Stability of Quaternionic Linear Systems

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***Abstract*— The main goal of this paper is to characterize stability and BIBO-stability of quaternionic dynamical systems. After defining the quaternion skew-field, algebraic properties of quaternionic polynomials such as divisibility and coprimeness are investigated. Having established these results, the Smith and the Smith-McMillan forms of quaternionic matrices are introduced and studied. Finally, all the tools that were developed are used to analyze stability of quaternionic linear systems in a behavioral framework.**

***Index Terms*— Stability, Quaternions, Behaviors**

I. Introduction)

This paper deals with stability, which is a very common issue in many areas of applied mathematics. In particular, for input/output dynamical control systems, it will focus on BIBO (bounded input-bounded output) stability which is especially important for control systems in the presence of disturbances: roughly speaking, it ensures that small perturbations in the control do not cause diverging errors in the output.

The systems which are here considered take values in the quaternion skew-field $\mathbb{H}$, that was discovered by Sir Rowan Hamilton in 1843. These *hypercomplex* numbers may be favorably used to describe phenomena occurring in areas such as electromagnetism and quantum physics [1] by means of a compact notation that leads to a higher efficiency in computational terms [2].

In particular, they are a powerful tool in the description of rotations. Indeed, by identifying $\mathbb{R}^3$ with a subset of $\mathbb{H}$, the expression qvq^{-1} represents the rotation of a vector $v \in \mathbb{R}^3$ by an angle and about a direction that are specified by $q \in \mathbb{H}$ (see, e.g., [3]). It is not uncommon to find situations, especially in robotics, where the rotation of a rigid body depends on time, and this dynamics is advantageously written in terms of quaternionic differential or difference equations. The effort to control the rotation dynamics motivates the study of these equations from a system theoretic point of view (see, for instance, [4]).

In general, a dynamical system represented by four units interconnected as in Fig. 1 presents a "quaternionic symmetry" [5] and can be modeled by differential equations with quaternionic coefficients. Such a system is a generalization of the twin-lift problem [6].

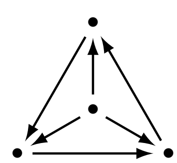

Fig. 1. Dynamical system with quaternionic symmetry

In the context of quantum mechanics, a possible quaternionic formulation of the Schrödinger equation has been proposed since the sixties as well as experiments to check the existence of quaternionic potentials (see, for instance, [7]). This theory leads to differential equations with quaternionic coefficients [8] which are the subject of this paper.

Here, the behavioral approach to dynamical systems is adopted, which was introduced by J. C. Willems in the eighties [9]. It essentially consists in extracting all the knowledge about a dynamical system from the set of admissible trajectories, called *behavior*. In this context, state-space models or input/output structures are not to be considered to be present in a first instance but rather identified later from the analysis of the system behavior.

Within this framework, usual techniques to check stability are based on determinants of polynomial matrices. Since there is not a unique definition of determinant for quaternionic polynomial matrices, another characterization, which uses Smith and Smith-McMillan forms of matrices, will be generalized to the quaternionic case. To do this, many new algebraic tools have to be introduced. In particular, quaternionic polynomials will be thoroughly investigated along with their properties regarding divisibility and coprimeness.

Original results are stated, which concern the relation between a quaternionic polynomial or rational matrix and its complex adjoint.

The structure of the paper is the following. After introducing quaternions and quaternionic polynomials in Section II, quaternionic behaviors are defined in Section III. Then, in Section IV, the properties of quaternionic Smith and Smith-McMillan forms are studied. Finally, Section V deals with the characterization of stability and BIBO-stability of quaternionic dynamical systems.

II. Quaternions

A. Quaternionic skew field

The set $\mathbb{H} = \{a + bi + cj + dk : \, a, b, c, d \in \mathbb{R}\}$, where the *imaginary units* i, j and k commute with real numbers and satisfy $i = jk = -kj$, $j = ki = -ik$ and $k = ij = -ji$, is an associative but noncommutative division algebra over $\mathbb{R}$ called quaternionic skew field. Given $\eta = a + bi + cj + dk \in \mathbb{H}$, its real part is $\operatorname{Re}\eta = a$, its *conjugate* is $\overline{\eta} = a - bi - cj - dk$, and its *norm* is $|\eta| = \sqrt{\eta\overline{\eta}} = \sqrt{a^2 + b^2 + c^2 + d^2}$. The usual Euclidean norm is used for vectors. The complex field $\mathbb{C}$ is identified with $\{a + bi : \, a, b \in \mathbb{R}\} \subseteq \mathbb{H}$.

Being a multiplicative group, quaternions can be partitioned into conjugacy classes $[\nu] = \{\alpha\nu\alpha^{-1} : \, 0 \neq \alpha \in \mathbb{H}\}$. Two quaternions $\eta, \nu \in \mathbb{H}$ are *conjugated* if $[\eta] = [\nu]$ (or $\eta \in [\nu]$). Note that $\overline{\nu} \in [\nu]$, as a consequence of the following theorem (see [10]).

Theorem 2.1: Given two quaternions $\eta, \nu \in \mathbb{H}$, $\eta \in [\nu]$ if and only if $\operatorname{Re}\eta = \operatorname{Re}\nu$ and $|\eta| = |\nu|$. Thus, $[\lambda] \cap \mathbb{C} \neq \emptyset$, $\forall \lambda \in \mathbb{H}$.

B. Quaternionic polynomials

The set of *quaternionic polynomials* is defined by

$$\mathbb{H}[s] = \left\{ p(s) = \sum_{l=0}^{N} p_l s^l, \; p_l \in \mathbb{H}, \; N \in \mathbb{N} \right\}.$$

Sum and product of polynomials are defined as in the commutative case with the additional rule $(as^n)(bs^m) = abs^{n+m}$, as if the indeterminate *commuted* with constant values. To simplify the notation, we omit the indeterminate and write $p \in \mathbb{H}[s]$ instead of $p(s)$, if no ambiguity arises.

