D_{q,\omega}[A](t)\,\eta^\sigma(t)\,d_{q,\omega}t \\ &= -\int_a^b f(t)\,\eta^\sigma(t)\,d_{q,\omega}t. \end{aligned}$$

Hence,

$$\begin{aligned} &\int_a^b \left(f(t)\,\eta^\sigma(t) + g(t)\,D_{q,\omega}[\eta](t)\right) d_{q,\omega}t = 0 \\ \Leftrightarrow &\int_a^b \left(-A(t) + g(t)\right) D_{q,\omega}[\eta](t)\,d_{q,\omega}t = 0. \end{aligned}$$

By Lemma 4.2.6 there is a $c \in \mathbb{R}$ such that $-A(t) + g(t) = c$ for all $t \in [a,b]_{q,\omega}$. Hence $D_{q,\omega}[A](t) = D_{q,\omega}[g](t)$ for all $t \in [a,b]_{q,\omega}$, which provides the desired result:

$$D_{q,\omega}[g](t) = f(t)\ ,\ \forall t \in [a,b]_{q,\omega}.$$

□

We are now in conditions to deduce a higher-order fundamental lemma of the Hahn quantum variational calculus.

Lemma 4.2.8 (Higher-order fundamental lemma of the Hahn variational calculus)**.** *Let* $f_0, f_1, \ldots, f_r \in \mathcal{Y}^1([a,b],\mathbb{R})$*. If*

$$\int_a^b \left(\sum_{i=0}^{r} f_i(t)\, D_{q,\omega}^i\left[\eta^{\sigma^{r-i}}\right](t)\right) d_{q,\omega}t = 0$$

for any variation η*, then*

$$\sum_{i=0}^{r} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i\left[f_i\right](t) = 0$$

for all $t \in [a,b]_{q,\omega}$*.*

Proof. We proceed by mathematical induction. If $r = 1$ the result is true by Lemma 4.2.7. Assume that

$$\int_a^b \left(\sum_{i=0}^{r+1} f_i(t)\, D_{q,\omega}^i\left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t = 0$$

for all functions η such that $\eta(a) = \eta(b) = 0, \ldots, D_{q,\omega}^r[\eta](a) = D_{q,\omega}^r[\eta](b) = 0$. Note that

$$\begin{aligned}
&\int_a^b f_{r+1}(t)\, D_{q,\omega}^{r+1}[\eta](t)\, d_{q,\omega}t \\
&\qquad = f_{r+1}(t)\, D_{q,\omega}^r[\eta](t)\Big|_a^b - \int_a^b D_{q,\omega}[f_{r+1}](t)\left(D_{q,\omega}^r[\eta]\right)^\sigma(t)\, d_{q,\omega}t \\
&\qquad = -\int_a^b D_{q,\omega}[f_{r+1}](t)\left(D_{q,\omega}^r[\eta]\right)^\sigma(t)\, d_{q,\omega}t
\end{aligned}$$

and, by Lemma 4.2.4,

$$\int_a^b f_{r+1}(t)\, D_{q,\omega}^{r+1}[\eta](t)\, d_{q,\omega}t = -\int_a^b D_{q,\omega}[f_{r+1}](t)\left(\frac{1}{q}\right)^r D_{q,\omega}^r[\eta^\sigma](t)\, d_{q,\omega}t.$$

Therefore,

$$\begin{aligned}
&\int_a^b \left(\sum_{i=0}^{r+1} f_i(t)\, D_{q,\omega}^i\left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t \\
&\quad = \int_a^b \left(\sum_{i=0}^{r} f_i(t)\, D_{q,\omega}^i\left[\eta^{\sigma^{r+1-i}}\right](t)\right) d_{q,\omega}t \\
&\qquad - \int_a^b D_{q,\omega}[f_{r+1}](t)\left(\frac{1}{q}\right)^r D_{q,\omega}^r[\eta^\sigma](t)\, d_{q,\omega}t \\
&\quad = \int_a^b \Bigg[\sum_{i=0}^{r-1} f_i(t)\, D_{q,\omega}^i\left[(\eta^\sigma)^{\sigma^{r-i}}\right](t)\, d_{q,\omega}t \\
&\qquad + \left(f_r - \left(\frac{1}{q}\right)^r D_{q,\omega}[f_{r+1}]\right)(t)\, D_{q,\omega}^r[\eta^\sigma](t)\Bigg] d_{q,\omega}t.
\end{aligned}$$

By Lemma 4.2.5, η^σ is a variation. Hence, using the induction hypothesis,

$$\begin{aligned}
&\sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i [f_i](t) \\
&\qquad + (-1)^r \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} D_{q,\omega}^r \left[\left(f_r - \left(\frac{1}{q}\right)^r D_{q,\omega}[f_{r+1}]\right)\right](t) \\
&= \sum_{i=0}^{r-1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i [f_i](t) + (-1)^r \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} D_{q,\omega}^r [f_r](t) \\
&\qquad + (-1)^{r+1} \left(\frac{1}{q}\right)^{\frac{(r-1)r}{2}} \left(\frac{1}{q}\right)^r D_{q,\omega}^r [D_{q,\omega}[f_{r+1}]](t) \\
&= 0
\end{aligned}$$

for all $t \in [a,b]_{q,\omega}$, which leads to

$$\sum_{i=0}^{r+1} (-1)^i \left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i [f_i](t) = 0, \;\; t \in [a,b]_{q,\omega}.$$

□

4.2.2 Higher-order Hahn's quantum Euler–Lagrange equation

For a variation η and an admissible function y, we define the function $\phi :]-\bar{\epsilon}, \bar{\epsilon}[\to \mathbb{R}$ by

$$\phi(\epsilon) := \mathcal{L}[y + \epsilon\eta].$$

The first variation of the variational problem (4.2.2) is defined by

$$\delta\mathcal{L}[y,\eta] := \phi'(0).$$

Observe that

$$\begin{aligned}
\mathcal{L}[y+\epsilon\eta] &= \int_a^b L\Bigg(t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t), \\
&\qquad \ldots, D_{q,\omega}^r[y](t) + \epsilon D_{q,\omega}^r[\eta](t)\Bigg) d_{q,\omega}t \\
&= \mathcal{L}_b[y+\epsilon\eta] - \mathcal{L}_a[y+\epsilon\eta]
\end{aligned}$$

with

$$\begin{aligned}
\mathcal{L}_\xi[y+\epsilon\eta] = \int_{\omega_0}^{\xi} L\Bigg(&t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t), \\
&\ldots, D_{q,\omega}^r[y](t) + \epsilon D_{q,\omega}^r[\eta](t)\Bigg) d_{q,\omega}t,
\end{aligned}$$

$\xi \in \{a,b\}$. Therefore,

$$\delta\mathcal{L}[y,\eta] = \delta\mathcal{L}_b[y,\eta] - \delta\mathcal{L}_a[y,\eta]. \tag{4.2.3}$$

The following definition and lemma are important for our purposes.

Definition 4.2.9. *Let $s \in I$ and $g : I\times]-\bar{\theta},\bar{\theta}[\to \mathbb{R}$. We say that $g(t,\cdot)$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$0 < |\theta - \theta_0| < \delta \Rightarrow \left|\frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0)\right| < \varepsilon$$

for all $t \in [s]_{q,\omega}$, where $\partial_2 g = \dfrac{\partial g}{\partial \theta}$.

Lemma 4.2.10 (cf. [89])**.** *Let $s \in I$. Assume that $g : I\times]-\bar{\theta},\bar{\theta}[\to \mathbb{R}$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$, and $\int_{\omega_0}^{s} \partial_2 g(t,\theta_0)\, d_{q,\omega}t$ exist. Then,*

$$G(\theta) := \int_{\omega_0}^{s} g(t,\theta)\, d_{q,\omega}t,$$

for θ near θ_0, is differentiable at θ_0 with

$$G'(\theta_0) = \int_{\omega_0}^{s} \partial_2 g(t,\theta_0)\, d_{q,\omega}t.$$

Considering (4.2.3), the following lemma is a direct consequence of Lemma 4.2.10.

Lemma 4.2.11. *For a variation η and an admissible function y, let*

$$g(t,\epsilon) := L\Bigg(t, y^{\sigma^r}(t) + \epsilon\eta^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t) + \epsilon D_{q,\omega}\left[\eta^{\sigma^{r-1}}\right](t), \\ \dots, D_{q,\omega}^{r}[y](t) + \epsilon D_{q,\omega}^{r}[\eta](t)\Bigg),$$

$\epsilon \in]-\bar{\epsilon},\bar{\epsilon}[$. Assume that
(1) $g(t,\cdot)$ is differentiable at ω_0 uniformly in $[a,b]_{q,\omega}$;
(2) $\mathcal{L}_a[y+\epsilon\eta] = \int_{\omega_0}^{a} g(t,\epsilon)\, d_{q,\omega}t$ and $\mathcal{L}_b[y+\epsilon\eta] = \int_{\omega_0}^{b} g(t,\epsilon)\, d_{q,\omega}t$ exist for $\epsilon \approx 0$;
(3) $\int_{\omega_0}^{a} \partial_2 g(t,0)\, d_{q,\omega}t$ and $\int_{\omega_0}^{b} \partial_2 g(t,0)\, d_{q,\omega}t$ exist.
Then,

$$\phi'(0) = \delta\mathcal{L}[y,\eta] = \int_a^b \Bigg[\sum_{i=0}^{r} \partial_{i+2} L\left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t), \dots, D_{q,\omega}^{r}[y](t)\right) \\ \cdot D_{q,\omega}^{i}\left[\eta^{\sigma^{r-i}}\right](t)\Bigg] d_{q,\omega}t.$$

The following result gives a necessary condition of Euler–Lagrange type for an admissible function to be a local extremizer for (4.2.2).

Theorem 4.2.12 (Higher-order Hahn's quantum Euler–Lagrange equation)**.** *Under hypotheses (H1)–(H3) and conditions (1)–(3) of Lemma 4.2.11 on the Lagrangian L, if $y_* \in \mathcal{Y}^r$ is a local extremizer for problem* (4.2.2)*, then y_* satisfies the Hahn quantum Euler–Lagrange equation*

$$\sum_{i=0}^{r}(-1)^i\left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_{q,\omega}^i\left[\tau \to \partial_{i+2}L\left(\tau, y^{\sigma^r}(\tau), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](\tau), \dots, D_{q,\omega}^r[y](\tau)\right)\right](t) \\ = 0 \tag{4.2.4}$$

for all $t \in [a,b]_{q,\omega}$.

Proof. Let y_* be a local extremizer for problem (4.2.2) and η a variation. Define $\phi :]-\bar{\epsilon}, \bar{\epsilon}[\to \mathbb{R}$ by $\phi(\epsilon) := \mathcal{L}[y_* + \epsilon\eta]$. A necessary condition for y_* to be an extremizer is given by $\phi'(0) = 0$. By Lemma 4.2.11 we conclude that

$$\int_a^b \left[\sum_{i=0}^{r} \partial_{i+2}L\left(t, y^{\sigma^r}(t), D_{q,\omega}\left[y^{\sigma^{r-1}}\right](t), \dots, D_{q,\omega}^r[y](t)\right) \right. \\ \left. \cdot D_{q,\omega}^i\left[\eta^{\sigma^{r-i}}\right](t)\right] d_{q,\omega}t = 0$$

and equation (4.2.4) follows from Lemma 4.2.8. □

Remark 4.2.13. *In practical terms the hypotheses of Theorem 4.2.12 are not so easy to verify* a priori. *One can, however, assume that all hypotheses are satisfied and apply the Hahn quantum Euler–Lagrange equation* (4.2.4) *heuristically to obtain a* candidate. *If such a candidate is, or not, a solution to problem* (4.2.2) *is a different question that always requires further analysis (see an example in Section 4.2.3).*

When $\omega = 0$ one obtains from (4.2.4) the *higher-order q-Euler–Lagrange equation*:

$$\sum_{i=0}^{r}(-1)^i\left(\frac{1}{q}\right)^{\frac{(i-1)i}{2}} D_q^i\left[\tau \to \partial_{i+2}L\left(\tau, y^{\sigma^r}(\tau), D_q\left[y^{\sigma^{r-1}}\right](\tau), \dots, D_q^r[y](\tau)\right)\right](t) = 0$$

for all $t \in \{aq^n : n \in \mathbb{N}_0\} \cup \{bq^n : n \in \mathbb{N}_0\} \cup \{0\}$. The *higher-order h-Euler–Lagrange equation* is obtained from (4.2.4) taking the limit $q \to 1$:

$$\sum_{i=0}^{r}(-1)^i \Delta_h^i\left[\tau \to \partial_{i+2}L\left(\tau, y^{\sigma^r}(\tau), \Delta_h\left[y^{\sigma^{r-1}}\right](\tau), \dots, \Delta_h^r[y](\tau)\right)\right](t) = 0$$

for all $t \in \{a + nh : n \in \mathbb{N}_0\} \cup \{b + nh : n \in \mathbb{N}_0\}$. The *classical higher-order Euler–Lagrange equation* [120] is recovered when $(\omega, q) \to (0, 1)$:

$$\sum_{i=0}^{r}(-1)^i \frac{d^i}{dt^i}\left[\tau \to \partial_{i+2}L\left(\tau, y(\tau), y'(\tau), \dots, y^{(r)}(\tau)\right)\right](t) = 0$$

for all $t \in [a, b]$.

We now illustrate the usefulness of our Theorem 4.2.12 by means of an example that is not covered by previous available results in the literature.

4.2.3 An example

Let $q = \frac{1}{2}$ and $\omega = \frac{1}{2}$. Consider the following problem:

$$\mathcal{L}[y] = \int_{-1}^{1} \left(y^{\sigma}(t) + \frac{1}{2}\right)^2 \left((D_{q,\omega}[y](t))^2 - 1\right)^2 d_{q,\omega}t \longrightarrow \min \tag{4.2.5}$$

over all $y \in \mathcal{Y}^1$ satisfying the boundary conditions

$$y(-1) = 0 \quad \text{and} \quad y(1) = -1. \tag{4.2.6}$$

This is an example of problem (4.2.2) with $r = 1$. Our Hahn's quantum Euler–Lagrange equation (4.2.4) takes the form

$$D_{q,\omega}[\tau \to \partial_3 L(\tau, y^{\sigma}(\tau), D_{q,\omega}[y](\tau))](t) = \partial_2 L(t, y^{\sigma}(t), D_{q,\omega}[y](t)).$$

Therefore, we look for an admissible function y_* of (4.2.5)–(4.2.6) satisfying

$$\begin{aligned} D_{q,\omega}&\left[4\left(y^{\sigma} + \frac{1}{2}\right)^2 \left((D_{q,\omega}[y])^2 - 1\right) D_{q,\omega}[y]\right](t) \\ &= 2\left(y^{\sigma}(t) + \frac{1}{2}\right)\left((D_{q,\omega}[y](t))^2 - 1\right)^2 \end{aligned} \tag{4.2.7}$$

for all $t \in [-1, 1]_{q,\omega}$. It is easy to see that

$$y_*(t) = \begin{cases} -t & \text{if } t \in]-1, 0[\cup]0, 1] \\ 0 & \text{if } t = -1 \\ 1 & \text{if } t = 0 \end{cases}$$

is an admissible function for (4.2.5)–(4.2.6) with

$$D_{q,\omega}[y_*](t) = \begin{cases} -1 & \text{if } t \in]-1, 0[\cup]0, 1] \\ 1 & \text{if } t = -1 \\ -3 & \text{if } t = 0, \end{cases}$$

satisfying the Hahn quantum Euler–Lagrange equation (4.2.7). We now prove that the *candidate* y_* is indeed a minimizer for (4.2.5)–(4.2.6). Note that here $\omega_0 = 1$ and, by Lemma 2.3.9,

$$\mathcal{L}[y] = \int_{-1}^{1} \left(y^{\sigma}(t) + \frac{1}{2}\right)^2 \left((D_{q,\omega}[y](t))^2 - 1\right)^2 d_{q,\omega}t \geqslant 0$$

for all admissible functions $y \in \mathcal{Y}^1([-1, 1], \mathbb{R})$. Since $\mathcal{L}[y_*] = 0$, we conclude that y_* is a minimizer for problem (4.2.5)–(4.2.6).

It is worth to mention that the minimizer y_* of (4.2.5)–(4.2.6) is not continuous while the classical calculus of variations [120], the calculus of variations on time scales [56, 87, 95], or the nondifferentiable scale variational calculus [8, 9, 43], deal with functions which are necessarily continuous. As an open question, we pose the problem of determining conditions on the data of problem (4.2.2) ensuring, *a priori*, the minimizer to be regular.

4.3 State of the Art

The results of this chapter are already published in [33] and were presented by the author at EUROPT Workshop "Advances in Continuous Optimization", July 9-10, 2010, University of Aveiro, Portugal. It is worth mentioning that nowadays other researchers are dedicating their time to the development of the theory of Hahn's quantum calculus (see [3, 4, 76, 85] and references within). The theory of calculus of variations within the Hahn quantum calculus is quite new and was first presented by Malinowska and Torres in 2010 [89].

Chapter 5

A Symmetric Quantum Calculus

In this chapter we present a symmetric quantum calculus. We define and prove the properties of the α, β-symmetric derivative (Section 5.3.1); Section 5.3.2 is devoted to the development of the α, β-symmetric Nörlund sum as well to some of its properties. Section 5.3.3 is dedicated to mean value theorems for the α, β-symmetric calculus: we prove α, β-symmetric versions of Fermat's theorem for stationary points, Rolle's, Lagrange's, and Cauchy's mean value theorems. In Section 5.3.4 we present and prove α, β-symmetric versions of Hölder's, Cauchy–Schwarz's and Minkowski's inequalities.

5.1 Introduction

Quantum derivatives and integrals play a leading role in the understanding of complex physical systems. The subject has been under strong development since the beginning of the 20th century [17, 43, 46, 49, 50, 66, 71, 73, 79, 85, 89, 97, 99, 102]. Roughly speaking, two approaches to quantum calculus are available. The first considers the set of points of study to be the lattice $\overline{q^{\mathbb{Z}}}$ or $h\mathbb{Z}$ and is nowadays part of the more general time scale calculus [1, 27, 28, 67, 88]; the second uses the same formulas for the quantum derivatives but the set of study is the set $\mathbb{R}$ [3, 4, 5, 11, 33, 89]. Here we take the second perspective.

Given a function f and a positive real number h, the h-derivative of f at x is defined by the ratio

$$\frac{f(x+h) - f(x)}{h}.$$

When $h \to 0$, one obtains the usual derivative of f at x. The symmetric h-derivative of f at x $(h > 0)$ is defined by

$$\frac{f(x+h) - f(x-h)}{2h},$$

which coincides with the standard symmetric derivative [117] when we let $h \to 0$. The notion of symmetrically differentiable is interesting because if a function is differentiable at a point then it is also symmetrically differentiable, but the converse is not true. The best known example of this fact is the absolute value function: $f(x) = |x|$ is not differentiable at $x = 0$ but is symmetrically differentiable at $x = 0$ with symmetric derivative zero [117].

The aim of this chapter is to introduce the α, β-symmetric difference derivative and Nörlund sum, and then develop the associated calculus. Such α, β-symmetric calculus gives a generalization to (both forward and backward) quantum h-calculus.

5.2 Forward and backward Nörlund sums

In what follows we denote by $|I|$ the measure of the interval I.

Definition 5.2.1. *Let α and β be two positive real numbers, $I \subseteq \mathbb{R}$ be an interval with $|I| > \alpha$, and $f : I \to \mathbb{R}$. The* α-forward difference operator Δ_α *is defined by*

$$\Delta_\alpha [f](t) := \frac{f(t+\alpha) - f(t)}{\alpha}$$

for all $t \in I \backslash [\sup I - \alpha, \sup I]$, in case $\sup I$ is finite, or, otherwise, for all $t \in I$. Similarly, for $|I| > \beta$ the β-backward difference operator ∇_β *is defined by*

$$\nabla_\beta [f](t) := \frac{f(t) - f(t-\beta)}{\beta}$$

for all $t \in I \backslash [\inf I, \inf I + \beta]$, in case $\inf I$ is finite, or, otherwise, for all $t \in I$. We call to $\Delta_\alpha [f]$ the α-forward difference derivative *of f and to $\nabla_\beta [f]$ the* β-backward difference derivative *of f.*

This section is dedicated to the inverse operators of the α-forward and β-backward difference operators.

Definition 5.2.2. *Let $I \subseteq \mathbb{R}$ be such that $a, b \in I$ with $a < b$ and $\sup I = +\infty$. For $f : I \to \mathbb{R}$ and $\alpha > 0$ we define the* Nörlund sum *(the* α-forward integral *) of f from a to b by*

$$\int_a^b f(t)\,\Delta_\alpha t = \int_a^{+\infty} f(t)\,\Delta_\alpha t - \int_b^{+\infty} f(t)\,\Delta_\alpha t,$$

where

$$\int_x^{+\infty} f(t)\,\Delta_\alpha t = \alpha \sum_{k=0}^{+\infty} f(x + k\alpha),$$

provided the series converges at $x = a$ and $x = b$. In that case, f is said to be α-forward integrable on $[a, b]$. We say that f is α-forward integrable over I if it is α-forward integrable for all $a, b \in I$.

Remark 5.2.3. *If $f : I \to \mathbb{R}$ is a function such that $\sup I < +\infty$, then we can easily extend f to $\tilde{f} : \tilde{I} \to \mathbb{R}$ with $\sup \tilde{I} = +\infty$ by letting $\tilde{f}|_I = f$ and $\tilde{f}|_{\tilde{I} \backslash I} = 0$.*

Remark 5.2.4. *Definition 5.2.2 is valid for any two real points a, b and not only for points belonging to $\alpha\mathbb{Z}$. This is in contrast with the theory of time scales [1, 27, 28].*

Using the same techniques that Aldwoah used in his Ph.D. thesis [3], it can be proved that the α-forward integral has the following properties.

Theorem 5.2.5. *If $f, g : I \to \mathbb{R}$ are α-forward integrable on $[a, b]$, $c \in [a, b]$, $k \in \mathbb{R}$, then*

1. $\displaystyle\int_a^a f(t)\,\Delta_\alpha t = 0$;

2. $\displaystyle\int_a^b f(t)\,\Delta_\alpha t = \int_a^c f(t)\,\Delta_\alpha t + \int_c^b f(t)\,\Delta_\alpha t$, *when the integrals exist;*

3. $\displaystyle\int_a^b f(t)\,\Delta_\alpha t = -\int_b^a f(t)\,\Delta_\alpha t$;

4. kf *is* α*-forward integrable on* $[a,b]$ *and* $\displaystyle\int_a^b kf(t)\,\Delta_\alpha t = k\int_a^b f(t)\,\Delta_\alpha t$;

5. $f+g$ *is* α*-forward integrable on* $[a,b]$ *and*

$$\int_a^b (f+g)(t)\,\Delta_\alpha t = \int_a^b f(t)\,\Delta_\alpha t + \int_a^b g(t)\,\Delta_\alpha t;$$

6. *if* $f \equiv 0$*, then* $\displaystyle\int_a^b f(t)\,\Delta_\alpha t = 0$.

Remark 5.2.6. *Since for* $a > b$ *we have*

$$\begin{aligned}\int_a^b f(t)\,\Delta_\alpha t &= -\left(\int_b^a f(t)\,\Delta_\alpha t\right)\\ &= -\left(\int_b^{+\infty} f(t)\,\Delta_\alpha t - \int_a^{+\infty} f(t)\,\Delta_\alpha t\right)\\ &= \int_a^{+\infty} f(t)\,\Delta_\alpha t - \int_b^{+\infty} f(t)\,\Delta_\alpha t,\end{aligned}$$

then we could defined the Nörlund sum (Definition 5.2.2) for any $a, b \in I$ *instead for* $a < b$*.*

Theorem 5.2.7. *Let* $f : I \to \mathbb{R}$ *be* α*-forward integrable on* $[a,b]$*. If* $g : I \to \mathbb{R}$ *is a nonnegative* α*-forward integrable function on* $[a,b]$*, then* fg *is* α*-forward integrable on* $[a,b]$*.*

Proof. Since g is α-forward integrable, then both series

$$\alpha\sum_{k=0}^{+\infty} g(a+k\alpha) \quad\text{and}\quad \alpha\sum_{k=0}^{+\infty} g(b+k\alpha)$$

converge. We want to study the nature of series

$$\alpha\sum_{k=0}^{+\infty} fg(a+k\alpha) \quad\text{and}\quad \alpha\sum_{k=0}^{+\infty} fg(b+k\alpha).$$

Since there exists an order $N \in \mathbb{N}$ such that

$$|fg(b+k\alpha)| \leqslant g(b+k\alpha) \quad\text{and}\quad |fg(a+k\alpha)| \leqslant g(a+k\alpha)$$

for all $k > N$, then both

$$\alpha\sum_{k=0}^{+\infty} fg(a+k\alpha) \quad\text{and}\quad \alpha\sum_{k=0}^{+\infty} fg(b+k\alpha)$$

converge absolutely. The intended conclusion follows. □

Theorem 5.2.8. *Let* $f : I \to \mathbb{R}$ *and* $p > 1$*. If* $|f|$ *is* α*-forward integrable on* $[a,b]$*, then* $|f|^p$ *is also* α*-forward integrable on* $[a,b]$*.*

Proof. There exists $N \in \mathbb{N}$ such that

$$|f(b+k\alpha)|^p \leqslant |f(b+k\alpha)|$$

and

$$|f(a+k\alpha)|^p \leqslant |f(a+k\alpha)|$$

for all $k > N$. Therefore, $|f|^p$ is α-forward integrable on $[a,b]$. □

Theorem 5.2.9. *Let $f, g : I \to \mathbb{R}$ be α-forward integrable on $[a,b]$. If $|f(t)| \leqslant g(t)$ for all $t \in \{a+k\alpha : k \in \mathbb{N}_0\}$, then for $b \in \{a+k\alpha : k \in \mathbb{N}_0\}$ one has*

$$\left|\int_a^b f(t)\,\Delta_\alpha t\right| \leqslant \int_a^b g(t)\,\Delta_\alpha t.$$

Proof. Since $b \in \{a+k\alpha : k \in \mathbb{N}_0\}$, there exists k_1 such that $b = a + k_1\alpha$. Thus,

$$\begin{aligned}
\left|\int_a^b f(t)\,\Delta_\alpha t\right| &= \left|\alpha\sum_{k=0}^{+\infty} f(a+k\alpha) - \alpha\sum_{k=0}^{+\infty} f(a+(k_1+k)\alpha)\right| \\
&= \left|\alpha\sum_{k=0}^{+\infty} f(a+k\alpha) - \alpha\sum_{k=k_1}^{+\infty} f(a+k\alpha)\right| = \left|\alpha\sum_{k=0}^{k_1-1} f(a+k\alpha)\right| \\
&\leqslant \alpha\sum_{k=0}^{k_1-1} |f(a+k\alpha)| \leqslant \alpha\sum_{k=0}^{k_1-1} g(a+k\alpha) \\
&= \alpha\sum_{k=0}^{+\infty} g(a+k\alpha) - \alpha\sum_{k=k_1}^{+\infty} g(a+k\alpha) = \int_a^b g(t)\,\Delta_\alpha t.
\end{aligned}$$

□

Corollary 5.2.10. *Let $f, g : I \to \mathbb{R}$ be α-forward integrable on $[a,b]$ with $b = a+k\alpha$ for some $k \in \mathbb{N}_0$.*

1. *If $f(t) \geqslant 0$ for all $t \in \{a+k\alpha : k \in \mathbb{N}_0\}$, then*

$$\int_a^b f(t)\,\Delta_\alpha t \geqslant 0;$$

2. *If $g(t) \geqslant f(t)$ for all $t \in \{a+k\alpha : k \in \mathbb{N}_0\}$, then*

$$\int_a^b g(t)\,\Delta_\alpha t \geqslant \int_a^b f(t)\,\Delta_\alpha t.$$

We can now prove the following fundamental theorem of the α-forward integral calculus.

