Reprinted from JOURNAL OF DIFFERENTIAL EQUATIONS
Vol. 90, No. 1, March 1991
Printed in Belgium

On Evolution Equations Having Monotonicities of Opposite Sign

ARRIGO CELLINA AND VASILE STAICU

International School for Advanced Studies, Strada Costiera, 11, 34014 Trieste, Italy

Received December 21, 1989

DEDICATED TO PROFESSOR ARISTIDE HALANAY FOR HIS 65TH BIRTHDAY

1. INTRODUCTION

Differential inclusions of the form

$$\dot{x}(t) \in -Ax(t) + f(t), \tag{1}$$

where A is a maximal monotone (in general, unbounded) map, have been extensively studied (see Brezis [6]), both in finite and in infinite dimensional spaces.

Existence of solutions follows, to some extent, from the basic relation

$$\frac{d}{dt}\|x(t)\|^2 = 2\langle \dot{x}(t), x(t)\rangle \tag{2}$$

(whenever meaningful), that applied to two solutions of (1), by the monotonicity of A and the minus sign on the right hand side, yields that their distance is nonincreasing. This reasoning allows us the construction of a Cauchy sequence of approximate solutions, converging to a solution.

The existence of the right approximate solutions is supplied by the maximality of A, that permits the use of the Yosida approximations. Hence existence is a result of completeness, of having the sign minus at the right hand side, and of maximality.

The same conditions have allowed to prove existence for several classes of perturbations of (1) to

$$\dot{x}(t) \in -Ax(t) + F(t, x(t))$$

in [4, 2, 9, 10, 14, 13, 11, 12, 15, 17].

0022-0396/91 $3.00

On the other hand, in [5] the problem

$$\dot{x}(t) \in F(x(t)) \subset \partial V(x(t))$$
$$x(0) = x_0 \tag{3}$$

has been considered, where F is a monotonic upper semicontinuous (not necessarily convex-valued, hence not maximal) map contained in the subdifferential of a locally bounded convex function, and x is in a finite dimensional space.

Here the basic relation (2) (and the lack of the minus sign in (3)) yields that the distance among solutions increases and typically there is no uniqueness. Existence of solutions depends on arguments of convex analysis.

This result has been generalized by Ancona–Colombo [1] to cover perturbations of the kind

$$\dot{x}(t) \in F(x(t)) + f(t, x(t))$$

with f satisfying Carathéodory conditions.

The purpose of the present paper is to study a Cauchy problem of the form

$$\dot{x}(t) \in -\partial V(x(t)) + F(x(t)), \qquad F(x) \subset \partial W(x)$$
$$x(0) = x_0, \tag{P}$$

where x is in a finite dimensional space, V is a lower semicontinuous proper convex function (hence ∂V is a maximal monotone map), W is a lower semicontinuous proper convex function and F is an upper semicontinuous compact-valued map defined over some neighborhood of x_0 (this last assumption implies [5] that locally W is Lipschitzian). Hence at the right hand side of (P) there are a maximal monotone map with a minus sign and a bounded monotone but not maximal monotone map with a plus sign.

We prove (local) existence of solutions in the future.

2. Assumptions and the Statement of the Main Result

In what follows $V: R^n \to (-\infty, +\infty]$ is a proper convex lower semicontinuous function, $D(V) = \{x \in R^n: V(x) < \infty\}$, $\partial V: R^n \to 2^{R^n}$ is the subdifferential of V defined by

$$\partial V(x) = \{\xi \in R^n: V(y) - V(x) \geqslant \langle \xi, y - x \rangle, \forall y \in R^n\} \tag{2.1}$$

and $D(\partial V) := \{x \in R^n: \partial V(x) \neq \varnothing\}$.

Then $D(\partial V) \subset D(V)$ and it is known that $x \to \partial V(x)$ is a maximal monotone map.