Conjugacy is extended to quaternionic polynomials by linearity and by the rule $\overline{as^n} = \overline{a}s^n$, $\forall a \in \mathbb{H}$. As a consequence, $\overline{pq} = \overline{q}\,\overline{p}$ for every $p, q \in \mathbb{H}[s]$ (see [11]).

The relation between degree, zeros and factors of polynomials in $\mathbb{H}[s]$ is not straightforward. We recall here some basic facts and address the interested reader to [12, §16] for a more detailed exposition.

Zeros of polynomials are only related to right factors: $\lambda \in \mathbb{H}$ is a zero of $p \in \mathbb{H}[s]$ if $p(\lambda) = 0$ or, equivalently, if $s - \lambda$ is a right divisor of p. A pair $(p, q) \in \mathbb{H}[s]^2$ is zero coprime (or right coprime) if p and q do not have common zeros. The degree of right factors of p whose unique zero is λ can vary from 1 to a maximum value $\mu_\lambda(p)$,

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which is the multiplicity of λ as a zero of p. As usual, $\mu_\lambda(p) = 0$ when $p(\lambda) \neq 0$. Furthermore, if $p(\lambda) = 0$, for every $q \in \mathbb{H}[s]$ there exists $\nu \in [\lambda]$ which is a zero of pq.

Every real irreducible monic polynomial is the minimal polynomial of a conjugacy class $[\lambda]$ and is denoted by $\psi_{[\lambda]} \in \mathbb{R}[s]$. So, by definition, $\psi_{[\lambda]}(\nu) = 0$ if and only if $\nu \in [\lambda]$. If $\lambda \in \mathbb{R}$ then $\psi_{[\lambda]}(s) = s - \lambda$, otherwise $\psi_{[\lambda]}(s) = (s-\overline{\lambda})(s-\lambda) = s^2 - 2\operatorname{Re}\lambda + |\lambda|^2$ (cfr. Theorem 2.1). Moreover, if $p(\lambda) = p(\nu) = 0$, with $[\nu] = [\lambda]$ but $\nu \neq \lambda$, then p is a multiple of $\psi_{[\lambda]}$.

Example 2.2: The polynomial $\psi_{[i]}(s) = s^2+1$ has infinitely many zeros, $\lambda \in [i]$, all with multiplicity one. Conversely, both $(s-j)(s-i)$ and $(s-k)(s-i)$ have a *unique* zero, $\lambda = i$, with multiplicity two.

C. Quaternionic rational functions

Since the polynomial ring $\mathbb{H}[s]$ is both a right and a left Ore ring [13], the field of left and right fractions of $\mathbb{H}[s]$ can be properly defined. In this paper, only left quaternionic rational functions are used: $\mathbb{H}(s) = \{p^{-1}q : \ p,q \in \mathbb{H}[s],\ p \neq 0\}$. For the sake of simplicity, the fraction form is used to indicate elements of $\mathbb{H}(s)$: $\frac{q}{p} = p^{-1}q$.

In the fraction $\frac{q}{p}$, only common left divisors of p and q can be *simplified* and so the fraction is irreducible if and only if (p,q) are left coprime, i.e., if p and q only have trivial (constant) common left divisors.

D. Quaternionic matrices

As usual, $\mathbb{H}^{g\times n}[s]$ and $\mathbb{H}^{g\times n}(s)$ denote the sets of $g \times n$ polynomial and rational matrices, respectively. The notion of left (right) coprimeness is defined as in the commutative case: (P,Q) are left (right) coprime matrices if and only if every common left (right) common factor of P and Q is unimodular, i.e., admits a polynomial inverse.

Any matrix $A \in \mathbb{H}^{g\times n}(s)$ may be uniquely written as $A = A_1 + A_2 j$, where $A_1, A_2 \in \mathbb{C}^{g\times n}(s)$. Thus an injective homomorphism of real algebras: $\mathbb{H}^{g\times n}(s) \to \mathbb{C}^{2g\times 2n}(s)$ can be defined such that

$$A \mapsto A^c = \begin{bmatrix} A_1 & A_2 \\ -\overline{A}_2 & \overline{A}_1 \end{bmatrix}. \tag{1}$$

The matrix A^c is called the *complex adjoint matrix* of A. In general, any complex matrix with the structure (1) is said to be a *complex adjoint matrix*. A bijective $\mathbb{R}$-linear map: $\mathbb{H}^{g\times n}(s) \to \mathbb{C}^{2g\times n}(s)$ may be as well defined by

$$A \mapsto A^{\mathbb{C}} = \begin{bmatrix} A_1 \\ -\overline{A}_2 \end{bmatrix}. \tag{2}$$

This is an isometry of the vector spaces $\mathbb{H}^g$ and $\mathbb{C}^{2g}$, i.e., $||v|| = ||v^{\mathbb{C}}||$, $\forall v \in \mathbb{H}^g$.

E. Further results on divisibility

We write $p|q$ if $p \in \mathbb{H}[s]$ is both a right and a left divisor of $q \in \mathbb{H}[s]$. A stronger divisibility property, that will be now defined and characterized (see also [14]), is essential to the construction of the Smith and Smith-McMillan forms.

Definition 2.3: $p \in \mathbb{H}[s]$ is a *total divisor* of $q \in \mathbb{H}[s]$, denoted by $p\|q$, if p is both a left and a right divisor of *every* multiple of q.

In [11] an equivalent definition is given and it is proved that $p\|q$ if and only if there exist $x \in \mathbb{R}[s]$ such that $p|x$ and $x|q$. This will be stated in a more compact way in Proposition 2.6 with the following notation.

Definition 2.4: Given $p \in \mathbb{H}[s]$, $p_\bullet \in \mathbb{R}[s]$ is the greatest monic real factor of p and $p^\bullet \in \mathbb{R}[s]$ is the least monic real multiple of p. We also denote by $p_\circ \in \mathbb{H}[s]$ the polynomial such that $p = p_\bullet p_\circ$.

