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TWO NONTRIVIAL SOLUTIONS OF A CLASS OF ELLIPTIC EQUATIONS WITH SINGULAR TERM

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Abstract. We consider the existence of nontrivial solutions of the equation

$$-\Delta u - \frac{\lambda}{|x|^2}u = |u|^{2^*-2}u + \mu|x|^{\alpha-2}u + f(x)|u|^{\gamma}, \quad x \in \Omega\backslash\{0\}, \quad u \in H_0^1(\Omega),$$

where $0 \in \Omega$ is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 3$). By variational methods and Nehari set techniques, we show that this equation has at least two nontrivial solutions in $H_0^1(\Omega)$, under some additional hypotheses on $\lambda > 0$, $\mu > 0$, $\alpha > 0$, $0 \leq \gamma < 1$ and $f \in L^{\infty}(\Omega)$, which may be sign-changing. If $f > 0$ then the solutions are positive.

1. **Introduction.** Let $0 \in \Omega \subset \mathbb{R}^N$ ($N \geq 3$) be a bounded domain with smooth boundary, and $2^* \doteq 2N/(N-2)$ denote the critical Sobolev exponent. Here, we study the existence of nontrivial $u \in H_0^1(\Omega)$ that satisfies the following problem

$$\begin{cases} -\Delta u(x) - \frac{\lambda}{|x|^2}u(x) &= |u(x)|^{2^*-2}u(x) + \mu|x|^{\alpha-2}u(x) + f(x)|u(x)|^{\gamma} \quad \text{in} \quad \Omega\backslash\{0\}, \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

and we prove that there are (at least) two solutions both positive for suitable values of $\lambda, \alpha, \mu, \gamma$ and hypotheses on the nonlinearity f.

Classes of elliptic equations which include Eq.(1) has a lost of compactness phenomena, since the nonlinearity has a critical growth imposed by the critical exponent 2^* of the Sobolev embedding $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$. This means that we could not use standard variational methods. On the other hand, due to the presence of the singular term $\frac{\lambda}{|x|^2}$, the problem has a strong singularity at $0 \in \Omega$.

Elliptic equations with critical exponent have been studied by many authors (e.g., see [9, 2]). For $\lambda = 0$, $\mu = 0$, and odd nonlinearity, Li-Zou [10] obtained infinitely many solutions of Eq.(1). For more related results, we refer the interested readers to [6, 11, 12].

Elliptic equations containing simultaneously the critical exponent and a singular term ($\lambda \neq 0$), which are particular cases of Eq.(1), were considered in the literature as Ferrero-Gazzola [7], which established the existence of solutions whenever $\alpha = 2$, $f \equiv 0$, μ belongs to a left neighborhood constant width of any eigenvalue, and

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suitable restrictions on the spatial dimension N exists. Note that in our case, the presence of suitable unbounded coefficients $|x|^{\alpha}$ will allow us to release the restriction on the spatial dimension N. Other relevant studies are the works of He-Zou [8], for $\mu = 0$ and under some conditions on $f(x,u)$, Tarantello [13] and Chen [3], for $\alpha = 2$ and $f \equiv 0$. For $\alpha = 2$, $\lambda = 0$ and $\gamma = 0$, Tarantello [14] proved the Eq.(1) with Neumann condition has three solutions, one of which necessarily changes sign. When $\alpha = 2$, $\gamma = 0$ and $N \geq 7$, Kang-Deng [9] proved the existence of two nontrivial solutions of Eq.(1) provided f satisfies some additional conditions. However, to our knowledge, there are no results containing both singular term and critical Sobolev exponent for the nonlinearity $f(x)|u|^{\gamma}$, with $\mu \neq 0$ and $\alpha \neq 2$. However, for $\gamma = 0$, Chen-Rocha [5] recently showed the existence of four solutions for Eq.(1), one of which changes sign.

In the present paper, motivated by overcoming the difficulties above, and the results of Chen-Rocha [5], we will show using variational methods and Nehari set techniques that Eq.(1) has at least two nontrivial solutions in the Sobolev space $H_0^1(\Omega)$, which are positive when $f > 0$.