Theorem 5.2.11 (Fundamental theorem of Nörlund calculus)**.** *Let $f : I \to \mathbb{R}$ be α-forward integrable over I. Let $a, b, x \in I$ and define*

$$F(x) := \int_a^x f(t)\,\Delta_\alpha t.$$

Then,

$$\Delta_\alpha [F] (x) = f (x) .$$

Conversely,

$$\int_a^b \Delta_\alpha [f] (t) \Delta_\alpha t = f (b) - f (a) .$$

Proof. If

$$G (x) = - \int_x^{+\infty} f (t) \Delta_\alpha t,$$

then

$$\begin{aligned} \Delta_\alpha [G] (x) &= \frac{G (x + \alpha) - G (x)}{\alpha} \\ &= \frac{-\alpha \sum_{k=0}^{+\infty} f (x + \alpha + k\alpha) + \alpha \sum_{k=0}^{+\infty} f (x + k\alpha)}{\alpha} \\ &= \sum_{k=0}^{+\infty} f (x + k\alpha) - \sum_{k=0}^{+\infty} f (x + (k+1) \alpha) = f (x) . \end{aligned}$$

Therefore,

$$\Delta_\alpha [F] (x) = \Delta_\alpha \left(\int_a^{+\infty} f (t) \Delta_\alpha t - \int_x^{+\infty} f (t) \Delta_\alpha t \right) = f (x) .$$

Using the definition of α-forward difference operator, the second part of the theorem is also a consequence of the properties of Mengoli's series. Since

$$\begin{aligned} \int_a^{+\infty} \Delta_\alpha [f] (t) \Delta_\alpha t &= \alpha \sum_{k=0}^{+\infty} \Delta_\alpha [f] (a + k\alpha) \\ &= \alpha \sum_{k=0}^{+\infty} \frac{f (a + k\alpha + \alpha) - f (a + k\alpha)}{\alpha} \\ &= \sum_{k=0}^{+\infty} \Big(f (a + (k+1) \alpha) - f (a + k\alpha) \Big) \\ &= -f (a) \end{aligned}$$

and

$$\int_b^{+\infty} \Delta_\alpha [f] (t) \Delta_\alpha t = -f (b) ,$$

it follows that

$$\int_a^b \Delta_\alpha [f] (t) \Delta_\alpha t = \int_a^{+\infty} f (t) \Delta_\alpha t - \int_b^{+\infty} f (t) \Delta_\alpha t = f (b) - f (a) .$$

□

Corollary 5.2.12 (α-forward integration by parts)**.** *Let $f, g : I \to \mathbb{R}$. If fg and $f\Delta_\alpha [g]$ are α-forward integrable on $[a, b]$, then*

$$\int_a^b f (t) \Delta_\alpha [g] (t) \Delta_\alpha t = f (t) g (t) \Big|_a^b - \int_a^b \Delta_\alpha [f] (t) g (t + \alpha) \Delta_\alpha t$$

Proof. Since

$$\Delta_\alpha [fg](t) = \Delta_\alpha [f](t)\, g(t+\alpha) + f(t)\, \Delta_\alpha [g](t),$$

then

$$\begin{aligned}
\int_a^b f(t)\, \Delta_\alpha [g](t)\, \Delta_\alpha t &= \int_a^b \Big(\Delta_\alpha [fg](t) - \Delta_\alpha [f](t)\, g(t+\alpha) \Big) \Delta_\alpha t \\
&= \int_a^b \Delta_\alpha [fg](t)\, \Delta_\alpha t - \int_a^b \Delta_\alpha [f](t)\, g(t+\alpha)\, \Delta_\alpha t \\
&= f(t)\, g(t) \Big|_a^b - \int_a^b \Delta_\alpha [f](t)\, g(t+\alpha)\, \Delta_\alpha t.
\end{aligned}$$

□

Remark 5.2.13. *Our study of the Nörlund sum is in agreement with the Hahn quantum calculus [3, 33, 89]. In [73] the Nörlund sum is defined by*

$$\int_a^b f(t)\, \Delta_\alpha t = \alpha \left[f(a) + f(a+\alpha) + \cdots + f(b-\alpha) \right]$$

for $a < b$ such that $b - a \in \alpha\mathbb{Z}$, $\alpha \in \mathbb{R}^+$. In contrast with [73], our definition is valid for any two real points a, b and not only for those points belonging to the time scale $\alpha\mathbb{Z}$. The definitions (only) coincide if the function f is α-forward integrable on $[a, b]$.

Similarly, one can introduce the β-backward integral.

Definition 5.2.14. *Let I be an interval of $\mathbb{R}$ such that $a, b \in I$ with $a < b$ and $\inf I = -\infty$. For $f : I \to \mathbb{R}$ and $\beta > 0$ we define the* β-backward integral *of f from a to b by*

$$\int_a^b f(t)\, \nabla_\beta t = \int_{-\infty}^b f(t)\, \nabla_\beta t - \int_{-\infty}^a f(t)\, \nabla_\beta t,$$

where

$$\int_{-\infty}^x f(t)\, \nabla_\beta t = \beta \sum_{k=0}^{+\infty} f(x - k\beta),$$

provided the series converges at $x = a$ and $x = b$. In that case, f is called β-backward integrable on $[a, b]$. We say that f is β-backward integrable over I if it is β-backward integrable for all $a, b \in I$.

The β-backward Nörlund sum has similar results and properties as the α-forward Nörlund sum. In particular, the β-backward integral is the inverse operator of ∇_β.

5.3 The α, β-symmetric quantum calculus

We begin by introducing in Section 5.3.1 the α, β-symmetric derivative; in Section 5.3.2 we define the α, β-symmetric Nörlund sum; Section 5.3.3 is dedicated to mean value theorems for the new α, β-symmetric calculus and in the last Section 5.3.4 we prove some α, β-Symmetric integral inequalities.

5.3.1 The α,β-symmetric derivative

In what follows, $\alpha,\beta \in \mathbb{R}_0^+$ with at least one of them positive and I is an interval such that $|I| > \max\{\alpha,\beta\}$. We denote by I_β^α the set

$$I_\beta^\alpha = \begin{cases} I \backslash \left([\inf I, \inf I + \beta] \cup [\sup I - \alpha, \sup I]\right) & \text{if} \quad \inf I \neq -\infty \wedge \sup I \neq +\infty \\ I \backslash \left([\inf I, \inf I + \beta]\right) & \text{if} \quad \inf I \neq -\infty \wedge \sup I = +\infty \\ I \backslash \left([\sup I - \alpha, \sup I]\right) & \text{if} \quad \inf I = -\infty \wedge \sup I \neq +\infty \\ I & \text{if} \quad \inf I = -\infty \wedge \sup I = +\infty. \end{cases}$$

Definition 5.3.1. *The α,β-*symmetric difference derivative *of $f : I \to \mathbb{R}$ is given by*

$$D_{\alpha,\beta}[f](t) = \frac{f(t+\alpha) - f(t-\beta)}{\alpha+\beta}$$

for all $t \in I_\beta^\alpha$.

Remark 5.3.2. *The α,β-symmetric difference operator is a generalization of both the α-forward and the β-backward difference operators. Indeed, the α-forward difference operator is obtained for $\alpha > 0$ and $\beta = 0$; while for $\alpha = 0$ and $\beta > 0$ we obtain the β-backward difference operator.*

Remark 5.3.3. *The classical symmetric derivative [117] is obtained by choosing $\beta = \alpha$ and taking the limit $\alpha \to 0$. When $\alpha = \beta = h > 0$, we call h-symmetric derivative to the α,β-symmetric difference operator.*

Remark 5.3.4. *If $\alpha,\beta \geqslant 0$ with $\alpha+\beta > 0$, then*

$$D_{\alpha,\beta}[f](t) = \frac{\alpha}{\alpha+\beta}\Delta_\alpha[f](t) + \frac{\beta}{\alpha+\beta}\nabla_\beta[f](t),$$

where Δ_α and ∇_β are, respectively, the α-forward and the β-backward difference operators.

The symmetric difference operator has the following properties.

Theorem 5.3.5. *Let $f,g : I \to \mathbb{R}$ and $c,\lambda \in \mathbb{R}$. For all $t \in I_\beta^\alpha$ one has:*

1. $D_{\alpha,\beta}[c](t) = 0$;
2. $D_{\alpha,\beta}[f+g](t) = D_{\alpha,\beta}[f](t) + D_{\alpha,\beta}[g](t)$;
3. $D_{\alpha,\beta}[\lambda f](t) = \lambda D_{\alpha,\beta}[f](t)$;
4. $D_{\alpha,\beta}[fg](t) = D_{\alpha,\beta}[f](t)\, g(t+\alpha) + f(t-\beta)\, D_{\alpha,\beta}[g](t)$;
5. $D_{\alpha,\beta}[fg](t) = D_{\alpha,\beta}[f](t)\, g(t-\beta) + f(t+\alpha)\, D_{\alpha,\beta}[g](t)$;
6. $D_{\alpha,\beta}\left[\frac{f}{g}\right](t) = \frac{D_{\alpha,\beta}[f](t)\, g(t-\beta) - f(t-\beta)\, D_{\alpha,\beta}[g](t)}{g(t+\alpha)\, g(t-\beta)}$

   provided $g(t+\alpha)\, g(t-\beta) \neq 0$;

7. $D_{\alpha,\beta}\left[\frac{f}{g}\right](t)=\frac{D_{\alpha,\beta}[f](t)\,g(t+\alpha)-f(t+\alpha)\,D_{\alpha,\beta}[g](t)}{g(t+\alpha)\,g(t-\beta)}$
provided $g(t+\alpha)\,g(t-\beta)\neq 0$.

Proof. Property 1 is a trivial consequence of Definition 5.3.1. Properties 2, 3 and 4 follow by direct computations:

$$\begin{aligned}D_{\alpha,\beta}[f+g](t)&=\frac{(f+g)(t+\alpha)-(f+g)(t-\beta)}{\alpha+\beta}\\&=\frac{f(t+\alpha)-f(t-\beta)}{\alpha+\beta}+\frac{g(t+\alpha)-g(t-\beta)}{\alpha+\beta}\\&=D_{\alpha,\beta}[f](t)+D_{\alpha,\beta}[g](t);\end{aligned}$$

$$\begin{aligned}D_{\alpha,\beta}[\lambda f](t)&=\frac{(\lambda f)(t+\alpha)-(\lambda f)(t-\beta)}{\alpha+\beta}\\&=\lambda\frac{f(t+\alpha)-f(t-\beta)}{\alpha+\beta}\\&=\lambda D_{\alpha,\beta}[f](t);\end{aligned}$$

$$\begin{aligned}D_{\alpha,\beta}[fg](t)&=\frac{(fg)(t+\alpha)-(fg)(t-\beta)}{\alpha+\beta}\\&=\frac{f(t+\alpha)\,g(t+\alpha)-f(t-\beta)\,g(t-\beta)}{\alpha+\beta}\\&=\frac{f(t+\alpha)-f(t-\beta)}{\alpha+\beta}g(t+\alpha)+\frac{g(t+\alpha)-g(t-\beta)}{\alpha+\beta}f(t-\beta)\\&=D_{\alpha,\beta}[f](t)\,g(t+\alpha)+f(t-\beta)\,D_{\alpha,\beta}[g](t).\end{aligned}$$

Property 5 is obtained from property 4 interchanging the role of f and g. To prove property 6 we begin by noting that

$$\begin{aligned}D_{\alpha,\beta}\left[\frac{1}{g}\right](t)&=\frac{\frac{1}{g}(t+\alpha)-\frac{1}{g}(t-\beta)}{\alpha+\beta}=\frac{\frac{1}{g(t+\alpha)}-\frac{1}{g(t-\beta)}}{\alpha+\beta}\\&=\frac{g(t-\beta)-g(t+\alpha)}{(\alpha+\beta)\,g(t+\alpha)\,g(t-\beta)}=-\frac{D_{\alpha,\beta}[g](t)}{g(t+\alpha)\,g(t-\beta)}.\end{aligned}$$

Hence,

$$\begin{aligned}D_{\alpha,\beta}\left[\frac{f}{g}\right](t)&=D_{\alpha,\beta}\left[f\frac{1}{g}\right](t)=D_{\alpha,\beta}[f](t)\,\frac{1}{g}(t+\alpha)+f(t-\beta)\,D_{\alpha,\beta}\left[\frac{1}{g}\right](t)\\&=\frac{D_{\alpha,\beta}[f](t)}{g(t+\alpha)}-f(t-\beta)\,\frac{D_{\alpha,\beta}[g](t)}{g(t+\alpha)\,g(t-\beta)}\\&=\frac{D_{\alpha,\beta}[f](t)\,g(t-\beta)-f(t-\beta)\,D_{\alpha,\beta}[g](t)}{g(t+\alpha)\,g(t-\beta)}.\end{aligned}$$

Property 7 follows from simple calculations:

$$\begin{aligned} D_{\alpha,\beta}\left[\frac{f}{g}\right](t) &= D_{\alpha,\beta}\left[f\frac{1}{g}\right](t) = D_{\alpha,\beta}[f](t)\,\frac{1}{g}(t-\beta) + f(t+\alpha)\,D_{\alpha,\beta}\left[\frac{1}{g}\right](t) \\ &= \frac{D_{\alpha,\beta}[f](t)}{g(t-\beta)} - f(t+\alpha)\,\frac{D_{\alpha,\beta}[g](t)}{g(t+\alpha)\,g(t-\beta)} \\ &= \frac{D_{\alpha,\beta}[f](t)\,g(t+\alpha) - f(t+\alpha)\,D_{\alpha,\beta}[g](t)}{g(t+\alpha)\,g(t-\beta)}. \end{aligned}$$

□

5.3.2 The α, β-symmetric Nörlund sum

Having in mind Remark 5.3.4, we define the α, β-symmetric integral as a linear combination of the α-forward and the β-backward integrals.

Definition 5.3.6. *Let $f : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$, $a < b$. If f is α-forward and β-backward integrable on $[a, b]$, $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, then we define the α, β-*symmetric integral *of f from a to b by*

$$\int_a^b f(t)\, d_{\alpha,\beta}t = \frac{\alpha}{\alpha+\beta}\int_a^b f(t)\,\Delta_\alpha t + \frac{\beta}{\alpha+\beta}\int_a^b f(t)\,\nabla_\beta t.$$

Function f is α, β-symmetric integrable if it is α, β-symmetric integrable for all $a, b \in \mathbb{R}$.

Remark 5.3.7. *Note that if $\alpha \in \mathbb{R}^+$ and $\beta = 0$, then*

$$\int_a^b f(t)\, d_{\alpha,\beta}t = \int_a^b f(t)\,\Delta_\alpha t$$

and we do not need to assume in Definition 5.3.6 that f is β-backward integrable; if $\alpha = 0$ and $\beta \in \mathbb{R}^+$, then

$$\int_a^b f(t)\, d_{\alpha,\beta}t = \int_a^b f(t)\,\nabla_\beta t$$

and we do not need to assume that f is α-forward integrable.

Example 5.3.8. *Let* $f(t) = \dfrac{1}{t^2}$.

$$\begin{aligned}
\int_1^3 \frac{1}{t^2} d_{2,2}t &= \frac{1}{2}\int_1^3 \frac{1}{t^2}\Delta_2 t + \frac{1}{2}\int_1^3 \frac{1}{t^2}\nabla_2 t \\
&= \frac{1}{2}\left(2\sum_{k=0}^{+\infty} f(1+2k) - 2\sum_{k=0}^{+\infty} f(3+2k)\right) \\
&+ \frac{1}{2}\left(2\sum_{k=0}^{+\infty} f(3-2k) - 2\sum_{k=0}^{+\infty} f(1-2k)\right) \\
&= \left(\sum_{k=0}^{+\infty} f(1+2k) - \sum_{k=0}^{+\infty} f(1+2(k+1))\right) \\
&+ \left(\sum_{k=0}^{+\infty} f(3-2k) - \sum_{k=0}^{+\infty} f(3-2(k+1))\right) \\
&= f(1) + f(3) \\
&= 1 + \frac{1}{9} = \frac{10}{9}.
\end{aligned}$$

The α, β-symmetric integral has the following properties.

Theorem 5.3.9. *Let $f, g : \mathbb{R} \to \mathbb{R}$ be α, β-symmetric integrable on $[a, b]$. Let $c \in [a, b]$ and $k \in \mathbb{R}$. Then,*

1. $\int_a^a f(t)\, d_{\alpha,\beta}t = 0$;
2. $\int_a^b f(t)\, d_{\alpha,\beta}t = \int_a^c f(t)\, d_{\alpha,\beta}t + \int_c^b f(t)\, d_{\alpha,\beta}t$, *when the integrals exist;*
3. $\int_a^b f(t)\, d_{\alpha,\beta}t = -\int_b^a f(t)\, d_{\alpha,\beta}t$;
4. kf *is α, β-symmetric integrable on $[a, b]$ and* $\int_a^b kf(t)\, d_{\alpha,\beta}t = k\int_a^b f(t)\, d_{\alpha,\beta}t$;
5. $f + g$ *is α, β-symmetric integrable on $[a, b]$ and*
$$\int_a^b (f+g)(t)\, d_{\alpha,\beta}t = \int_a^b f(t)\, d_{\alpha,\beta}t + \int_a^b g(t)\, d_{\alpha,\beta}t;$$
6. *if g is a nonnegative function, then fg is α, β-symmetric integrable on $[a, b]$.*

Proof. These results are easy consequences of the α-forward and β-backward integral properties. □

The properties of the α, β-symmetric integral follow from the corresponding α-forward and β-backward integral properties. It should be noted, however, that the equality

$$D_{\alpha,\beta}\left[s \mapsto \int_a^s f(\tau)\, d_{\alpha,\beta}\tau\right](t) = f(t)$$

is not always true in the α,β-symmetric calculus, despite both forward and backward integrals satisfy the corresponding fundamental theorem of calculus. Indeed, let

$$f(t)=\begin{cases}\frac{1}{2^t} & \text{if } t\in\mathbb{N},\\ \\ 0 & \text{otherwise.}\end{cases}$$

Then, for a fixed $t\in\mathbb{N}$,

$$\begin{aligned}\int_0^t \frac{1}{2^\tau}d_{1,1}\tau &= \frac{1}{2}\int_0^t \frac{1}{2^\tau}\Delta_1\tau+\frac{1}{2}\int_0^t\frac{1}{2^\tau}\nabla_1\tau\\ &=\frac{1}{2}\left(\sum_{k=0}^{+\infty}f(0+k)-\sum_{k=0}^{+\infty}f(t+k)\right)+\frac{1}{2}\left(\sum_{k=0}^{+\infty}f(t-k)-\sum_{k=0}^{+\infty}f(0-k)\right)\\ &=\frac{1}{2}\left(1+\frac{1}{2}+\cdots+\frac{1}{2^{t-1}}\right)+\frac{1}{2}\left(\frac{1}{2^t}+\frac{1}{2^{t-1}}+\cdots+\frac{1}{2}\right)\\ &=\frac{1}{2}\frac{1-\frac{1}{2^t}}{1-\frac{1}{2}}+\frac{1}{4}\frac{1-\frac{1}{2^t}}{1-\frac{1}{2}}=\frac{3}{2}\left(1-\frac{1}{2^t}\right)\end{aligned}$$

and

$$D_{1,1}\left[s\mapsto\int_0^s\frac{1}{2^\tau}d_{1,1}\tau\right](t)=\frac{3}{2}D_{1,1}\left[s\mapsto 1-\frac{1}{2^s}\right](t)=-\frac{3}{2}\frac{\frac{1}{2^{t+1}}-\frac{1}{2^{t-1}}}{2}=\frac{9}{2^{t+3}}.$$

Therefore,

$$D_{1,1}\left[s\mapsto\int_0^s\frac{1}{2^\tau}d_{1,1}\tau\right](t)\neq\frac{1}{2^t}.$$

The next result follows immediately from Theorem 5.2.8 and the corresponding β-backward version.

Theorem 5.3.10. *Let $f:\mathbb{R}\to\mathbb{R}$ and $p>1$. If $|f|$ is α,β-symmetric integrable on $[a,b]$, then $|f|^p$ is also α,β-symmetric integrable on $[a,b]$.*

Theorem 5.3.11. *Let $f,g:\mathbb{R}\to\mathbb{R}$ be α,β-symmetric integrable functions on $[a,b]$, $\mathcal{A}:=\{a+k\alpha:k\in\mathbb{N}_0\}$ and $\mathcal{B}:=\{b-k\beta:k\in\mathbb{N}_0\}$. For $b\in\mathcal{A}$ and $a\in\mathcal{B}$ one has:*

1. *if $|f(t)|\leqslant g(t)$ for all $t\in\mathcal{A}\cup\mathcal{B}$, then $\left|\int_a^b f(t)\,d_{\alpha,\beta}t\right|\leqslant\int_a^b g(t)\,d_{\alpha,\beta}t$;*
2. *if $f(t)\geqslant 0$ for all $t\in\mathcal{A}\cup\mathcal{B}$, then $\int_a^b f(t)\,d_{\alpha,\beta}t\geqslant 0$;*
3. *if $g(t)\geqslant f(t)$ for all $t\in\mathcal{A}\cup\mathcal{B}$, then $\int_a^b g(t)\,d_{\alpha,\beta}t\geqslant\int_a^b f(t)\,d_{\alpha,\beta}t$.*

Proof. It follows from Theorem 5.2.9 and Corollary 5.2.10 and the corresponding β-backward versions. □

5.3.3 Mean value theorems

We begin by remarking that if f assumes its local maximum at t_0, then there exist $\alpha, \beta \in \mathbb{R}_0^+$ with at least one of them positive, such that $f(t_0+\alpha) \leqslant f(t_0)$ and $f(t_0) \geqslant f(t_0-\beta)$. If $\alpha, \beta \in \mathbb{R}^+$ this means that $\Delta_\alpha[f](t) \leqslant 0$ and $\nabla_\beta[f](t) \geqslant 0$. Also, we have the corresponding result for a local minimum: if f assumes its local minimum at t_0, then there exist $\alpha, \beta \in \mathbb{R}^+$ such that $\Delta_\alpha[f](t) \geqslant 0$ and $\nabla_\beta[f](t) \leqslant 0$.