Remark 2.5: If p is monic, then $p^\bullet = p\overline{p}_\circ = p_\bullet p_\circ \overline{p}_\circ$.

Proposition 2.6: If $p,q \in \mathbb{H}[s]$, $p\|q \Leftrightarrow p|q_\bullet \Leftrightarrow p^\bullet|q_\bullet \Leftrightarrow p^\bullet|q$.

More properties of the polynomials $p_\bullet$, $p^\bullet$, and $p_\circ$ are stated in the following propositions.

Proposition 2.7: Given two quaternionic polynomials $p,q \in \mathbb{H}[s]$,
1) $(pq)_\bullet = p_\bullet q_\bullet \Leftrightarrow (pq)^\bullet = p^\bullet q^\bullet \Leftrightarrow (\overline{p}_\circ, q_\circ)$ are left coprime;
2) $pq \in \mathbb{R}[s] \Leftrightarrow \overline{p}_\circ = q_\circ$.

Proof: 1) The first two conditions are easily proved to be equivalent to $(p_\circ q_\circ)_\bullet = 1$, and so we only show that this one is equivalent to left coprimeness of $(\overline{p}_\circ, q_\circ)$. If $\overline{p}_\circ$ and $q_\circ$ have a non-trivial left common factor, say x, then $p_\circ q_\circ$ is a multiple of the real factor $\overline{x}x$, which proves one implication.

On the other hand, suppose that $\psi_{[\lambda]}|p_\circ q_\circ$. By definition, $\lambda \notin \mathbb{R}$ and if by contradiction $p_\circ(\nu) \neq 0$ for any $\nu \in [\lambda]$ then it would be $\psi_{[\lambda]}|q_\circ$, which is impossible by definition of $q_\circ$. Therefore, $p_\circ = ax$ with $x(s) = s - \nu$ for some $\nu \in [\lambda]$ and $a \in \mathbb{H}[s]$. By the division algorithm, there exist $y \in \mathbb{H}[s]$ and $\eta \in \mathbb{H}$ such that $\overline{q}_\circ = yx + \eta$ and therefore $p_\circ q_\circ = ax\overline{x}\,\overline{y} + p_\circ\overline{\eta} = a\overline{y}\psi_{[\lambda]} + p_\circ\overline{\eta}$. So, $\psi_{[\lambda]}|p_\circ\overline{\eta}$ which is possible if and only if $\eta = 0$. Therefore, x is a common left factor of $p_\circ$ and $\overline{q}_\circ$.

2) The implication '$\Leftarrow$' is trivial. Now, let x be the greatest monic left common factor of $\overline{p}_\circ$ and $q_\circ$ such that $\overline{p}_\circ = xa$ and $q_\circ = xb$ where (a,b) are left coprime. By the first part, $(\overline{a}b)_\bullet = 1$ and so $pq = (pq)_\bullet = (p_\bullet \overline{a}\,\overline{x} q_\bullet xb)_\bullet = p_\bullet \overline{x} q_\bullet x$, i.e., $a = b = 1$, and the result follows. ∎

Corollary 2.8: If p and q are monic quaternionic polynomials such that $pq \in \mathbb{R}[s]$, then $pq = qp = p_\bullet q^\bullet = p^\bullet q_\bullet$.

Let us now define $\mu^\bullet_\lambda(p) = \max\{\mu_\nu(p) : \nu \in [\lambda]\}$.

Proposition 2.9: For any $p \in \mathbb{H}[s]$ and $\nu \in [\lambda]$, $\mu^\bullet_\lambda(p) = \mu_\nu(p^\bullet)$.

Proof: The fact is trivial if $\lambda \in \mathbb{R}$. If it is not, suppose, without loss of generality, that p is monic and let $\mu = \mu^\bullet_\lambda(p)$. By definition of $\mu^\bullet$, p has a right factor γ such that $\gamma\overline{\gamma} = \psi^\mu_{[\lambda]}$ which can be decomposed as $\gamma = \alpha\beta$ where $p_\bullet = a\alpha\overline{\alpha}$ and $p_\circ = b\beta$, for some $a \in \mathbb{R}[s]$ and $b \in \mathbb{H}[s]$. Note that a and b cannot have zeros in $[\lambda]$. By Remark 2.5, $p^\bullet = a\alpha\overline{\alpha}b\beta\overline{\beta}\,\overline{b} = ab\overline{b}\alpha\beta\overline{\beta}\,\overline{\alpha} = ab\overline{b}\psi^\mu_{[\lambda]}$, and so $\mu_\nu(p^\bullet) = \mu_\nu(ab\overline{b}\psi^\mu_{[\lambda]}) = \mu_\nu(\psi^\mu_{[\lambda]}) = \mu$ for any $\nu \in [\lambda]$. ∎

III. Quaternionic behaviors

Following [15] (which contains a much more detailed exposition of the concepts that are here just outlined), a *behavior* $\mathcal{B}$ is a set of functions, called *trajectories*, having the same domain $\mathbb{T}$, called *time set*, and the same codomain W, i.e., $\mathcal{B} \subseteq W^{\mathbb{T}} = \{w : \mathbb{T} \to W\}$.

In this paper, behaviors are solution sets of linear systems of quaternionic difference or differential equations. In other words, we will deal with discrete-time systems, where $\mathbb{T} = \mathbb{Z}$ and

$$\mathcal{B} = \left\{ w : \mathbb{Z} \to \mathbb{H}^n \text{ such that } \sum_{l=M}^{N} R_l w(t+l) = 0, \forall t \in \mathbb{Z} \right\}, \tag{3}$$

and with continuous-time systems, where $\mathbb{T} = \mathbb{R}$ and

$$\mathcal{B} = \left\{ w : \mathbb{R} \to \mathbb{H}^n \text{ such that } \sum_{l=0}^{N} R_l w^{(l)}(t) = 0, \forall t \in \mathbb{R} \right\}. \tag{4}$$

The systems are time-invariant, i.e., $R_l \in \mathbb{H}^{g\times n}$ are constant matrices, and in the continuous case, where $w^{(l)}$ is the l-th order derivative of w, trajectories are supposed to be sufficiently smooth, otherwise equations have to be intended in a distributional sense (see [16]).

