Dynamical properties of quaternionic behavioral systems

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Abstract

In this paper we study behavioral systems whose trajectories are given as solutions of quaternionic difference equations. As happens in the commutative case, it turns out that quaternionic polynomial matrices play an important role in this context. Therefore we focus our attention on such matrices and derive new results concerning their Smith form. Based on these results, we obtain characterizations of system theoretic properties of quaternionic behaviors.

1 Introduction

In the eighties, J. C. Willems introduced the rather innovative behavioral approach to dynamical systems [9, 10], which essentially consists in extracting all the knowledge about a system from its *behavior*, i.e., the set of its admissible trajectories. Unlike the classical approaches, in the behavioral approach one looks at the set of trajectories without imposing any structure, that is, without speaking of inputs and outputs or of causes and effects at an early stage. This point of view does not only unify the previous approaches, fitting them within an elegant theory, but it also permits to study a larger class of dynamical systems including situations where it is not possible or desirable to make any distinction between input and output variables.

During the last two decades the importance of the noncommutative quaternion algebra has been widely recognized. In fact, using this algebra, phenomena occurring in areas such as electromagnetism, quantum physics and robotics may be described by a more compact notation that leads to a higher efficiency in computational terms [2, 4].

Systems with quaternionic signals were already investigated in the classic state-space approach [1]. Here we study quaternionic behavioral systems. As we will show, quaternionic polynomial matrices, and in particular their Smith form, play an important role in this context. Therefore, a considerable part of our work is devoted to the study of such matrices.

The structure of the paper is as follows. In Section 2, after introducing the quaternionic skew-field, we define and state some properties of quaternionic polynomials. Thereafter, in Section 3

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we give some fundamental definitions of behavioral theory, showing how to extend the usual concepts based on commutative linear algebra to the quaternionic algebra. In Section 4, we define the quaternionic Smith form and characterize the (complex) Smith form of a class of complex matrices which can be used to represent quaternionic matrices, and make its relation to the quaternionic Smith form explicit. Finally, Section 5 is devoted to the characterization of dynamical properties of quaternionic behaviors. Proofs of results which are not given in the paper can be found in [5].

2 Quaternions

The real and complex fields are here denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The set

$$\mathbb{H} = \{a + bi + cj + dk \,:\, a, b, c, d \in \mathbb{R}\},$$

where i, j, k are called imaginary units and are defined by the relations

$$i^2 = j^2 = k^2 = ijk = -1,$$

is an associative but noncommutative algebra over $\mathbb{R}$ called quaternionic skew-field. For any $\eta = a + bi + cj + dk \in \mathbb{H}$, its *conjugate* is $\overline{\eta} = a - bi - cj - dk$ and its *norm* is $|\eta| = \sqrt{\eta\overline{\eta}} = \sqrt{a^2 + b^2 + c^2 + d^2}$.

Definition 2.1. The set of *quaternionic polynomials* is defined by

$$\mathbb{H}[s] = \left\{ p(s) = \sum_{l=0}^{N} p_l s^l,\ p_l \in \mathbb{H},\ N \in \mathbb{N} \right\}.$$

Sum and product of polynomials are defined as in the commutative case with the additional rule $(as^n)(bs^m) = abs^{n+m}$, i.e., roughly speaking, s commutes with constant values.

We shall use the more general algebra $\mathbb{H}[s, s^{-1}]$ of *quaternionic Laurent polynomials*, or *L-polynomials*, i.e., polynomials with positive and negative powers of s.

To simplify the notation, we will indicate the product of polynomials $p(s)$ and $q(s)$ as $pq(s)$. We may also omit the indeterminate s and write $p \in \mathbb{H}[s]$ if no ambiguity arises.

As usual, $\mathbb{H}^{g\times r}[s]$ is the set of $g \times r$ polynomial matrices. Since each matrix $A \in \mathbb{H}^{g\times r}[s]$ may be uniquely written as $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathbb{C}^{g\times r}[s]$, an injective $\mathbb{R}$-linear map: $\mathbb{H}^{g\times r}[s] \to \mathbb{C}^{2g\times 2r}[s]$ can be defined such that

$$A \mapsto A^c = \begin{bmatrix} A_1 & A_2 \\ -\overline{A}_2 & \overline{A}_1 \end{bmatrix}. \tag{2.1}$$

The matrix A^c is called the *complex adjoint matrix* of A. In general, any complex matrix with the structure (2.1) is said to be a *complex adjoint matrix*.

A bijective $\mathbb{R}$-linear map: $\mathbb{H}^{g\times r}[s] \to \mathbb{C}^{2g\times r}[s]$ may be as well defined such that

$$A \mapsto A^{\mathbb{C}} = \begin{bmatrix} A_1 \\ -\overline{A}_2 \end{bmatrix}, \tag{2.2}$$

which, in particular, maps column vectors into column vectors.