Theorem 5.3.12 (The α, β-symmetric Fermat theorem for stationary points)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. If f assumes a local extremum at $t_0 \in \,]a,b[$, then there exist two positive real numbers α and β such that*

$$D_{\alpha,\beta}[f](t_0) = 0.$$

Proof. We prove the case where f assumes a local maximum at t_0. Then there exist $\alpha_1, \beta_1 \in \mathbb{R}^+$ such that $\Delta_{\alpha_1}[f](t_0) \leqslant 0$ and $\nabla_{\beta_1}[f](t_0) \geqslant 0$. If $f(t_0+\alpha_1) = f(t_0-\beta_1)$, then $D_{\alpha_1,\beta_1}[f](t_0) = 0$. If $f(t_0+\alpha_1) \neq f(t_0-\beta_1)$, then let us choose $\gamma = \min\{\alpha_1, \beta_1\}$. Suppose (without loss of generality) that $f(t_0-\gamma) > f(t_0+\gamma)$. Then, $f(t_0) \geqslant f(t_0-\gamma) > f(t_0+\gamma)$ and, since f is continuous, by the intermediate value theorem there exists ρ such that $0 < \rho < \gamma$ and $f(t_0+\rho) = f(t_0-\gamma)$. Therefore, $D_{\rho,\gamma}[f](t_0) = 0$. □

Theorem 5.3.13 (The α, β-symmetric Rolle mean value theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function with $f(a) = f(b)$. Then there exist α, $\beta \in \mathbb{R}^+$ and $c \in \,]a,b[$ such that*

$$D_{\alpha,\beta}[f](c) = 0.$$

Proof. If $f = const$, then the result is obvious. If f is not a constant function, then there exists $t \in \,]a,b[$ such that $f(t) \neq f(a)$. Since f is continuous on the compact set $[a,b]$, f has an extremum $M = f(c)$ with $c \in \,]a,b[$. Since c is also a local extremizer, then, by Theorem 5.3.12, there exist α, $\beta \in \mathbb{R}^+$ such that $D_{\alpha,\beta}[f](c) = 0$. □

Theorem 5.3.14 (The α, β-symmetric Lagrange mean value theorem)**.** *Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then there exist $c \in \,]a,b[$ and α, $\beta \in \mathbb{R}^+$ such that*

$$D_{\alpha,\beta}[f](c) = \frac{f(b)-f(a)}{b-a}.$$

Proof. Let function g be defined on $[a,b]$ by $g(t) = f(a) - f(t) + (t-a)\frac{f(b)-f(a)}{b-a}$. Clearly, g is continuous on $[a,b]$ and $g(a) = g(b) = 0$. Hence, by Theorem 5.3.13, there exist α, $\beta \in \mathbb{R}^+$ and $c \in \,]a,b[$ such that $D_{\alpha,\beta}[g](c) = 0$. Since

$$\begin{aligned}
D_{\alpha,\beta}[g](t) &= \frac{g(t+\alpha)-g(t-\beta)}{\alpha+\beta} \\
&= \frac{1}{\alpha+\beta}\left(f(a) - f(t+\alpha) + (t+\alpha-a)\frac{f(b)-f(a)}{b-a}\right) \\
&- \frac{1}{\alpha+\beta}\left(f(a) - f(t-\beta) + (t-\beta-a)\frac{f(b)-f(a)}{b-a}\right) \\
&= \frac{1}{\alpha+\beta}\left(f(t-\beta) - f(t+\alpha) + (\alpha+\beta)\frac{f(b)-f(a)}{b-a}\right) \\
&= \frac{f(b)-f(a)}{b-a} - D_{\alpha,\beta}[f](t),
\end{aligned}$$

we conclude that

$$D_{\alpha,\beta}[f](c)=\frac{f(b)-f(a)}{b-a}.$$

□

Theorem 5.3.15 (The α,β-symmetric Cauchy mean value theorem)**.** *Let $f,g:[a,b]\to\mathbb{R}$ be continuous functions. Suppose that $D_{\alpha,\beta}[g](t)\neq 0$ for all $t\in\,]a,b[$ and all $\alpha,\ \beta\in\mathbb{R}^+$. Then there exist $\bar{\alpha},\bar{\beta}\in\mathbb{R}^+$ and $c\in\,]a,b[$ such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{D_{\bar{\alpha},\bar{\beta}}[f](c)}{D_{\bar{\alpha},\bar{\beta}}[g](c)}.$$

Proof. From condition $D_{\alpha,\beta}[g](t)\neq 0$ for all $t\in\,]a,b[$ and all $\alpha,\beta\in\mathbb{R}^+$ and the α,β-symmetric Rolle mean value theorem (Theorem 5.3.13), it follows that $g(b)\neq g(a)$. Let us consider function F defined on $[a,b]$ by

$$F(t)=f(t)-f(a)-\frac{f(b)-f(a)}{g(b)-g(a)}\left[g(t)-g(a)\right].$$

Clearly, F is continuous on $[a,b]$ and $F(a)=F(b)$. Applying the α,β-symmetric Rolle mean value theorem to function F, we conclude that there exist $\bar{\alpha},\bar{\beta}\in\mathbb{R}^+$ and $c\in\,]a,b[$ such that

$$0=D_{\bar{\alpha},\bar{\beta}}[F](c)\ =D_{\bar{\alpha},\bar{\beta}}[f](c)-\frac{f(b)-f(a)}{g(b)-g(a)}D_{\bar{\alpha},\bar{\beta}}[g](c),$$

proving the intended result. □

Theorem 5.3.16 (Mean value theorem for the α,β- integral)**.** *Let $a,b\in\mathbb{R}$ such that $a<b$ and $b\in\mathcal{A}:=\{a+k\alpha:k\in\mathbb{N}_0\}$ and $a\in\mathcal{B}:=\{b-k\beta:k\in\mathbb{N}_0\}$, where $\alpha,\beta\in\mathbb{R}_0^+$, $\alpha+\beta\neq 0$. Let $f,g:\mathbb{R}\to\mathbb{R}$ be bounded and α,β-symmetric integrable on $[a,b]$ with g nonnegative. Let m and M be the infimum and the supremum, respectively, of function f. Then, there exists a real number K satisfying the inequalities*

$$m\leqslant K\leqslant M$$

such that

$$\int_a^b f(t)\,g(t)\,d_{\alpha,\beta}t=K\int_a^b g(t)\,d_{\alpha,\beta}t.$$

Proof. Since

$$m\leqslant f(t)\leqslant M$$

for all $t\in\mathbb{R}$ and

$$g(t)\geqslant 0,$$

then

$$mg(t)\leqslant f(t)\,g(t)\leqslant Mg(t)$$

for all $t\in\mathcal{A}\cup\mathcal{B}$. All functions mg, fg and Mg are α,β-symmetric integrable on $[a,b]$. By Theorems 5.3.9 and 5.3.11,

$$m\int_a^b g(t)\,d_{\alpha,\beta}t\leqslant\int_a^b f(t)\,g(t)\,d_{\alpha,\beta}t\leqslant M\int_a^b g(t)\,d_{\alpha,\beta}t.$$

If

$$\int_a^b g(t)\, d_{\alpha,\beta}t = 0,$$

then

$$\int_a^b f(t)\, g(t)\, d_{\alpha,\beta}t = 0;$$

if

$$\int_a^b g(t)\, d_{\alpha,\beta}t > 0,$$

then

$$m \leqslant \frac{\int_a^b f(t)\, g(t)\, d_{\alpha,\beta}t}{\int_a^b g(t)\, d_{\alpha,\beta}t} \leqslant M.$$

Therefore, the middle term of these inequalities is equal to a number K, which yields the intended result. □

5.3.4 α, β-Symmetric integral inequalities

Inspired in the work by Agarwal et al. [2], we now present α, β-symmetric versions of Hölder's, Cauchy–Schwarz's and Minkowski's inequalities.

Theorem 5.3.17 (α, β-symmetric Hölder's inequality). *Let $f, g : \mathbb{R} \to \mathbb{R}$ and $a, b \in \mathbb{R}$ with $a < b$ such that $b \in \mathcal{A} := \{a + k\alpha : k \in \mathbb{N}_0\}$ and $a \in \mathcal{B} := \{b - k\beta : k \in \mathbb{N}_0\}$, where $\alpha, \beta \in \mathbb{R}_0^+$, $\alpha + \beta \neq 0$. If $|f|$ and $|g|$ are α, β-symmetric integrable on $[a, b]$, then*

$$\int_a^b |f(t)\, g(t)|\, d_{\alpha,\beta}t \leqslant \left(\int_a^b |f(t)|^p\, d_{\alpha,\beta}t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\, d_{\alpha,\beta}t\right)^{\frac{1}{q}}, \tag{5.3.1}$$

where $p > 1$ and $q = p/(p-1)$.

Proof. For $\alpha, \beta \in \mathbb{R}_0^+$, $\alpha + \beta \neq 0$, the following inequality holds (Young's inequality):

$$\alpha^{\frac{1}{p}} \beta^{\frac{1}{q}} \leqslant \frac{\alpha}{p} + \frac{\beta}{q}.$$

Without loss of generality, suppose that

$$\left(\int_a^b |f(t)|^p\, d_{\alpha,\beta}t\right) \left(\int_a^b |g(t)|^q\, d_{\alpha,\beta}t\right) \neq 0$$

(note that both integrals exist by Theorem 5.3.10). Set

$$\xi(t) := \frac{|f(t)|^p}{\int_a^b |f(\tau)|^p\, d_{\alpha,\beta}\tau}$$

and

$$\gamma(t) := \frac{|g(t)|^q}{\int_a^b |g(\tau)|^q\, d_{\alpha,\beta}\tau}.$$

Since both functions ξ and γ are α,β-symmetric integrable on $[a,b]$, then

$$\begin{aligned}
&\int_a^b \frac{|f(t)|}{\left(\int_a^b |f(\tau)|^p d_{\alpha,\beta}\tau\right)^{\frac{1}{p}}} \frac{|g(t)|}{\left(\int_a^b |g(\tau)|^q d_{\alpha,\beta}\tau\right)^{\frac{1}{q}}} d_{\alpha,\beta}t = \int_a^b \xi(t)^{\frac{1}{p}} \gamma(t)^{\frac{1}{q}} d_{\alpha,\beta}t \\
&\leqslant \int_a^b \left(\frac{\xi(t)}{p} + \frac{\gamma(t)}{q}\right) d_{\alpha,\beta}t \\
&= \frac{1}{p}\int_a^b \left(\frac{|f(t)|^p}{\int_a^b |f(\tau)|^p d_{\alpha,\beta}\tau}\right) d_{\alpha,\beta}t + \frac{1}{q}\int_a^b \left(\frac{|g(t)|^q}{\int_a^b |g(\tau)|^q d_{\alpha,\beta}\tau}\right) d_{\alpha,\beta}t = 1
\end{aligned}$$

proving the intended result. □

The particular case $p = q = 2$ of (5.3.1) gives the α,β-symmetric Cauchy–Schwarz's inequality.

Corollary 5.3.18 (α,β-symmetric Cauchy–Schwarz's inequality)**.** *Let $f,g : \mathbb{R} \to \mathbb{R}$ and $a,b \in \mathbb{R}$ with $a < b$ such that $b \in \mathcal{A} := \{a + k\alpha : k \in \mathbb{N}_0\}$ and $a \in \mathcal{B} := \{b - k\beta : k \in \mathbb{N}_0\}$, where $\alpha,\beta \in \mathbb{R}_0^+$, $\alpha + \beta \neq 0$. If f and g are α,β-symmetric integrable on $[a,b]$, then*

$$\int_a^b |f(t)\, g(t)|\, d_{\alpha,\beta}t \leqslant \sqrt{\left(\int_a^b |f(t)|^2 d_{\alpha,\beta}t\right)\left(\int_a^b |g(t)|^2 d_{\alpha,\beta}t\right)}.$$

We prove the α,β-symmetric Minkowski inequality using α,β-symmetric Hölder's inequality.

Theorem 5.3.19 (α,β-symmetric Minkowski's inequality)**.** *Let $f,g : \mathbb{R} \to \mathbb{R}$ and $a,b \in \mathbb{R}$ with $a < b$ such that $b \in \mathcal{A} := \{a + k\alpha : k \in \mathbb{N}_0\}$ and $a \in \mathcal{B} := \{b - k\beta : k \in \mathbb{N}_0\}$, where $\alpha,\beta \in \mathbb{R}_0^+$, $\alpha + \beta \neq 0$. If f and g are α,β-symmetric integrable on $[a,b]$, then, for any $p > 1$,*

$$\left(\int_a^b |f(t) + g(t)|^p d_{\alpha,\beta}t\right)^{\frac{1}{p}} \leqslant \left(\int_a^b |f(t)|^p d_{\alpha,\beta}t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p d_{\alpha,\beta}t\right)^{\frac{1}{p}}. \qquad (5.3.2)$$

Proof. If $\int_a^b |f(t) + g(t)|^p d_{\alpha,\beta}t = 0$, the result is trivial. Suppose that

$$\int_a^b |f(t) + g(t)|^p d_{\alpha,\beta}t \neq 0.$$

Since

$$\begin{aligned}
\int_a^b |f(t) + g(t)|^p d_{\alpha,\beta}t &= \int_a^b |f(t) + g(t)|^{p-1} |f(t) + g(t)|\, d_{\alpha,\beta}t \\
&\leqslant \int_a^b |f(t)|\, |f(t) + g(t)|^{p-1} d_{\alpha,\beta}t + \int_a^b |g(t)|\, |f(t) + g(t)|^{p-1} d_{\alpha,\beta}t,
\end{aligned}$$

then applying α, β-symmetric Hölder's inequality (Theorem 5.3.17) with $q = p/(p-1)$, we obtain

$$\int_a^b |f(t) + g(t)|^p \, d_{\alpha,\beta}t \leqslant \left(\int_a^b |f(t)|^p \, d_{\alpha,\beta}t\right)^{\frac{1}{p}} \left(\int_a^b |f(t) + g(t)|^{(p-1)q} \, d_{\alpha,\beta}t\right)^{\frac{1}{q}}$$
$$+ \left(\int_a^b |g(t)|^p \, d_{\alpha,\beta}t\right)^{\frac{1}{p}} \left(\int_a^b |f(t) + g(t)|^{(p-1)q} \, d_{\alpha,\beta}t\right)^{\frac{1}{q}}$$
$$= \left[\left(\int_a^b |f(t)|^p \, d_{\alpha,\beta}t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \, d_{\alpha,\beta}t\right)^{\frac{1}{p}}\right] \left(\int_a^b |f(t) + g(t)|^p \, d_{\alpha,\beta}t\right)^{\frac{1}{q}}.$$

and therefore we get

$$\frac{\int_a^b |f(t) + g(t)|^p \, d_{\alpha,\beta}t}{\left(\int_a^b |f(t) + g(t)|^p \, d_{\alpha,\beta}t\right)^{\frac{1}{q}}} \leqslant \left(\int_a^b |f(t)|^p \, d_{\alpha,\beta}t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \, d_{\alpha,\beta}t\right)^{\frac{1}{p}},$$

proving inequality (5.3.2). □

To conclude this section we remark that our α, β-symmetric calculus is more general than the standard h-calculus and h-symmetric calculus. In particular, all our results give, as corollaries, results in the classical quantum h-calculus by choosing $\alpha = h > 0$ and $\beta = 0$, and results in the h-symmetric calculus by choosing $\alpha = \beta = h > 0$.

5.4 State of the Art

The results of this chapter were accepted to be published on the Proceedings of the International Conference on Differential & Difference Equations and Applications [34, 35]. Quantum calculus is receiving an increase of interest due to its applications in physics, economics and calculus of variations. In this chapter we have developed new results in this field: we introduced a new symmetric quantum calculus.

Chapter 6

The q-Symmetric Variational Calculus

In this chapter we bring a new approach to the study of quantum calculus and introduce the q-symmetric variational calculus. We prove a necessary optimality condition of Euler–Lagrange type and a sufficient optimality condition for symmetric quantum variational problems. The results are illustrated with an example.

6.1 Introduction

Quantum calculus, also known as calculus without limits, is a very interesting field in mathematics. Moreover, it plays an important role in several fields of physics such as cosmic strings and black holes [115], conformal quantum mechanics [123], nuclear and high energy physics [80], just to name a few. For a deeper understanding of quantum calculus we refer the reader to [30, 50, 73, 76].

Usually we are concern with two types of quantum calculus: the q-calculus and the h-calculus. In this chapter we are concerned with the q-calculus. Historically, the q-calculus was first introduced by Jackson [71] and is a calculus based on the notion of the q-derivative

$$\frac{f(qt)-f(t)}{(q-1)t},$$

where q is a fixed number different from 1, $t \neq 0$ and f is a real function. In contrast to the classical derivative, which measures the rate of change of the function of an incremental translation of its argument, the q-derivative measures the rate of change with respect to a dilatation of its argument by a factor q. It is clear that if f is differentiable at $t \neq 0$, then

$$f'(t) = \lim_{q \to 1} \frac{f(qt)-f(t)}{(q-1)t}.$$

For a fixed $q \in \,]0,1[$ and $t \neq 0$ the q-symmetric derivative of a function f at point t is defined by

$$\frac{f\left(qt\right)-f\left(q^{-1}t\right)}{\left(q-q^{-1}\right)t}.$$

The q-symmetric quantum calculus has proven to be useful in several fields, in particular in quantum mechanics [79]. As noticed in [79], the q-symmetric derivative has important properties for the q-exponential function which turns out to be not true with the usual derivative.

It is well known that the derivative of a differentiable real function f at a point $t \neq 0$ can be approximated by the q-symmetric derivative. We believe that the q-symmetric derivative has, in general, better convergence properties than the h-derivative

$$\frac{f(t+h)-f(t)}{h}$$

and the q-derivative

$$\frac{f(qt)-f(t)}{(q-1)t},$$

but this requires additional investigation.

Our goal is to establish a necessary optimality condition and a sufficient optimality condition for the q-symmetric variational problem

$$\begin{cases} \mathcal{L}[y] = \displaystyle\int_a^b L\left(t, y(qt), \tilde{D}_q[y](t)\right) \tilde{d}_q t \to \text{extremize} \\ y \in \mathcal{Y}^1\left([a,b]_q, \mathbb{R}\right) \\ y(a) = \alpha \\ y(b) = \beta, \end{cases} \tag{6.1.1}$$

where α, β are fixed real numbers and $\tilde{D}_q$ denotes the q-symmetric derivative operator (see 6.2.1). By extremize we mean maximize or minimize. Also we must assume that the Lagrangian function L satisfies the following hypotheses:

(H_q1) $(u,v) \to L(t,u,v)$ is a $C^1\left(\mathbb{R}^2, \mathbb{R}\right)$ function for any $t \in [a,b]$;

(H_q2) $t \to L\left(t, y(qt), \tilde{D}_q[y](t)\right)$ is continuous at 0 for any admissible function y;

(H_q3) functions $t \to \partial_{i+2} L\left(t, y(qt), \tilde{D}_q[y](t)\right)$, $i = 0, 1$ belong to $\mathcal{Y}^1\left([a,b]_q, \mathbb{R}\right)$ for all admissible functions y, where $\partial_i L$ denotes the partial derivative of L with respect to its ith argument.

This chapter is organized as follows. In Section 6.2 we review the necessary definitions and prove some new results of the q-symmetric calculus. Usually the set of study in q-calculus is the lattice $\left\{a, aq, aq^2, aq^3, \ldots\right\}$. In this chapter we work in an arbitrary real interval containing 0. This new approach follows the ideas that Aldwoah used in his Ph.D. thesis [3] (see also [5]). In Section 6.3 we formulate and prove our results for the q-symmetric variational calculus.

6.2 Preliminaries

Let $q \in \,]0,1[$ and let I be an interval (bounded or unbounded) of $\mathbb{R}$ containing 0. We denote by I^q the set

$$I^q := qI := \{qx : x \in I\}.$$

Note that $I^q \subseteq I$.

Definition 6.2.1 (cf. [73]). *Let f be a real function defined on I. The q-*symmetric difference operator* of f is defined by*

$$\tilde{D}_q[f](t) = \frac{f(qt) - f\left(q^{-1}t\right)}{\left(q - q^{-1}\right)t}, \quad \text{if } t \in I^q \backslash \{0\},$$

*and $\tilde{D}_q[f](0) := f'(0)$, provided f is differentiable at 0. We usually call $\tilde{D}_q[f]$ the q-*symmetric derivative *of f.*

Remark 6.2.2. *Notice that if f is differentiable (in the classical sense) at $t \in I^q$, then*

$$\lim_{q \to 1} \tilde{D}_q[f](t) = f'(t).$$

The q-symmetric difference operator has the following properties.

Theorem 6.2.3 (cf. [73]). *Let f and g be q-symmetric differentiable on I, let $\alpha, \beta \in \mathbb{R}$ and $t \in I^q$. One has*

1. $\tilde{D}_q[f] \equiv 0$ *if and only if f is constant on I;*
2. $\tilde{D}_q[\alpha f + \beta g](t) = \alpha \tilde{D}_q[f](t) + \beta \tilde{D}_q[g](t)$;
3. $\tilde{D}_q[fg](t) = \tilde{D}_q[f](t)\, g(qt) + f\left(q^{-1}t\right) \tilde{D}_q[g](t)$;
4. $\tilde{D}_q\left[\frac{f}{g}\right](t) = \frac{\tilde{D}_q[f](t)\, g\left(q^{-1}t\right) - f\left(q^{-1}t\right) \tilde{D}_q[g](t)}{g(qt)\, g\left(q^{-1}t\right)}$ *if $g(qt)\, g\left(q^{-1}t\right) \neq 0$.*

Proof. If f is constant, then it is clear that $\tilde{D}_q[f] \equiv 0$. For each $t \in I$ if $\tilde{D}_q[f](qt) = 0$, then $f(t) = f\left(q^2 t\right)$ and one has, for $n \in \mathbb{N}$,

$$f(t) = f\left(q^2 t\right) = \cdots = f\left(q^{2n} t\right).$$

Since

$$\lim_{n \to +\infty} f(t) = \lim_{n \to +\infty} f\left(q^{2n} t\right)$$

and f is continuous at 0, then

$$f(t) = f(0), \; \forall t \in I.$$

The other properties are trivial for $t = 0$ and for $t \neq 0$ see [73]. □

Definition 6.2.4 (cf. [73]). *Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ and for $q \in \,]0, 1[$ the q-*symmetric integral* of f from a to b is given by*

$$\int_a^b f(t)\, \tilde{d}_q t = \int_0^b f(t)\, \tilde{d}_q t - \int_0^a f(t)\, \tilde{d}_q t,$$

where

$$\begin{aligned} \int_0^x f(t)\, \tilde{d}_q t &= \left(q^{-1} - q\right) x \sum_{n=0}^{+\infty} q^{2n+1} f\left(q^{2n+1} x\right) \\ &= \left(1 - q^2\right) x \sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1} x\right), \qquad x \in I, \end{aligned}$$

provided that the series converges at $x = a$ and $x = b$. In that case, f is called q-symmetric integrable on $[a, b]$. We say that f is q-integrable on I if it is q-integrable on $[a, b]$ for all $a, b \in I$.

We now present two technical results that are useful to prove the Fundamental Theorem of the q-Symmetric Integral Calculus (Theorem 6.2.7).

Lemma 6.2.5 ([3]). *Let $a, b \in I$, $a < b$ and $f : I \to \mathbb{R}$ continuous at 0. Then for $s \in [a, b]$ the sequence $\left(f\left(q^{2n+1}s\right)\right)_{n\in\mathbb{N}}$ converges uniformly to $f(0)$ on I.*

Corollary 6.2.6 ([3]). *If $f : I \to \mathbb{R}$ is continuous at 0, then for $s \in [a, b]$ the series $\sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1}s\right)$ is uniformly convergent on I, and consequently, f is q-symmetric integrable on $[a, b]$.*

Theorem 6.2.7 (Fundamental theorem of the q-symmetric integral calculus). *Assume that $f : I \to \mathbb{R}$ is continuous at 0 and, for each $x \in I$, define*

$$F(x) := \int_0^x f(t)\, \tilde{d}_q t.$$

Then F is continuous at 0. Furthermore, $\tilde{D}_q[F](x)$ exists for every $x \in I^q$ with

$$\tilde{D}_q[F](x) = f(x).$$

Conversely,

$$\int_a^b \tilde{D}_q[f](t)\, \tilde{d}_q t = f(b) - f(a)$$

for all $a, b \in I$.

Proof. By Corollary 6.2.6, the function F is continuous at 0. If $x \in I^q \setminus \{0\}$, then

$$\begin{aligned}
&\tilde{D}_q\left(\int_0^x f(t)\, \tilde{d}_q t\right) = \frac{\int_0^{qx} f(t)\, \tilde{d}_q t - \int_0^{q^{-1}x} f(t)\, \tilde{d}_q t}{\left(q - q^{-1}\right) x} \\
&= \frac{q}{\left(q^2 - 1\right) x}\left[\left(1 - q^2\right) qx \sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1}qx\right) - \left(1 - q^2\right) q^{-1}x \sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1}q^{-1}x\right)\right] \\
&= \sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n}x\right) - \sum_{n=0}^{+\infty} q^{2n+2} f\left(q^{2n+2}x\right) \\
&= \sum_{n=0}^{+\infty}\left[q^{2n} f\left(q^{2n}x\right) - q^{2(n+1)} f\left(q^{2(n+1)}x\right)\right] \\
&= f(x).
\end{aligned}$$

If $x = 0$, then

$$\begin{aligned}
\tilde{D}_q[F](0) &= \lim_{h\to 0} \frac{F(h) - F(0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left(1-q^2\right) h \sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1}h\right) \\
&= \lim_{h\to 0} \left(1-q^2\right) \sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1}h\right) \\
&= \left(1-q^2\right) \sum_{n=0}^{+\infty} q^{2n} f(0) \ \text{ (by the continuity of } f \text{ at } 0) \\
&= \left(1-q^2\right) \frac{1}{1-q^2} f(0) \\
&= f(0)\ .
\end{aligned}$$

Finally, since for $x \in I$,

$$\begin{aligned}
\int_0^x \tilde{D}_q[f](t)\,\tilde{d}_q t &= \left(1-q^2\right) x \sum_{n=0}^{+\infty} q^{2n} \tilde{D}_q[f]\left(q^{2n+1}x\right) \\
&= \left(1-q^2\right) x \sum_{n=0}^{+\infty} q^{2n} \frac{f\left(qq^{2n+1}x\right) - f\left(q^{-1}q^{2n+1}x\right)}{\left(q-q^{-1}\right)\left(q^{2n+1}x\right)} \\
&= \sum_{n=0}^{+\infty} \left[f\left(q^{2n}x\right) - f\left(q^{2(n+1)}x\right)\right] \\
&= f(x) - f(0)\,,
\end{aligned}$$

we have

$$\begin{aligned}
\int_a^b \tilde{D}_q[f](t)\,\tilde{d}_q t &= \int_0^b \tilde{D}_q[f](t)\,\tilde{d}_q t - \int_0^a \tilde{D}_q[f](t)\,\tilde{d}_q t \\
&= f(b) - f(a)\,.
\end{aligned}$$

□

The q-symmetric integral has the following properties.

Theorem 6.2.8. *Let $f, g : I \to \mathbb{R}$ be q-symmetric integrable on I, $a, b, c \in I$ and $\alpha, \beta \in \mathbb{R}$. Then*

1. $\displaystyle\int_a^a f(t)\,\tilde{d}_q t = 0;$

2. $\displaystyle\int_a^b f(t)\,\tilde{d}_q t = -\int_b^a f(t)\,\tilde{d}_q t;$

3. $\displaystyle\int_a^b f(t)\,\tilde{d}_q t = \int_a^c f(t)\,\tilde{d}_q t + \int_c^b f(t)\,\tilde{d}_q t;$

4. $\displaystyle\int_a^b (\alpha f + \beta g)(t)\, \tilde{d}_q t = \alpha \int_a^b f(t)\, \tilde{d}_q t + \beta \int_a^b g(t)\, \tilde{d}_q t$;

5. *If $\tilde{D}_q[f]$ and $\tilde{D}_q[g]$ are continuous at* 0, *then*

$$\int_a^b f\left(q^{-1}t\right) \tilde{D}_q[g](t)\, \tilde{d}_q t = f(t)\, g(t)\Big|_a^b - \int_a^b \tilde{D}_q[f](t)\, g(qt)\, \tilde{d}_q t.$$

We call this formula q-symmetric integration by parts.