It is possible to treat discrete and continuous linear systems in a unified fashion by means of polynomial operators. Define, in the discrete-time case, the *backward shift operator* by $(\sigma^\tau w)(t) = w(t+\tau)$, for any $t, \tau \in \mathbb{Z}$. Then the condition defining $\mathcal{B}$ in (3) is $\sum_{l=M}^{N} R_l w(t+l) = \sum_{l=M}^{N} R_l \sigma^l w(t) = R(\sigma) w = 0$, where $R(s) = \sum_{l=M}^{N} R_l s^l \in \mathbb{H}^{g\times n}[ß1]$ is a quaternionic Laurent polynomial matrix (i.e., a polynomial with both positive and negative powers of s) acting on w as a *linear difference operator*.

For the sake of simplicity we will suppose, without loss of generality, that $R \in \mathbb{H}^{g\times n}[s]$. Indeed, by definition (3), $w \in \mathcal{B}$ if and only if $\sigma^\tau w \in \mathcal{B}$, i.e., $R(\sigma)\sigma^\tau w = 0$, for any $t \in \mathbb{Z}$. So, if we take $\tau = -M$, the behavior $\mathcal{B}$ can be equivalently defined by $R(s)s^{-M}$, which is a polynomial matrix (see also [11, Corollary 3.12]).

Analogously, if $R(s) = \sum_{l=0}^{N} R_l s^l \in \mathbb{H}^{g\times n}[s]$, the condition in (4) can be written in the operator form $R\left(\frac{d}{dt}\right) w(t) = \sum_{l=0}^{N} R_l \frac{d^l}{dt^l} w(t) = 0$.

Eventually, both in the discrete and in the continuous case, the behavior is the kernel of the operator R, $\mathcal{B} = \ker R$, where $R(\sigma)$ is a difference operator when $\mathbb{T} = \mathbb{Z}$ and $R(\frac{d}{dt})$ is a differential operator when $\mathbb{T} = \mathbb{R}$. The polynomial matrix $R(s)$ is a *kernel representation* of $\mathcal{B}$. Note that different representations may give rise to the same behavior. In particular $\ker R = \ker UR$ for any unimodular matrix U [11].

Example 3.1: Consider the equations $(\sigma - \alpha)^n w(t) = 0$ and $\left(\frac{d}{dt} - \alpha\right)^n w(t) = 0$ where $\alpha \in \mathbb{H}$. Their solutions are the kernel of operators represented by the polynomial $p(s) = (s - \alpha)^n$. It is not difficult to check that the solutions are, as in the commutative case, $w(t) = t^l \alpha^t q$ and $w(t) = t^l e^{\alpha t} q$, respectively, for every $l = 0, \ldots, n-1$ and $q \in \mathbb{H}$. However, in this case, the position of the constant q cannot be changed due to noncommutativity.

The representation of a behavior as a kernel is very general but sometimes it is possible and desirable to use other representations as, for instance, input/output (i/o) representations.

To introduce the class of i/o systems in a proper way, we need the following preliminary definition.

Definition 3.2: Let $\mathcal{B} \subseteq \left\{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] : \mathbb{T} \to \mathbb{H}^{p+m} \right\}$. Then u is an *input variable* and y is an *output variable* of $\mathcal{B}$ if

1) u is *free* in $\mathcal{B}$: $\forall u \in (\mathbb{H}^m)^{\mathbb{T}}$, $\exists y \in (\mathbb{H}^p)^{\mathbb{T}}$ such that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B}$;
2) once u is fixed, no component of y is free in $\{y : \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B}\}$.

The notation $\mathcal{B}_{\text{i/o}}$ is used to denote behaviors which satisfy Definition 3.2. In general, $\mathcal{B}$ is an *i/o behavior* if the components of its trajectories w can be partitioned into input and output variables, i.e., a permutation of coordinates transforms it into a behavior $\mathcal{B}_{\text{i/o}}$.

A partition of any kernel representation $R = [P \ \ -Q]$ of $\mathcal{B}_{\text{i/o}}$ is naturally induced, which is made explicit by the i/o representation

$$\mathcal{B}_{\text{i/o}} = \left\{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] : \mathbb{T} \to \mathbb{H}^{p+m} \ : \ Py = Qu \right\}. \tag{5}$$

As we will show in Proposition 4.2, we may assume that P is full row rank. Therefore, since condition 2 of Definition 3.2 is equivalent to saying that P has full column rank, in the following we only consider i/o representations (5) where P is invertible over the field of rational matrices.

Remark 3.3: If P is invertible then (5) is an i/o behavior, independently of Q. Indeed, P is an surjective operator, as in the commutative case, and therefore freeness of u, i.e., condition 1 of Definition 3.2, is guaranteed.

We will only deal with proper systems, i.e., we also assume that the *transfer matrix* $P^{-1}Q$ of the behavior (5) is a proper rational matrix [9].

Definition 3.4: A dynamical system defined by the equation

$$Py = Qu, \tag{6}$$

where $P \in \mathbb{H}^{p\times p}[s]$ and $Q \in \mathbb{H}^{p\times m}[s]$, is a *(proper) quaternionic i/o system*, with behavior $\mathcal{B}_{\text{i/o}}$ defined by equation (5), if P admits a rational inverse and its transfer matrix $P^{-1}Q \in \mathbb{H}^{p\times m}(s)$ is proper.

IV. Quaternionic Smith and Smith-McMillan forms

In this section we define and characterize the Smith and Smith-McMillan forms of quaternionic polynomial and rational matrices.

The notation $\operatorname{diag}(\alpha_1, \ldots, \alpha_n)$ denotes a matrix (a_{hl}) with suitable size, not necessarily square, such that $a_{hl} = \alpha_h$ if $h = l = 1, \ldots, n$ and $a_{hl} = 0$ otherwise.