3 Quaternionic Behavioral Systems

According to [6, Def. 1.3.1], a *dynamical system* Σ is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$, where $\mathbb{T}$ is a set called *time axis*, $\mathbb{W}$ a set called *signal space*, and $\mathcal{B}$, called the *behavior*, is a subset of $\mathbb{W}^{\mathbb{T}} = \{w : \mathbb{T} \to \mathbb{W}\}$.

Here we only consider $\mathbb{T} = \mathbb{Z}$ and $\mathbb{W} = \mathbb{H}^r$, for some $r \in \mathbb{N}$. This class of systems is called *discrete-time quaternionic systems*.

We assume that the system behavior $\mathcal{B}$ can be described by means of a matrix difference equation, i.e., the trajectories w in $\mathcal{B}$ are the solutions of an equation of the form

$$R_N w(t+N) + \cdots + R_{M+1} w(t+M+1) + R_M w(t+M) = 0,\ \forall\, t \in \mathbb{Z}, \tag{3.1}$$

where $R_p \in \mathbb{H}^{g\times r}$, $p = M, \ldots, N$, $N \geq M$, $M, N \in \mathbb{Z}$.

If we define the *shift operator* by $(\sigma^\tau w)(t) = w(t+\tau)$, for every $t, \tau \in \mathbb{Z}$, the left-hand side of equation (3.1) can be written in the more compact form

$$R(\sigma, \sigma^{-1}) w(t) = \sum_{l=M}^{N} R_l \sigma^l w(t) = \sum_{l=M}^{N} R_l w(t+l). \tag{3.2}$$

This notation reveals that $\mathcal{B}$ may be described as the kernel of the difference operator $R(\sigma, \sigma^{-1}) \in \mathbb{H}^{g\times r}[\sigma, \sigma^{-1}]$ acting on $(\mathbb{H}^r)^{\mathbb{Z}}$, i.e.,

$$\mathcal{B} = \ker R(\sigma, \sigma^{-1}) = \left\{ w \in (\mathbb{H}^r)^{\mathbb{Z}} : R(\sigma, \sigma^{-1}) w = 0 \right\}. \tag{3.3}$$

Note that if $\mathcal{B}$ is the kernel of a difference operator, it is *linear on the right*, i.e., for any $w_1, w_2 \in \mathcal{B}$ and $\alpha_1, \alpha_2 \in \mathbb{H}$, $w_1\alpha_1 + w_2\alpha_2 \in \mathcal{B}$, and *shift-invariant*, i.e., $\sigma^\tau \mathcal{B} = \mathcal{B}$ for all $\tau \in \mathbb{Z}$.

The shift operator σ commutes with any quaternionic value and this fact induces the isomorphism $\mathbb{H}[s, s^{-1}] \cong \mathbb{H}[\sigma, \sigma^{-1}]$. This suggests, as it is usual within the behavioral approach, to consider the L-polynomial matrix

$$R(s, s^{-1}) = \sum_{l=M}^{N} R_l s^l, \tag{3.4}$$

which is a *kernel representation* of the behavior (3.3), and try to relate its algebraic properties to dynamical properties of $\mathcal{B}$.

Notice that, unlike the real or complex case, there is not a unique way to define quaternionic polynomials. However, other definitions (see, e.g., [7]) are apparently useless here, while the one we chose fits well into this context.

By extending to sequences the map (2.2), we define for any behavior $\mathcal{B}$ the complex behavior $\mathcal{B}^{\mathbb{C}} = \left\{ w^{\mathbb{C}} : w \in \mathcal{B} \right\}$, where $w^{\mathbb{C}}(t) = (w(t))^{\mathbb{C}}$. $\mathcal{B}^{\mathbb{C}}$ is called the *complex form* of $\mathcal{B}$ and, as the following proposition shows, admits a kernel representation which can be derived from any kernel representation of $\mathcal{B}$.

Proposition 3.1. *Let* $R \in \mathbb{H}^{m\times n}[s, s^{-1}]$*. Then* $(\ker R(\sigma, \sigma^{-1}))^{\mathbb{C}} = \ker R^c(\sigma, \sigma^{-1})$.

Proof. Let $v \in (\ker R(\sigma, \sigma^{-1}))^{\mathbb{C}}$. Then, by definition there exists $w \in \ker R(\sigma, \sigma^{-1})$ such that $v = w^{\mathbb{C}}$. Since $Rw = 0$ then $R^c v = R^c w^{\mathbb{C}} = (Rw)^{\mathbb{C}} = 0$. Hence $v \in \ker R^c(\sigma, \sigma^{-1})$. Conversely, let $v \in \ker R^c(\sigma, \sigma^{-1})$. This uniquely determines w (see formula (2.2)) such that $v = w^{\mathbb{C}}$. Then $Rw = 0$, since $(Rw)^{\mathbb{C}} = R^c w^{\mathbb{C}} = R^c v = 0$, and so $v \in (\ker R(\sigma, \sigma^{-1}))^{\mathbb{C}}$. $\square$

It can be proved too, that if $\mathcal{B}^{\mathbb{C}} = \ker \widetilde{R}(\sigma, \sigma^{-1})$ then there exists a quaternionic matrix R such that $\mathcal{B} = \ker R(\sigma, \sigma^{-1})$. This confirms the equivalence of $\mathcal{B}$ and $\mathcal{B}^{\mathbb{C}}$, thus showing that there is no loss of generality in studying only kernel representations, since this is the standard representation of the most studied class of real and complex behaviors – i.e., the linear, shift-invariant and complete ones (see [10]).