Consider on R^n the Euclidean norm $\|\cdot\|$ and, for $x \in R^n$ and $r > 0$, set $B(x, r) = \{y \in R^n: \|y - x\| < r\}$, $B[x, r] = \{y \in R^n: \|y - x\| \leqslant r\}$, and for a closed subset A of R^n, $B(A, r) = \{y \in R^n: d(y, A) < r\}$. Denote by cl A the closure of A, and, if A is a closed and convex set, let $m(A)$ be the element of A such that

$$\|m(A)\| = \inf\{\|y\|: y \in A\}.$$

$F: R^n \to 2^{R^n}$ is called upper semicontinuous if for every x and for every $\varepsilon > 0$ there exists $\delta > 0$ such that $y \in B(x, \delta)$ implies $F(y) \subset B(F(x), \varepsilon)$. F is called cyclically monotone if for every cyclical sequence

$$x_0, x_1, ..., x_N = x_0 \qquad (N \text{ arbitrary})$$

and every sequence $y_i \in F(x_i)$, $i = 1, ..., N$, we have

$$\sum_{i=1}^{N} \langle y_i, x_i - x_{i-1} \rangle \geqslant 0.$$

Let $\mathscr{L}$ be the σ-algebra of Lebesgue measurable subsets of $[0, T]$. A multivalued map $G: [0, T] \to 2^{R^n}$ is called measurable if for any closed subset C of R^n:

$$\{t \in [0, T]: G(t) \cap C \neq \varnothing\} \in \mathscr{L}.$$

About cyclically monotone maps we have the following proposition.

Proposition 2.1 [6, Theorem 2.5]. *F is cyclically monotone if and only if there exists $W: R^n \to (-\infty, +\infty]$, a proper convex lower semicontinuous function, such that for every x: $F(x) \subset \partial W(x)$.*

Solutions of an evolution equation are meant in the following sense.

Definition 2.2. Let $f \in L^1([0, T], R^n)$ and $x_0 \in \text{cl}\, D(\partial V)$. $x: [0, T] \to R^n$ is called a (strong) solution to the problem

$$\dot{x}(t) \in -\partial V(x(t)) + f(t), \qquad x(0) = x_0 \tag{P_f}$$

if x is continuous on $[0, T]$ and absolutely continuous on every compact subset of $(0, T)$, $x(0) = x_0$ and for a.e. $t \in [0, T]$:

$$x(t) \in D(\partial V) \qquad \text{and} \qquad \dot{x}(t) \in -\partial V(x(t)) + f(t).$$

DEFINITION 2.3. $x: [0, T] \to R^n$ is called a solution to the Cauchy problem

$$\dot{x}(t) \in -\partial V(x(t)) + F(x(t)), \qquad x(0) = x_0 \in \text{cl } D(\partial V) \qquad \text{(P)}$$

if there exists $f \in L^1([0, T], R^n)$, a selection of $F(x(\cdot))$ (i.e., $f(t) \in F(x(t))$ for a.e. $t \in [0, T]$), such that x is a solution to (P_f).

Our main result is the following:

THEOREM. *Let V be a proper convex lower semicontinuous function and x_0 be in $D(\partial V)$; let F be an upper semicontinuous cyclically monotone map with compact nonempty values defined on a neighborhood of x_0. Then there exist $T > 0$ and $x: [0, T] \to R^n$, a solution to the Cauchy problem*

$$\dot{x}(t) \in -\partial V(x(t)) + F(x(t)), \qquad x(0) = x_0. \qquad \text{(P)}$$

EXAMPLE. As an illustration of the previous theorem in the case $n = 2$, let V be indicator function of the closed unit disk D; let $F(x_1, x_2)$ be $\{(\text{sign } x_1, 0)\}$, where

$$\text{sign } x = \begin{cases} -1 & \text{if } x < 0 \\ \{-1, 1\} & \text{if } x = 0 \\ 1 & \text{if } x > 0. \end{cases}$$

Since F is uniformly bounded on D solutions exist on $[0, +\infty)$, and converge to either $(-1, 0)$ or $(1, 0)$. So the two invariant points $(-1, 0)$ and $(1, 0)$ attract solutions from every initial point in the disk D.