The Smith form has been already studied in [14]. The following theorem states its defining properties, using our notation.

Theorem 4.1: Let $R \in \mathbb{H}^{g\times n}[s]$. Then there exist unimodular quaternionic polynomial matrices U and V such that

$$URV = \operatorname{diag}(\gamma_1, \ldots, \gamma_r) \in \mathbb{H}^{g\times n}[s], \tag{7}$$

where r is the rank of R, γ_l are monic and $\gamma_l \| \gamma_{l+1}$ for any l.

Matrix (7) is a *quaternionic Smith form* of R. Unlike the real or complex case, it is not unique. Uniqueness can be stated in some cases, see [17].

As an application, we use now the Smith form to prove the existence of a representation of an i/o behavior which is *minimal* in the number of rows.

Proposition 4.2: If u is free in $\mathcal{B}_{\text{i/o}}$, defined as in (5), then there exist $\tilde{P}$, with full row rank, and $\tilde{Q}$ such that $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B}_{\text{i/o}} \Leftrightarrow \tilde{P}y = \tilde{Q}u$.

Proof: Let $R = [P \ \ -Q]$ and S be a full row rank matrix such that $URV = \left[\begin{smallmatrix} S \\ 0 \end{smallmatrix}\right]$ is a Smith form of R. Then $\tilde{R} = SV^{-1}$ has full row rank and $\ker \tilde{R} = \ker \left[\begin{smallmatrix} \tilde{R} \\ 0 \end{smallmatrix}\right] = \ker UR = \ker R = \mathcal{B}_{\text{i/o}}$. By partitioning suitably $\tilde{R} = [\tilde{P} \ \ -\tilde{Q}]$, the claim is proved if $\tilde{P}$ is full row rank. Let $a \in \mathbb{H}^{1\times r}[s]$ and suppose that $a\tilde{P} = 0$. For every u we can write $0 = a\tilde{P}y = a\tilde{Q}u$ and so $a\tilde{Q} = 0$. Thus $a\tilde{R} = 0$, hence $a = 0$, which concludes the proof. ■

We are now in a position to define the Smith-McMillan form of quaternionic rational matrices.

Theorem 4.3: Let $R \in \mathbb{H}^{g\times n}(s)$ with rank r. Then there exist unimodular polynomial matrices U and V such that

$$URV = \operatorname{diag}\left(\frac{\epsilon_1}{\psi_1}, \ldots, \frac{\epsilon_r}{\psi_r}\right) \in \mathbb{H}^{g\times n}(s), \tag{8}$$

where ϵ_l and ψ_l are monic polynomials that satisfy the following conditions for any l:

- the fraction $\frac{\epsilon_l}{\psi_l}$ is irreducible, i.e., (ϵ_l, ψ_l) are left coprime;
- $\epsilon_l \| \epsilon_{l+1}$ and $\psi_{l+1} \| \psi_l$.

The matrix (8) is a *quaternionic Smith-McMillan form* of R and is not unique.

Proof: Let $d \in \mathbb{R}[s]$, monic, be such that $M = dR \in \mathbb{H}^{g\times n}[s]$ is a polynomial matrix (for instance, let d be the least common real multiple of the denominators of the entries of R). By Theorem 4.1 there exist a quaternionic Smith form of M, $\operatorname{diag}(\gamma_1, \ldots, \gamma_r)$, and unimodular polynomial matrices U and V such that

$$URV = Ud^{-1}MV = d^{-1}UMV = \operatorname{diag}\left(\frac{\gamma_1}{d}, \ldots, \frac{\gamma_r}{d}\right),$$

since d, being a real polynomial, commutes with U. Therefore, by eliminating the common left factors of the fractions, we obtain the matrix (8) with irreducible fractions. We only have to show that the numerators and the denominators verify the required properties.

Let $\alpha_l \in \mathbb{H}[s]$ be the common left factor of γ_l and of d. Then we can write $\gamma_l = \alpha_l \epsilon_l$ and $d = \alpha_l \psi_l$. By Proposition 2.7.2, we obtain ${\psi_l}_\circ = \overline{\alpha_l}_\circ$ and so, since (ϵ_l, ψ_l) are left coprime, $\gamma_l^\bullet = (\alpha_l \epsilon_l)^\bullet = \alpha_l^\bullet \epsilon_l^\bullet$ by Proposition 2.7.1. Since $\gamma_l \| \gamma_{l+1}$, from Proposition 2.6 it follows that $\gamma_{l+1} = \gamma_l^\bullet \beta$ for some $\beta \in \mathbb{H}[s]$ and, by Corollary 2.8, $d = \alpha_l^\bullet {\psi_l}_\bullet$. Therefore,

$$\frac{\gamma_{l+1}}{d} = \frac{\alpha_l^\bullet \epsilon_l^\bullet \beta}{\alpha_l^\bullet {\psi_l}_\bullet} = \frac{\epsilon_l^\bullet \beta}{{\psi_l}_\bullet} = \frac{\epsilon_{l+1}}{\psi_{l+1}}.$$

Note that in the last passage simplifications can only occur between β and ${\psi_l}_\bullet$ since $(\epsilon_l^\bullet, {\psi_l}_\bullet)$ are coprime. This clearly shows that $\epsilon_l^\bullet | \epsilon_{l+1}$

and that $\psi_{l+1}|\psi_{l_\bullet}$, i.e., that the required conditions are satisfied, by Proposition 2.6. ∎

Remark 4.4: In (8), (ϵ_l, ψ_l) may not be zero coprime. For example, $\frac{j(s-i)}{s-i}$ is a Smith-McMillan form. Actually, $j(s-i)$ and $(s-i)$ are left coprime but are not zero coprime.

In [11], the special structure of the complex Smith form of any complex adjoint matrix has been investigated, leading to the following statement.