At this point it is natural to ask what algebraic properties of a quaternionic matrix are preserved passing to its complex adjoint. In the following, *unimodular* matrices are defined analogously to the commutative case and *full row rank* (frr) matrices are L-polynomial matrices R such that for any L-polynomial row vector X, $XR = 0$ implies $X = 0$. A matrix is *full column rank* if its transpose if frr.

Lemma 3.2. *A quaternionic L-polynomial matrix* R *is frr if and only if* R^c *is frr. More generally, for every quaternionic L-polynomial matrix* R, $\operatorname{rank} R = n$ *if and only if* $\operatorname{rank} R^c = 2n$.

Proposition 3.3. *Given two quaternionic L-polynomial matrices* A *and* B*, if the equation*

$$A^c = MB^c \tag{3.5}$$

holds with a complex L-polynomial matrix M*, then there exists a quaternionic L-polynomial matrix* T *such that* $A = TB$*. Moreover, if* B *is frr then* $M = T^c$*.*

Corollary 3.4. *Let* $U \in \mathbb{H}^{r\times r}[s, s^{-1}]$*. Then* U *is unimodular if and only if* $U^c \in \mathbb{C}^{2r\times 2r}[s, s^{-1}]$ *is unimodular.*

In the sequel we investigate a fundamental equivalence relation for kernel representations.

Definition 3.5. Let $R_l \in \mathbb{H}^{g_l\times r}[s, s^{-1}]$, $l = 1, 2$. Then R_1 and R_2 are said to be *equivalent representations* if $\ker R_1(\sigma, \sigma^{-1}) = \ker R_2(\sigma, \sigma^{-1})$.

Example 3.6. Consider the following quaternionic polynomial matrices

$$R_1 = \begin{bmatrix} s & -i \\ 0 & s-k \end{bmatrix}, \quad R_2 = \begin{bmatrix} s+k & 0 \\ j & 1 \end{bmatrix}. \tag{3.6}$$

These are equivalent representations of the same behavior which, as it is easy to check, is

$$\ker R_1(\sigma, \sigma^{-1}) = \ker R_2(\sigma, \sigma^{-1}) = \left\{ w(t) = \begin{bmatrix} j \\ 1 \end{bmatrix} k^t q,\ q \in \mathbb{H} \right\}.$$

A straightforward calculation shows that $R_2 = UR_1$, where

$$U = \begin{bmatrix} 1 & -i \\ -j & s-k \end{bmatrix}$$

is an unimodular L-polynomial matrix.

We will show that, as in the real and complex case, two representations are equivalent if and only if each one is a left multiple of the other, as in the previous example. This main result is a consequence of the following statement.

Theorem 3.7. *Let R_1 and R_2 be two kernel representations of $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. Then $\mathcal{B}_1 \subseteq \mathcal{B}_2$ if and only if $XR_1 = R_2$ for some quaternionic L-polynomial matrix X.*

Proof. By Proposition 3.1,

$$\mathcal{B}_1 \subseteq \mathcal{B}_2 \Leftrightarrow \ker R_1^c(\sigma, \sigma^{-1}) \subseteq \ker R_2^c(\sigma, \sigma^{-1})$$

which, as stated in [8], holds if and only if there exists a complex matrix Y such that $YR_1^c = R_2^c$. However, from Proposition 3.3, this is equivalent to saying that $XR_1 = R_2$ for some quaternionic matrix X, thus proving the theorem. $\square$

Corollary 3.8. *Two quaternionic representations R_1 and R_2 are equivalent if and only if there exist X_1 and X_2 such that $R_1 = X_1R_2$ and $R_2 = X_2R_1$. Moreover, if both matrices are frr then $X_1 = X_2^{-1}$, i.e., X_1 and X_2 are unimodular matrices.*

Remark 3.9. Since s^l is an invertible element in $\mathbb{H}[s, s^{-1}]$, it follows that, for any $l \in \mathbb{Z}$,

$$\ker R(\sigma, \sigma^{-1}) = \ker \sigma^l R(\sigma, \sigma^{-1}).$$

As a consequence, it is always possible to choose a polynomial kernel representation of a behavior. Indeed, if $R \in \mathbb{H}^{g\times r}[s, s^{-1}]$ is a representation of $\mathcal{B}$ then, for an adequate integer $M > 0$, $s^M R(s, s^{-1}) \in \mathbb{H}^{g\times r}[s]$ is still a representation of $\mathcal{B}$. Therefore, without loss of generality, we shall always choose polynomial kernel representations.