3. SOME PRELIMINARY RESULTS

In this section we present some statements that will be used to prove the main result. Let $V: R^n \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function and ∂V its subdifferential.

Given any compact set K containing x_0, $\inf\{V(x): x \in K\} = V(x^*)$ for some $x^* \in K$. Since $\partial(V(x) - V(x^*)) = \partial V(x)$ we will assume that $V \geq 0$.

LEMMA 3.1 [2, Theorem 1.3]. (i) *for every $x_0 \in D(\partial V)$ and $f \in L^2([0, T], R^n)$ there exists a unique solution $x^f: [0, T] \to R^n$ to the problem*

$$\dot{x}(t) \in -\partial V(x(t)) + f(t), \qquad x(0) = x_0. \qquad (P_f)$$

(ii) *$t \to V(x^f(t))$ is absolutely continuous on $[0, T]$ (hence it is almost everywhere differentiable on $[0, T]$)*

(iii) $\|dx^f(t)/dt\|^2 = -(d/dt)\, V(x^f(t)) + \langle f(t), dx^f(t)/dt \rangle$

(iv) *$dx^f/dt \in L^2([0, T], R^n)$ and*

$$\left[\int_0^T \left\|\frac{dx^f(t)}{dt}\right\|^2 dt\right]^{1/2} \leq \left(\int_0^T \|f(t)\|^2\, dt\right)^{1/2} + \sqrt{V(x_0)}. \qquad (3.1)$$

Let $x^0: [0, \infty) \to R^n$ be the unique solution to the problem (P_f) with $f = 0$. Then, by [3, Theorem 3.2.1] for a.e. $t \in (0, T)$: $(d/dt)\, x^0(t) = -m(\partial V(x(t)))$ and $t \to \|m(\partial V(x^0(t)))\|$ is nonincreasing. Therefore, for any $t \in [0, T]$, $\|x^0(t) - x_0\| = \|\int_0^t \dot{x}^0(s)\, ds\| \leq \int_0^t \|m(\partial V(x^0(s)))\|\, ds \leq \int_0^t \|m(\partial V(x^0(0)))\|\, ds$, hence

$$\|x^0(t) - x_0\| \leq t\, \|m(\partial V(x_0))\|. \qquad (3.2)$$

On the other hand if $f \in L^2([0, T], R^n)$ then [2, Theorem 1.2] for any $0 \leq s \leq t \leq T$:

$$\|x^f(t) - x^0(t)\| \leq \|x^f(s) - x^0(s)\| + \int_s^t \|f(s)\|\, ds, \qquad (3.3)$$

and by (3.2) and (3.3) it follows that

$$\|x^f(t) - x_0\| \leq \int_0^t \|f(s)\|\, ds + t\, \|m(\partial V(x_0))\|. \qquad (3.4)$$

LEMMA 3.2. *Let $\{\delta_n(\cdot): n \in N\}$ be a sequence of measurable functions, $\delta_n: [0, T] \to R^n$ and let $k \in L^1([0, T], R^n)$ be such that for a.e. $t \in [0, T]$*

$$\|\delta_n(t)\| \leq k(t).$$

Then

(i) *$t \to \psi_i(t) := \text{cl}(\bigcup_{n \geq i} \{\delta_n(t)\})$ is measurable, $\forall i \in N$.*

(ii) *for a.e. $t \in [0, T]$, $\psi_*(t) = \bigcap_{i \in N} \psi_i(t)$ is nonempty compact and $t \to \psi_*(t)$ is measurable.*

(iii) *Assume moreover that for a.e. $t \in [0, T]$, $G(t)$ is closed and $d(\delta_n(t), G(t)) \to 0$ for $n \to \infty$. Then $\psi_*(t) \subset G(t)$.*

Proof. (i) Since $t \to \{\delta_n(t)\}$ is measurable and since $\psi_i(t)$ is compact, by [8, Proposition III.4] it follows that $t \to \psi_i(t)$ is measurable.