Let us introduce some notation and remarks. Define the best constant in the Hardy inequality by $\Lambda \doteq (N-2)^2/4$, and, for convenience of presentation, define the functionals

$$T(u) \doteq \int_{\Omega} |\nabla u|^2 - \left(\frac{\lambda}{|x|^2} + \mu|x|^{\alpha-2}\right)|u|^2\,dx, \quad U(u) \doteq \|u\|_{2^*}^{2^*}, \quad F(u) \doteq \int_{\Omega} f|u|^{\gamma}u\,dx.$$

Since Eq.(1) is variational, mainly because of the Hardy inequality, we say that $u \in H_0^1(\Omega)$ is a (weak) solution of Eq.(1) if and only if u is a critical point of the Euler functional

$$I(u) \doteq \frac{1}{2}T(u) - \frac{1}{2^*}U(u) - \frac{1}{\gamma+1}F(u),$$

i.e. for any $v \in H_0^1(\Omega)$ there holds

$$\int_{\Omega}(\nabla u \nabla v - \frac{\lambda}{|x|^2}uv - \mu|x|^{\alpha-2}uv - |u|^{2^*-2}uv - f|u|^{\gamma}v)dx = 0.$$

We also define the functionals (derivatives of I)

$$Q(u) \doteq T(u) - U(u) - F(u) \quad \text{and} \quad J(u) \doteq 2\,T(u) - 2^*\,U(u) - (\gamma+1)F(u).$$

So, Q and J are well defined C^1-functionals on $H_0^1(\Omega)$. Define the Nehari set

$$M \doteq \{u \in H_0^1(\Omega)\backslash\{0\} : Q(u) = 0\}.$$

and the subsets of M defined by the sign of J (second derivative of I)

$$M^+ \doteq \{u \in M : J(u) > 0\}, \quad M^0 \doteq \{u \in M : J(u) = 0\}, \quad M^- \doteq \{u \in M : J(u) < 0\}.$$

From the work of Chaudhuri-Ramaswamy [2], we know that

$$\mu_1 \doteq \inf\left\{\int_{\Omega}\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}|u|^2\right)dx : \int_{\Omega}|x|^{\alpha-2}|u|^2dx = 1\right\} > 0.$$

Define the value

$$\mathbf{S} \doteq \inf\left\{\Big(T(u)\Big)^{\frac{1}{2}} : \int_{\Omega}|u|^{2^*}dx = 1\right\}.$$

Remark 1. If $0 \leq \lambda < \Lambda$, and $0 < \mu < \mu_1$. Note that, using Hardy inequality and Sobolev embedding, we have $\mathbf{S} > 0$, $T(u) > 0$ for all $u \in H_0^1(\Omega)\backslash\{0\}$ and $T(0) = 0$.

For any $u \in H_0^1(\Omega)\backslash\{0\}$, define the positive value

$$t_{\max} \equiv t_{\max}(u) \doteq \left(\frac{1-\gamma}{2^*-\gamma-1}\, T(u)\, U(u)^{-1}\right)^{\frac{N-2}{4}}$$

and the functional $\Phi_* : H_0^1(\Omega)\backslash\{0\} \to \mathbb{R}$ by

$$\Phi_*(u) \doteq t_{\max}(u)^{1-\gamma}\, T(u) - t_{\max}(u)^{2^*-\gamma-1}\, U(u) = C_{(\gamma,N)} T(u)^{\frac{2^*-\gamma-1}{2^*-2}} U(u)^{-\frac{1-\gamma}{2^*-2}},$$

where $C_{(\gamma,N)} \doteq \left(\frac{1-\gamma}{2^*-\gamma-1}\right)^{\frac{1-\gamma}{2^*-2}} \left(\frac{2^*-2}{2^*-\gamma-1}\right)$. Let $B_\epsilon \doteq \left\{w \in H_0^1(\Omega) : \|w\| < \epsilon\right\}$, $\widetilde{\mu}_f \doteq \inf_{u \in H_0^1(\Omega)} \{\Phi_*(u) - |F(u)|\}$ and the infimum introduced by Tarantello [14]

$$\mu_f \doteq \inf_{U(u)=1} \left\{C_{(\gamma,N)} T(u)^{\frac{2^*-\gamma-1}{2^*-2}} - F(u)\right\}.$$

Remark 2. If $\widetilde{\mu}_f > 0$ then $\mu_f > 0$.