As in the commutative case, the quaternionic Smith form plays an important role in the study of quaternionic behavioral systems, in particular in the characterization of controllability and stability. Thus, we dedicate the following section to a detailed analysis of this form.

4 Quaternionic Smith Form

The main result of this section is the characterization of the Smith form of complex adjoint matrices and its relation to the quaternionic Smith form. We assume that the reader is already familiar with the Smith form for real and complex L-polynomial matrices.

Before tackling this subject, it is necessary to state some basic, but rather surprising, properties of quaternionic polynomials.

Conjugacy is extended to quaternionic polynomials by linearity and by the rule $\overline{as^n} = \overline{a}s^n$, $\forall a \in \mathbb{H}$. With this definition, the following properties hold [5].

Proposition 4.1. *Let $p, q \in \mathbb{H}[s]$. Then*

1. $\overline{pq} = \overline{q}\,\overline{p}$.

2. $p\overline{p} = \overline{p}p \in \mathbb{R}[s]$.

3. *If* $pq \in \mathbb{R}[s]$*, then* $pq = qp$.

A polynomial d is a *divisor* of the polynomial p, $d \mid p$, if it divides p on the right and on the left, i.e., if there exist polynomials r and l such that $p = dr$ and $p = ld$. It turns out that, to define the Smith form in the quaternionic case, an even stronger concept of divisibility has to be used.

Endow the algebra $\mathbb{H}[s]$ with a similarity relation $\sim$ which induces equivalence classes

$$[q] = \left\{p \in \mathbb{H}[s] : \exists\, \alpha \in \mathbb{H},\ p(s) = \alpha q(s) \alpha^{-1}\right\}.$$

Definition 4.2. The polynomial $d \in \mathbb{H}[s]$ is a *total divisor* of $p \in \mathbb{H}[s]$ if $[d] \mid [p]$, i.e., if for any $d' \in [d]$ and $p' \in [p]$, $d' \mid p'$. The *greatest real factor* of p, $r = \operatorname{grf} p$, is the (unique) highest degree monic real factor of the polynomial p.

The concept of total divisor has been introduced long ago by Jacobson [3], but the definition given in this paper is new as well as the characterizations presented by the following proposition.

Proposition 4.3. *Let* $p, d \in \mathbb{H}[s]$*. Then the following conditions are equivalent [5]:*

1. $[d] \mid [p]$*;*

2. $d \mid \operatorname{grf} p$*;*

3. $p = dab$ *with* $da \in \mathbb{R}[s]$ *and* $a, b \in \mathbb{H}[s]$.

Factors of a polynomial p are usually related to its zeros that, also in the quaternionic case, are defined as those values $\lambda \in \mathbb{H}$ such that $p(\lambda) = 0$. Unfortunately, the relation between factors and zeros of p is not as simple as for real or complex polynomials. Indeed, if $r = pq \in \mathbb{H}[s]$, then in general $r(\lambda) \neq p(\lambda)q(\lambda)$. However, if $q(\lambda) = 0$ then $r(\lambda) = 0$ but zeros of p are not necessarily zeros of r. For example, $p(s) = (s - i)$ and $q(s) = j$ are factors of $r(s) = pq(s) = js - k$ but, while $p(i) = 0$, $r(i) = ji - k = -2k \neq 0$.

The following lemma collects some basic results about zeros of quaternionic polynomials. First, define the *minimal polynomial of the equivalence class* $[\lambda]$, $\lambda \in \mathbb{H}$, as the real polynomial

$$\Psi_{[\lambda]} = (s - \lambda)(s - \overline{\lambda}) = s^2 - 2(\operatorname{Re}\lambda)s + |\lambda|^2. \tag{4.1}$$

Lemma 4.4. *Let* $p \in \mathbb{H}[s]$*. Then*

1. $\Psi_{[\nu]} = \Psi_{[\lambda]}$ *if and only if* $\nu \sim \lambda$.

2. *If* $p(\nu) = p(\lambda) = 0$ *with* $\lambda \neq \nu \sim \lambda$ *then* $\Psi_{[\lambda]} \mid p$*. If* $\Psi_{[\lambda]} \mid p$ *then* $p(\nu) = 0$ *for every* $\nu \sim \lambda$.

3. *If* $p(\lambda) = 0$ *then* $\Psi_{[\lambda]} \mid \overline{p}p$*. If* $\Psi_{[\lambda]} \mid \overline{p}p$ *then* $p(\nu) = 0$ *for some* $\nu \sim \lambda$.

In the following this notation is used: $\operatorname{diag}(a_1, \ldots, a_n)$ is a (not necessarily square) matrix with suitable dimensions whose first elements on the main diagonal are $a_1, \ldots, a_n$ and all the other entries are zero.