(ii) $\{\psi_i(t): i \in N\}$ is a decreasing sequence of compact subsets of R^n hence $\psi_*(t)$ is compact nonempty. The measurability of $t \to \psi_*(t)$ follows from (i) and [8, Proposition III.4].

(iii) Fix $\varepsilon > 0$. Then there exists $n_\varepsilon \in N$ such that for $n \geqslant n_\varepsilon$, $\delta_n(t) \in \mathrm{cl}\, B(G(t), \varepsilon)$, hence

$$\psi_{n_\varepsilon}(t) \subset \mathrm{cl}\, B(G(t), \varepsilon) \quad \text{and} \quad \psi_*(t) \subset \mathrm{cl}\, B(G(t), \varepsilon). \quad ∎$$

4. Proof of the Main Result

Let $x_0 \in D(\partial V)$ and let $W: R^n \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function such that for every x: $F(x) \subset \partial W(x)$.

As in [5] we can assume that there exist $r > 0$ and $M < \infty$ such that W is Lipschitzian with Lipschitz constant M on $B(x_0, r)$. It follows that F is bounded by M on $B(x_0, r)$. Let $m(\partial V(x_0))$ be the element of minimal norm of $\partial V(x_0)$ and set

$$T < \frac{r}{M + \|m(\partial V(x_0))\|}.$$

Our purpose is to prove that there exists $x: [0, T] \to B(x_0, r)$, a solution to the Cauchy problem (P).

Let $n \in N$ and set for $k = 0, 1, ..., n$, $t_k^n = kT/n$. Take $y_0^n \in F(x_0)$ and define $f_1^n: [t_0^n, t_1^n] \to R^n$ by $f_1^n(t) = y_0^n$. Then $f_1^n \in L^2([t_0^n, t_1^n], R^n)$ and by Lemma 3.1(ii), there exists $x_1^n: [t_0^n, t_1^n] \to R^n$, the unique solution to the problem

$$\dot{x}(t) \in -\partial V(x(t)) + f_1^n(t), \qquad x(0) = x_0. \tag{P_1^n}$$

By (3.4) we obtain that for any $t \in [t_0^n, t_1^n]$, $\|x_1^n(t) - x_0\| \leqslant \int_0^t \|f_1^n(s)\|\, ds + t\|m(\partial V(x_0))\| \leqslant (T/n)(M + \|m(\partial V(x_0))\|) < r/n$, hence $x_1^n(t) \in B(x_0, r/n)$.

Analogously for $k = 2, ..., n$ take $y_{k-1}^n \in F(x_{k-1}^n(t_{k-1}^n))$ set $I_k^n = (t_{k-1}^n, t_k^n]$; define $f_k^n: I_k^n \to R^n$ by $f_k^n(t) = y_{k-1}^n$ and set $x_k^n: I_k^n \to B(x_0, kr/n)$ to be the unique solution to the problem

$$\dot{x}(t) \in -\partial V(x(t)) + f_k^n(t), \qquad x(t_{k-1}^n) = x_{k-1}^n(t_{k-1}^n). \tag{P_k^n}$$

Remark that from $x_{k-1}^n(t_{k-1}^n) \in B(x_0, (k-1)r/n)$, (3.2) and from (3.3) applied for $s = t_{k-1}^n$ it follows that $x_k^n(I_k^n) \subset B(x_0, kr/n)$.