Proof. Properties 1-4 are trivial. Property 5 follows from Theorem 6.2.3 and Theorem 6.2.7:

$$\begin{aligned}
&\tilde{D}_q[fg](t) = \tilde{D}_q[f](t)\, g(qt) + f\left(q^{-1}t\right) \tilde{D}_q[g](t) \\
&\Leftrightarrow f\left(q^{-1}t\right) \tilde{D}_q[g](t) = \tilde{D}_q[fg](t) - \tilde{D}_q[f](t)\, g(qt) \\
&\Rightarrow \int_a^b f\left(q^{-1}t\right) \tilde{D}_q[g](t)\, \tilde{d}_q t = f(t)\, g(t)\Big|_a^b - \int_a^b \tilde{D}_q[f](t)\, g(qt)\, \tilde{d}_q t.
\end{aligned}$$

□

Remark 6.2.9. *Let us consider the function σ defined by $\sigma(t) := qt$ and use the notation $f^\sigma(t) := f(qt)$. Since, for each $t \in I^q \setminus \{0\}$,*

$$\tilde{D}_q[f^\sigma](t) = \frac{f^\sigma(qt) - f^\sigma\left(q^{-1}t\right)}{\left(q - q^{-1}\right)t} = \frac{f\left(q^2 t\right) - f(t)}{\left(q - q^{-1}\right)t}$$

and

$$\tilde{D}_q[f](qt) = \frac{f\left(q^2 t\right) - f(t)}{\left(q - q^{-1}\right)qt}$$

then we may conclude that

$$\tilde{D}_q[f^\sigma](t) = q\tilde{D}_q[f](qt), \qquad \forall t \in I^q \setminus \{0\}.$$

Since $\tilde{D}_q[f^\sigma](0) = q\tilde{D}_q[f](0)$, we may conclude that

$$\tilde{D}_q[f^\sigma](t) = q\tilde{D}_q[f](qt), \qquad \forall t \in I^q. \tag{6.2.1}$$

Hence, an analogue formula for q-symmetric integration by parts *is given by*

$$\int_a^b f(t)\, \tilde{D}_q[g](t)\, \tilde{d}_q t = f(qt)\, g(t)\Big|_a^b - q\int_a^b \tilde{D}_q[f](qt)\, g(qt)\, \tilde{d}_q t. \tag{6.2.2}$$

Proposition 6.2.10. *Let $c \in I$ and let f and g be q-symmetric integrable on I. Suppose that*

$$|f(t)| \leqslant g(t)$$

for all $t \in \left\{q^{2n+1}c : n \in \mathbb{N}_0\right\} \cup \{0\}$.

1. *If $c \geqslant 0$, then*

$$\left|\int_0^c f(t)\, \tilde{d}_q t\right| \leqslant \int_0^c g(t)\, \tilde{d}_q t.$$

2. *If $c < 0$, then*
$$\left|\int_c^0 f(t)\,\tilde{d}_q t\right| \leqslant \int_c^0 g(t)\,\tilde{d}_q t.$$

Proof. If $c \geqslant 0$, then

$$\begin{aligned}\left|\int_0^c f(t)\,\tilde{d}_q t\right| &= \left|\left(1-q^2\right)c\sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1}c\right)\right| \leqslant \left(1-q^2\right)c\sum_{n=0}^{+\infty} q^{2n}\left|f\left(q^{2n+1}c\right)\right| \\ &\leqslant \left(1-q^2\right)c\sum_{n=0}^{+\infty} q^{2n} g\left(q^{2n+1}c\right) = \int_0^c g(t)\,\tilde{d}_q t.\end{aligned}$$

If $c < 0$, then

$$\begin{aligned}\left|\int_c^0 f(t)\,\tilde{d}_q t\right| &= \left|\left(1-q^2\right)c\sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1}c\right)\right| \leqslant -\left(1-q^2\right)c\sum_{n=0}^{+\infty} q^{2n}\left|f\left(q^{2n+1}c\right)\right| \\ &\leqslant -\left(1-q^2\right)c\sum_{n=0}^{+\infty} q^{2n} g\left(q^{2n+1}c\right) = -\int_0^c g(t)\,\tilde{d}_q t = \int_c^0 g(t)\,\tilde{d}_q t\end{aligned}$$

proving the desired result. □

As an immediate consequence, we have the following result.

Corollary 6.2.11. *Let $c \in I$ and f be q-symmetric integrable on I. Suppose that*

$$f(t) \geqslant 0,\ \forall t \in \left\{q^{2n+1}c : n \in \mathbb{N}_0\right\} \cup \{0\}.$$

1. *If $c \geqslant 0$, then*
$$\int_0^c f(t)\,\tilde{d}_q t \geqslant 0.$$

2. *If $c < 0$, then*
$$\int_c^0 f(t)\,\tilde{d}_q t \geqslant 0.$$

Remark 6.2.12. *In general it is not true that if f is a nonnegative function on $[a,b]$, then*

$$\int_a^b f(t)\,\tilde{d}_q t \geqslant 0.$$

For example, consider the function f defined in $[-1,1]$ by

$$f(t) = \begin{cases} 1 & \text{if} \quad t = \frac{1}{2} \\ 6 & \text{if} \quad t = \frac{1}{6} \\ 0 & \text{if} \quad t \in [-1,1] \setminus \left\{\frac{1}{6}, \frac{1}{2}\right\}. \end{cases}$$

For $q=\frac{1}{2}$ this function is q-symmetric integrable because is continuous at $t=0$ and

$$\begin{aligned}\int_{\frac{1}{3}}^{1} f(t)\,\tilde{d}_q t &= \int_0^1 f(t)\,\tilde{d}_q t - \int_0^{\frac{1}{3}} f(t)\,\tilde{d}_q t \\ &= \frac{3}{4}\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^{2n} f\left(\left(\frac{1}{2}\right)^{2n+1}\right) - \frac{3}{4}\left(\frac{1}{3}\right)\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^{2n} f\left(\frac{1}{3}\cdot\left(\frac{1}{2}\right)^{2n+1}\right) \\ &= \frac{3}{4}\times 1 - \frac{1}{4}\times 6 = -\frac{3}{4}.\end{aligned}$$

This example also proves that in general it is not true that, for any $a,b\in I$,

$$\left|\int_a^b f(t)\,\tilde{d}_q t\right| \leqslant \int_a^b |f(t)|\,\tilde{d}_q t.$$

We conclude this section with some useful definitions and notations. For $s\in I$ we set

$$[s]_q := \left\{q^{2n+1}s : n\in\mathbb{N}_0\right\}\cup\{0\}.$$

Let $a,b\in I$ with $a<b$. We define the q-symmetric interval from a to b by

$$[a,b]_q := \left\{q^{2n+1}a : n\in\mathbb{N}_0\right\}\cup\left\{q^{2n+1}b : n\in\mathbb{N}_0\right\}\cup\{0\},$$

that is,

$$[a,b]_q = [a]_q\cup[b]_q.$$

We introduce the linear spaces

$$\begin{aligned}\mathcal{Y}^0\left([a,b]_q,\mathbb{R}\right) &:= \left\{y: I\to\mathbb{R}\mid y \text{ is bounded on } [a,b]_q \text{ and continuous at } 0\right\} \\ \mathcal{Y}^1\left([a,b]_q,\mathbb{R}\right) &:= \left\{y\in\mathcal{Y}^0\left([a,b]_q,\mathbb{R}\right)\mid \tilde{D}_q[y] \text{ is bounded on } [a,b]_q \text{ and continuous at } 0\right\}\end{aligned}$$

and, for $r=0,1$, we endow these spaces with the norm

$$\|y\|_r = \sum_{i=0}^{r}\sup_{t\in[a,b]_q}\left|\tilde{D}_q^i[y](t)\right|$$

where $\tilde{D}_q^0[y]\equiv y$.

Definition 6.2.13. *We say that y is an admissible function to problem* (6.1.1) *if $y\in\mathcal{Y}^1\left([a,b]_q,\mathbb{R}\right)$ and y satisfies the boundary conditions $y(a)=\alpha$ and $y(b)=\beta$.*

Definition 6.2.14. *We say that y_* is a local minimizer (resp. local maximizer) to problem* (6.1.1) *if y_* is an admissible function and there exists $\delta>0$ such that*

$$\mathcal{L}[y_*]\leqslant\mathcal{L}[y] \ \ (\text{resp. } \mathcal{L}[y_*]\geqslant\mathcal{L}[y]\,)$$

for all admissible y with $\|y_*-y\|_1<\delta$.

Definition 6.2.15. *We say that $\eta\in\mathcal{Y}^1\left([a,b]_q,\mathbb{R}\right)$ is an admissible variation to problem* (6.1.1) *if $\eta(a)=0=\eta(b)$.*

6.3 Main results

In this section we apply similar techniques than those used in Hahn's quantum variational calculus [33, 89].

6.3.1 Basic lemmas

We now present some lemmas which are useful to achieve our main results.

Lemma 6.3.1 (Fundamental Lemma of the q-symmetric variational calculus)**.** *Let* $f \in \mathcal{Y}^0\left([a,b]_q, \mathbb{R}\right)$. *One has*

$$\int_a^b f(t)\, h(qt)\, \tilde{d}_q t = 0$$

for all functions $h \in \mathcal{Y}^0\left([a,b]_q, \mathbb{R}\right)$ *with*

$$h(a) = h(b) = 0$$

if, and only if,

$$f(t) = 0$$

for all $t \in [a,b]_q$.

Proof. The implication "$\Leftarrow$" is obvious. Let us prove the implication "$\Rightarrow$". Suppose, by contradiction, that there exists $p \in [a,b]_q$ such that $f(p) \neq 0$.

1. If $p \neq 0$, then $p = q^{2k+1}a$ or $p = q^{2k+1}b$ for some $k \in \mathbb{N}_0$.

   (a) Suppose that $a \neq 0$ and $b \neq 0$. In this case we can assume, without loss of generality that $p = q^{2k+1}a$. Define

   $$h(t) = \begin{cases} f\left(q^{2k+1}a\right) & \text{if} \quad t = q^{2k+2}a \\ 0 & \text{otherwise.} \end{cases}$$

   Then

   $$\begin{aligned}\int_a^b f(t)\, h(qt)\, \tilde{d}_q t &= \int_0^b f(t)\, h(qt)\, \tilde{d}_q t - \int_0^a f(t)\, h(qt)\, \tilde{d}_q t \\ &= 0 - \left(1-q^2\right) a \sum_{n=0}^{+\infty} q^{2n} f\left(q^{2n+1}a\right) h\left(q^{2n+2}a\right) \\ &= -\left(1-q^2\right) a q^{2k} \left[f\left(q^{2k+1}a\right)\right]^2 \neq 0,\end{aligned}$$

   which is a contradiction.

   (b) If $a = 0$ and $b \neq 0$, then $p = q^{2k+1}b$ for some $k \in \mathbb{N}_0$. Define

   $$h(t) = \begin{cases} f\left(q^{2k+1}b\right) & \text{if} \quad t = q^{2k+2}b \\ 0 & \text{otherwise} \end{cases}$$

   and with a similar proof to (a) we obtain a contradiction.

(c) The case $b=0$ and $a \neq 0$ is similar to the previous one.

2. If $p=0$, without loss of generality, we can assume $f(p)>0$. Since

$$\lim_{n\to+\infty} q^{2n+1}a = \lim_{n\to+\infty} q^{2n+1}b = 0$$

and f is continuous at 0 we have

$$\lim_{n\to+\infty} f\left(q^{2n+1}a\right) = \lim_{n\to+\infty} f\left(q^{2n+1}b\right) = f(0).$$

Therefore, there exists an order $n_0 \in \mathbb{N}$ such that for all $n > n_0$ the inequalities

$$f\left(q^{2n+1}a\right) > 0 \text{ and } f\left(q^{2n+1}b\right) > 0$$

hold.

(a) If $a, b \neq 0$, then for some $k > n_0$ we define

$$h(t) = \begin{cases} f\left(q^{2k+1}b\right) & \text{if} \quad t = q^{2k+2}a \\ f\left(q^{2k+1}a\right) & \text{if} \quad t = q^{2k+2}b \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\int_a^b f(t)\, h(qt)\, \tilde{d}_q t = \left(1-q^2\right)(b-a)\, q^{2k} \left[f\left(q^{2k+1}a\right) f\left(q^{2k+1}b\right)\right] \neq 0.$$

(b) If $a=0$, then we define

$$h(t) = \begin{cases} f\left(q^{2k+1}b\right) & \text{if} \quad t = q^{2k+2}b \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\int_0^b f(t)\, h(qt)\, \tilde{d}_q t = \left(1-q^2\right) b q^{2k} \left[f\left(q^{2k+1}b\right)\right]^2 \neq 0.$$

(c) If $b=0$, the proof is similar to the previous case.

$\square$

Definition 6.3.2 ([89]). *Let $s \in I$ and $g : I \times \left]-\bar{\theta}, \bar{\theta}\right[\to \mathbb{R}$. We say that $g(t,\cdot)$ is differentiable at θ_0 uniformly in $[s]_q$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$0 < |\theta - \theta_0| < \delta \Rightarrow \left| \frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0) \right| < \varepsilon$$

for all $t \in [s]_q$, where $\partial_2 g = \frac{\partial g}{\partial \theta}$.

Lemma 6.3.3 (cf. [89])**.** *Let $s \in I$ and assume that $g : I \times \left]-\bar{\theta},\bar{\theta}\right[\to \mathbb{R}$ is differentiable at θ_0 uniformly in $[s]_q$. If $\int_0^s g(t,\theta_0)\,\tilde{d}_q t$ exists, then $G(\theta) := \int_0^s g(t,\theta)\,\tilde{d}_q t$ for θ near θ_0, is differentiable at θ_0 and*

$$G'(\theta_0) = \int_0^s \partial_2 g(t,\theta_0)\,\tilde{d}_q t.$$

Proof. For $s > 0$ the proof is similar to the proof given in Lemma 3.2 of [89]. The result is trivial for $s = 0$. For $s < 0$, let $\varepsilon > 0$ be arbitrary. Since $g(t,\cdot)$ is differentiable at θ_0 uniformly in $[s]_q$, then there exists $\delta > 0$ such that, for all $t \in [s]_q$ and for $0 < |\theta - \theta_0| < \delta$,

$$\left|\frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0)\right| < -\frac{\varepsilon}{2s}.$$

Applying Proposition 6.2.10, for $0 < |\theta - \theta_0| < \delta$, we have

$$\begin{aligned}
\left|\frac{G(\theta) - G(\theta_0)}{\theta - \theta_0} - \int_0^s \partial_2 g(t,\theta_0)\,\tilde{d}_q t\right| &= \left|\frac{\int_0^s g(t,\theta)\,\tilde{d}_q t - \int_0^s g(t,\theta_0)\,\tilde{d}_q t}{\theta - \theta_0} - \int_0^s \partial_2 g(t,\theta_0)\,\tilde{d}_q t\right| \\
&= \left|\int_0^s \left[\frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0)\right]\tilde{d}_q t\right| \\
&< \int_s^0 -\frac{\varepsilon}{2s}\tilde{d}_q t = -\frac{\varepsilon}{2s}\int_s^0 1\tilde{d}_q t = \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

proving that

$$G'(\theta_0) = \int_0^s \partial_2 g(t,\theta_0)\,\tilde{d}_q t.$$

□

For an admissible variation η and an admissible function y, we define the real function ϕ by

$$\phi(\epsilon) := \mathcal{L}[y + \epsilon\eta].$$

The first variation of the functional $\mathcal{L}$ of the problem (6.1.1) is defined by

$$\delta\mathcal{L}[y,\eta] := \phi'(0).$$

Note that

$$\begin{aligned}
\mathcal{L}[y + \epsilon\eta] &= \int_a^b L\left(t, y(qt) + \epsilon\eta(qt), \tilde{D}_q[y](t) + \epsilon\tilde{D}_q[\eta](t)\right)\tilde{d}_q t \\
&= \mathcal{L}_b[y + \epsilon\eta] - \mathcal{L}_a[y + \epsilon\eta],
\end{aligned}$$

where

$$\mathcal{L}_\xi[y + \epsilon\eta] = \int_0^\xi L\left(t, y(qt) + \epsilon\eta(qt), \tilde{D}_q[y](t) + \epsilon\tilde{D}_q[\eta](t)\right)\tilde{d}_q t$$

with $\xi \in \{a, b\}$. Therefore,

$$\delta\mathcal{L}[y,\eta] = \delta\mathcal{L}_b[y,\eta] - \delta\mathcal{L}_a[y,\eta].$$

The following lemma is a direct consequence of Lemma 6.3.3.

Lemma 6.3.4. *For an admissible variation η and an admissible function y, let*

$$g(t,\epsilon) := L\left(t, y(qt) + \epsilon\eta(qt), \tilde{D}_q[y](t) + \epsilon\tilde{D}_q[\eta](t)\right).$$

Assume that

1. *$g(t,\cdot)$ is differentiable at 0 uniformly in $[a,b]_q$;*
2. *$\mathcal{L}_a[y+\epsilon\eta] = \int_0^a g(t,\epsilon)\,\tilde{d}_q t$ and $\mathcal{L}_b[y+\epsilon\eta] = \int_0^b g(t,\epsilon)\,\tilde{d}_q t$ exist for $\epsilon \approx 0$;*
3. *$\int_0^a \partial_2 g(t,0)\,\tilde{d}_q t$ and $\int_0^b \partial_2 g(t,0)\,\tilde{d}_q t$ exist.*

Then

$$\begin{aligned}\phi'(0) &= \delta\mathcal{L}[y,\eta]\\ &= \int_a^b \left(\partial_2 L\left(t, y(qt), \tilde{D}_q[y](t)\right)\eta(qt) + \partial_3 L\left(t, y(qt), \tilde{D}_q[y](t)\right)\tilde{D}_q[\eta](t)\right)\tilde{d}_q t.\end{aligned}$$

6.3.2 Optimality conditions

In this section we present a necessary condition (the q-symmetric Euler–Lagrange equation) and a sufficient condition to our problem (6.1.1).

Theorem 6.3.5 (The q-symmetric Euler–Lagrange equation)**.** *Under hypotheses (H_q1)-(H_q3) and conditions 1-3 of Lemma 6.3.4 on the Lagrangian L, if $y_* \in \mathcal{Y}^1\left([a,b]_q, \mathbb{R}\right)$ is a local extremizer for problem* (6.1.1)*, then y_* satisfies the q-symmetric Euler–Lagrange equation*

$$\partial_2 L\left(t, y(qt), \tilde{D}_q[y](t)\right) = \tilde{D}_q\left[\tau \to \partial_3 L\left(q\tau, y\left(q^2\tau\right), \tilde{D}_q[y](q\tau)\right)\right](t) \tag{6.3.1}$$

for all $t \in [a,b]_q$.

Proof. Let y_* be a local minimizer (resp. maximizer) to problem (6.1.1) and η an admissible variation. Define $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(\epsilon) := \mathcal{L}[y_* + \epsilon\eta].$$

A necessary condition for y_* to be an extremizer is given by $\phi'(0) = 0$. By Lemma 6.3.4 we conclude that

$$\int_a^b \left(\partial_2 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right)\eta(qt) + \partial_3 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right)\tilde{D}_q[\eta](t)\right)\tilde{d}_q t = 0.$$

By integration by parts (equation (6.2.2)) we get

$$\begin{aligned}&\int_a^b \partial_3 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right)\tilde{D}_q[\eta](t)\,\tilde{d}_q t\\ =&\partial_3 L\left(qt, y_*\left(q^2 t\right), \tilde{D}_q[y_*](qt)\right)\eta(t)\Big|_a^b\\ &- q\int_a^b \tilde{D}_q\left[\tau \to \partial_3 L\left(\tau, y_*(q\tau), \tilde{D}_q[y_*](\tau)\right)\right](qt)\,\eta(qt)\,\tilde{d}_q t.\end{aligned}$$

Since $\eta(a) = \eta(b) = 0$, then

$$\int_a^b \left(\partial_2 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right) - q\tilde{D}_q\left[\tau \to \partial_3 L\left(\tau, y_*(q\tau), \tilde{D}_q[y_*](\tau)\right)\right](qt)\right)\eta(qt)\,\tilde{d}_q t = 0.$$

Finally, by Lemma 6.3.1 and equation (6.2.1), for all $t \in [a,b]_q$

$$\begin{aligned}
&\partial_2 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right) = q\tilde{D}_q\left[\tau \to \partial_3 L\left(\tau, y_*(q\tau), \tilde{D}_q[y_*](\tau)\right)\right](qt) \\
\Leftrightarrow &\partial_2 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right) = \tilde{D}_q\left[\tau \to \partial_3 L\left(q\tau, y_*\left(q^2\tau\right), \tilde{D}_q[y_*](q\tau)\right)\right](t).
\end{aligned}$$

□

Observe that the q-symmetric Euler–Lagrange equation (6.3.1) is a second-order dynamic equation. To the best of our knowledge, there is no general method to solve this type of equation. We believe this is an interesting open problem.

To conclude this section we prove a sufficient optimality condition for problem (6.1.1).

Definition 6.3.6. *Given a function L, we say that $L(t,u,v)$ is* jointly convex *(resp.* jointly concave*) in (u,v), if, and only if, $\partial_i L$, $i = 2, 3$, exist and are continuous and verify the following condition:*

$$L(t, u+u_1, v+v_1) - L(t,u,v) \underset{(resp. \leqslant)}{\geqslant} \partial_2 L(t,u,v)\,u_1 + \partial_3 L(t,u,v)\,v_1$$

for all $(t,u,v), (t, u+u_1, v+v_1) \in I \times \mathbb{R}^2$.

Theorem 6.3.7. *Suppose that $a, b \in I$, $a < b$, and $a, b \in [c]_q$ for some $c \in I$. Also assume that L is a jointly convex (resp. concave) function in (u,v). If y_* satisfies the q-symmetric Euler–Lagrange equation (6.3.1), then y_* is a global minimizer (resp. maximizer) to the problem (6.1.1).*

Proof. Let L be a jointly convex function in (u,v) (the concave case is similar). Then, for any admissible variation η, we have

$$\begin{aligned}
&\mathcal{L}[y_* + \eta] - \mathcal{L}[y_*] \\
&= \int_a^b \left[L\left(t, y_*(qt) + \eta(qt), \tilde{D}_q[y_*](t) + \tilde{D}_q[\eta](t)\right) - L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right)\right]\tilde{d}_q t \\
&\geqslant \int_a^b \left[\partial_2 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right)\eta(qt) + \partial_3 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right)\tilde{D}_q[\eta](t)\right]\tilde{d}_q t
\end{aligned}$$

Using integrations by parts, formula (6.2.2), we get

$$\begin{aligned}
\mathcal{L}[y_* + \eta] - \mathcal{L}[y_*] \geqslant\ &\partial_3 L\left(qt, y_*\left(q^2 t\right), \tilde{D}_q[y_*](qt)\right)\eta(t)\Big|_a^b \\
&+ \int_a^b \Big(\partial_2 L\left(t, y_*(qt), \tilde{D}_q[y_*](t)\right) \\
&- \tilde{D}_q\left[\tau \to \partial_3 L\left(q\tau, y_*\left(q^2\tau\right), \tilde{D}_q[y_*](q\tau)\right)\right](t)\Big)\eta(qt)\,\tilde{d}_q t.
\end{aligned}$$

Since y_* satisfies (6.3.1) and η is an admissible variation, we obtain

$$\mathcal{L}\left[y_*+\eta\right]-\mathcal{L}\left[y_*\right]\geqslant 0$$

proving that y_* is a minimizer of problem (6.1.1). $\square$

Example 6.3.8. *Let $q\in\left]0,1\right[$ be a fixed real number and $I\subseteq\mathbb{R}$ be an interval such that $0,1\in I$. Consider the problem*

$$\begin{cases}\mathcal{L}\left[y\right]=\displaystyle\int_0^1\left(1+\left(\tilde{D}_q\left[y\right]\left(t\right)\right)^2\right)\tilde{d}_qt\rightarrow \textit{minimize}\\ y\in\mathcal{Y}^1\left(\left[0,1\right]_q,\mathbb{R}\right)\\ y\left(0\right)=0\\ y\left(1\right)=1.\end{cases}\tag{6.3.2}$$

If y_* is a local minimizer of the problem, then y_* satisfies the q-symmetric Euler–Lagrange equation

$$\tilde{D}_q\left[\tau\rightarrow 2\tilde{D}_q\left[y\right]\left(q\tau\right)\right]\left(t\right)=0 \textit{ for all } t\in\left[0,1\right]_q.$$

It's simple to see that the function

$$y_*\left(t\right)=t$$

is a candidate to the solution of problem (6.3.2)*. Since the Lagrangian function is jointly convex in (u,v), then we may conclude that the function y_* is a minimizer of problem* (6.3.2)*.*

6.4 State of the Art

The study of the q-quantum calculus of variations and its applications is under current research [17, 18]. In this chapter not only we developed it with new techniques, but also we introduced the q-symmetric quantum calculus of variations. These results were presented by the author at the international conference Progress on Difference Equations 2011, Dublin City University, Ireland, and are published in [32].

Chapter 7

Hahn's Symmetric Quantum Variational Calculus

In this chapter we introduce and develop the Hahn symmetric quantum calculus with applications to the calculus of variations. Namely, we obtain a necessary optimality condition of Euler–Lagrange type and a sufficient optimality condition for variational problems within the context of Hahn's symmetric calculus. Moreover, we show the effectiveness of Leitmann's direct method when applied to Hahn's symmetric variational calculus. Illustrative examples are provided.

It should be noted that the q-symmetric derivative is a particular case of Hahn's symmetric derivative, and so this chapter is a generalization of Chapter 6.

7.1 Introduction

Due to its many applications, quantum operators are recently subject to an increase number of investigations [89, 97, 99]. The use of quantum differential operators, instead of classical derivatives, is useful because they allow to deal with sets of nondifferentiable functions [11, 43]. Applications include several fields of physics, such as cosmic strings and black holes [115], quantum mechanics [58, 123], nuclear and high energy physics [80], just to mention a few. In particular, the q-symmetric quantum calculus has applications in quantum mechanics [79].

In 1949, Hahn introduced his quantum difference operator [66], which is a generalization of the quantum q-difference operator defined by Jackson [71]. However, only in 2009, Aldwoah [3] defined the inverse of Hahn's difference operator, and short after, Malinowska and Torres [89] introduced and investigated the Hahn quantum variational calculus. For a deep understanding of quantum calculus, we refer the reader to [5, 30, 33, 50, 73, 76] and references therein.