Theorem 4.5. *Let $R \in \mathbb{H}^{g\times r}[s, s^{-1}]$. Then there exist L-polynomial unimodular matrices U and V such that*

$$URV = \Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_n) \in \mathbb{H}^{g\times r}[s],$$

where n is the rank of R and γ_l, $l = 1, \dots, n$, are monic polynomials such that $\gamma_l(0) \neq 0$ and $[\gamma_l] \,|\, [\gamma_{l+1}]$, $l = 1, \dots, n-1$. If $R \in \mathbb{H}^{g\times r}[s]$, hence U and V are polynomial matrices too, then it is not possible to guarantee that $\gamma_l(0) \neq 0$.

The matrix Γ introduced in Theorem 4.5 is a *quaternionic Smith form* of R. Note that, unless it is real, the quaternionic Smith form is not unique.

Before stating the main theorem about quaternionic and complex Smith forms, we give an auxiliary result. As in the commutative case, two matrices R and S are said to be equivalent if there exist unimodular matrices U and V such that $UR = SV$.

Proposition 4.6. *For all monic $q \in \mathbb{H}[s]$ there exists $p \in \mathbb{C}[s]$ such that q^c and p^c are equivalent and $\mathrm{grf}(q) = \mathrm{grf}(p)$. Furthermore, for all monic $p \in \mathbb{C}[s]$, the complex Smith form of p^c is $\mathrm{diag}(r, rc\overline{c})$, where $p = rc$ and $r = \mathrm{grf}(p)$.*

The following theorem characterizes the complex Smith form of polynomial complex adjoint matrices and gives its relation to their quaternionic Smith forms. The result is trivially generalized to L-polynomial matrices.

Theorem 4.7.

1. *A polynomial matrix*

$$\Delta = \mathrm{diag}(\delta_1, \delta_1', \dots, \delta_n, \delta_n') \in \mathbb{C}^{2g\times 2r}[s],$$

is the complex Smith form of the complex adjoint matrix R^c, for some $R \in \mathbb{H}^{g\times r}[s]$, if and only if it is a real matrix, $\delta_1|\delta_1'|\cdots|\delta_n|\delta_n'$ and, for every $l = 1, \dots, n$, δ_l, δ_l' are monic polynomials which share exactly the same real zeros.

2. *If $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_m) \in \mathbb{H}^{g\times r}[s]$ is a quaternionic Smith form of R, then $m = n$ and, for every $l = 1, \dots, n$,*

$$\delta_l = \mathrm{grf}(\gamma_l) \text{ and } \gamma_l\overline{\gamma}_l = \delta_l\delta_l'.$$

Proof. 1. "If" part. It follows from the hypothesis that there exist complex polynomials c_l, with no real zeros, such that $\delta_l' = \delta_l c_l \overline{c}_l$,. Therefore, since $\delta_l = \mathrm{grf}(\delta_l c_l)$, $\mathrm{diag}(\delta_l, \delta_l') = \mathrm{diag}(\delta_l, \delta_l c_l \overline{c}_l)$ is equivalent to $\mathrm{diag}(\delta_l c_l, \overline{\delta_l c_l})$ by Proposition 4.6. Hence, Δ is equivalent to

$$\mathrm{diag}(\delta_1 c_1, \overline{\delta_1 c_1}, \dots, \delta_n c_n, \overline{\delta_n c_n}) \in \mathbb{C}^{2g\times 2r}[s],$$

which, in turn, is equivalent to the complex adjoint matrix, R^c, of

$$R = \mathrm{diag}(\delta_1 c_1, \dots, \delta_n c_n) \in \mathbb{H}^{g\times r}[s].$$

"Only if" part. Let Δ be the complex Smith form of R^c. Suppose that $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_n) \in \mathbb{H}^{g\times r}[s]$ is a quaternionic Smith form of R. By Lemma 3.2 it is clear that $m = n$. Let $\gamma_l = r_l d_l$, where $r_l = \text{grf}(\gamma_l)$. By Proposition 4.6, there exists $c_l \in \mathbb{C}[s]$ with no real zeros such that γ_l^c is equivalent to $\text{diag}(r_l, r_l c_l \overline{c}_l)$ and consequently, Γ^c is equivalent to

$$\Delta' = \text{diag}(r_1, r_1 c_1 \overline{c}_1, \dots, r_n, r_n c_n \overline{c}_n). \tag{4.2}$$

Next we show that Δ' is the complex Smith form of R^c, and hence $\Delta = \Delta'$. Since Δ' is equivalent to R^c, we only need to show that it satisfies the required division properties. Obviously, $r_l \mid r_l c_l \overline{c}_l,\ l = 1, \dots, n$.