In what follows, we state the main result (Theorem 1.1), for such we consider the following hypotheses:

(H_0) $0 \le \lambda < \Lambda$, $0 < \mu < \mu_1$, $0 < \alpha < \sqrt{\Lambda - \lambda}$, $f \in L^\infty(\Omega)$, and $\widetilde{\mu}_f > 0$;
(H_1) $\frac{N-\sqrt{\Lambda}}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < \gamma < 1$, f is continuous at $0 \in \Omega$ and $f(0) > 0$;
(H_2) $0 \le \gamma < 1$ and $f > 0$.

We say that hypotheses (H) hold if hypotheses (H_0) hold and one of the hypotheses (H_1) or (H_2) hold.

We will prove the following result:

Theorem 1.1. *Suppose hypotheses (H) hold, then Eq.(1) has two nontrivial solutions in $H_0^1(\Omega)$. Moreover, if (H_2) then both solutions are positive.*

Notation. In what follows, we denote the norm in $H_0^1(\Omega)$ by $\|\cdot\|$, the integral $\int_\Omega \cdot\, dx$ by $\int \cdot$. We use $\doteq$ to emphasize a new definition. Additionally, $O(\varepsilon^\beta)$ means that $|O(\varepsilon^\beta)\varepsilon^{-\beta}| \le K$ for some constant $K > 0$, $o(\varepsilon^\beta)$ means $|o(\varepsilon^\beta)\varepsilon^{-\beta}| \to 0$ as $\varepsilon \to 0$, $o(1)$ is just an infinitesimal value, and $\to$ (respectively, $\rightharpoonup$) will denote strongly (respectively, weakly) convergence.

2. **Preliminaries results.**

In this section, we give some preliminaries which play an important role in the variational methods used to study Eq.(1).

Proposition 1 (see [1]). *For $0 < \lambda < \Lambda \doteq (\frac{N-2}{2})^2$, the problem*

$$-\Delta u - \frac{\lambda}{|x|^2} u = |u|^{2^*-2} u \qquad x \in \mathbb{R}^N\backslash\{0\}, \quad u(x) \to 0 \text{ as } |x| \to +\infty, \tag{2}$$

has a family of solutions

$$U_\varepsilon(x) = \frac{[4\varepsilon(\Lambda-\lambda)N/(N-2)]^{\frac{N-2}{4}}}{[\varepsilon|x|^{\gamma_1/\sqrt{\Lambda}} + |x|^{\gamma_2/\sqrt{\Lambda}}]^{\frac{N-2}{2}}} \quad \text{for } \varepsilon > 0,$$

where $\gamma_1 = \sqrt{\Lambda} - \sqrt{\Lambda - \lambda}$, $\gamma_2 = \sqrt{\Lambda} + \sqrt{\Lambda - \lambda}$. Moreover, U_ε is the extremal function of the minimization problem

$$S_\lambda = \inf\left\{ \int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2} u^2\right) dx \,:\, u \in \mathcal{D}^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2^*} dx = 1\right\}.$$

Clearly,

$$\int_{\mathbb{R}^N} |U_\varepsilon(x)|^{2^*} dx = \int_{\mathbb{R}^N} \left(|\nabla U_\varepsilon|^2 - \frac{\lambda}{|x|^2} U_\varepsilon^2\right) dx = S_\lambda^{\frac{N}{2}}.$$

The following integral estimates are also relevant. Define a cut-off function $\phi(x) = 1$ if $|x| \leq \delta$, $\phi(x) = 0$ if $|x| \geq 2\delta$, $\phi(x) \in C_0^1(\Omega)$ and $|\phi(x)| \leq 1$, $|\nabla\phi(x)| \leq C$. Let $v_\varepsilon(x) = \phi(x)U_\varepsilon(x)$.