For a fixed $q \in \,]0,1[$ and an $\omega \geqslant 0$, we introduce here the Hahn symmetric difference operator of function f at point $t \neq \dfrac{\omega}{1-q}$ by

$$\tilde{D}_{q,\omega}\left[y\right](t) = \frac{f\left(qt+\omega\right) - f\left(q^{-1}\left(t-\omega\right)\right)}{\left(q-q^{-1}\right)t + \left(1+q^{-1}\right)\omega}.$$

Our main aim is to establish a necessary optimality condition and a sufficient optimality condition for the Hahn symmetric variational problem

$$\begin{cases} \mathcal{L}(y) = \displaystyle\int_a^b L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right) \tilde{d}_{q,\omega} t \longrightarrow \text{extremize} \\ y \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right) \\ y(a) = \alpha, \quad y(b) = \beta, \end{cases} \tag{7.1.1}$$

where α and β are fixed real numbers, and extremize means maximize or minimize. Problem (7.1.1) will be clear and precise after definitions of Section 7.2. We assume that the Lagrangian L satisfies the following hypotheses:

($\mathrm{H}_{q,\omega}1$) $(u,v) \to L(t,u,v)$ is a $C^1(\mathbb{R}^2, \mathbb{R})$ function for any $t \in I$;

($\mathrm{H}_{q,\omega}2$) $t \to L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right)$ is continuous at ω_0 for any admissible function y;

($\mathrm{H}_{q,\omega}3$) functions $t \to \partial_{i+2} L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right)$ belong to $\mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ for all admissible y, $i = 0, 1$;

where I is an interval of $\mathbb{R}$ containing $\omega_0 := \dfrac{\omega}{1-q}$, $a, b \in I$, $a < b$, and $\partial_j L$ denotes the partial derivative of L with respect to its jth argument.

In Section 7.2 we introduce the necessary definitions and prove some basic results for the Hahn symmetric calculus. In Section 7.3 we formulate and prove our main results for the Hahn symmetric variational calculus. New results include a necessary optimality condition (Theorem 7.3.8) and a sufficient optimality condition (Theorem 7.3.10) to problem (7.1.1). In Section 7.3.3 we show that Leitmann's direct method can also be applied to variational problems within Hahn's symmetric variational calculus. Leitmann introduced his direct method in the sixties of the 20th century [81], and the approach has recently proven to be universal: see, e.g., [10, 40, 41, 82, 83, 84, 90, 119].

7.2 Hahn's symmetric calculus

Let $q \in \,]0,1[$ and $\omega \geqslant 0$ be real fixed numbers. Throughout this chapter, we make the assumption that I is an interval (bounded or unbounded) of $\mathbb{R}$ containing $\omega_0 := \dfrac{\omega}{1-q}$. We denote by $I^{q,\omega}$ the set $I^{q,\omega} := qI + \omega := \{qt + \omega : t \in I\}$. Note that $I^{q,\omega} \subseteq I$ and, for all $t \in I^{q,\omega}$, one has $q^{-1}(t-\omega) \in I$. For $k \in \mathbb{N}_0$,

$$[k]_q := \frac{1-q^k}{1-q}.$$

Definition 7.2.1. *Let f be a real function defined on I. The* Hahn symmetric difference operator *of f at a point $t \in I^{q,\omega} \setminus \{\omega_0\}$ is defined by*

$$\tilde{D}_{q,\omega}[f](t) = \frac{f(qt+\omega) - f\left(q^{-1}(t-\omega)\right)}{(q-q^{-1})\,t + (1+q^{-1})\,\omega},$$

while $\tilde{D}_{q,\omega}[f](\omega_0) := f'(\omega_0)$, provided f is differentiable at ω_0 (in the classical sense). We call to $\tilde{D}_{q,\omega}[f]$ the Hahn symmetric derivative *of f.*

Remark 7.2.2. *If $\omega = 0$, then the Hahn symmetric difference operator $\tilde{D}_{q,\omega}$ coincides with the q-symmetric difference operator $\tilde{D}_q$ (see Definition 6.2.1): if $t \neq 0$, then*

$$\tilde{D}_{q,0}[f](t) = \frac{f(qt) - f\left(q^{-1}t\right)}{(q - q^{-1})\,t} =: \tilde{D}_q[f](t);$$

for $t = 0$ and f differentiable at 0, $\tilde{D}_{q,0}[f](0) = f'(0) =: \tilde{D}_q[f](0)$.

Remark 7.2.3. *If $\omega > 0$ and we let $q \to 1$ in Definition 7.2.1, then we obtain the well known symmetric difference operator $\tilde{D}_\omega$:*

$$\tilde{D}_\omega[f](t) := \frac{f(t+\omega) - f(t-\omega)}{2\omega}.$$

Remark 7.2.4. *If f is differentiable at $t \in I^{q,\omega}$ in the classical sense, then*

$$\lim_{(q,\omega)\to(1,0)} \tilde{D}_{q,\omega}[f](t) = f'(t).$$

In what follows we make use of the operator σ defined by $\sigma(t) := qt + \omega$, $t \in I$. Note that the inverse operator of σ, σ^{-1}, is defined by $\sigma^{-1}(t) := q^{-1}(t - \omega)$. Moreover, Aldwoah [3, Lemma 6.1.1] proved the following useful result.

Lemma 7.2.5 ([3])**.** *Let $k \in \mathbb{N}$ and $t \in I$. Then,*

1. $\sigma^k(t) = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{k \text{ times}}(t) = q^k t + \omega\,[k]_q$;
2. $\left(\sigma^k(t)\right)^{-1} = \sigma^{-k}(t) = q^{-k}\left(t - \omega\,[k]_q\right)$.

Furthermore, $\{\sigma^k(t)\}_{k=1}^{\infty}$ is a decreasing (resp. an increasing) sequence in k when $t > \omega_0$ (resp. $t < \omega_0$) with

$$\omega_0 = \inf_{k\in\mathbb{N}} \sigma^k(t) \qquad \left(\textit{resp.}\ \ \omega_0 = \sup_{k\in\mathbb{N}} \sigma^k(t)\right).$$

The sequence $\{\sigma^{-k}(t)\}_{k=1}^{\infty}$ is increasing (resp. decreasing) when $t > \omega_0$ (resp. $t < \omega_0$) with

$$+\infty = \sup_{k\in\mathbb{N}} \sigma^{-k}(t) \qquad \left(\textit{resp.}\ \ -\infty = \inf_{k\in\mathbb{N}} \sigma^{-k}(t)\right).$$

For simplicity of notation, we write $f(\sigma(t)) := f^\sigma(t)$.

Remark 7.2.6. *With above notations, if $t \in I^{q,\omega} \setminus \{\omega_0\}$, then the Hahn symmetric difference operator of f at point t can be written as*

$$\tilde{D}_{q,\omega}[f](t) = \frac{f^\sigma(t) - f^{\sigma^{-1}}(t)}{\sigma(t) - \sigma^{-1}(t)}.$$

Lemma 7.2.7. *Let $n \in \mathbb{N}_0$ and $t \in I$. Then,*

$$\sigma^{n+1}(t) - \sigma^{n-1}(t) = q^n\left(\sigma(t) - \sigma^{-1}(t)\right),$$

where $\sigma^0 \equiv id$ is the identity function.

Proof. The equality follows by direct calculations:

$$\begin{aligned}\sigma^{n+1}(t)-\sigma^{n-1}(t) &= q^{n+1}t+\omega\,[n+1]_q - q^{n-1}t-\omega\,[n-1]_q\\ &= q^n\left(q-q^{-1}\right)t+\omega\left(q^n+q^{n-1}\right)\\ &= q^n\left(qt+\omega-q^{-1}t+q^{-1}\omega\right)\\ &= q^n\left(\sigma(t)-\sigma^{-1}(t)\right).\end{aligned}$$

□

The Hahn symmetric difference operator has the following properties.

Theorem 7.2.8. *Let $\alpha,\beta\in\mathbb{R}$ and $t\in I^{q,\omega}$. If f and g are Hahn symmetric differentiable on I, then*

1. $\tilde{D}_{q,\omega}[\alpha f+\beta g](t)=\alpha\tilde{D}_{q,\omega}[f](t)+\beta\tilde{D}_{q,\omega}[g](t)$;
2. $\tilde{D}_{q,\omega}[fg](t)=\tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t)+f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t)$;
3. $\tilde{D}_{q,\omega}\left[\frac{f}{g}\right](t)=\dfrac{\tilde{D}_{q,\omega}[f](t)\,g^{\sigma^{-1}}(t)-f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t)}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)}$ *if* $g^{\sigma}(t)\,g^{\sigma^{-1}}(t)\neq 0$;
4. $\tilde{D}_{q,\omega}[f]\equiv 0$ *if, and only if, f is constant on I.*

Proof. For $t=\omega_0$ the equalities are trivial (note that $\sigma(\omega_0)=\omega_0=\sigma^{-1}(\omega_0)$). We do the proof for $t\neq\omega_0$:

1.

$$\begin{aligned}\tilde{D}_{q,\omega}[\alpha f+\beta g](t) &= \frac{(\alpha f+\beta g)^{\sigma}(t)-(\alpha f+\beta g)^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\ &= \alpha\frac{f^{\sigma}(t)-f^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}+\beta\frac{g^{\sigma}(t)-g^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\ &= \alpha\tilde{D}_{q,\omega}[f](t)+\beta\tilde{D}_{q,\omega}[g](t).\end{aligned}$$

2.

$$\begin{aligned}\tilde{D}_{q,\omega}[fg](t) &= \frac{(fg)^{\sigma}(t)-(fg)^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\ &= \frac{f^{\sigma}(t)-f^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}g^{\sigma}(t)+f^{\sigma^{-1}}(t)\,\frac{g^{\sigma}(t)-g^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\ &= \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t)+f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t).\end{aligned}$$

3. Because

$$\begin{aligned}\tilde{D}_{q,\omega}\left[\frac{1}{g}\right](t) &= \frac{\frac{1}{g^{\sigma}(t)}-\frac{1}{g^{\sigma^{-1}}(t)}}{\sigma(t)-\sigma^{-1}(t)}\\ &= -\frac{1}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)}\frac{g^{\sigma}(t)-g^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\ &= -\frac{\tilde{D}_{q,\omega}[g](t)}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)},\end{aligned}$$

one has

$$\begin{aligned}
\tilde{D}_{q,\omega}\left[\frac{f}{g}\right](t) = \tilde{D}_{q,\omega}\left[f\frac{1}{g}\right](t) \\
&= \tilde{D}_{q,\omega}[f](t)\frac{1}{g^{\sigma}(t)} + f^{\sigma^{-1}}(t)\tilde{D}_{q,\omega}\left[\frac{1}{g}\right](t) \\
&= \frac{\tilde{D}_{q,\omega}[f](t)}{g^{\sigma}(t)} - f^{\sigma^{-1}}(t)\frac{\tilde{D}_{q,\omega}[g](t)}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)} \\
&= \frac{\tilde{D}_{q,\omega}[f](t)\,g^{\sigma^{-1}}(t) - f^{\sigma^{-1}}(t)\tilde{D}_{q,\omega}[g](t)}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)}.
\end{aligned}$$

4. If f is constant on I, then it is clear that $\tilde{D}_{q,\omega}[f] \equiv 0$. Suppose now that $\tilde{D}_{q,\omega}[f] \equiv 0$. Then, for each $t \in I$, $\left(\tilde{D}_{q,\omega}[f]\right)^{\sigma}(t) = 0$ and, therefore, $f(t) = f^{\sigma^2}(t)$. Hence,

$$f(t) = f^{\sigma^2}(t) = \cdots = f^{\sigma^{2n}}(t)$$

for each $n \in \mathbb{N}$ and $t \in I$. Because

$$\lim_{n\to+\infty} f(t) = \lim_{n\to+\infty} f^{\sigma^{2n}}(t),$$

$$\lim_{n\to+\infty} \sigma^{2n}(t) = \omega_0 \quad \text{(by Lemma 7.2.5)},$$

and f is continuous at ω_0, then

$$f(t) = f(\omega_0)$$

for all $t \in I$.

□

Lemma 7.2.9. *For $t \in I$ one has $\tilde{D}_{q,\omega}[f^{\sigma}](t) = q\tilde{D}_{q,\omega}[f](\sigma(t))$.*

Proof. For each $t \in I\backslash\{\omega_0\}$ we have

$$\tilde{D}_{q,\omega}[f^{\sigma}](t) = \frac{f^{\sigma^2}(t) - f(t)}{\sigma(t) - \sigma^{-1}(t)}$$

and

$$\begin{aligned}
\tilde{D}_{q,\omega}[f](\sigma(t)) &= \frac{f^{\sigma^2}(t) - f(t)}{\sigma^2(t) - t} \\
&= \frac{f^{\sigma^2}(t) - f(t)}{q(\sigma(t) - \sigma^{-1}(t))} \quad \text{(see Lemma 7.2.7)}.
\end{aligned}$$

We conclude that

$$\tilde{D}_{q,\omega}[f^{\sigma}](t) = q\tilde{D}_{q,\omega}[f](\sigma(t)).$$

Finally, the intended result follows from the fact that

$$\tilde{D}_{q,\omega}[f^{\sigma}](\omega_0) = q\tilde{D}_{q,\omega}[f](\omega_0).$$

□

Definition 7.2.10. *Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ the* Hahn symmetric integral *of f from a to b is given by*

$$\int_a^b f(t)\, \tilde{d}_{q,\omega} t = \int_{\omega_0}^b f(t)\, \tilde{d}_{q,\omega} t - \int_{\omega_0}^a f(t)\, \tilde{d}_{q,\omega} t,$$

where

$$\int_{\omega_0}^x f(t)\, \tilde{d}_{q,\omega} t = \left(\sigma^{-1}(x) - \sigma(x)\right) \sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(x), \quad x \in I,$$

provided the series converges at $x = a$ and $x = b$. In that case, f is said to be Hahn symmetric integrable on $[a, b]$. We say that f is Hahn symmetric integrable on I if it is Hahn symmetric integrable over $[a, b]$ for all $a, b \in I$.

Remark 7.2.11. *If $\omega = 0$, then the Hahn symmetric integral of f from a to b coincides with the q-symmetric integral of f from a to b (see Definition 6.2.4) given by*

$$\int_a^b f(t)\, \tilde{d}_q t := \int_a^b f(t)\, \tilde{d}_{q,0} t = \int_0^b f(t)\, \tilde{d}_{q,0} t - \int_0^a f(t)\, \tilde{d}_{q,0} t,$$

where

$$\int_0^x f(t)\, \tilde{d}_q t := \int_0^x f(t)\, \tilde{d}_{q,0} t = \left(q^{-1} - q\right) x \sum_{n=0}^{+\infty} q^{2n+1} f\left(q^{2n+1} x\right), \quad x \in I,$$

provided the series converges at $x = a$ and $x = b$.

We now present two technical results that are useful to prove the fundamental theorem of Hahn's symmetric integral calculus (Theorem 7.2.14).

Lemma 7.2.12 (cf. [3])**.** *Let $a, b \in I$, $a < b$. If $f : I \to \mathbb{R}$ is continuous at ω_0, then, for $s \in [a, b]$, the sequence $\left(f^{\sigma^{2n+1}}(s)\right)_{n\in\mathbb{N}}$ converges uniformly to $f(\omega_0)$ on I.*

The next result tell us that if a function f is continuous at ω_0, then f is Hahn's symmetric integrable.

Corollary 7.2.13 (cf. [3])**.** *Let $a, b \in I$, $a < b$, and $f : I \to \mathbb{R}$ be continuous at ω_0. Then, for $s \in [a, b]$, the series $\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(s)$ is uniformly convergent on I.*

Theorem 7.2.14 (Fundamental theorem of the Hahn symmetric integral calculus)**.** *Assume that $f : I \to \mathbb{R}$ is continuous at ω_0 and, for each $x \in I$, define*

$$F(x) := \int_{\omega_0}^x f(t)\, \tilde{d}_{q,\omega} t.$$

Then F is continuous at ω_0. Furthermore, $\tilde{D}_{q,\omega}[F](x)$ exists for every $x \in I^{q,\omega}$ with

$$\tilde{D}_{q,\omega}[F](x) = f(x).$$

Conversely,

$$\int_a^b \tilde{D}_{q,\omega}[f](t)\, \tilde{d}_{q,\omega} t = f(b) - f(a)$$

for all $a, b \in I$.

Proof. We note that function F is continuous at ω_0 by Corollary 7.2.13. Let us begin by considering $x \in I^{q,\omega}\backslash\{\omega_0\}$. Then,

$$\begin{aligned}
&\tilde{D}_{q,\omega}\left[\tau \mapsto \int_0^\tau f(t)\,\tilde{d}_{q,\omega}t\right](x) \\
&= \frac{\int_{\omega_0}^{\sigma(x)} f(t)\,\tilde{d}_{q,\omega}t - \int_{\omega_0}^{\sigma^{-1}(x)} f(t)\,\tilde{d}_{q,\omega}t}{\sigma(x) - \sigma^{-1}(x)} \\
&= \frac{1}{\sigma(x) - \sigma^{-1}(x)}\left\{ \left[\sigma^{-1}(\sigma(x)) - \sigma(\sigma(x))\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\sigma(x)) \right. \\
&\qquad \left. - \left[\sigma^{-1}\left(\sigma^{-1}(x)\right) - \sigma\left(\sigma^{-1}(x)\right)\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}\left(\sigma^{-1}(x)\right) \right\} \\
&= \sum_{n=0}^{+\infty} q^{2n} f^{\sigma^{2n}}(x) - \sum_{n=0}^{+\infty} q^{2n+2} f^{\sigma^{2n+2}}(x) \\
&= f(x).
\end{aligned}$$

If $x = \omega_0$, then

$$\begin{aligned}
&\tilde{D}_{q,\omega}[F](\omega_0) \\
&= \lim_{h\to 0} \frac{F(\omega_0 + h) - F(\omega_0)}{h} \\
&= \lim_{h\to 0} \frac{1}{h}\left[\sigma^{-1}(\omega_0 + h) - \sigma(\omega_0 + h)\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0 + h) \\
&= \lim_{h\to 0} \frac{1}{h}\left[q^{-1}(\omega_0 + h - \omega) - q(\omega_0 + h) - \omega\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0 + h) \\
&= \lim_{h\to 0} \frac{1}{h}\left[\left(q^{-1} - q\right)\omega_0 + \left(-q^{-1} - 1\right)\omega + \left(q^{-1} - q\right)h\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0 + h) \\
&= \lim_{h\to 0} \frac{1}{h}\left[\frac{\left(q^{-1} - q\right)\omega}{1-q} + \left(-q^{-1} - 1\right)\omega + \left(q^{-1} - q\right)h\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0 + h) \\
&= \lim_{h\to 0} \frac{1}{h}\left[\left(\frac{1+q}{q} + \frac{-1-q}{q}\right)\omega + \left(q^{-1} - q\right)h\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0 + h) \\
&= \lim_{h\to 0} \frac{1-q^2}{q}\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0 + h) \\
&= \left(1-q^2\right)\sum_{n=0}^{+\infty} q^{2n} f(\omega_0) \\
&= \left(1-q^2\right)\frac{1}{1-q^2} f(\omega_0) \\
&= f(\omega_0).
\end{aligned}$$

Finally, since for $x \in I \setminus \{\omega_0\}$ we have

$$\begin{aligned}
\int_{\omega_0}^{x} \tilde{D}_{q,\omega}[f](t)\,\tilde{d}_{q,\omega}t &= \left[\sigma^{-1}(x) - \sigma(x)\right] \sum_{n=0}^{+\infty} q^{2n+1} \tilde{D}_{q,\omega}[f]^{\sigma^{2n+1}}(x) \\
&= \left[\sigma^{-1}(x) - \sigma(x)\right] \sum_{n=0}^{+\infty} q^{2n+1} \frac{f^{\sigma}\left(\sigma^{2n+1}(x)\right) - f^{\sigma^{-1}}\left(\sigma^{2n+1}(x)\right)}{\sigma\left(\sigma^{2n+1}(x)\right) - \sigma^{-1}\left(\sigma^{2n+1}(x)\right)} \\
&= \left[\sigma^{-1}(x) - \sigma(x)\right] \sum_{n=0}^{+\infty} q^{2n+1} \frac{f^{\sigma}\left(\sigma^{2n+1}(x)\right) - f^{\sigma^{-1}}\left(\sigma^{2n+1}(x)\right)}{q^{2n+1}\left(\sigma(x) - \sigma^{-1}(x)\right)} \\
&= \sum_{n=0}^{+\infty} \left[f^{\sigma^{2n}}(x) - f^{\sigma^{2(n+1)}}(x)\right] \\
&= f(x) - f(\omega_0),
\end{aligned}$$

where in the third equality we use Lemma 7.2.7, then

$$\begin{aligned}
\int_{a}^{b} \tilde{D}_{q,\omega}[f](t)\,\tilde{d}_{q,\omega}t &= \int_{\omega_0}^{b} \tilde{D}_{q,\omega}[f](t)\,\tilde{d}_{q,\omega}t - \int_{\omega_0}^{a} \tilde{D}_{q,\omega}[f](t)\,\tilde{d}_{q,\omega}t \\
&= f(b) - f(a).
\end{aligned}$$

□

The Hahn symmetric integral has the following properties.

Theorem 7.2.15. *Let $f, g : I \to \mathbb{R}$ be Hahn's symmetric integrable on I, $a, b, c \in I$, and $\alpha, \beta \in \mathbb{R}$. Then,*

1. $\displaystyle\int_{a}^{a} f(t)\,\tilde{d}_{q,\omega}t = 0;$
2. $\displaystyle\int_{a}^{b} f(t)\,\tilde{d}_{q,\omega}t = -\int_{b}^{a} f(t)\,\tilde{d}_{q,\omega}t;$
3. $\displaystyle\int_{a}^{b} f(t)\,\tilde{d}_{q,\omega}t = \int_{a}^{c} f(t)\,\tilde{d}_{q,\omega}t + \int_{c}^{b} f(t)\,\tilde{d}_{q,\omega}t;$
4. $\displaystyle\int_{a}^{b} (\alpha f + \beta g)(t)\,\tilde{d}_{q,\omega}t = \alpha \int_{a}^{b} f(t)\,\tilde{d}_{q,\omega}t + \beta \int_{a}^{b} g(t)\,\tilde{d}_{q,\omega}t;$
5. *if $\tilde{D}_{q,\omega}[f]$ and $\tilde{D}_{q,\omega}[g]$ are continuous at ω_0, then*

$$\int_{a}^{b} f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t)\,\tilde{d}_{q,\omega}t = f(t)\,g(t)\Big|_{a}^{b} - \int_{a}^{b} \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t)\,\tilde{d}_{q,\omega}t. \tag{7.2.1}$$

Proof. Properties 1 to 4 are trivial. Property 5 follows from Theorem 7.2.8 and Theorem 7.2.14: since

$$\tilde{D}_{q,\omega}[fg](t) = \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t) + f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t),$$

then

$$f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t) = \tilde{D}_{q,\omega}[fg](t) - \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t)$$

and hence,

$$\int_a^b f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t)\,\tilde{d}_{q,\omega}t = f(t)\,g(t)\Big|_a^b - \int_a^b \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t)\,\tilde{d}_{q,\omega}t.$$

□

Remark 7.2.16. *Relation* (7.2.1) *gives a* Hahn's symmetric integration by parts *formula.*

Remark 7.2.17. *Using Lemma 7.2.9 and the Hahn symmetric integration by parts formula* (7.2.1)*, we conclude that*

$$\int_a^b f(t)\,\tilde{D}_{q,\omega}[g](t)\,\tilde{d}_{q,\omega}t = f^{\sigma}(t)\,g(t)\Big|_a^b - q\int_a^b \left(\tilde{D}_{q,\omega}[f]\right)^{\sigma}(t)\,g^{\sigma}(t)\,\tilde{d}_{q,\omega}t. \qquad (7.2.2)$$

Proposition 7.2.18. *Let $c \in I$, f and g be Hahn's symmetric integrable on I. Suppose that*

$$|f(t)| \leqslant g(t)$$

for all $t \in \left\{\sigma^{2n+1}(c) : n \in \mathbb{N}_0\right\} \cup \{\omega_0\}$.

1. *If $c \geqslant \omega_0$, then*
$$\left|\int_{\omega_0}^{c} f(t)\,\tilde{d}_{q,\omega}t\right| \leqslant \int_{\omega_0}^{c} g(t)\,\tilde{d}_{q,\omega}t.$$

2. *If $c < \omega_0$, then*
$$\left|\int_{c}^{\omega_0} f(t)\,\tilde{d}_{q,\omega}t\right| \leqslant \int_{c}^{\omega_0} g(t)\,\tilde{d}_{q,\omega}t.$$

Proof. If $c \geqslant \omega_0$, then

$$\begin{aligned}
\left|\int_{\omega_0}^{c} f(t)\,\tilde{d}_{q,\omega}t\right| &= \left|\left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(c)\right| \\
&\leqslant \left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1}\left|f^{\sigma^{2n+1}}(c)\right| \\
&\leqslant \left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1} g^{\sigma^{2n+1}}(c) \\
&= \int_{\omega_0}^{c} g(t)\,\tilde{d}_{q,\omega}t.
\end{aligned}$$

If $c < \omega_0$, then

$$\begin{aligned}\left|\int_c^{\omega_0} f(t)\,\tilde{d}_{q,\omega}t\right| &= \left|-\left[\sigma^{-1}(c)-\sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(c)\right| \\ &\leqslant \left|\sigma^{-1}(c)-\sigma(c)\right|\sum_{n=0}^{+\infty} q^{2n+1}\left|f^{\sigma^{2n+1}}(c)\right| \\ &= -\left[\sigma^{-1}(c)-\sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1}\left|f^{\sigma^{2n+1}}(c)\right| \\ &\leqslant -\left[\sigma^{-1}(c)-\sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1} g^{\sigma^{2n+1}}(c) \\ &= -\int_{\omega_0}^{c} g(t)\,\tilde{d}_{q,\omega}t \\ &= \int_c^{\omega_0} g(t)\,\tilde{d}_{q,\omega}t\end{aligned}$$

providing the desired equality. □

As an immediate consequence, we have the following result.

Corollary 7.2.19. *Let $c \in I$ and f be Hahn's symmetric integrable on I. Suppose that*

$$f(t) \geqslant 0$$

for all $t \in \left\{\sigma^{2n+1}(c) : n \in \mathbb{N}_0\right\} \cup \{\omega_0\}$.