We will prove that $r_l c_l \overline{c}_l \mid r_{l+1}$. By Proposition 4.3 we know that

$$\gamma_{l+1} = ab\gamma_l,\ b\gamma_l \in \mathbb{R}[s],\ a, b \in \mathbb{H}[s]. \tag{4.3}$$

The fact that $\gamma_l = r_l d_l$ divides $b\gamma_l \in \mathbb{R}[s]$ implies that also the least real multiple of γ_l, i.e., $r_l d_l \overline{d}_l$, is a factor of $b\gamma_l$, and hence, by (4.3), a factor of γ_{l+1}. Note that $a \mid b \Rightarrow \text{grf}(a) \mid \text{grf}(b)$ and therefore we have that $r_l d_l \overline{d}_l \mid \text{grf}(\gamma_{l+1}) = r_{l+1}$. However, by Proposition 4.6, we know that the matrices γ_l^c and $(r_l c_l)^c$ are similar and must have the same determinant

$$r_l^2 d_l \overline{d}_l = r_l^2 c_l \overline{c}_l, \tag{4.4}$$

and thus $r_l c_l \overline{c}_l = r_l d_l \overline{d}_l \mid r_{l+1}$. Therefore, $\Delta = \Delta'$, i.e., $\delta_l = r_l$ and $\delta_l' = r_l c_l \overline{c}_l$, $l = 1, \dots, n$, and consequently $\delta_1 | \delta_1' | \cdots | \delta_n | \delta_n'$. It is obvious that Δ is a real matrix. Moreover, since the polynomials c_l have no real zeros, we have that δ_l and δ_l' do have the same real zeros.

2. In the previous point we have seen that $m = n$, and $\delta_l = r_l = \text{grf}(\gamma_l)$. Finally, note that equation (4.4) states exactly that $\delta_l \delta_l' = \gamma_l \overline{\gamma}_l$. □

Remark 4.8. Since the complex Smith form is unique, it follows from Theorem 4.7 that if

$$\Gamma = \text{diag}(\gamma_1, \dots, \gamma_m) \quad \text{and} \quad \Gamma' = \text{diag}(\gamma_1', \dots, \gamma_m')$$

are quaternionic Smith forms of a quaternionic matrix R, then $\gamma_l \overline{\gamma}_l = \gamma_l' \overline{\gamma'}_l$, $l = 1, \dots, m$.

However, the reciprocal fact is not true. For instance, let $\gamma = s^2 + 1$ and $\gamma' = (s + i)(s + j)$. It is easily checked that $\gamma\overline{\gamma} = \gamma'\overline{\gamma'} = (s^2 + 1)^2$ but, since $\gamma \not\sim \gamma'$, they are not equivalent and cannot be quaternionic Smith forms of the same polynomial.

5 Dynamical properties of quaternionic behaviors

Being isomorphic, $\mathcal{B}$ and $\mathcal{B}^{\mathbb{C}}$ share the same dynamical properties (the definitions for real or complex systems may be found in [6]). Therefore it is possible to study $\mathcal{B}$ using a representation of $\mathcal{B}^{\mathbb{C}}$ at the cost of an increased size and, consequently, of a lower computational efficiency.

In this section it is shown how basic but fundamental dynamical properties of a quaternionic behavior can be characterized in terms of its kernel representations.

Autonomy

We start by introducing the concept of autonomous behaviors, i.e., the ones whose trajectories are completely determined once their ‘past’ is known.

Definition 5.1. A behavior $\mathcal{B}$ is called *autonomous* if for all $w_1, w_2 \in \mathcal{B}$

$$w_1(t) = w_2(t) \text{ for } t < 0 \;\Rightarrow\; w_1 \equiv w_2.$$

Clearly, if $\mathcal{B}$ is a linear behavior then $\mathcal{B}$ is autonomous if and only if $w(t) = 0,\ t < 0$ implies that $w(t) = 0$ for every t. As in the commutative case the following proposition holds.

Proposition 5.2. *Let $R \in \mathbb{H}^{g\times r}[s]$ and $\mathcal{B} = \ker R(\sigma)$. Then these conditions are equivalent:*

(i) *$\mathcal{B}$ is autonomous;*

(ii) *R is full column rank;*

(iii) *$\mathcal{B}$ is a finite dimensional vector space.*

Controllability

The ‘opposite’ of autonomous behaviors are the controllable ones in which it is possible to switch freely from one to another of its trajectories in finite time.

Definition 5.3. A behavior $\mathcal{B}$ of a time-invariant dynamical system is called *controllable* if for any two trajectories $w_1, w_2 \in \mathcal{B}$, and any time instant t_1, there exists $t_2 > t_1$ and a trajectory $w \in \mathcal{B}$ such that

$$w(t) = \begin{cases} w_1(t), & t \leq t_1; \\ w_2(t), & t \geq t_2. \end{cases} \tag{5.1}$$

When property (5.1) holds, w_1 and w_2 are said to be *concatenable* in $\mathcal{B}$. Therefore $\mathcal{B}$ is controllable if all its trajectories are concatenable in $\mathcal{B}$.