Define for $t \in [0, T]$:

$$x_n(t) = \sum_{k=1}^n x_k^n(t)\, \chi_{I_k^n}(t), \qquad f_n(t) = \sum_{k=1}^n f_k^n(t)\, \chi_{I_k^n}(t),$$

$$a_n(t) = \sum_{k=1}^n t_{k-1}^n\, \chi_{I_k^n}(t).$$

By construction we have

$$\dot{x}_n(t) \in -\partial V(x_n(t)) + f_n(t) \qquad \text{a.e. on } [0, T], \tag{4.1}$$

$$f_n(t) \in F(x_n(a_n(t))) \qquad \text{a.e. on } [0, T] \tag{4.2}$$

$$x_n(t) \in D(\partial V) \cap B(x_0, r) \qquad \text{a.e. on } [0, T], \tag{4.3}$$

and by (3.1) we obtain

$$\left(\int_0^T \left\|\frac{dx_n(t)}{dt}\right\|^2 dt\right)^{1/2} \leqslant \left(\int_0^T \|f_n(t)\|^2\, dt\right)^{1/2} + \sqrt{V(x_0)}$$

$$\leqslant M\sqrt{T} + \sqrt{V(x_0)} =: N.$$

It follows that $\|dx_n/dt\|_{L^2} \leqslant N$ and since $\|x_n\|_\infty \leqslant r + \|x_0\|$, we can assume that $(x_n, \dot{x}_n)$ is precompact in $C([0, T], R^n) \times L^2([0, T], R^n)$, the first space with the sup norm and the second with the weak topology. Therefore there exists a subsequence (again denoted by) x_n and an absolutely continuous function $x: [0, T] \to B[x_0, r]$ such that

$$x_n \text{ converges to } x \text{ uniformly on compact subsets on } [0, T] \tag{4.4}$$

$$\dot{x}_n \text{ converges weakly in } L^2 \text{ to } \dot{x}. \tag{4.5}$$

Since $\|f_n(t)\| \leqslant M$ on $[0, T]$, we can assume that

$$f_n \text{ converges weakly in } L^2 \text{ to } f. \tag{4.6}$$

By (4.2) we have that

$$d((x_n(t), f_n(t)), \text{graph } F) \leqslant \|x_n(a_n(t)) - x_n(t)\|$$

and, since $a_n(t) \to t$ and $x_n \to x$ uniformly, we obtain that

$$d((x_n(t), f_n(t)), \text{graph } F)) \to 0 \text{ for } n \to \infty.$$

Then by (4.4), (4.6), and the Convergence theorem [3, p. 60] it follows that $f(t) \in \mathrm{co}\, F(x(t)) \subset \partial V(x(t))$ and by [6, p. 73, Lemma 3.3] we obtain that $(d/dt)\, W(x(t)) = \langle \dot{x}(t), f(t) \rangle$, i.e.,

$$\int_0^T \langle \dot{x}(s), f(s) \rangle\, ds = W(x(T)) - W(x_0). \tag{4.7}$$

By the definition of ∂W,

$$W(x_n(t_k^n)) - W(x_n(t_{k-1}^n)) \geqslant \left\langle y_{k-1}^n, \int_{t_{k-1}^n}^{t_k^n} \dot{x}_n(s)\, ds \right\rangle$$

$$= \int_{t_{k-1}^n}^{t_k^n} \langle f_n(s), \dot{x}_n(s) \rangle\, ds.$$

Adding for $k = 1, ..., n$ we obtain

$$W(x_n(T)) - W(x_0) \geq \int_0^T \langle f_n(s), \dot{x}_n(s) \rangle \, ds. \tag{4.8}$$

Comparing (4.7) and (4.8), using the continuity of W in $x(T)$ and the convergence of x_n to x, it follows

$$\limsup_{n \to \infty} \int_0^T \langle \dot{x}_n(s), f_n(s) \rangle \, ds \leq \int_0^T \langle \dot{x}(s), f(s) \rangle \, ds. \tag{4.9}$$

By (4.1) and Lemma 3.1(iii) we obtain

$$\int_0^T \|\dot{x}_n(s)\|^2 \, ds = \int_0^T \langle f_n(s), \dot{x}_n(s) \rangle \, ds - V(x_n(T)) + V(x_0). \tag{4.10}$$