1. *If $c \geqslant \omega_0$, then*
$$\int_{\omega_0}^{c} f(t)\,\tilde{d}_{q,\omega}t \geqslant 0;$$
2. *If $c < \omega_0$, then*
$$\int_c^{\omega_0} f(t)\,\tilde{d}_{q,\omega}t \geqslant 0.$$

Remark 7.2.20. *In general it is not true that if f is a nonnegative function on $[a,b]$, then*

$$\int_a^b f(t)\,\tilde{d}_{q,\omega}t \geqslant 0.$$

As an example, consider the function f defined in $[-5,5]$ by

$$f(t) = \begin{cases} 6 & \textit{if} \quad t = 3 \\ 1 & \textit{if} \quad t = 4 \\ 0 & \textit{if} \quad t \in [-5,5]\setminus\{3,4\}. \end{cases}$$

For $q=\frac{1}{2}$ and $\omega=1$, this function is Hahn's symmetric integrable because is continuous at $\omega_0=2$. However,

$$\begin{aligned}\int_4^6 f(t)\,\tilde{d}_{q,\omega}t &= \int_2^6 f(t)\,\tilde{d}_{q,\omega}t - \int_2^4 f(t)\,\tilde{d}_{q,\omega}t\\ &= (10-4)\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^{2n+1} f^{\sigma^{2n+1}}(6) - (6-3)\sum_{n=0}^{+\infty}\left(\frac{1}{2}\right)^{2n+1} f^{\sigma^{2n+1}}(4)\\ &= 6\left(\frac{1}{2}\right)\times 1 - 3\left(\frac{1}{2}\right)\times 6\\ &= -6.\end{aligned}$$

This example also proves that, in general, it is not true that

$$\left|\int_a^b f(t)\,\tilde{d}_{q,\omega}t\right| \leqslant \int_a^b |f(t)|\,\tilde{d}_{q,\omega}t$$

for any $a,b\in I$.

7.3 Hahn's symmetric variational calculus

We begin this section with some useful definitions and notations. For $s\in I$ we set

$$[s]_{q,\omega} := \left\{\sigma^{2n+1}(s) : n\in\mathbb{N}_0\right\}\cup\{\omega_0\}.$$

Let $a,b\in I$ with $a<b$. We define the Hahn symmetric interval from a to b by

$$[a,b]_{q,\omega} := \left\{\sigma^{2n+1}(a) : n\in\mathbb{N}_0\right\}\cup\left\{\sigma^{2n+1}(b) : n\in\mathbb{N}_0\right\}\cup\{\omega_0\},$$

that is,

$$[a,b]_{q,\omega} = [a]_{q,\omega}\cup[b]_{q,\omega}.$$

Let $r\in\{0,1\}$. We denote the linear space

$$\left\{y : I\to\mathbb{R} \mid \tilde{D}^i_{q,\omega}[y], i=0,r, \text{ are bounded on } [a,b]_{q,\omega} \text{ and continuous at } \omega_0\right\}$$

endowed with the norm

$$\|y\|_r = \sum_{i=0}^{r}\sup_{t\in[a,b]_{q,\omega}}\left|\tilde{D}^i_{q,\omega}[y](t)\right|,$$

where $\tilde{D}^0_{q,\omega}[y]=y$, by $\mathcal{Y}^r\left([a,b]_{q,\omega},\mathbb{R}\right)$.

Definition 7.3.1. *We say that y is an admissible function to problem* (7.1.1) *if $y\in\mathcal{Y}^1\left([a,b]_{q,\omega},\mathbb{R}\right)$ and y satisfies the boundary conditions $y(a)=\alpha$ and $y(b)=\beta$.*

Definition 7.3.2. *We say that y_* is a local minimizer (resp. local maximizer) to problem* (7.1.1) *if y_* is an admissible function and there exists $\delta>0$ such that*

$$\mathcal{L}(y_*)\leqslant\mathcal{L}(y) \qquad \textit{(resp. } \mathcal{L}(y_*)\geqslant\mathcal{L}(y)\textit{)}$$

for all admissible y with $\|y_*-y\|_1<\delta$.

Definition 7.3.3. *We say that $\eta\in\mathcal{Y}^1\left([a,b]_{q,\omega},\mathbb{R}\right)$ is an admissible variation to problem* (7.1.1) *if $\eta(a)=0=\eta(b)$.*

Before proving our main results, we begin with three basic lemmas.

7.3.1 Basic lemmas

The following results are useful to prove Theorem 7.3.8.

Lemma 7.3.4 (Fundamental lemma of the Hahn symmetric variational calculus)**.** *Let* $f \in \mathcal{Y}^0\left([a,b]_{q,\omega}, \mathbb{R}\right)$*. One has*

$$\int_a^b f(t)\, h^\sigma(t)\, \tilde{d}_{q,\omega}t = 0$$

for all $h \in \mathcal{Y}^0\left([a,b]_{q,\omega}, \mathbb{R}\right)$ *with* $h(a) = h(b) = 0$ *if, and only if,* $f(t) = 0$ *for all* $t \in [a,b]_{q,\omega}$*.*

Proof. The implication "$\Leftarrow$" is obvious. Let us prove the implication "$\Rightarrow$". Suppose, by contradiction, that exists $p \in [a,b]_{q,\omega}$ such that $f(p) \neq 0$.

1. If $p \neq \omega_0$, then $p = \sigma^{2k+1}(a)$ or $p = \sigma^{2k+1}(b)$ for some $k \in \mathbb{N}_0$.

   (a) Suppose that $a \neq \omega_0$ and $b \neq \omega_0$. In this case we can assume, without loss of generality, that $p = \sigma^{2k+1}(a)$. Define

   $$h(t) = \begin{cases} f^{\sigma^{2k+1}}(a) & \text{if } t = \sigma^{2k+2}(a) \\ \\ 0 & \text{otherwise.} \end{cases}$$

   Then,

   $$\begin{aligned} &\int_a^b f(t)\, h^\sigma(t)\, \tilde{d}_{q,\omega}t \\ &= \left[\sigma^{-1}(b) - \sigma(b)\right] \sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(b)\, h^{\sigma^{2n+2}}(b) \\ &\quad - \left[\sigma^{-1}(a) - \sigma(a)\right] \sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(a)\, h^{\sigma^{2n+2}}(a) \\ &= -\left[\sigma^{-1}(a) - \sigma(a)\right] q^{2k+1} \left[f^{\sigma^{2k+1}}(a)\right]^2 \neq 0, \end{aligned}$$

   which is a contradiction.

   (b) Suppose that $a \neq \omega_0$ and $b = \omega_0$. Therefore, $p = \sigma^{2k+1}(a)$ for some $k \in \mathbb{N}_0$. Define

   $$h(t) = \begin{cases} f^{\sigma^{2k+1}}(a) & \text{if } t = \sigma^{2k+2}(a) \\ \\ 0 & \text{otherwise.} \end{cases}$$

   We obtain a contradiction with a similar proof as in case (a).

   (c) The case $a = \omega_0$ and $b \neq \omega_0$ is similar to (b).

2. If $p = \omega_0$, we assume, without loss of generality, that $f(p) > 0$. Since

   $$\lim_{n\to+\infty} \sigma^{2k+2}(a) = \lim_{n\to+\infty} \sigma^{2k+2}(b) = \omega_0$$

and f is continuous at ω_0,

$$\lim_{n\to+\infty} f^{\sigma^{2k+1}}(a) = \lim_{n\to+\infty} f^{\sigma^{2k+1}}(b) = f(\omega_0).$$

Therefore, there exists an order $n_0 \in \mathbb{N}$ such for all $n > n_0$ the inequalities

$$f^{\sigma^{2k+1}}(a) > 0 \quad \text{and} \quad f^{\sigma^{2k+1}}(b) > 0$$

hold.

(a) If $a, b \neq \omega_0$, then for some $k > n_0$ we define

$$h(t) = \begin{cases} -\frac{f^{\sigma^{2k+1}}(b)}{\sigma^{-1}(a)-\sigma(a)} & \text{if } t = \sigma^{2k+2}(a) \\ \\ \frac{f^{\sigma^{2k+1}}(a)}{\sigma^{-1}(b)-\sigma(b)} & \text{if } t = \sigma^{2k+2}(b) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\int_a^b f(t)\, h^{\sigma}(t)\, \tilde{d}_{q,\omega}t = 2q^{2k+1} f^{\sigma^{2k+1}}(a)\, f^{\sigma^{2k+1}}(b) > 0.$$

(b) If $a = \omega_0$, then we define

$$h(t) = \begin{cases} f^{\sigma^{2k+1}}(b) & \text{if } t = \sigma^{2k+2}(b) \\ \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\int_{\omega_0}^b f(t)\, h^{\sigma}(t)\, \tilde{d}_{q,\omega}t = \left[\sigma^{-1}(b) - \sigma(b)\right] q^{2k+1} \left[f^{\sigma^{2k+1}}(b)\right]^2 \neq 0.$$

(c) If $b = \omega_0$, the proof is similar to the previous case.

□

Definition 7.3.5 ([89])**.** *Let $s \in I$ and $g : I \times \left]-\bar{\theta}, \bar{\theta}\right[\to \mathbb{R}$. We say that $g(t, \cdot)$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$0 < |\theta - \theta_0| < \delta \Rightarrow \left| \frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0) \right| < \varepsilon$$

for all $t \in [s]_{q,\omega}$, where $\partial_2 g = \frac{\partial g}{\partial \theta}$.

Lemma 7.3.6 (cf. [89])**.** *Let $s \in I$ and assume that $g : I \times \left]-\bar{\theta}, \bar{\theta}\right[\to \mathbb{R}$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$. If $\int_{\omega_0}^s g(t,\theta_0)\, \tilde{d}_{q,\omega}t$ exists, then $G(\theta) := \int_{\omega_0}^s g(t,\theta)\, \tilde{d}_{q,\omega}t$ for θ near θ_0, is differentiable at θ_0 with*

$$G'(\theta_0) = \int_{\omega_0}^s \partial_2 g(t,\theta_0)\, \tilde{d}_{q,\omega}t.$$

Proof. For $s > \omega_0$ the proof is similar to the proof given in Lemma 3.2 of [89]. The result is trivial for $s = \omega_0$. Suppose that $s < \omega_0$ and let $\varepsilon > 0$ be arbitrary. Since $g(t, \cdot)$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$, then there exists $\delta > 0$ such that for all $t \in [s]_{q,\omega}$ and for $0 < |\theta - \theta_0| < \delta$ the following inequality holds:

$$\left| \frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0) \right| < \frac{\varepsilon}{2(\omega_0 - s)}. \tag{7.3.1}$$

Since, for $0 < |\theta - \theta_0| < \delta$, we have

$$\begin{aligned}
&\left| \frac{G(\theta) - G(\theta_0)}{\theta - \theta_0} - \int_{\omega_0}^{s} \partial_2 g(t,\theta_0)\, \tilde{d}_{q,\omega}t \right| \\
&= \left| \frac{\int_{\omega_0}^{s} g(t,\theta)\, \tilde{d}_{q,\omega}t - \int_{\omega_0}^{s} g(t,\theta_0)\, \tilde{d}_{q,\omega}t}{\theta - \theta_0} - \int_{\omega_0}^{s} \partial_2 g(t,\theta_0)\, \tilde{d}_{q,\omega}t \right| \\
&= \left| \int_{\omega_0}^{s} \left[\frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0) \right] \tilde{d}_{q,\omega}t \right| \\
&< \int_{s}^{\omega_0} \frac{\varepsilon}{2(\omega_0 - s)} \tilde{d}_{q,\omega}t \quad \text{(using Proposition 7.2.18 and inequality (7.3.1))} \\
&= \frac{\varepsilon}{2(\omega_0 - s)} \int_{s}^{\omega_0} 1 \tilde{d}_{q,\omega}t \\
&= \frac{\varepsilon}{2} \\
&< \varepsilon
\end{aligned}$$

then we can conclude that

$$G'(\theta) = \int_{\omega_0}^{s} \partial_2 g(t,\theta_0)\, \tilde{d}_{q,\omega}t.$$

□

For an admissible variation η and an admissible function y, we define $\phi :]-\bar{\epsilon}, \bar{\epsilon}[\to \mathbb{R}$ by $\phi(\epsilon) := \mathcal{L}[y + \epsilon\eta]$. The first variation of functional $\mathcal{L}$ of problem (7.1.1) is defined by $\delta\mathcal{L}[y, \eta] := \phi'(0)$. Note that

$$\begin{aligned}
\mathcal{L}[y + \epsilon\eta] &= \int_a^b L\left(t, y^\sigma(t) + \epsilon\eta^\sigma(t), \tilde{D}_{q,\omega}[y](t) + \epsilon\tilde{D}_{q,\omega}[\eta](t)\right) \tilde{d}_{q,\omega}t \\
&= \mathcal{L}_b[y + \epsilon\eta] - \mathcal{L}_a[y + \epsilon\eta],
\end{aligned}$$

where

$$\mathcal{L}_\xi[y + \epsilon\eta] = \int_{\omega_0}^{\xi} L\left(t, y^\sigma(t) + \epsilon\eta^\sigma(t), \tilde{D}_{q,\omega}[y](t) + \epsilon\tilde{D}_{q,\omega}[\eta](t)\right) \tilde{d}_{q,\omega}t$$

with $\xi \in \{a, b\}$. Therefore, $\delta\mathcal{L}[y, \eta] = \delta\mathcal{L}_b[y, \eta] - \delta\mathcal{L}_a[y, \eta]$.

The following lemma is a direct consequence of Lemma 7.3.6.

Lemma 7.3.7. *For an admissible variation η and an admissible function y, let*

$$g(t, \epsilon) := L\left(t, y^\sigma(t) + \epsilon\eta^\sigma(t), \tilde{D}_{q,\omega}[y](t) + \epsilon\tilde{D}_{q,\omega}[\eta](t)\right).$$

Assume that

1. *$g(t,\cdot)$ is differentiable at ω_0 uniformly in $[a,b]_{q,\omega}$;*

2. *$\mathcal{L}_\xi[y+\epsilon\eta]=\int_{\omega_0}^{\xi} g(t,\epsilon)\,\tilde{d}_{q,\omega}t$, $\xi\in\{a,b\}$, exist for $\epsilon\approx 0$;*

3. *$\int_{\omega_0}^{a}\partial_2 g(t,0)\,\tilde{d}_{q,\omega}t$ and $\int_{\omega_0}^{b}\partial_2 g(t,0)\,\tilde{d}_{q,\omega}t$ exist.*

Then,

$$\phi'(0):=\delta\mathcal{L}[y,\eta]=\int_a^b\Bigg[\partial_2 L\left(t,y^\sigma(t),\tilde{D}_{q,\omega}[y](t)\right)\eta^\sigma(t)\\ +\partial_3 L\left(t,y^\sigma(t),\tilde{D}_{q,\omega}[y](t)\right)\tilde{D}_{q,\omega}[\eta](t)\Bigg]\tilde{d}_{q,\omega}t.$$

7.3.2 Optimality conditions

In this section we present a necessary optimality condition (the Hanh symmetric Euler–Lagrange equation) and a sufficient optimality condition to problem (7.1.1).

Theorem 7.3.8 (The Hahn symmetric Euler–Lagrange equation)**.** *Under hypotheses ($H_{q,\omega}$1)-($H_{q,\omega}$3) and conditions 1 to 3 of Lemma 7.3.7 on the Lagrangian L, if $y_*\in\mathcal{Y}^1\left([a,b]_{q,\omega},\mathbb{R}\right)$ is a local extremizer to problem* (7.1.1)*, then y_* satisfies the Hahn symmetric Euler–Lagrange equation*

$$\partial_2 L\left(t,y^\sigma(t),\tilde{D}_{q,\omega}[y](t)\right)=\tilde{D}_{q,\omega}\left[\tau\mapsto\partial_3 L\left(\sigma(\tau),y^{\sigma^2}(\tau),\left(\tilde{D}_{q,\omega}[y]\right)^\sigma(\tau)\right)\right](t)\qquad(7.3.2)$$

for all $t\in[a,b]_{q,\omega}$.

Proof. Let y_* be a local minimizer (resp. maximizer) to problem (7.1.1) and η an admissible variation. Define $\phi:\mathbb{R}\to\mathbb{R}$ by $\phi(\epsilon):=\mathcal{L}[y_*+\epsilon\eta]$. A necessary condition for y_* to be an extremizer is given by $\phi'(0)=0$. By Lemma 7.3.7,

$$\int_a^b\Bigg[\partial_2 L\left(t,y_*^\sigma(t),\tilde{D}_{q,\omega}[y_*](t)\right)\eta^\sigma(t)\\ +\partial_3 L\left(t,y_*^\sigma(t),\tilde{D}_{q,\omega}[y_*](t)\right)\tilde{D}_{q,\omega}[\eta](t)\Bigg]\tilde{d}_{q,\omega}t=0.$$

Using the integration by parts formula (7.2.2), we get

$$\int_a^b\partial_3 L\left(t,y_*^\sigma(t),\tilde{D}_{q,\omega}[y_*](t)\right)\tilde{D}_{q,\omega}[\eta](t)\,\tilde{d}_{q,\omega}t\\ =\partial_3 L\left(\sigma(t),y_*^{\sigma^2}(t),\left(\tilde{D}_{q,\omega}[y_*]\right)^\sigma(t)\right)\eta(t)\Big|_a^b\\ -q\int_a^b\left(\tilde{D}_{q,\omega}\left[\tau\mapsto\partial_3 L\left(\tau,y_*^\sigma(\tau),\left(\tilde{D}_{q,\omega}[y_*]\right)(\tau)\right)\right]\right)^\sigma(t)\,\eta^\sigma(t)\,\tilde{d}_{q,\omega}t.$$

Since $\eta(a)=\eta(b)=0$, then

$$\int_a^b \left[\partial_2 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right) - q\left(\tilde{D}_{q,\omega}\left[\tau \mapsto \partial_3 L\left(\tau, y_*^\sigma(\tau), \left(\tilde{D}_{q,\omega}[y_*]\right)(\tau)\right)\right]\right)^\sigma(t)\right]\eta^\sigma(t)\,\tilde{d}_{q,\omega}t = 0$$

and by Lemma 7.3.4 we get

$$\partial_2 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right) = q\left(\tilde{D}_{q,\omega}\left[\tau \mapsto \partial_3 L\left(\tau, y_*^\sigma(\tau), \tilde{D}_{q,\omega}[y_*](\tau)\right)\right]\right)^\sigma(t)$$

for all $t \in [a,b]_{q,\omega}$. Finally, using Lemma 7.2.9, we conclude that

$$\partial_2 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right) = \tilde{D}_{q,\omega}\left[\tau \mapsto \partial_3 L\left(\sigma(\tau), y_*^{\sigma^2}(\tau), \left(\tilde{D}_{q,\omega}[y_*]\right)^\sigma(\tau)\right)\right](t).$$

□

The particular case $\omega = 0$ gives the q-symmetric Euler–Lagrange equation (see Theorem 6.3.5).

Corollary 7.3.9 (The q-symmetric Euler–Lagrange equation [32])**.** *Let $\omega = 0$. Under hypotheses ($H_{q,\omega}$1)–($H_{q,\omega}$3) and conditions 1 to 3 of Lemma 7.3.7 on the Lagrangian L, if $y_* \in \mathcal{Y}^1\left([a,b]_{q,0}, \mathbb{R}\right)$ is a local extremizer to problem* (7.1.1) *(with $\omega = 0$), then y_* satisfies the q-symmetric Euler–Lagrange equation*

$$\partial_2 L\left(t, y(qt), \tilde{D}_q[y](t)\right) = \tilde{D}_q\left[\tau \mapsto \partial_3 L\left(q\tau, y\left(q^2\tau\right), \tilde{D}_q[y](q\tau)\right)\right](t)$$

for all $t \in [a,b]_q$.

To conclude this section, we prove a sufficient optimality condition to (7.1.1).

Theorem 7.3.10. *Suppose that $a < b$ and $a, b \in [c]_{q,\omega}$ for some $c \in I$. Also, assume that L is a jointly convex (resp. concave) function in (u,v). If y_* satisfies the Hahn symmetric Euler–Lagrange equation* (7.3.2)*, then y_* is a global minimizer (resp. maximizer) to problem* (7.1.1)*.*

Proof. Let L be a jointly convex function in (u,v) (the concave case is similar). Then, for any admissible variation η, we have

$$\begin{aligned}
&\mathcal{L}[y_*+\eta] - \mathcal{L}[y_*] \\
&\quad = \int_a^b \Bigg(L\left(t, y_*^\sigma(t) + \eta^\sigma(t), \tilde{D}_{q,\omega}[y_*](t) + \tilde{D}_{q,\omega}[\eta](t)\right) \\
&\qquad\qquad - L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right)\Bigg)\tilde{d}_{q,\omega}t \\
&\quad \geqslant \int_a^b \Bigg(\partial_2 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right)\eta^\sigma(t) \\
&\qquad\qquad + \partial_3 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right)\tilde{D}_{q,\omega}[\eta](t)\Bigg)\tilde{d}_{q,\omega}t.
\end{aligned}$$

Using the integration by parts formula (7.2.2) and Lemma 7.2.9, we get

$$\begin{aligned}\mathcal{L}[y_*+\eta]-\mathcal{L}[y_*] &\geqslant \partial_3 L\left(\sigma(t), y_*^{\sigma^2}(t), \left(\tilde{D}_{q,\omega}[y_*]\right)^{\sigma}(t)\right)\eta(t)\Big|_a^b \\ &+\int_a^b \left[\partial_2 L\left(t, y_*^{\sigma}(t), \tilde{D}_{q,\omega}[y_*](t)\right)\right. \\ &\left. -\tilde{D}_{q,\omega}\left[\tau\mapsto \partial_3 L\left(\sigma(\tau), y_*^{\sigma^2}(\tau), \left(\tilde{D}_{q,\omega}[y_*]\right)^{\sigma}(\tau)\right)(t)\right]\right]\eta^{\sigma}(t)\,\tilde{d}_{q,\omega}t.\end{aligned}$$

Since y_* satisfies (7.3.2) and η is an admissible variation, we obtain

$$\mathcal{L}[y_*+\eta]-\mathcal{L}[y_*]\geqslant 0,$$

proving that y_* is a minimizer to problem (7.1.1). □

Example 7.3.11. *Let $q\in\,]0,1[$ and $\omega\geqslant 0$ be fixed real numbers. Also, let $I\subseteq\mathbb{R}$ be an interval such that $a:=\omega_0, b\in I$ and $a<b$. Consider the problem*

$$\begin{cases}\mathcal{L}[y]=\displaystyle\int_a^b\sqrt{1+\left(\tilde{D}_{q,\omega}[y](t)\right)^2}\,\tilde{d}_{q,\omega}t\longrightarrow\min\\[2ex] y\in\mathcal{Y}^1\left([a,b]_{q,\omega},\mathbb{R}\right)\\[2ex] y(a)=a,\quad y(b)=b.\end{cases}\tag{7.3.3}$$

If y_* is a local minimizer to the problem, then y_* satisfies the Hahn symmetric Euler–Lagrange equation

$$\tilde{D}_{q,\omega}\left[\tau\mapsto\frac{\left(\tilde{D}_{q,\omega}[y]\right)^{\sigma}(\tau)}{\sqrt{1+\left(\left(\tilde{D}_{q,\omega}[y]\right)^{\sigma}(\tau)\right)^2}}\right](t)=0\quad\text{for all}\quad t\in[a,b]_{q,\omega}.\tag{7.3.4}$$

It is simple to check that function $y_*(t)=t$ is a solution to (7.3.4) satisfying the given boundary conditions. Since the Lagrangian is jointly convex in (u,v), then we conclude from Theorem 7.3.10 that function $y_*(t)=t$ is indeed a minimizer to problem (7.3.3).

7.3.3 Leitmann's direct method

Similarly to Malinowska and Torres [89], we show that Leitmann's direct method [81] has also applications in the Hahn symmetric variational calculus. Consider the variational functional integral

$$\bar{\mathcal{L}}[\bar{y}]=\int_a^b\bar{L}\left(t,\bar{y}^{\sigma}(t),\tilde{D}_{q,\omega}[\bar{y}](t)\right)\tilde{d}_{q,\omega}t.$$

As before, we assume that function $\bar{L}:I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the following hypotheses:

($\overline{\mathrm{H}_{q,\omega}1}$) $(u,v)\to\bar{L}(t,u,v)$ is a $C^1\left(\mathbb{R}^2,\mathbb{R}\right)$ function for any $t\in I$;

($\overline{\mathrm{H}_{q,\omega}2}$) $t\to\bar{L}\left(t,\bar{y}^{\sigma}(t),\tilde{D}_{q,\omega}[\bar{y}](t)\right)$ is continuous at ω_0 for any admissible function $\bar{y}$;

($\overline{\mathrm{H}_{q,\omega}3}$) functions $t \to \partial_{i+2}\bar{L}\left(t, \bar{y}^{\sigma}(t), \tilde{D}_{q,\omega}[\bar{y}](t)\right)$ belong to $\mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ for all admissible $\bar{y}$, $i = 0, 1$.

Lemma 7.3.12 (Leitmann's fundamental lemma via Hahn's symmetric quantum operator). *Let $y = z(t, \bar{y})$ be a transformation having a unique inverse $\bar{y} = \bar{z}(t, y)$ for all $t \in [a,b]_{q,\omega}$, such that there is a one-to-one correspondence*

$$y(t) \leftrightarrow \bar{y}(t)$$

for all functions $y \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ satisfying the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ and all functions $\bar{y} \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ satisfying

$$\bar{y}(a) = \bar{z}(a, \alpha) \ \text{ and } \bar{y}(b) = \bar{z}(b, \beta). \tag{7.3.5}$$

If the transformation $y = z(t, \bar{y})$ is such that there exists a function $G : I \times \mathbb{R} \to \mathbb{R}$ satisfying the identity

$$L\left(t, y^{\sigma}(t), \tilde{D}_{q,\omega}[y](t)\right) - \bar{L}\left(t, \bar{y}^{\sigma}(t), \tilde{D}_{q,\omega}[\bar{y}](t)\right) = \tilde{D}_{q,\omega}[\tau \mapsto G(\tau, \bar{y}(\tau))](t),$$

for all $t \in [a,b]_{q,\omega}$, then if $\bar{y}_*$ is a maximizer (resp. a minimizer) of $\bar{\mathcal{L}}$ with $\bar{y}_*$ satisfying (7.3.5), $y_* = z(t, \bar{y}_*)$ is a maximizer (resp. a minimizer) of $\mathcal{L}$ for y_* satisfying $y_*(a) = \alpha$ and $y_*(b) = \beta$.