In the commutative case there are many characterizations of controllability. Some of them still hold in the quaternionic case and are collected in the following proposition. We recall that a matrix is left prime if it admits only unimodular left factors.

Proposition 5.4. *Let $R \in \mathbb{H}^{g\times r}[s]$ be frr and $\mathcal{B} = \ker R(\sigma)$. Then the following conditions are equivalent:*

(i) *$\mathcal{B}$ is controllable;*

(ii) *R is left prime;*

(iii) *the quaternionic Smith form of R is $\begin{bmatrix} I & 0 \end{bmatrix}$;*

(iv) *there exists an image representation, i.e., $\exists M \in \mathbb{H}^{r\times m}[s]$ such that $\mathcal{B} = \operatorname{Im} M(\sigma)$.*

However, the most well-known characterization of controllability, which corresponds to the Hautus criterion for state-space models, does not hold in the quaternionic case. Namely, even if $\ker R(\sigma)$ is controllable, the rank of $R(\lambda)$ may depend on $0 \neq \lambda \in \mathbb{H}$.

For instance, any unimodular matrix U is a kernel representation of the (trivially) controllable behavior $\mathcal{B} = \{0\}$ but $U(\lambda)$ is not necessarily invertible for all $0 \neq \lambda \in \mathbb{H}$. Let, for example,

$$U = \begin{bmatrix} -is + k & js \\ -i & j \end{bmatrix} \text{ and } V = \begin{bmatrix} -k & ks \\ 1 & -s - j \end{bmatrix}.$$

Since $UV = I$, U and V are unimodular matrices. However, $U\left(\frac{1}{2}j\right)$ is not invertible. Indeed,

$$U\left(\tfrac{1}{2}j\right)\begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} \frac{1}{2}k & -\frac{1}{2} \\ -i & j \end{bmatrix}\begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

As in the commutative case every behavior can be decomposed into a (unique) controllable and an autonomous part.

Theorem 5.5. *Every quaternionic behavior $\mathcal{B}$ contains a unique controllable subbehavior $\mathcal{B}_c$ and in any decomposition*

$$\mathcal{B} = \mathcal{B}_c \oplus \mathcal{B}_a,$$

$\mathcal{B}_a$ is an autonomous subbehavior of $\mathcal{B}$.

Stabilizability

A property which is weaker than controllability is stabilizability. In a stabilizable behavior, instead of switching, we may steer asymptotically, i.e., in infinite time, from one trajectory to any other within the behavior.

Definition 5.6. A dynamical system with behavior $\mathcal{B}$ is called *stabilizable* if for every trajectory $w \in \mathcal{B}$,there exists a trajectory $w' \in \mathcal{B}$ such that

$$w'(t) = w(t)\ ,\ t < 0 \quad \text{and} \quad \lim_{t\to+\infty} w'(t) = 0.$$

The characterization of stabilizability for a complex behavior $\mathcal{B} \subseteq (\mathbb{C}^r)^{\mathbb{Z}}$ is given by the next result, which is the discrete version of [6, Thm. 5.2.30].

Theorem 5.7. *Let $\mathcal{B}$ be a complex behavior with kernel representation $R \in \mathbb{C}^{g\times r}[s]$. Then $\mathcal{B}$ is stabilizable if and only if $\operatorname{rank} R(\lambda)$ is constant for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geq 1$.*

For quaternionic behaviors the following result holds.

Theorem 5.8. *Let $\mathcal{B}$ be a quaternionic behavior with kernel representation $R \in \mathbb{H}^{g\times r}[s]$ and let $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$ be a quaternionic Smith form of R. Then*

$$\mathcal{B} \textit{ is stabilizable } \Leftrightarrow \gamma_n(\lambda) = 0 \Rightarrow |\lambda| < 1,\ \lambda \in \mathbb{H}.$$

Proof. As we mentioned, $\mathcal{B}$ is stabilizable if and only if $\mathcal{B}^{\mathbb{C}}$ is stabilizable and so, to check this property, we may analyze the complex Smith form of R^c, $\Delta = \mathrm{diag}(\delta_1, \delta_1' \dots, \delta_n, \delta_n') \in \mathbb{R}^{2g\times 2r}[s]$. Since R^c and Δ are equivalent, by Theorem 5.7 $\mathcal{B}$ is stable if and only if $\delta_n'(\mu) = 0$ with $\mu \in \mathbb{C} \Rightarrow |\mu| < 1$.

We first show that this is equivalent to $\delta_n'(\lambda) = 0$ with $\lambda \in \mathbb{H} \Rightarrow |\lambda| < 1$. One implication is obvious. On the other side, let $\lambda \in \mathbb{H} \setminus \mathbb{C}$ be such that $\delta_n'(\lambda) = 0$. By Lemma 4.4.1 and the definition (4.1) of $\Psi_{[\lambda]}$, it follows that there exists $\mu \in [\lambda] \cap \mathbb{C}$ and that $|\mu| = |\lambda|$. Since $\delta_n' \in \mathbb{R}[s]$, also $\delta_n'(\overline{\lambda}) = 0$ and, since $\lambda \neq \overline{\lambda} \sim \lambda$, by Lemma 4.4.2 it follows that $\delta_n'(\mu) = 0$ too and therefore $|\lambda| = |\mu| < 1$.