Let

$$\delta_n(t) := f_n(t) - \dot{x}_n(t) \tag{4.11}$$

$$\delta(t) := f(t) - \dot{x}(t). \tag{4.12}$$

Then δ_n converges to δ weakly in L^2 (by (4.5) and (4.6)), $\delta_n(t) \in \partial V(x_n(t))$ and, since x_n converges to x, it follows that $\delta(t) \in \partial V(x(t))$, hence, by [6, Lemma 3.3],

$$\frac{d}{dt} V(x(t)) = \langle f(t), \dot{x}(t) \rangle - \|\dot{x}(t)\|^2.$$

By integrating we obtain

$$\int_0^T \|\dot{x}(s)\|^2 \, ds = \int_0^T \langle f(s), \dot{x}(s) \rangle \, ds - V(x(T)) + V(x_0). \tag{4.13}$$

By (4.10), (4.9), and the lower semicontinuity of V it follows that

$$\limsup_{n \to \infty} \int_0^T \|\dot{x}_n(s)\|^2 \, ds$$

$$\leq \limsup_{n \to \infty} \int_0^T \langle f_n(s), \dot{x}_n(s) \rangle \, ds - \liminf_{n \to \infty} V(x_n(T)) + V(x_0)$$

$$\leq \int_0^T \langle f(s), \dot{x}(s) \rangle \, ds - V(x(T)) + V(x_0) = \int_0^T \|\dot{x}(s)\| \, ds.$$

Hence

$$\limsup_{n \to \infty} \|\dot{x}_n\|_{L^2} \leq \|\dot{x}\|_{L^2}.$$

Since, by the weak convergence of $\dot{x}_n$ to $\dot{x}$, $\limsup_{n \to \infty} \|\dot{x}_n\|_{L^2} \geq \|\dot{x}\|_{L^2}$ we obtain that

$$\lim_{n \to \infty} \|\dot{x}_n\|_{L^2} = \|\dot{x}\|.$$

Hence $\dot{x}_n$ converges to $\dot{x}$ in L^2-norm, and [7, p. 58, Theorem IV.9] a subsequence (denoted again by) $\dot{x}_n$ converges pointwise almost everywhere on $[0, T]$ to $\dot{x}$ and there exists $\lambda \in L^2([0, T], R^n)$ such that $\|\dot{x}_n(t)\| \leq \lambda(t)$.

Now we apply Lemma 3.2 for δ_n given by $\delta_n(t) = f_n(t) - \dot{x}_n(t)$. By construction, $\delta_n(t) \in F(x_n(a_n(t))) - \dot{x}_n(t)$, hence $\|\delta_n(t)\| \leq M + \lambda(t) =: k(t)$. Set $G(t) := F(x(t)) - \dot{x}(t)$ and obtain

$$d(\delta_n(t), G(t)) = d(\delta_n(t) + \dot{x}(t), F(x(t)))$$

$$\leq \|\dot{x}_n(t) - \dot{x}(t)\| + d^*(F(x_n(a_n(t))), F(x(t))),$$

where $d^*(A, B) = \sup\{d(a, B) : a \in A\}$.

Since $\dot{x}_n(t) \to \dot{x}(t)$, $x_n(a_n(t)) \to x(t)$ and F is upper semicontinuous we have that

$$d(\delta_n(t), G(t)) \to 0 \qquad \text{for} \quad n \to \infty.$$

Then, Lemma 3.2 implies that

$$\psi_*(t) = \bigcap_{l \in N} \operatorname{cl}\left(\bigcup_{n \geq l} \{\delta_n(t)\} \right)$$

is nonempty, compact, contained in $G(t)$ and $t \to \psi_*(t)$ is measurable.

Taking $G^*(t) = \partial V(x(t)) \cap B(0, k(t))$ we have that $\delta_n(t) \in \partial V(x_n(t)) \cap B(0, k(t))$ and since $x \to \partial V(x) \cap B(0, k(t))$ is upper semicontinuous, it follows that

$$d(\delta_n(t), G^*(t)) \to 0 \qquad \text{for} \quad n \to \infty$$

and, by Lemma 3.2, $\psi_*(t) \subset \partial V(x(t)) \cap B(0, k(t))$.