Proof. Suppose $y \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ satisfies the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$. Define function $\bar{y} \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ through the formula $\bar{y} = \bar{z}(t, y)$, $t \in [a,b]_{q,\omega}$. Then, $\bar{y}$ satisfies (7.3.5) and

$$\begin{aligned}
&\mathcal{L}[y] - \bar{\mathcal{L}}[\bar{y}] \\
&= \int_a^b L\left(t, y^{\sigma}(t), \tilde{D}_{q,\omega}[y](t)\right) \tilde{d}_{q,\omega}t - \int_a^b \bar{L}\left(t, \bar{y}^{\sigma}(t), \tilde{D}_{q,\omega}[\bar{y}](t)\right) \tilde{d}_{q,\omega}t \\
&= \int_a^b \tilde{D}_{q,\omega}[\tau \mapsto G(\tau, \bar{y}(\tau))](t)\, \tilde{d}_{q,\omega}t \\
&= G(b, \bar{y}(b)) - G(a, \bar{y}(a)) \\
&= G(b, \bar{z}(b, \beta)) - G(a, \bar{z}(a, \alpha)).
\end{aligned}$$

The desired result follows immediately because the right-hand side of the above equality is a constant, depending only on the fixed-endpoint conditions $y(a) = \alpha$ and $y(b) = \beta$. □

Example 7.3.13. *Let $q \in]0,1[$, $\omega \geqslant 0$, and $a := \omega_0, b$ with $a < b$ be fixed real numbers. Also, let I be an interval of $\mathbb{R}$ such that $a, b \in I$. We consider the problem*

$$\begin{cases}
\mathcal{L}[y] = \displaystyle\int_a^b \left(\left(\tilde{D}_{q,\omega}[y](t)\right)^2 + q y^{\sigma}(t) + t\tilde{D}_{q,\omega}[y](t)\right) \tilde{d}_{q,\omega}t \longrightarrow \min \\
y \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right) \\
y(a) = \alpha, \quad y(b) = \beta,
\end{cases} \tag{7.3.6}$$

where $\alpha, \beta \in \mathbb{R}$ *and* $\alpha \neq \beta$. *We transform problem* (7.3.6) *into the trivial problem*

$$\begin{cases} \bar{\mathcal{L}}[\bar{y}] = \int_a^b \left(\tilde{D}_{q,\omega}[\bar{y}](t)\right)^2 \tilde{d}_{q,\omega}t \longrightarrow \min \\ \bar{y} \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right) \\ \bar{y}(a) = 0, \quad \bar{y}(b) = 0, \end{cases}$$

which has solution $\bar{y} \equiv 0$. *For that we consider the transformation*

$$y(t) = \bar{y}(t) + ct + d,$$

where c, d *are real constants that will be chosen later. Since*

$$y^\sigma(t) = \bar{y}^\sigma(t) + c\sigma(t) + d$$

and

$$\tilde{D}_{q,\omega}[y](t) = \tilde{D}_{q,\omega}[\bar{y}](t) + c,$$

we have

$$\begin{aligned} &\left(\tilde{D}_{q,\omega}[y](t)\right)^2 + qy^\sigma(t) + t\tilde{D}_{q,\omega}[y](t) \\ &= \left(\tilde{D}_{q,\omega}[\bar{y}](t)\right)^2 + 2c\tilde{D}_{q,\omega}[\bar{y}](t) + c^2 + qd + q\bar{y}^\sigma(t) + t\tilde{D}_{q,\omega}[\bar{y}](t) + c\left(q\sigma(t) + t\right). \end{aligned}$$

Therefore,

$$\begin{aligned} &\left[\left(\tilde{D}_{q,\omega}[y](t)\right)^2 + qy^\sigma(t) + t\tilde{D}_{q,\omega}[y](t)\right] - \left(\tilde{D}_{q,\omega}[\bar{y}](t)\right)^2 \\ &= \tilde{D}_{q,\omega}[2c\bar{y}](t) + \tilde{D}_{q,\omega}\left[\left(c^2 + qd\right)id\right](t) + \tilde{D}_{q,\omega}[\sigma \cdot \bar{y}](t) + c\tilde{D}_{q,\omega}[\sigma \cdot id](t) \\ &= \tilde{D}_{q,\omega}\left[2c\bar{y} + \left(c^2 + qd\right)id + \sigma \cdot \bar{y} + c\left(\sigma \cdot id\right)\right](t), \end{aligned}$$

where id *represents the identity function. In order to obtain the solution to the original problem, it suffices to choose* c *and* d *such that*

$$\begin{cases} ca + d = \alpha \\ cb + d = \beta. \end{cases} \tag{7.3.7}$$

Solving the system of equations (7.3.7), *we obtain*

$$c = \frac{\alpha - \beta}{a - b}$$

and

$$d = \frac{a\beta - b\alpha}{a - b}.$$

Hence, the global minimizer to problem (7.3.6) *is*

$$y(t) = \frac{\alpha - \beta}{a - b}t + \frac{a\beta - b\alpha}{a - b}.$$

7.4 State of the Art

Since the recent construction of the inverse operator of Hahn's derivative [3, 4], the calculus of variations within Hahn's quantum calculus is under current research. We provide here some references within this subject: [33, 85, 89].

The results of this chapter were presented by the author at the International Conference on Differential & Difference Equations and Applications 2011, University of Azores, Portugal and are published in [36].

Chapter 8

The Symmetric Calculus on Time Scales

In this chapter we define a symmetric derivative on time scales and derive some of its properties. A diamond integral, which is a refined version of the diamond-α integral, is also introduced. A mean value theorem is proved for the diamond integral as well as versions of Holder's, Cauchy-Schwarz's and Minkowski's inequalities.

8.1 Introduction

Symmetric properties of functions are very useful in a large number of problems. In particular, in the theory of trigonometric series, applications of these properties are well known [124]. Differentiability of a function is one of the most important properties in the theory of functions of real variables. However, even simple functions such as

$$\begin{aligned} &f(x) = |x| \\ &g(x) = \begin{cases} x \sin \frac{1}{x} & , \quad x \neq 0 \\ 0 & , \quad x = 0 \end{cases} \\ &h(x) = \tfrac{1}{x^2}\ , \quad x \neq 0 \end{aligned} \tag{8.1.1}$$

do not have (classical) derivative at $x = 0$. Authors like Riemann, Schwarz, Peano, Dini and de la Vallée-Poussin extended the notion of the classical derivative in different ways, depending on the purpose [124]. One of such generalizations is the symmetric derivative defined by

$$f^s(x) = \lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}.$$

As notice before, the functions in (8.1.1) do not have ordinary derivative at $x = 0$, however they have symmetric derivative at $x = 0$ and $f^s(0) = 0$, $g^s(0) = 0$ and $h^s(0) = 0$. For a deeper understanding of the symmetric derivative and its properties, we refer the reader to the book [117].

On the other hand, the symmetric quotient

$$\frac{f(x+h)-f(x-h)}{2h}$$

has, in general, better convergence properties than the ordinary difference quotient [110]. The study of the symmetric quotient naturally lead us to the quantum calculus. Kac and Cheung [73] defined the h-symmetric difference and the q-symmetric difference operators, respectively, by

$$\tilde{D}_h(x)=\frac{f(x+h)-f(x-h)}{2h}$$

and

$$\tilde{D}_q(x)=\frac{f(qx)-f\left(q^{-1}x\right)}{\left(q-q^{-1}\right)x}$$

where $h\neq 0$ and $q\neq 1$.

In 1988, Hilger introduced the theory of time scales, which is a theory that was created in order to unify, and extend, discrete and continuous analysis into a single theory [67]. In Chapter 1 we reviewed the necessary definitions and results on time scales [27, 28]. Namely, we presented the nabla and the delta calculus. We also presented the diamond-α dynamic derivative, which was introduced by Sheng, Fadag, Henderson and Davis, as a linear combination of the delta and nabla derivatives [112].

Our goal in this chapter is to define the symmetric derivative on time scales and to develop the symmetric time scale calculus. This chapter is organized as follows. In Section 8.2 we define the time scale symmetric derivative and derive some of its properties. In some special cases we will show that the diamond-α derivative coincides with the symmetric derivative on time scales, however in general this is not true. In Section 8.3 we introduce the diamond integral which is an attempt to obtain a symmetric integral and whose construction is similar to the diamond-α integral. Finally, we prove some inequalities for the diamond integral.

In what follows, $\mathbb{T}$ denotes a time scale with operators σ, ρ, Δ and ∇ (see Chapter 1). We recall that $\mathbb{T}$ has the topology inherited from $\mathbb{R}$ with the standard topology.

8.2 Symmetric differentiation

Definition 8.2.1. *We say that a function $f:\mathbb{T}\to\mathbb{R}$ is* symmetric continuous *at $t\in\mathbb{T}$ if, for any $\varepsilon>0$, there exists a neighborhood U_t of t such that for all $s\in U_t$ for which $2t-s\in U_t$ one has*

$$|f(s)-f(2t-s)|\leqslant\varepsilon.$$

It is easy to see that continuity implies symmetric continuity.

Proposition 8.2.2. *Let $\mathbb{T}$ be a time scale. If $f:\mathbb{T}\to\mathbb{R}$ is a continuous function, then f is symmetric continuous.*

Proof. Since f is continuous at $t\in\mathbb{T}$, then, for any $\varepsilon>0$, there exists an neighborhood U_t such that

$$|f(s)-f(t)|<\frac{\varepsilon}{2}$$

and

$$|f(2t-s)-f(t)|<\frac{\varepsilon}{2}$$

for all $s \in U_t$ for which $2t - s \in U_t$. Thus,

$$\begin{aligned} |f(s) - f(2t-s)| &\leqslant |f(s) - f(t)| + |f(t) - f(2t-s)| \\ &< \varepsilon. \end{aligned}$$

□

The next example shows that symmetric continuity does not imply continuity.

Example 8.2.3. *Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$f(t) = \begin{cases} 0 & if \quad t \neq 0 \\ 1 & if \quad t = 0. \end{cases}$$

Function f is symmetric continuous at 0: for any $\varepsilon > 0$, there exists a neighborhood U_t of $t = 0$ such that

$$|f(s) - f(-s)| = 0 < \varepsilon$$

for all $s \in U_t$ for which $-s \in U_t$. However, f is not continuous at 0.

Now we define the symmetric derivative on time scales.

Definition 8.2.4. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}^{\kappa}_{\kappa}$. The* symmetric derivative *of f at t, denoted by $f^{\diamondsuit}(t)$, is the real number (provided that exists) with the property that, for any $\varepsilon > 0$, there exists a neighborhood U of t such that, for all $s \in U_t$ for which $2t - s \in \mathbb{T}$, we have*

$$\left| [f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)] - f^{\diamondsuit}(t) [\sigma(t) + 2t - 2s - \rho(t)] \right| \leqslant \varepsilon |\sigma(t) + 2t - 2s - \rho(t)| . \quad (8.2.1)$$

A function f is said to be symmetric differentiable provided $f^{\diamondsuit}(t)$ exists for all $t \in \mathbb{T}^{\kappa}_{\kappa}$.

Some useful properties are given next.

Theorem 8.2.5. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}^{\kappa}_{\kappa}$.*

1. *Then f has at most one symmetric derivative at t;*
2. *If f is symmetric differentiable at t, then f is symmetric continuous at t;*
3. *If f is continuous at t and if t is not dense, then f is symmetric differentiable at t and*

$$f^{\diamondsuit}(t) = \frac{f^{\sigma}(t) - f^{\rho}(t)}{\sigma(t) - \rho(t)};$$

4. *If t is dense, then f is symmetric differentiable at t if and only if the limit*

$$\lim_{s \to t} \frac{-f(s) + f(2t-s)}{2t - 2s}$$

exists as a finite number. In this case

$$\begin{aligned} f^{\diamondsuit}(t) &= \lim_{s \to t} \frac{-f(s) + f(2t-s)}{2t - 2s} \\ &= \lim_{h \to 0} \frac{f(t+h) - f(t-h)}{2h}; \end{aligned}$$

5. *If f is symmetric differentiable and continuous at t, then*

$$f^{\sigma}(t) = f^{\rho}(t) + f^{\diamondsuit}(t)\left[\sigma(t) - \rho(t)\right].$$

Proof. 1. Let us suppose that f has two symmetric derivatives at t, $f_1^{\diamondsuit}(t)$ and $f_2^{\diamondsuit}(t)$. Then there exists a neighborhood U_1 of t such that

$$\begin{aligned}
&\left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f_1^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \\
\leqslant\ & \frac{\varepsilon}{2}\left|\sigma(t) + 2t - 2s - \rho(t)\right|
\end{aligned}$$

for all $s \in U_1$ for which $2t - s \in \mathbb{T}$, and there exists a neighborhood U_2 of t such that

$$\begin{aligned}
&\left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f_2^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \\
\leqslant\ & \frac{\varepsilon}{2}\left|\sigma(t) + 2t - 2s - \rho(t)\right|
\end{aligned}$$

for all $s \in U_2$ for which $2t - s \in \mathbb{T}$. Therefore, for all $s \in U_1 \cap U_2$ for which $2t - s \in \mathbb{T}$,

$$\begin{aligned}
&\left|f_1^{\diamondsuit}(t) - f_2^{\diamondsuit}(t)\right| = \left|\left[f_1^{\diamondsuit}(t) - f_2^{\diamondsuit}(t)\right]\frac{\sigma(t) + 2t - 2s - \rho(t)}{\sigma(t) + 2t - 2s - \rho(t)}\right| \\
=\ & \Big|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f_2^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right] \\
& -\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] + f_1^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\Big| \\
& \times\frac{1}{\left|\sigma(t) + 2t - 2s - \rho(t)\right|} \\
\leqslant\ & \Big(\left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f_2^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \\
& +\left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f_1^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right|\Big) \\
& \times\frac{1}{\left|\sigma(t) + 2t - 2s - \rho(t)\right|} \\
\leqslant\ & \varepsilon
\end{aligned}$$

proving the desired result.

2. By definition of symmetric derivative, for any $\epsilon^* > 0$ there exists a neighborhood U of t such that

$$\begin{aligned}
&\left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \\
&\qquad \leqslant \varepsilon^*\left|\sigma(t) + 2t - 2s - \rho(t)\right|
\end{aligned}$$

for all $s \in U$ for which $2t - s \in \mathbb{T}$. Therefore we have for all $s \in U \cap]t - \varepsilon^*, t + \varepsilon^*[$

$$\begin{array}{ll} & \left|-f(s) + f(2t-s)\right| \\ \leqslant & \left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \\ & + \left|\left[f^{\sigma}(t) - f^{\rho}(t)\right] - f^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \\ \leqslant & \left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \\ & + \left|\left[f^{\sigma}(t) - f(t) + f(t) - f^{\rho}(t)\right] - f^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2t - \rho(t)\right]\right| \\ & +2\left|f^{\diamondsuit}(t)\right| |t-s| \\ \leqslant & \varepsilon^* \left|\sigma(t) + 2t - 2s - \rho(t)\right| + \varepsilon^* \left|\sigma(t) + 2t - 2t - \rho(t)\right| + 2\left|f^{\diamondsuit}(t)\right| |t-s| \\ \leqslant & \varepsilon^* \left|\sigma(t) - \rho(t)\right| + 2\varepsilon^* |t-s| + \varepsilon^* \left|\sigma(t) - \rho(t)\right| + 2\left|f^{\diamondsuit}(t)\right| |t-s| \\ = & 2\varepsilon^* \left|\sigma(t) - \rho(t)\right| + 2\left(\varepsilon^* + \left|f^{\diamondsuit}(t)\right|\right) |t-s| \\ \leqslant & 2\varepsilon^* \left|\sigma(t) - \rho(t)\right| + 2\left(\varepsilon^* + \left|f^{\diamondsuit}(t)\right|\right) \varepsilon^* \\ = & 2\varepsilon^* \left[\left|\sigma(t) - \rho(t)\right| + \varepsilon^* + \left|f^{\diamondsuit}(t)\right|\right] \end{array}$$

proving that f is symmetric continuous at t.

3. Suppose that $t \in \mathbb{T}^{\kappa}_{\kappa}$ is not dense and f is continuous at t. Then,

$$\lim_{s \to t} \frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} = \frac{f^{\sigma}(t) - f^{\rho}(t)}{\sigma(t) - \rho(t)}.$$

Hence, for any $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\left|\frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} - \frac{f^{\sigma}(t) - f^{\rho}(t)}{\sigma(t) - \rho(t)}\right| \leqslant \varepsilon$$

for all $s \in U$ for which $2t - s \in \mathbb{T}$. It follows that

$$\left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - \frac{f^{\sigma}(t) - f^{\rho}(t)}{\sigma(t) - \rho(t)}\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \leqslant \varepsilon\left[\sigma(t) + 2t - 2s - \rho(t)\right],$$

which proves that

$$f^{\diamondsuit}(t) = \frac{f^{\sigma}(t) - f^{\rho}(t)}{\sigma(t) - \rho(t)}.$$

4. Assume that f is symmetric differentiable at t and t is dense. Let $\varepsilon > 0$ be given. Hence there exists a neighborhood U of t such that

$$\left|\left[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)\right] - f^{\diamondsuit}(t)\left[\sigma(t) + 2t - 2s - \rho(t)\right]\right| \leqslant \varepsilon\left|\sigma(t) + 2t - 2s - \rho(t)\right|$$

for all $s \in U$ for which $2t - s \in \mathbb{T}$. Since t is dense we have that

$$\left|[-f(s) + f(2t-s)] - f^{\diamondsuit}(t)[2t-2s]\right| \leqslant \varepsilon |2t - 2s|$$

for all $s \in U$ for which $2t - s \in \mathbb{T}$. It follows that

$$\left|\frac{-f(s) + f(2t-s)}{2t-2s} - f^{\diamondsuit}(t)\right| \leqslant \varepsilon$$

for all $s \in U$ with $s \neq t$. Therefore we get the desired result:

$$f^{\diamondsuit}(t) = \lim_{s \to t} \frac{-f(s) + f(2t-s)}{2t-2s}.$$

Conversely, let us suppose that t is dense and the limit

$$\lim_{s \to t} \frac{-f(s) + f(2t-s)}{2t-2s}$$

exists. Let $L := \lim_{s \to t} \frac{-f(s) + f(2t-s)}{2t-2s}$. Hence, there exists a neighborhood U of t such that

$$\left|\frac{-f(s) + f(2t-s)}{2t-2s} - L\right| \leqslant \varepsilon$$

for all $s \in U$ for which $2t - s \in \mathbb{T}$, and since t is dense, we have

$$\left|\frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} - L\right| \leqslant \varepsilon$$

and hence

$$|[f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)] - L[\sigma(t) + 2t - 2s - \rho(t)]| \leqslant \varepsilon |\sigma(t) + 2t - 2s - \rho(t)|$$

which lead us to the conclusion that f is symmetric differentiable and

$$f^{\diamondsuit}(t) = \lim_{s \to t} \frac{-f(s) + f(2t-s)}{2t-2s}.$$

Note that if we use the substitution $s = t + h$, then

$$\begin{aligned} f^{\diamondsuit}(t) &= \lim_{s \to t} \frac{-f(s) + f(2t-s)}{2t-2s} \\ &= \lim_{h \to 0} \frac{-f(t+h) + f(2t-(t+h))}{2t - 2(t+h)} \\ &= \lim_{h \to 0} \frac{f(t+h) - f(t-h)}{2h}. \end{aligned}$$

5. If t is a dense point, then $\sigma(t) = \rho(t)$ and

$$\begin{aligned} & f^{\sigma}(t) - f^{\rho}(t) = f^{\diamondsuit}(t)[\sigma(t) - \rho(t)] \\ \Leftrightarrow\ & f^{\sigma}(t) = f^{\rho}(t) + f^{\diamondsuit}(t)[\sigma(t) - \rho(t)]. \end{aligned}$$

If t is not dense and since f is continuous, then

$$\begin{aligned} f^{\diamondsuit}(t) &= \frac{f^{\sigma}(t) - f^{\rho}(t)}{\sigma(t) - \rho(t)} \\ \Leftrightarrow \quad f^{\sigma}(t) &= f^{\rho}(t) + f^{\diamondsuit}(t)\left[\sigma(t) - \rho(t)\right]. \end{aligned}$$

□

Example 8.2.6. *1. If $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$, then the symmetric derivative is the symmetric difference operator given by*

$$f^{\diamondsuit}(t) = \frac{f(t+h) - f(t-h)}{2h}.$$

2. If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ and $0 < q < 1$, then the symmetric derivative is the q-symmetric difference operator given by

$$f^{\diamondsuit}(t) = \frac{f(qt) - f\left(q^{-1}t\right)}{\left(q - q^{-1}\right)t}, \quad \text{for } t \neq 0.$$

3. If $\mathbb{T} = \mathbb{R}$, then the symmetric derivative is the classic symmetric derivative given by

$$f^{\diamondsuit}(t) = \lim_{h \to 0} \frac{f(t+h) - f(t-h)}{2h}.$$

Remark 8.2.7. *It is clear that the symmetric derivative of a constant function is zero and the symmetric derivative of the identity functions is one.*

Remark 8.2.8. *An alternative way to define the symmetric derivative of f at $t \in \mathbb{T}^{\kappa}_{\kappa}$ is saying that the following limit exists:*

$$\begin{aligned} f^{\diamondsuit}(t) &= \lim_{s \to t} \frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} \\ &= \lim_{h \to 0} \frac{f^{\sigma}(t) - f(t+h) + f(t-h) - f^{\rho}(t)}{\sigma(t) - 2h - \rho(t)}. \end{aligned}$$

Example 8.2.9. *Let $f : \mathbb{T} \to \mathbb{R}$ be defined by $f(t) = |t|$. Then*

$$\begin{aligned} f^{\diamondsuit}(0) &= \lim_{h \to 0} \frac{f^{\sigma}(0) - f(0+h) + f(0-h) - f^{\rho}(0)}{\sigma(0) - 2h - \rho(0)} \\ &= \lim_{h \to 0} \frac{\sigma(0) - |h| + |-h| + \rho(0)}{\sigma(0) - 2h - \rho(0)} \\ &= \lim_{h \to 0} \frac{\sigma(0) + \rho(0)}{\sigma(0) - 2h - \rho(0)}. \end{aligned}$$

In the particular case $\mathbb{T} = \mathbb{R}$, we have

$$f^{\diamondsuit}(0) = 0.$$

Proposition 8.2.10. *If f is delta and nabla differentiable, then f is symmetric differentiable and, for each $t \in \mathbb{T}_\kappa^\kappa$,*

$$f^{\diamondsuit}(t) = \gamma_1(t) f^{\Delta}(t) + \Big(1 - \gamma_1(t)\Big) f^{\nabla}(t),$$

where

$$\gamma_1(t) = \lim_{s \to t} \frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)}.$$

Proof. Note that

$$\begin{aligned}
f^{\diamondsuit}(t) &= \lim_{s \to t} \frac{f^{\sigma}(t) - f(s) + f(2t - s) - f^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} \\
&= \lim_{s \to t} \left(\frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)} \frac{f^{\sigma}(t) - f(s)}{\sigma(t) - s} \right. \\
&\quad \left. + \frac{(2t - s) - \rho(t)}{\sigma(t) + 2t - 2s - \rho(t)} \frac{f(2t - s) - f^{\rho}(t)}{(2t - s) - \rho(t)} \right) \\
&= \lim_{s \to t} \left(\frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)} f^{\Delta}(t) + \frac{(2t - s) - \rho(t)}{\sigma(t) + 2t - 2s - \rho(t)} f^{\nabla}(t) \right).
\end{aligned}$$

Let

$$\gamma_2(t) := \lim_{s \to t} \frac{(2t - s) - \rho(t)}{\sigma(t) + 2t - 2s - \rho(t)}.$$

It is clear that

$$\gamma_1(t) + \gamma_2(t) = 1.$$

Note that if $t \in \mathbb{T}$ is dense, then

$$\begin{aligned}
\gamma_1(t) &= \lim_{s \to t} \frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)} \\
&= \lim_{s \to t} \frac{t - s}{2t - 2s} \\
&= \frac{1}{2}
\end{aligned}$$

and therefore

$$\gamma_2(t) = \frac{1}{2}.$$

On the other hand, if $t \in \mathbb{T}$ is not dense, then

$$\begin{aligned}
\gamma_1(t) &= \lim_{s \to t} \frac{\sigma(t) - s}{\sigma(t) + 2t - 2s - \rho(t)} \\
&= \frac{\sigma(t) - t}{\sigma(t) - \rho(t)}
\end{aligned}$$

and

$$\gamma_2(t) = \frac{t - \rho(t)}{\sigma(t) - \rho(t)}.$$

Hence, the functions $\gamma_1,\,\gamma_2:\mathbb{T}\to\mathbb{R}$ are well defined and if f is delta and nabla differentiable we have

$$\begin{aligned} f^{\diamondsuit}(t) &= \gamma_1(t)\,f^{\Delta}(t)+\gamma_2(t)\,f^{\nabla}(t)\\ &= \gamma_1(t)\,f^{\Delta}(t)+\Big(1-\gamma_1(t)\Big)f^{\nabla}(t)\,. \end{aligned}$$

□

Remark 8.2.11. *1. Suppose that f is delta and nabla differentiable. When a point $t\in\mathbb{T}^{\kappa}_{\kappa}$ is right-scattered and left-dense, then its symmetric derivative is equal to its delta derivative and when t is left-scattered and right-dense, its symmetric derivative is equal to its nabla derivative.*

2. Note that if all the points in $\mathbb{T}^{\kappa}_{\kappa}$ are dense and f is delta (or nabla) differentiable, then

$$f^{\diamondsuit}(t)=\frac{1}{2}f^{\Delta}(t)+\frac{1}{2}f^{\nabla}(t)\;=f'(t)\,,\;\;t\in\mathbb{T}^{\kappa}_{\kappa}.$$

3. If f is delta and nabla differentiable and if γ_1 is constant, then the symmetric derivative coincides with the diamond-α derivative. The symmetric derivative tell us exactly the weight of the delta and nabla derivative at each point. In the definition of the diamond-α derivative we choose the influence of the nabla and delta derivative. Moreover, this influence of the nabla and delta derivative does not depend of the point we choose in the diamond-α derivative.