Proposition 2 (see [5]). *Let $0 \leq \lambda < \Lambda$ and $w \in H_0^1(\Omega)$ be a solution of Eq.(1). Then for $\varepsilon > 0$ small enough we have that*
(i) $\int w^{2^*-1}v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}})$ *and* $\int wv_\varepsilon^{2^*-1}dx = O(\varepsilon^{\frac{N-2}{4}})$;
(ii) $\int(|\nabla v_\varepsilon|^2 - \frac{\lambda}{|x|^2}v_\varepsilon^2) = S_\lambda^{\frac{N}{2}} + O(\varepsilon^{\frac{N}{2}}) + O(\varepsilon^{\frac{N-2}{2}})$;
(iii) $\int v_\varepsilon^{2^*} = S_\lambda^{\frac{N}{2}} - O(\varepsilon^{\frac{N}{2}})$;
(iv) $\int |x|^{\alpha-2}v_\varepsilon^2 = O(\varepsilon^{\frac{\alpha\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}})$, *when* $0 < \alpha < 2\sqrt{\Lambda-\lambda}$.

Note that, for all $u \in H_0^1(\Omega)$,

$$F(u) \leq \left|\int f|u|^\gamma u\right| \leq \|f\|_\infty \|u\|_{\gamma+1}^{\gamma+1} \leq (\|f\|_\infty K_{\gamma+1})\,\|u\|^{\gamma+1} \doteq K^T\|u\|^{\gamma+1}, \tag{3}$$

since $f \in L^\infty$, using Hölder inequality and the Sobolev embedding of $H_0^1(\Omega)$ in $L^{\gamma+1}(\Omega)$ with constant $K_{\gamma+1} > 0$.

For $u \in M$, the functionals I and J, can be rewritten as

$$I_M(u) = -\frac{1-\gamma}{2(\gamma+1)}T(u) + \frac{2^*-\gamma-1}{2^*(\gamma+1)}U(u),$$
$$J_M(u) = (1-\gamma)\,T(u) - (2^*-\gamma-1)\,U(u),$$

where we have denoted the restrictions of I and J, to the set M, by I_M and J_M, respectively.

Remark 3. (a) $I_M(u)$ is bounded from below in M; (b) For any $u \in H_0^1(\Omega)\backslash\{0\}$, we have $I(tu) \to -\infty$ as $|t| \to \infty$.

The following Lemma is a generalization of Lemma 2.1 of Tarantello [14]:

Lemma 2.1. *Suppose the hypotheses (H_0) hold and $0 \leq \gamma < 1$. For any $u \in H_0^1(\Omega)\backslash\{0\}$,*
define $s_f \equiv s_f(u) \doteq sign\, F(u) \in \{-1, +1\}$. Then there exist three unique values $t_0 \equiv t_0(u), t_- \equiv t_-(u), t_+ \equiv t_+(u) \in \mathbb{R}$ such that:
(i) $s_f\, t_+ > 0$, $t_+u \in M^-$, $s_f\, t_+ > t_{\max}$ *and* $I(t_+u) = \max_{s_f t \geq -t_{\max}} I(tu)$;
(ii) $s_f\, t_- > 0$, $t_-u \in M^+$, $0 < s_f\, t_- < t_{\max}$ *and* $I(t_-u) = \min_{-t_{\max} \leq t \leq t_{\max}} I(tu)$;
(iii) $s_f\, t_0 < 0$, $t_0u \in M^-$, $s_f\, t_0 < -t_{\max}$ *and* $I(t_0u) = \max_{s_f t \leq t_{\max}} I(tu)$.