Theorem 4.5: If $\operatorname{diag}(\gamma_1, \ldots, \gamma_r) \in \mathbb{H}^{g\times n}[s]$ is a quaternionic Smith form of $R \in \mathbb{H}^{g\times n}[s]$, then the complex Smith form of its complex adjoint R^c is $\operatorname{diag}(\gamma_{1_\bullet}, \gamma_1^\bullet, \ldots, \gamma_{r_\bullet}, \gamma_r^\bullet) \in \mathbb{R}^{2g\times 2n}[s]$.

In general, if $\delta_1|\delta_1'|\cdots|\delta_r|\delta_r'$ and the polynomials are all monic,

$$\operatorname{diag}(\delta_1, \delta_1', \ldots, \delta_r, \delta_r') \in \mathbb{C}^{2g\times 2n}[s]$$

is the complex Smith form of a complex adjoint matrix if and only if, for any l, the polynomials δ_l and δ_l' are real and have the same real zeros with equal multiplicities, i.e., $\mu_\lambda(\delta_l) = \mu_\lambda(\delta_l')$, $\forall\lambda \in \mathbb{R}$.

An analogous property can be stated for Smith-McMillan forms.

Theorem 4.6: If $R \in \mathbb{H}^{g\times n}(s)$ has quaternionic Smith-McMillan form (8), then the complex Smith-McMillan form of R^c is

$$\operatorname{diag}\left(\frac{\epsilon_{1_\bullet}}{\psi_1^\bullet}, \frac{\epsilon_1^\bullet}{\psi_{1_\bullet}}, \ldots, \frac{\epsilon_{r_\bullet}}{\psi_r^\bullet}, \frac{\epsilon_r^\bullet}{\psi_{r_\bullet}}\right) \in \mathbb{R}^{2g\times 2n}(s). \tag{9}$$

If $\theta_1|\theta_1'|\cdots|\theta_r|\theta_r'$, $\omega_r'|\omega_r|\cdots|\omega_1'|\omega_1$, the polynomials are all monic and the fractions $\frac{\theta_l}{\omega_l}$ and $\frac{\theta_l'}{\omega_l'}$ are irreducible for every l, then

$$\operatorname{diag}\left(\frac{\theta_1}{\omega_1}, \frac{\theta_1'}{\omega_1'}, \ldots, \frac{\theta_r}{\omega_r}, \frac{\theta_r'}{\omega_r'}\right) \in \mathbb{C}^{2g\times 2n}(s), \tag{10}$$

is a complex Smith-McMillan form of a complex adjoint matrix if and only if it is real and, for every l, $\mu_\lambda(\theta_l) = \mu_\lambda(\theta_l')$ and $\mu_\lambda(\omega_l) = \mu_\lambda(\omega_l')$, $\forall\lambda \in \mathbb{R}$.

Proof: With the notation used in the proof of Theorem 4.3, let $\frac{\epsilon_l}{\psi_l} = \frac{\alpha_l\epsilon_l}{\alpha_l\psi_l} = \frac{\gamma_l}{d}$ where $\operatorname{diag}(\gamma_1, \ldots, \gamma_r)$ is a quaternionic Smith form and d a real monic polynomial. By Proposition 2.7 it follows that $\gamma_l^\bullet = \alpha_l^\bullet\epsilon_l^\bullet$ and that $\gamma_{l_\bullet} = \alpha_{l_\bullet}\epsilon_{l_\bullet}$ and, by Corollary 2.8, that $d = \alpha_l^\bullet\psi_{l_\bullet} = \alpha_{l_\bullet}\psi_l^\bullet$. This proves the first statement and the "only if" implication of the second one.

On the other hand, let $d \in \mathbb{R}[s]$ be the least common multiple of the denominators ω_l, ω_l' of (10) and define d_l, d_l' by the relations $d_l\omega_l = d_l'\omega_l' = d$ for any $l = 1, \ldots, n$. It follows that $\mu_\lambda(d_l) = \mu_\lambda(d_l')$ for any $\lambda \in \mathbb{R}$ and so the same condition holds true for $\delta_l = d_l\theta_l$ and $\delta_l' = d_l'\theta_l'$ too. By Theorem 4.5, $\operatorname{diag}(\delta_1, \delta_1', \ldots, \delta_r, \delta_r') \in \mathbb{R}^{2g\times 2n}[s]$ is the complex Smith form of M^c, for some $M \in \mathbb{H}^{g\times n}[s]$, and therefore (10) is the complex Smith-McMillan form of the complex adjoint matrix of $d^{-1}M$. ∎

V. Stability of Quaternionic Systems

In this section different concepts of stability are analyzed. Simple and asymptotic stability for a generic behavior are defined and characterized first. Then, BIBO-stability of i/o systems is investigated.

Definition 5.1: A linear dynamical system with behavior $\mathcal{B}$ is *stable* if for every $w \in \mathcal{B}$, $||w(t)||$ is bounded for all $t > 0$. If, in addition, $\lim_{t\to+\infty} w(t) = 0$, the behavior is *asymptotically stable*.

Remark 5.2: A stable system cannot contain free variables and therefore it only admits full column rank kernel representations.

The characterization of an (asymptotically) stable real behavior $\mathcal{B} = \ker R$ has been given in terms of determinants of minors of R (see for instance [16, Theorem 7.2.2]). Unfortunately, there is not a unique definition of such a determinant in the noncommutative case. Therefore, an alternative characterization, based on Smith and Smith-McMillan forms, will be here extended to the quaternionic case. Before, we introduce the necessary terminology and preliminary results.

Consider first the *stability regions* $\mathcal{S}_\mathbb{Z} = \{q \in \mathbb{H} : |q| < 1\}$ and $\mathcal{S}_\mathbb{R} = \{q \in \mathbb{H} : \operatorname{Re} q < 0\}$, which extend to the quaternionic case the usual complex stability regions used for discrete and, respectively, for continuous-time real systems. These regions are *conjugacy-invariant*, i.e., they satisfy the condition $\lambda \in \mathcal{S}_\mathbb{T} \Rightarrow [\lambda] \subseteq \mathcal{S}_\mathbb{T}$, both for $\mathbb{T} = \mathbb{Z}$ and for $\mathbb{T} = \mathbb{R}$, as stated by Theorem 2.1.