Proposition 8.2.12. *The functions $\gamma_1,\,\gamma_2:\mathbb{T}\to\mathbb{R}$ are bounded and nonnegative. Moreover,*

$$0\leqslant\gamma_i\leqslant1\;,\;\;i=1,2.$$

Proof. The result follows from the fact that

$$\rho(t)\leqslant t\leqslant\sigma(t)$$

for every $t\in\mathbb{T}$. □

Theorem 8.2.13. *Let $f,g:\mathbb{T}\to\mathbb{R}$ be functions symmetric differentiable at $t\in\mathbb{T}^{\kappa}_{\kappa}$ and let $\lambda\in\mathbb{R}$. Then:*

1. *$f+g$ is symmetric differentiable at t with*

$$(f+g)^{\diamondsuit}(t)=f^{\diamondsuit}(t)+g^{\diamondsuit}(t)\,;$$

2. *λf is symmetric differentiable at t with*

$$(\lambda f)^{\diamondsuit}(t)=\lambda f^{\diamondsuit}(t)\,;$$

3. *fg is symmetric differentiable at t with*

$$(fg)^{\diamondsuit}(t)=f^{\diamondsuit}(t)\,g^{\sigma}(t)+f^{\rho}(t)\,g^{\diamondsuit}(t)\,,$$

provided that f and g are continuous;

4. $\frac{1}{f}$ *is symmetric differentiable at* t *with*

$$\left(\frac{1}{f}\right)^{\diamondsuit}(t) = -\frac{f^{\diamondsuit}(t)}{f^{\sigma}(t)\, f^{\rho}(t)},$$

provided that f *is continuous and* $f^{\sigma}(t)\, f^{\rho}(t) \neq 0$*;*

5. $\frac{f}{g}$ *is symmetric differentiable at* t *with*

$$\left(\frac{f}{g}\right)^{\diamondsuit}(t) = \frac{f^{\diamondsuit}(t)\, g^{\rho}(t) - f^{\rho}(t)\, g^{\diamondsuit}(t)}{g^{\sigma}(t)\, g^{\rho}(t)}$$

provided that f *and* g *are continuous and* $g^{\sigma}(t)\, g^{\rho}(t) \neq 0$*.*

Proof. 1. For $t \in \mathbb{T}_{\kappa}^{\kappa}$ we have

$$\begin{aligned}
(f+g)^{\diamondsuit}(t) &= \lim_{s\to t} \frac{(f+g)^{\sigma}(t) - (f+g)(s) + (f+g)(2t-s) - (f+g)^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} \\
&= \lim_{s\to t} \frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} \\
&\quad + \lim_{s\to t} \frac{g^{\sigma}(t) - g(s) + g(2t-s) - g^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} \\
&= f^{\diamondsuit}(t) + g^{\diamondsuit}(t).
\end{aligned}$$

2. Let $t \in \mathbb{T}_{\kappa}^{\kappa}$ and $\lambda \in \mathbb{R}$, then

$$\begin{aligned}
(\lambda f)^{\diamondsuit}(t) &= \lim_{s\to t} \frac{(\lambda f)^{\sigma}(t) - (\lambda f)(s) + (\lambda f)(2t-s) - (\lambda f)^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} \\
&= \lambda \lim_{s\to t} \frac{f^{\sigma}(t) - f(s) + f(2t-s) - f^{\rho}(t)}{\sigma(t) + 2t - 2s - \rho(t)} \\
&= \lambda f^{\diamondsuit}(t).
\end{aligned}$$

3. Let us assume that $t \in \mathbb{T}_{\kappa}^{\kappa}$ and f and g are continuous. If t is dense, then

$$\begin{aligned}
(fg)^{\diamondsuit}(t) &= \lim_{h\to 0} \frac{(fg)(t+h) - (fg)(t-h)}{2h} \\
&= \lim_{h\to 0} \left(\frac{f(t+h) - f(t-h)}{2h} g(t+h)\right) + \lim_{h\to 0} \left(\frac{g(t+h) - g(t-h)}{2h} f(t-h)\right) \\
&= f^{\diamondsuit}(t)\, g(t) + f(t)\, g^{\diamondsuit}(t) \\
&= f^{\diamondsuit}(t)\, g^{\sigma}(t) + f^{\rho}(t)\, g^{\diamondsuit}(t).
\end{aligned}$$

If t is not dense, then

$$\begin{aligned}
(fg)^{\diamondsuit}(t) &= \frac{(fg)^{\sigma}(t) - (fg)^{\rho}(t)}{\sigma(t) - \rho(t)} \\
&= \frac{f^{\sigma}(t) - f^{\rho}(t)}{\sigma(t) - \rho(t)} g^{\sigma}(t) + \frac{g^{\sigma}(t) - g^{\rho}(t)}{\sigma(t) - \rho(t)} f^{\rho}(t) \\
&= f^{\diamondsuit}(t)\, g^{\sigma}(t) + f^{\rho}(t)\, g^{\diamondsuit}(t)
\end{aligned}$$

proving the intended result.

4. Because
$$\left(\frac{1}{f}\times f\right)(t)=1$$
one has
$$\begin{aligned} 0 &= \left(\frac{1}{f}\times f\right)^{\diamondsuit}(t) \\ &= f^{\diamondsuit}(t)\left(\frac{1}{f}\right)^{\sigma}(t)+f^{\rho}(t)\left(\frac{1}{f}\right)^{\diamondsuit}(t). \end{aligned}$$
Therefore,
$$\left(\frac{1}{f}\right)^{\diamondsuit}(t)=-\frac{f^{\diamondsuit}(t)}{f^{\sigma}(t)\,f^{\rho}(t)}.$$

5. Let $t\in\mathbb{T}_{\kappa}^{\kappa}$, then
$$\begin{aligned} \left(\frac{f}{g}\right)^{\diamondsuit}(t) &= \left(f\times\frac{1}{g}\right)^{\diamondsuit}(t) \\ &= f^{\diamondsuit}(t)\left(\frac{1}{g}\right)^{\sigma}(t)+f^{\rho}(t)\left(\frac{1}{g}\right)^{\diamondsuit}(t) \\ &= \frac{f^{\diamondsuit}(t)}{g^{\sigma}(t)}+f^{\rho}(t)\left(-\frac{g^{\diamondsuit}(t)}{g^{\sigma}(t)\,g^{\rho}(t)}\right) \\ &= \frac{f^{\diamondsuit}(t)\,g^{\rho}(t)-f^{\rho}(t)\,g^{\diamondsuit}(t)}{g^{\sigma}(t)\,g^{\rho}(t)}. \end{aligned}$$

□

Example 8.2.14. *1. The symmetric derivative of $f(t)=t^2$ is*
$$f^{\diamondsuit}(t)=\sigma(t)+\rho(t).$$

2. The symmetric derivative of $f(t)=1/t$ is
$$f^{\diamondsuit}(t)=-\frac{1}{\sigma(t)\,\rho(t)}.$$

Remark 8.2.15. *In the classical case, it can be proved that "A continuous function is necessarily increasing in any interval in which its symmetric derivative exists and is positive" [117]. However, it should be noted that this result is not valid for the symmetric derivative on time scales. For instance, consider the time scale $\mathbb{T}=\mathbb{N}$ and the function*
$$f(n)=\begin{cases} n & \text{if} \quad n \text{ is odd} \\ 10n & \text{if} \quad n \text{ is even.} \end{cases}$$
The symmetric derivative of f for n odd is given by
$$\begin{aligned} f^{\diamondsuit}(n) &= \frac{f^{\sigma}(n)-f^{\rho}(n)}{\sigma(n)-\rho(n)} \\ &= \frac{10(n+1)-10(n-1)}{(n+1)-(n-1)} \\ &= 10 \end{aligned}$$

and for n even is given by

$$\begin{aligned} f^{\diamondsuit}(n) &= \frac{f^{\sigma}(n) - f^{\rho}(n)}{\sigma(n) - \rho(n)} \\ &= \frac{(n+1)-(n-1)}{(n+1)-(n-1)} \\ &= 1. \end{aligned}$$

Clearly, the function is not increasing although its symmetric derivative is always positive.

In this example the symmetric derivative coincides with the diamond-α derivative with $\alpha = \frac{1}{2}$, and this shows that there is an inconsistency in [106]. Indeed, Corollary 2.1. of [106] is not valid.

8.3 The diamond integral

In the classical calculus, there are some attempts to define a symmetric integral: see, for example, [44]. However, those integrals invert only "approximately" the symmetric derivatives. In discrete time we have some examples of symmetric integrals, namely the q-symmetric integral: see, for example, [32, 73]. In the context of quantum calculus, in [36] the authors introduced the Hahn symmetric integral that inverts the Hahn symmetric derivative. In time scale calculus, the problem of determining a symmetric integral is an interesting open question. In our opinion, the diamond-α integral is by now the nearest idea to construct a symmetric integral. Regarding the similarities and the advantages of the symmetric derivative towards the diamond-α derivative, we think that the construction of the diamond integral bring us closer to the construction of a genuine symmetric integral on time scales.

Definition 8.3.1. *Let $f : \mathbb{T} \to \mathbb{R}$ and $a, b \in \mathbb{T}$, $a < b$. We define the* diamond integral *(or* $\diamondsuit$-integral*) of f from a to b (or on $[a,b]_{\mathbb{T}}$) by*

$$\int_a^b f(t)\diamondsuit t = \int_a^b \gamma_1(t) f(t)\Delta t + \int_a^b \gamma_2(t) f(t)\nabla t,$$

provided $\gamma_1 f$ is delta integrable and $\gamma_2 f$ is nabla integrable on $[a,b]_{\mathbb{T}}$. We say that the function f is diamond integrable (or $\diamondsuit$-integrable) on $\mathbb{T}$ if it is $\diamondsuit$-integrable for all $a, b \in \mathbb{T}$.

Example 8.3.2. *1. Let $f : \mathbb{Z} \to \mathbb{R}$ be defined by*

$$f(t) = t^2.$$

Then,

$$\begin{aligned} \int_0^2 f(t)\diamondsuit t &= \frac{1}{2}\int_0^2 f(t)\Delta t + \frac{1}{2}\int_0^2 f(t)\nabla t \\ &= \frac{1}{2}\sum_{t=0}^{1} f(t) + \frac{1}{2}\sum_{t=1}^{2} f(t) \\ &= \frac{1}{2}(0+1) + \frac{1}{2}(1+4) \\ &= 3 \end{aligned}$$

2. *Let $f : [0,1] \cup \{2,4\} \to \mathbb{R}$ be defined by*

$$f(t) = 1.$$

Then

$$\begin{aligned}
\int_0^4 f(t) \diamondsuit t &= \int_0^1 f(t) \diamondsuit t + \int_1^4 f(t) \diamondsuit t \\
&= \int_0^1 1 dt + \int_1^4 \gamma_1(t) \Delta t + \int_1^4 \gamma_2(t) \nabla t \\
&= 1 + \gamma_1(1)(\sigma(1) - 1) + \gamma_1(2)(\sigma(2) - 2) \\
&\quad + \gamma_2(2)(2 - \rho(2)) + \gamma_2(4)(4 - \rho(4)) \\
&= 1 + 1 + \frac{4}{3} + \frac{1}{3} + 2 \\
&= \frac{17}{3}.
\end{aligned}$$

Note that the diamond-α integral of the same function is

$$\begin{aligned}
\int_0^4 f(t) \diamondsuit_\alpha t &= \int_0^1 f(t) \diamondsuit t + \alpha \int_1^4 1 \Delta t + (1 - \alpha) \int_1^4 1 \nabla t \\
&= 1 + 3\alpha + (1 - \alpha) 3 \\
&= 4.
\end{aligned}$$

The $\diamondsuit$-integral has the following properties.

Theorem 8.3.3. *Let $f, g : \mathbb{T} \to \mathbb{R}$ be $\diamondsuit$-integrable on $[a,b]_{\mathbb{T}}$. Let $c \in [a,b]_{\mathbb{T}}$ and $\lambda \in \mathbb{R}$. Then,*

1. $\displaystyle\int_a^a f(t) \diamondsuit t = 0$;
2. $\displaystyle\int_a^b f(t) \diamondsuit t = \int_a^c f(t) \diamondsuit t + \int_c^b f(t) \diamondsuit t$;
3. $\displaystyle\int_a^b f(t) \diamondsuit t = -\int_b^a f(t) \diamondsuit t$;
4. *$f + g$ is $\diamondsuit$-integrable on $[a,b]_{\mathbb{T}}$ and*
$$\int_a^b (f+g)(t) \diamondsuit t = \int_a^b f(t) \diamondsuit t + \int_a^b g(t) \diamondsuit t;$$
5. *λf is $\diamondsuit$-integrable on $[a,b]_{\mathbb{T}}$ and*
$$\int_a^b \lambda f(t) \diamondsuit t = \lambda \int_a^b f(t) \diamondsuit t;$$
6. *fg is $\diamondsuit$-integrable on $[a,b]_{\mathbb{T}}$;*
7. *for $p > 0$, $|f|^p$ is $\diamondsuit$-integrable on $[a,b]_{\mathbb{T}}$;*

8. *if* $f(t) \leqslant g(t)$ *for all* $t \in [a,b]_{\mathbb{T}}$, *then*

$$\int_a^b f(t)\diamondsuit t \leqslant \int_a^b g(t)\diamondsuit t;$$

9. $|f|$ *is* $\diamondsuit$*-integrable on* $[a,b]_{\mathbb{T}}$ *and*

$$\left|\int_a^b f(t)\diamondsuit t\right| \leqslant \int_a^b |f(t)|\diamondsuit t.$$

Proof. The results follow straightforwardly from the properties of the nabla and delta integrals. □

Remark 8.3.4. *It is clear that the* $\diamondsuit$*-integral coincides with the* $\diamondsuit_\alpha$*-integral when the functions* γ_1 *and* γ_2 *are constant and we choose a proper* α*. There are several and important time scales where this happens. For instance, when* $\mathbb{T} = \mathbb{R}$ *or* $\mathbb{T} = h\mathbb{Z}$, $h > 0$, *we have that the* $\diamondsuit$*-integral is equal to the* $\diamondsuit_{\frac{1}{2}}$*-integral. Since the Fundamental Theorem of Calculus is not valid for the* $\diamondsuit_\alpha$*-integral [112], then it is clear that the Fundamental Theorem of Calculus is also not valid for the* $\diamondsuit$*-integral.*

Next we prove some integral inequalities which are similar to the $\diamondsuit_\alpha$-integral counterparts found in [13, 88].

Theorem 8.3.5 (Mean value theorem for the diamond integral)**.** *Let* $f, g : \mathbb{T} \to \mathbb{R}$ *be bounded and* $\diamondsuit$*-integrable functions on* $[a,b]_{\mathbb{T}}$, *and let* g *be nonnegative or nonpositive on* $[a,b]_{\mathbb{T}}$*. Let* m *and* M *be the infimum and supremum, respectively, of function* f*. Then, there exists a real number* K *satisfying the inequalities*

$$m \leqslant K \leqslant M$$

such that

$$\int_a^b (fg)(t)\diamondsuit t = K\int_a^b g(t)\diamondsuit t.$$

Proof. Without loss of generality, we suppose that g is nonnegative on $[a,b]_{\mathbb{T}}$. Since, for all $t \in [a,b]_{\mathbb{T}}$

$$m \leqslant f(t) \leqslant M$$

and $g(t) \geqslant 0$, then

$$mg(t) \leqslant f(t)g(t) \leqslant Mg(t)$$

for all $t \in [a,b]_{\mathbb{T}}$. Each of the functions mg, fg and Mg is $\diamondsuit$-integrable from a to b and, by Theorem 8.3.3, one has

$$m\int_a^b g(t)\diamondsuit t \leqslant \int_a^b f(t)g(t)\diamondsuit t \leqslant M\int_a^b g(t)\diamondsuit t.$$

If $\int_a^b g(t)\diamondsuit t = 0$, then $\int_a^b f(t)g(t)\diamondsuit t = 0$ and can choose any $K \in [m,M]$. If $\int_a^b g(t)\diamondsuit t > 0$, then

$$m \leqslant \frac{\int_a^b f(t)g(t)\diamondsuit t}{\int_a^b g(t)\diamondsuit t} \leqslant M$$

and we choose K as

$$K := \frac{\int_a^b f(t)\,g(t)\,\Diamond t}{\int_a^b g(t)\,\Diamond t}.$$

□

We now present $\Diamond$-versions of Hölder's, Cauchy-Schwarz's and Minkowski's inequalities.

Theorem 8.3.6 (Hölder's inequality for the diamond integral)**.** *Let $f, g : \mathbb{T} \to \mathbb{R}$ be functions $\Diamond$-integrable on $[a,b]_{\mathbb{T}}$. Then*

$$\int_a^b |f(t)\,g(t)|\,\Diamond t \leqslant \left(\int_a^b |f(t)|^p\,\Diamond t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q\,\Diamond t\right)^{\frac{1}{q}},$$

where $p > 1$ and $q = \dfrac{p}{p-1}$.

Proof. For $\lambda, \gamma \in \mathbb{R}_0^+$ and p, q such that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, the following inequality holds (Young's inequality):

$$\lambda^{\frac{1}{p}}\gamma^{\frac{1}{q}} \leqslant \frac{\lambda}{p} + \frac{\gamma}{q}.$$

Without loss of generality, let us suppose that

$$\left(\int_a^b |f(t)|^p\,\Diamond t\right)\left(\int_a^b |g(t)|^q\,\Diamond t\right) \neq 0$$

(note that both integrals exist by Theorem 8.3.3). Define

$$\lambda(t) := \frac{|f(t)|^p}{\int_a^b |f(\tau)|^p\,\Diamond\tau} \quad \text{and} \quad \gamma(t) := \frac{|g(t)|^q}{\int_a^b |g(\tau)|^q\,\Diamond\tau}.$$

Since both functions λ and γ are $\Diamond$-integrable on $[a,b]_{\mathbb{T}}$, then

$$\begin{aligned}
&\int_a^b \frac{|f(t)|}{\left(\int_a^b |f(\tau)|^p\,\Diamond\tau\right)^{\frac{1}{p}}} \frac{|g(t)|}{\left(\int_a^b |g(\tau)|^q\,\Diamond\tau\right)^{\frac{1}{q}}}\Diamond t \\
&= \int_a^b \left(\lambda(t)^{\frac{1}{p}}\right)\left(\gamma(t)^{\frac{1}{q}}\right)\Diamond t \\
&\leqslant \int_a^b \left(\frac{\lambda(t)}{p} + \frac{\gamma(t)}{q}\right)\Diamond t \\
&= \int_a^b \left(\frac{1}{p}\frac{|f(t)|^p}{\int_a^b |f(\tau)|^p\,\Diamond\tau} + \frac{1}{q}\frac{|g(t)|^q}{\int_a^b |g(\tau)|^q\,\Diamond\tau}\right)\Diamond t \\
&= \frac{1}{p}\int_a^b \left(\frac{|f(t)|^p}{\int_a^b |f(\tau)|^p\,\Diamond\tau}\right)\Diamond t + \frac{1}{q}\int_a^b \left(\frac{|g(t)|^q}{\int_a^b |g(\tau)|^q\,\Diamond\tau}\right)\Diamond t \\
&= \frac{1}{p} + \frac{1}{q} \\
&= 1.
\end{aligned}$$

Hence

$$\int_a^b |f(t)\, g(t)| \diamondsuit t \leqslant \left(\int_a^b |f(t)|^p \diamondsuit t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \diamondsuit t\right)^{\frac{1}{q}}.$$

□

Corollary 8.3.7 (Cauchy-Schwarz inequality for the diamond integral)**.** *If $f, g : \mathbb{T} \to \mathbb{R}$ are $\diamondsuit$-integrable on $[a, b]_{\mathbb{T}}$, then*

$$\int_a^b |f(t)\, g(t)| \diamondsuit t \leqslant \sqrt{\left(\int_a^b |f(t)|^2 \diamondsuit t\right)\left(\int_a^b |g(t)|^2 \diamondsuit t\right)}.$$

Proof. This is a particular case of Theorem 8.3.6 where $p = 2 = q$. □

Theorem 8.3.8 (Minkowski's inequality for diamond integral)**.** *If $f, g : \mathbb{T} \to \mathbb{R}$ are $\diamondsuit$-integrable on $[a, b]_{\mathbb{T}}$ and $p > 1$, then*

$$\left(\int_a^b |f(t) + g(t)|^p \diamondsuit t\right)^{\frac{1}{p}} \leqslant \left(\int_a^b |f(t)|^p \diamondsuit t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \diamondsuit t\right)^{\frac{1}{p}}.$$

Proof. If $\int_a^b |f(t) + g(t)|^p \diamondsuit t = 0$ the result is trivial. Suppose that $\int_a^b |f(t) + g(t)|^p \diamondsuit t \neq 0$. Since

$$\begin{aligned}
&\int_a^b |f(t) + g(t)|^p \diamondsuit t = \int_a^b |f(t) + g(t)|^{p-1} |f(t) + g(t)| \diamondsuit t \\
&\leqslant \int_a^b |f(t)| \, |f(t) + g(t)|^{p-1} \diamondsuit t + \int_a^b |g(t)| \, |f(t) + g(t)|^{p-1} \diamondsuit t,
\end{aligned}$$

then by Hölder's inequality (Theorem 8.3.6) with $q = \dfrac{p}{p-1}$, we obtain

$$\begin{aligned}
&\int_a^b |f(t) + g(t)|^p \diamondsuit t \\
&\leqslant \left(\int_a^b |f(t)|^p \diamondsuit t\right)^{\frac{1}{p}} \left(\int_a^b |f(t) + g(t)|^{(p-1)q} \diamondsuit t\right)^{\frac{1}{q}} \\
&\quad + \left(\int_a^b |g(t)|^p \diamondsuit t\right)^{\frac{1}{p}} \left(\int_a^b |f(t) + g(t)|^{(p-1)q} \diamondsuit t\right)^{\frac{1}{q}} \\
&= \left[\left(\int_a^b |f(t)|^p \diamondsuit t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \diamondsuit t\right)^{\frac{1}{p}}\right] \left(\int_a^b |f(t) + g(t)|^{(p-1)q} \diamondsuit t\right)^{\frac{1}{q}}.
\end{aligned}$$

Hence,

$$\int_a^b |f(t) + g(t)|^p \diamondsuit t \leqslant \left[\left(\int_a^b |f(t)|^p \diamondsuit t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \diamondsuit t\right)^{\frac{1}{p}}\right] \left(\int_a^b |f(t) + g(t)|^p \diamondsuit t\right)^{\frac{1}{q}}$$

and, dividing both sides by

$$\left(\int_a^b |f(t)+g(t)|^p \diamondsuit t\right)^{\frac{1}{q}},$$

we obtain

$$\left(\int_a^b |f(t)+g(t)|^p \diamondsuit t\right)^{\frac{1}{p}} \leqslant \left(\int_a^b |f(t)|^p \diamondsuit t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \diamondsuit t\right)^{\frac{1}{p}}.$$

□

8.4 State of the Art

The time scale theory is quite new and is under strong current research. We now summarize the references already given in the Chapter 1. For the time scale calculus we refer to [1, 2, 27, 28, 67], for the calculus of variations within the time scale setting we refer to [7, 20, 25, 55, 56, 57, 69, 86, 90, 95, 96, 100, 118], while for the diamond-α integral we suggest [13, 88, 109, 111, 112].

The results of this chapter were presented by the author at The International Meeting on Applied Mathematics in Errachidia, Morocco, April, 2012 and are published in [37].

Chapter 9

Conclusions and Future Work

The goal of this thesis was the development of a symmetric variational calculus. We studied some symmetric quantum calculus and the symmetric time scale calculus and, whenever possible, we introduced the calculus of variations within the set of study.

For a first experience on quantum calculus, we began our work on Hahn's (nonsymmetric) quantum calculus (see Chapter 4). We contributed with a necessary optimality condition of Euler–Lagrange type involving Hahn's derivatives of higher-order (Theorem 4.2.12).

For the quantum symmetric calculus, we established and proved results for the α, β-calculus, q-calculus and Hahn's calculus.

In the α, β-symmetric calculus, Chapter 5, we defined and proved some properties of the α, β-symmetric derivative and the α, β-symmetric Nörlund sum. In Section 5.3.3, we derived some mean value theorems for the α, β-symmetric derivative and we presented α, β-symmetric versions of Fermat's theorem for stationary points, Rolle's, Lagrange's, and Cauchy's mean value theorems. In Section 5.3.4 we obtained some α, β-symmetric Nörlund sum inequalities, namely α, β-symmetric versions of Hölder's, Cauchy-Scharwz's and Minkowski's integral inequalities.

In the symmetric q-calculus, Chapter 6, we were able not only to develop this calculus for any real interval, instead the quantum lattice $\{a, aq, aq^2, aq^2, \ldots\}$, but also to introduce the q-symmetric variational calculus. We proved a necessary optimality condition of Euler–Lagrange type (Theorem 6.3.5) and a sufficient optimality condition (Theorem 6.3.7) for q-symmetric variational problems. It should be noted that in the q-symmetric calculus we were able to introduce the calculus of variations because we proved the fundamental theorem for the q-symmetric calculus (Theorem 6.2.7).

The Hahn symmetric calculus, Chapter 7, is a generalization of the q-symmetric calculus. This calculus has good properties, like the fundamental theorem of Hahn's symmetric integral calculus (Theorem 7.2.14) and hence we were able to introduce the Hahn symmetric variational calculus. In Section 7.3.2 we presented necessary (Euler–Lagrange type equation) and sufficient optimality conditions for the Hahn symmetric calculus. Moreover, we were able to apply Leitmann's direct method in the Hahn symmetric variational calculus (see Section 7.3.3).

Right from the start, we wanted to define the symmetric calculus on time scales. In Chapter 8 we successfully defined a symmetric derivative on time scales and derived some of its properties. Although we did not define the symmetric integral, we defined the diamond integral, which is a refined version of the diamond-α integral. We proved a mean value theorem for the diamond integral and we proved versions of Hölder's, Cauchy-Schwarz's, and Minkowski's inequalities (see Section 8.3).

In this thesis we defined several new quantum calculus and, as in every new type of calculus, there are more questions than answers. Some possible directions for future work are:

- to develop and derive new properties for symmetric quantum derivatives and integrals, for example, to study first- and second-order equations, linear systems, and higher-order differential equations;
- to explore what good properties one gets if we study the limits of symmetric quantum calculus;
- to study optimality conditions for more general variable endpoint variational problems and isoperimetric problems;
- to extend the results on symmetric quantum variational problems for higher-order problems of the calculus of variations;
- to obtain a Legendre's necessary condition for Hahn's variational calculus and for the Hahn symmetric variational calculus.

We would like to be able to construct the symmetric integral for an arbitrary time scale. We trust that after this work we are several steps closer to a solution. However, such question remains an open problem, and it will be one direction for our future research.

This Ph.D. thesis comes to an end with the list of the author publications during his Ph.D.: [31, 32, 33, 34, 35, 36, 37].

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