Proof. Let $t \in \mathbb{R}$. Define $\phi_u(t) \doteq |t|^{-\gamma}\langle I'(tu), u\rangle + F(u)$, i.e.

$$\phi_u(t) = t\,|t|^{-\gamma}\,T(u) - t\,|t|^{2^*-\gamma-2}\,U(u). \tag{4}$$

From the definition of ϕ_u, we have $\phi_u(0) = \lim_{t\to 0^\pm}\phi_u(t) = 0$, $\lim_{t\to+\infty}\phi_u(t) = -\infty$, $\phi_u(-t) = -\phi_u(t)$ for all $t > 0$, and $\phi_u''(t) < 0$ for all $t > 0$, so ϕ_u (restricted to $t > 0$) is a concave function which attains its maximum at $t_{\max}$ and $\phi_u(t_{\max}) = \Phi_*(u) > 0$. For simplicity of presentation, we first assume $s_f = +1$.
(i) Since ϕ_u $(t > 0)$ is a concave and continuous function and $0 < F(u) < \phi_u(t_{\max})$, there exists an unique $t_+ > t_{\max}$ such that $\phi_u(t_+) = F(u) > 0$. This implies, from the definition of ϕ_u, that $|t_+|^{-\gamma}\langle I'(t_+u), u\rangle = 0$ so $Q(t_+u) = 0$ and $t_+u \in M$. Moreover, from $\phi_u'(t_+) < 0$ i.e. $T(u) < (2^*-\gamma-1)(1-\gamma)^{-1}|t_+|^{2^*-2}\,U(u)$, we have $J_M(t_+u) < 0$; thus $t_+u \in M^-$ and $I(t_+u) \geq I(tu)$ for all $t \geq t_{\max}$. The last statement is true because, if we set $r(t) = I(t\,u)$, then $r'(t) = t^{-1}Q(t\,u)$ so $r'(t_+) = 0$, and from $r'(t) = |t|^\gamma(\phi_u(t) - \phi_u(t_+))$ we have $r'(t) > 0$, when $t_{\max} \leq t < t_+$, and

$r'(t) < 0$, when $t > t_+$.
(ii) By similar arguments to the ones used in (i), there exists an unique $t_- > 0$ such that $-t_{\max} < 0 < t_- < t_{\max}$ and $\phi_u(t_-) = F(u) > 0$ so $t_- u \in M$ and, from $\phi'_u(t_-) > 0$, $t_- u \in M^+$. From $r'(t) = t^\gamma(\phi_u(t) - \phi_u(t_-))$, we have $r'(t) > 0$, when $t_- < t \leq t_{\max}$, and $r'(t) < 0$, when $-t_{\max} \leq t < t_-$. Therefore, at least, $I(t_- u) \leq I(tu)$ for all $-t_{\max} \leq t \leq t_{\max}$.
(iii) Note that $\lim_{t\to-\infty} \phi_u(t) = +\infty$, $\phi_u(-t_{\max}) = -\Phi_*(u) < 0$, $\phi'_u(t) < 0$ for all $t < -t_{\max}$, and $\phi''_u(t) > 0$ for all $t < 0$, hence there exists an unique $t_0 < -t_{\max} < 0$ such that $\phi_u(t_0) = F(u) > 0$ so $t_0 u \in M$ and, from $\phi'_u(t_0) < 0$, $t_0 u \in M^-$. From $r'(t) = t^\gamma(\phi_u(t) - \phi_u(t_0))$, we have $r'(t) > 0$, when $t < t_0$, and $r'(t) < 0$, when $t_0 < t < -t_{\max}$. Therefore, $I(t_0 u) \geq I(tu)$ for all $t < -t_{\max}$.
For the general situation $s_f \in \{-1, +1\}$, it is enough to observe that $(s_f)^{-1} = s_f$, $(s_f)^2 = 1$, $\phi_u(s_f\, t) = s_f\, \phi_u(t)$ for $t \in \mathbb{R}$, $F(s_f\, u) = s_f\, F(u)$, $J_M(s_f\, u) = J_M(u)$, and $r'(s_f\, t) = s_f\, r'(t)$ for $t \in \mathbb{R}$. □

Remark 4. The above Lemma can be further improved. In fact, $\phi'_u(\pm t_{\max}) = 0$, $\phi'_u(t) > 0$ when $-t_{\max} < t < t_{\max}$ and $\phi'_u(t) < 0$ otherwise. So, at least, we can say that: (i) $I(t_+ u) = \max_{s_f t \geq t_-} I(tu)$; (ii) $I(t_- u) = \min_{t_0 \leq s_f t \leq t_+} I(tu)$; (iii) $I(t_0 u) = \max_{s_f t \leq t_-} I(tu)$.