The notion of stable polynomial is generalized as follows, where $\overline{\mathcal{S}}$ denotes the closure of $\mathcal{S}$ and, by definition of multiplicity, $\mu_\lambda(p) > 0 \Leftrightarrow p(\lambda) = 0$.

Definition 5.3: When dealing with a system having time-set $\mathbb{T}$, $p \in \mathbb{H}[s]$ is

- asymptotically stable in $\mathbb{X} \subseteq \mathbb{H}$ if, for any $\lambda \in \mathbb{X}$, $\mu_\lambda(p) > 0 \Rightarrow \lambda \in \mathcal{S}_\mathbb{T}$;
- stable in $\mathbb{X} \subseteq \mathbb{H}$ if, for any $\lambda \in \mathbb{X}$, $\mu_\lambda(p) > 0 \Rightarrow \lambda \in \overline{\mathcal{S}_\mathbb{T}}$ and $\mu_\lambda(p) > 1 \Rightarrow \lambda \in \mathcal{S}_\mathbb{T}$.

In what follows we do not specify $\mathbb{X}$ when $\mathbb{X} = \mathbb{H}$.

Lemma 5.4: The polynomial $p \in \mathbb{H}[s]$ is (asymptotically) stable if and only if $p^\bullet$ is (asymptotically) stable in $\mathbb{C}$.

Proof: First, let us prove that if $\mathcal{S}$ is a conjugacy-invariant region, $(\mu_\lambda(p) > l \Rightarrow \lambda \in \mathcal{S}) \Leftrightarrow (\mu_\lambda^\bullet(p) > l \Rightarrow \lambda \in \mathcal{S})$. To show "$\Rightarrow$", suppose that $\mu_\lambda^\bullet(p) > l$. Then, by definition, $\exists\nu$ such that $\lambda \in [\nu]$ and $\mu_\nu(p) > l$ and so, by hypothesis, $\nu \in \mathcal{S}$. Hence, by conjugacy-invariance, $\lambda \in \mathcal{S}$. As for "$\Leftarrow$", note that $\mu_\lambda^\bullet(p) \geq \mu_\lambda(p)$. Thus, if $\mu_\lambda(p) > l$ then $\mu_\lambda^\bullet(p) > l$ and so $\lambda \in \mathcal{S}$.

Finally, by Theorem 2.1 there always exists $\nu \in [\lambda] \cap \mathbb{C}$ and, by Proposition 2.9, $\mu_\lambda^\bullet(p) = \mu_\nu(p^\bullet)$. The result then follows since we showed that in Definition 5.3 every condition about p with $\lambda \in \mathbb{H}$ can be equivalently written in terms of $p^\bullet$ with $\lambda \in \mathbb{C}$. ∎

Theorem 5.5: If $\Gamma = \operatorname{diag}(\gamma_1, \ldots, \gamma_r)$ is a quaternionic Smith form of a kernel representation of the behavior $\mathcal{B}$, this is (asymptotically) stable if and only if γ_r is (asymptotically) stable.

Proof: By extending map (2) to sequences, we define the complex adjoint behavior

$$\mathcal{B}^\mathbb{C} = \left\{w^\mathbb{C} : w \in \mathcal{B}\right\} \text{ where } w^\mathbb{C}(t) = (w(t))^\mathbb{C},\ \forall t \in \mathbb{T}. \tag{11}$$

Since the transformation in an isometry, stability of $\mathcal{B}$ is equivalent to stability of $\mathcal{B}^\mathbb{C}$. Moreover, in [11] it is proved that if $\mathcal{B} = \ker R$ and so $\mathcal{B}^\mathbb{C} = \ker R^c$ which, by Theorem 4.5, has highest degree invariant polynomial $\gamma_r^\bullet$. By the properties of Smith forms, the stability criterion provided by [16, Thm. 7.2.2] says that $\mathcal{B}^\mathbb{C}$ is (asymptotically) stable if and only if $\gamma_r^\bullet$ is (asymptotically) stable in $\mathbb{C}$. The result is then a consequence of Lemma 5.4. ∎

In the analysis of i/o systems, the most widely used concept is called *BIBO (bounded input-bounded output) stability* and is so defined.

Definition 5.6: An i/o behavior (5) is *BIBO-stable* if it does not contain trajectories with bounded input and unbounded output, i.e.,

$$\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B} \text{ and } ||u||_\infty < \infty \Rightarrow ||y||_\infty < \infty,$$

where $||u||_\infty = \sup\{||u(t)|| : t \in \mathbb{T}, t > 0\}$.

Remark 5.7: A state-space model is BIBO-stable in *classical* systems theory if bounded inputs generate bounded outputs *when the initial state is zero*. Clearly, if such a model is BIBO-stable in the *behavioral sense*, it is BIBO-stable in the *classical sense*. The reciprocal fact is not true.

Example 5.8: Consider the discrete-time i/o system

$$(\sigma - 2)y = (\sigma - 2)u. \tag{12}$$

The realization of the system given by

$$\begin{cases} \sigma x &= 2x \\ y &= x + u \end{cases}$$

easily shows that $x(t) = 2^t x(0)$ and therefore, if $x(0) = 0$, $y = u$. In the *classical sense* the system is BIBO-stable.

However, let $\mathcal{B}_{\text{i/o}}$ be the i/o behavior of (12). The trajectories $u = 0$, bounded, and $y(t) = 2^t$, unbounded, satisfy (12). Then $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B}_{\text{i/o}}$, which is not BIBO-stable from a behavioral point of view.