Now we just need to show that

$$\delta_n'(\nu) = 0 \text{ with } \nu \in \mathbb{H} \Rightarrow |\nu| < 1 \ \Leftrightarrow \ \gamma_n(\lambda) = 0 \text{ with } \lambda \in \mathbb{H} \Rightarrow |\lambda| < 1.$$

Recall that by Theorem 4.7 he have

$$\gamma_n \overline{\gamma}_n = \delta_n \delta_n'. \tag{5.2}$$

"$\Rightarrow$" Let $\lambda \in \mathbb{H}$ be such that $\gamma_n(\lambda) = 0$. By Lemma 4.4.3 we have that $\gamma_n \overline{\gamma}_n(\lambda) = 0$ which by (5.2) implies that $\delta_n \delta_n'(\lambda) = 0$. As $\delta_l, \delta_l' \in \mathbb{R}[s]$ for any l, then $\delta_n \delta_n'(\lambda) = \delta_n(\lambda)\delta_n'(\lambda)$ and thus $\delta_n(\lambda) = 0$ or $\delta_n'(\lambda) = 0$. Eventually, since $\delta_n \mid \delta_n'$, it must be $\delta_n'(\lambda) = 0$ and, by hypothesis, $|\lambda| < 1$.

"$\Leftarrow$" Let $\nu \in \mathbb{H}$ be such that $\delta_n'(\nu) = 0$. This implies that $\delta_n \delta_n'(\nu) = 0$ and by (5.2) we have that $\gamma_n \overline{\gamma}_n(\nu) = 0$. The same equation says that $\gamma_n \overline{\gamma}_n \in \mathbb{R}[s]$ and therefore, as it was shown in the first part of the proof, $\Psi_{[\nu]} \mid \gamma_n \overline{\gamma}_n$. By Lemma 4.4.3 there exists $\lambda \sim \nu$ such that $\gamma_n(\lambda) = 0$, and since $|\nu| = |\lambda| < 1$ the statement is proved. $\square$

Stability

Stability is a rather important property of dynamical systems. Roughly speaking, a dynamical system is said to be stable if small perturbations produce small effects.

Definition 5.9. A dynamical system with behavior $\mathcal{B}$ is *(asymptotically) stable* if for every trajectory $w \in \mathcal{B}$, $\lim_{t\to+\infty} w(t) = 0$.

As for stabilizability the following result holds. Note that the only difference is that in this case the behavior is autonomous, i.e., the representation matrix is full column rank.

Theorem 5.10. *Let $\mathcal{B}$ a quaternionic behavior with full column rank kernel representation $R \in \mathbb{H}^{g\times r}[s]$ and let $\Gamma = \mathrm{diag}(\gamma_1, \dots, \gamma_r)$ be a quaternionic Smith form of R. Then $\mathcal{B}$ is stable if and only if*

$$\gamma_r(\lambda) = 0 \textit{ with } \lambda \in \mathbb{H} \ \Rightarrow \ |\lambda| < 1.$$

Observability

Another dynamical property of a behavior is observability, which expresses the possibility of obtaining information concerning some components of a trajectory by observing the values of the other ones.

Definition 5.11. Let $\Sigma = (\mathbb{T}, \mathbb{W}, \mathcal{B})$ be a time-invariant dynamical system and suppose that the trajectories in $\mathcal{B}$ are partitioned as $w = (w_1, w_2)$. We say that w_2 is *observable* from w_1 if $(w_1, w_2), (w_1, w_2') \in \mathcal{B}$ implies that $w_2 = w_2'$.

Clearly, for linear behaviors $\mathcal{B}$, w_2 is observable from w_1 if and only if $(0, w_2) \in \mathcal{B}$ implies that $w_2 = 0$. In particular, if $\mathcal{B}$ is given as $R_1(\sigma)w_1 = R_2(\sigma)w_2$, then w_2 is observable from w_1 if and only if $\ker R_2(\sigma) = \{0\}$.

The following theorem characterizes observability. The proof is analogous to the commutative case [6].

Theorem 5.12. *Let $R_1 \in \mathbb{H}^{g \times r_1}[s]$ and let $R_2 \in \mathbb{H}^{g \times r_2}[s]$. Let $\mathcal{B}$ be the behavior defined by $R_1(\sigma)w_1 = R_2(\sigma)w_2$. Then the following conditions are equivalent:*

(i) w_2 *is observable from* w_1*;*

(ii) R_2 *is right prime;*

(iii) *the Smith form of* R_2 *is* $\begin{bmatrix} I \\ 0 \end{bmatrix}$.

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