3. **Proof of Theorem 1.1.** We now introduce some auxiliar results which are relevant to proof the main result of this work. Set

$$c_+ \doteq \inf_{u\in M^+} I(u) \quad \text{and} \quad c_- \doteq \inf_{u\in M^-} I(u).$$

Recall $M \neq \emptyset$ (since $M^- \neq \emptyset$; see Lemma 2.1), M is a manifold, and I is continuous and bounded from below on M. Ekeland's variational principle applied to the optimization problem

$$c_0 \doteq \inf_{u\in M} I(u) \tag{5}$$

gives a bounded minimizing sequence $(u_n)_{n\in\mathbb{N}} \subset M$ satisfying:

(E_a) $c_0 \leq I(u_n) < c_0 + \frac{1}{n}$;
(E_b) $I(u) \geq I(u_n) - \frac{1}{n}\|u - u_n\|$ for all $u \in M$.

The following result will be used below, in a contradiction argument, to show that the minimizing sequence converges strongly in $H_0^1(\Omega)$.

Proposition 3. *Assume hypotheses (H_0) hold and $0 \leq \gamma < 1$. Let $u \in H_0^1(\Omega)$, $(u_n)_{n\in\mathbb{N}} \subset M^-$ be such that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $I(u_n) \to c \in \mathbb{R}$ but u_n does not converge strongly to u in $H_0^1(\Omega)$. Recall the definitions of $s_f \equiv s_f(u)$, $t_+ \equiv t_+(u)$ and $t_- \equiv t_-(u)$ in Lemma 2.1. Then the following holds:*
(1) If $u \neq 0$ and $s_f t_+ \leq 1$, then $c > I(s_f t_+ u)$;
(2) If $u \neq 0$ and $s_f t_+ > 1$, then $c \geq I(s_f t_- u) + \frac{1}{N} S_\lambda^{\frac{N}{2}}$;
(3) If $u \equiv 0$, then $c \geq \frac{1}{N} S_\lambda^{\frac{N}{2}}$.

Proof. Note that $u_n \rightharpoonup u$ (see Chen-Li-Li [4], Lemma 2.6), and $\int |x|^{\alpha-2}|u_n - u|^2 \to 0$ as $n \to \infty$. We may assume that exist $a, b \geq 0$ such that $T(u_n - u) = \int(|\nabla u_n - \nabla u|^2 - \frac{\lambda}{|x|^2}|u_n - u|^2) + o(1) \to a^2$, and $\int |u_n - u|^{2^*} \to b^{2^*}$. Note that, since u_n does not converge strongly to u, we have $a \neq 0$. On the other hand, from $f \in L^\infty$ and the compactness of the Sobolev embedding, we have $\int f|u_n - u|^\gamma(u_n - u) \to 0$. For $t \in \mathbb{R}$, we set $r(t) = I(tu)$, $\beta(t) = \frac{a^2}{2}t^2 - \frac{b^{2^*}}{2^*}|t|^{2^*}$ and $\theta(t) = r(t) + \beta(t)$. We have $r'(t) = \langle I'(tu), u\rangle = |t|^\gamma\left(\phi_u(t) - \phi_u(t_+)\right)$. From

$$|I(tu_n) - \theta(t)| \le \left| \tfrac{1}{2}t^2 T(u_n - u) - \tfrac{|t|^{2^*}}{2^*} \|u_n - u\|_{2^*}^{2^*} - \beta(t) \right|$$

we see that $I(tu_n) \to \theta(t)$ as $n \to +\infty$. Now the proof of the three statements follow the scheme of the proof of Proposition 3.3 in [5]. □

Lemma 3.1. *Suppose hypotheses (H_0) hold and $0 \le \gamma < 1$, then*

(i) For every $u \in M$, $J_M(u) \doteq (1-\gamma)T(u) - (2^* - \gamma - 1)U(u) \neq 0$, i.e. $M^0 = \emptyset$;

(ii) For any sequence $(u_n)_{n\in\mathbb{N}} \subset M$, we have

$$\lim_{n\to+\infty} J_M(u_n) = 0 \quad \Rightarrow \quad \liminf_{n\to+\infty} \|u_n\| = 0;$$