Let $\sigma(\cdot)$ be a measurable (hence in $L^2([0, T], R^n)$) selection of $\psi_*(\cdot)$. Set $g(t) := \dot{x}(t) + \sigma(t)$. By definition of G, $g(t) \in F(x(t))$. Therefore $\dot{x}(t) = -\sigma(t) + g(t) \in -\partial V(x(t)) + g(t)$ and the proof is complete. ∎

REFERENCES

1. F. ANCONA AND G. COLOMBO, Existence of solutions for a class of nonconvex differential inclusions, *Rend. Sem. Mat. Univ. Padova* 83 (1990), 71–76.
2. H. ATTOUCH AND D. DAMLAMIAN, On multivalued evolution equations in Hilbert spaces, *Israel J. Math.* 12 (1972), 373–390.

3. J. P. AUBIN AND A. CELLINA, "Differential Inclusions," Springer-Verlag, Berlin, 1984.
4. P. BENILAN AND H. BREZIS, Solutions faibles d'équations d'évolution dans les espaces de Hilbert," *Ann. Inst. Fourier (Grenoble)* 22 (1972), 311–329.
5. A. BRESSAN, A. CELLINA, AND G. COLOMBO, Upper semicontinuous differential inclusions without convexity, *Proc. Amer. Math. Soc.* 106 (1989), 771–775.
6. H. BREZIS, "Opérateurs maximaux monotones et semigroupes de contractions dans les espaces de Hilbert," North-Holland, Amsterdam, 1973.
7. H. BREZIS, "Analyse fonctionelle, théorie et applications," Masson, Paris, 1983.
8. C. CASTAING AND M. VALADIER, "Convex Analysis and Measurable Multifunctions," Springer-Verlag, Berlin, 1977.
9. A. CELLINA AND M. V. MARCHI, Non-convex perturbations of maximal monotone differential inclusions, *Israel J. Math.* 46 (1983), 1–11.
10. G. COLOMBO, A. FONDA, AND A. ORNELAS, Lower semicontinuous perturbations of maximal monotone differential inclusions, *Israel J. Math.* 61 (1988), 211–218.
11. G. COLOMBO AND M. TOSQUES, Multivalued perturbations for a class of nonlinear evolution equations, *Ann. Mat. Pura. Appl.*, to appear.
12. D. KRAVVARITIS AND N. S. PAPAGEORGIOU, Multivalued perturbations of subdifferential type evolution equations in Hilbert spaces, *J. Differential Equations* 76 (1988), 238–255.
13. E. MITIDIERI AND M. TOSQUES, Nonlinear integrodifferential equations in Hilbert spaces: The variational case, *in* "Proceedings of Volterra Integrodifferential Equations in Banach Spaces and Applications," Pitman Res. Notes in Math., Ser., Longman Sci. Tech, Harlow, to appear.
14. E. MITIDIERI AND I. I. VRABIE, Differential inclusions governed by non-convex perturbations of m-accretive operators, *Differential Integral Equations* 2 (1989), 525–531.
15. N. S. PAPAGEORGIOU, Nonconvex and nonmonotone perturbations of evolution inclusions of subdifferential type, *Period. Math. Hungar.*, to appear.
16. R. T. ROCKAFELLAR, "Convex Analysis," Princeton, Univ. Press, Princeton, NJ, 1972.
17. A. A. TOLSTONOGOV, On the properties of solutions of evolutions equations of subdifferential type in a Hilbert space, *in* "Proceedings of EQUADIFF 7," Praha, 1989.
18. I. I. VRABIE, "Compactness Methods for Nonlinear Evolutions," Pitman Monographs Surveys Pure Appl. Math., Vol. 32, Longman Sci. Tech, Harlow, 1987.
19. K. YOSIDA, "Functional Analysis," Springer-Verlag, Berlin, 1980.

Printed by Catherine Press, Ltd., Tempelhof 41, B-8000 Brugge, Belgium