To generalize the situation evidenced by the latter example, consider an i/o quaternionic system with representation (5), and define

$$\tilde{\mathcal{B}}_{\text{i/o}} = \left\{ \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] : \mathbb{T} \to \mathbb{H}^{p+m} \;:\; \tilde{P}y = \tilde{Q}u \right\}, \tag{13}$$

where $P = L\tilde{P}$ has full rank, $Q = L\tilde{Q}$, and $(\tilde{P}, \tilde{Q})$ are left coprime.

Lemma 5.9: The behavior $\mathcal{B}_{\text{i/o}}$ is BIBO-stable if and only if $\tilde{\mathcal{B}}_{\text{i/o}}$ is BIBO-stable and $\ker L$ is stable.

Proof: "Only if". Since $\tilde{\mathcal{B}}_{\text{i/o}} \subseteq \mathcal{B}_{\text{i/o}}$, also $\tilde{\mathcal{B}}_{\text{i/o}}$ is BIBO-stable. If by contradiction $\ker L$ is unstable, there exists z unbounded such that $Lz = 0$. Since $\tilde{P}$ is surjective, there exists y, necessarily unbounded, such that $z = \tilde{P}y$ and $\left[\begin{smallmatrix} y \\ 0 \end{smallmatrix}\right] \in \mathcal{B}_{\text{i/o}}$, because $Py = L\tilde{P}y = Lz = 0$.

"If". If $\left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right] \in \mathcal{B}_{\text{i/o}}$, then $\tilde{P}y - \tilde{Q}u \in \ker L$ which is stable. Thus, $\tilde{P}y = \tilde{Q}u + v$ for some v bounded. By a standard argument [18], $(\tilde{P}, \tilde{Q})$ are left coprime matrices and therefore satisfy a Bézout equation, i.e., there exist polynomial matrices S and T such that $\tilde{P}S = \tilde{Q}T + I$. By applying these operators to v and subtracting the resulting equation from the previous one, we get $\tilde{P}(y - Sv) = \tilde{Q}(u - Tv)$. If u is bounded, so is $u - Tv$ and, by BIBO-stability of $\tilde{\mathcal{B}}_{\text{i/o}}$, also $y - Sv$ and, consequently, y. ∎

Using the results obtained so far, we can now characterize BIBO-stable quaternionic systems.

Theorem 5.10: Let $\mathcal{B}_{\text{i/o}}$ be the quaternionic i/o behavior (5) and

$$\operatorname{diag}(\gamma_1, \ldots, \gamma_p) \;\text{ and }\; \operatorname{diag}\left(\frac{\epsilon_1}{\psi_1}, \ldots, \frac{\epsilon_p}{\psi_p}\right)$$

be a quaternionic Smith form of $[P \;\; -Q]$ and a quaternionic Smith-McMillan form of $P^{-1}Q$, respectively. Then $\mathcal{B}_{\text{i/o}}$ is BIBO-stable if and only if γ_p is stable and ψ_1 is asymptotically stable.

Proof: Define $\tilde{\mathcal{B}}_{\text{i/o}}$, $\tilde{P}$, $\tilde{Q}$, and L as in (13) and $\tilde{\mathcal{B}}_{\text{i/o}}^{\mathbb{C}}$ as in (11). Notice that this is an i/o behavior which, after a suitable permutation of its variables, has kernel representation $[\tilde{P}^c \;\; -\tilde{Q}^c]$, i.e., transfer matrix $(\tilde{P}^c)^{-1}\tilde{Q}^c = (\tilde{P}^{-1}\tilde{Q})^c = (P^{-1}Q)^c$. Therefore, by Theorem 4.6, Lemma 5.4, and by [16, §7.6], the equivalent behaviors $\tilde{\mathcal{B}}_{\text{i/o}}$ and $\tilde{\mathcal{B}}_{\text{i/o}}^{\mathbb{C}}$ are BIBO-stable if and only if ψ_1 is asymptotically stable. By Lemma 5.9, the statement is proved if we show that $\ker L$ is stable if and only if γ_p is stable.

Being left prime, the Smith form of $[\tilde{P} \;\; -\tilde{Q}]$ is $[I \; 0]$ (see [11, Theorem 5.3]). So, if $[S \; 0]$ is a Smith form of $[P \;\; -Q]$, there are unimodular matrices U, V, $\tilde{U}$, and $\tilde{V}$ such that $[S \; 0] = U[P \;\; -Q]V = UL[\tilde{P} \;\; -\tilde{Q}]V = UL\tilde{U}^{-1}[I \; 0]\tilde{V}^{-1}V$. Let $[X \; Y]$, with X square, be the first p rows of $\tilde{V}^{-1}V$. Then

$$[S \; 0] = UL\tilde{U}^{-1}[X \; Y]. \tag{14}$$

From this, we obtain that $0 = UL\tilde{U}^{-1}Y$. However, L has full rank, and so $Y = 0$. It follows that $\tilde{V}^{-1}V$ is block triangular, hence X (block on its diagonal) must be unimodular. Now, from (14), $S = UL\tilde{U}^{-1}X$ is a Smith form of L and this, by Theorem 5.5, concludes the proof. ∎

Remark 5.11: By its definition, BIBO-stability in the *classical sense* only considers the i/o relation (i.e., the transfer matrix $P^{-1}Q$), thus ignoring the *internal behavior* of the state. Therefore, in the notation of Theorem 5.10, it is equivalent to asymptotic stability of ψ_1.

VI. Conclusions

In this paper, stability properties of linear quaternionic continuous and discrete-time dynamical systems have been defined and characterized within the behavioral framework using algebraic properties of their kernel representations.

After investigating quaternionic polynomials, Smith and Smith-McMillan forms of matrices were defined and important properties stated. It was then proved that stability or BIBO-stability of a quaternionic behavior can be checked by looking at the zeros of polynomials which appear in a Smith form of its kernel representation or in a Smith-McMillan form of its transfer matrix.

The difference in the definition and in the characterization of BIBO-stability for classical state-space models and for general behaviors has been stressed. To our knowledge, the characterization of BIBO-stability in the behavioral sense, which holds *mutatis mutandis* for complex and real systems too, is new in the literature.

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