(iii) Given $u \in M$, there exist $\varepsilon > 0$ and a differentiable function $t : H_0^1(\Omega) \to \mathbb{R}$, satisfying $t(w) > 0$ for all $w \in B_\varepsilon$, $t(0) = 1$, $t(w)(u - w) \in M$ for all $w \in B_\varepsilon$ and

$$\langle t'(0), w\rangle = \frac{\int \left(2\nabla u \nabla w - 2\frac{\lambda}{|x|^2} uw - 2\mu |x|^{\alpha-2} uw - 2^* |u|^{2^*-2} uw - (1+\gamma) f |u|^\gamma w \right)}{J_M(u)}. \tag{6}$$

Proof. (*i*) Assume, by contradiction, that $(1-\gamma)T(\bar u) - (2^* - \gamma - 1)U(\bar u) = 0$ for some $\bar u \in M$, then we have $s_{\bar u} \doteq U(\bar u)^{\frac{1}{2^*}} \ge \left(\frac{1-\gamma}{2^*-\gamma-1} C\right)^{\frac{1}{2^*-2}} > 0$ for some constant $C > 0$, by using the Gagliardo-Nirenberg-Sobolev inequality. On the other hand, since $\bar u \in M$, we have $F(\bar u) = \frac{2^*-2}{1-\gamma} U(\bar u)$. Recall the definition of Φ_* in Lemma 2.1, and define $\Psi_*(u) \doteq \Phi_*(u) - F(u)$ for all $u \in M$. Hence, $\Psi_*(su) = s^{1+\gamma}\Psi_*(u)$, for any $s > 0$ and $u \in M$, and

$$\Psi_*(\bar u) \ge \inf_{U(u)^{1/2^*} = s_{\bar u}} \Psi_*(u) = s_{\bar u}^{1+\gamma} \left(\inf_{U(v)^{1/2^*}=1} \Psi_*(v) \right) \ge s_{\bar u}^{1+\gamma} \mu_f.$$

Let $K \doteq \frac{2^*-\gamma-1}{1-\gamma}$. Thus, from $\mu_f > 0$, we have

$$0 < s_{\bar u}^{1+\gamma}\mu_f \le \Psi_*(\bar u) \le \left[K^{-\frac{1-\gamma}{2^*-2}} (1-K) K^{\frac{2^*-\gamma-1}{2^*-2}} - (K-1) \right] U(\bar u) < 0.$$

This is a contradiction. Therefore $(1-\gamma)T(u) - (2^*-\gamma-1)U(u) \neq 0$ for all $u \in M$.

(*ii*) Arguing by contradiction again, assume there exists a subsequence $(u_n)_{n\in\mathbb{N}} \subset M$ such that $(1-\gamma)T(u_n) - (2^*-\gamma-1)U(u_n) = o(1)$ and $\|u_n\| > s$ for all $n \in \mathbb{N}$ and some $s > 0$. Hence, $s_{u_n} \doteq U(u_n)^{\frac{1}{2^*}} > 0$ for all $n \in \mathbb{N}$. Since $u_n \in M$, we get

$$F(u_n) = T(u_n) - U(u_n) = [(2^*-2)/(1-\gamma)]\, U(u_n) + o(1).$$

These together with $\mu_f > 0$ and $\Psi_*(u_n) \ge \inf_{U(u)^{1/2^*} = s_{u_n}} \Psi_*(u) \ge s_{u_n}^{1+\gamma}\mu_f$ implies

$$0 < s_{u_n}^{1+\gamma}\mu_f \le \Psi_*(u_n) \le \left(1 - K^2\right) U(u_n) + o(1) < 0$$

which is a contradiction, so $(1-\gamma)T(u_n) - (2^*-\gamma-1)U(u_n) = o(1)$ and $\|u_n\| = o(1)$.

(*iii*) Let $u \in M$ and $\phi : \mathbb{R} \times H_0^1(\Omega) \to \mathbb{R}$ be defined by

$$\phi(t, w) \doteq t|t|^{-\gamma} T(u - w) - t|t|^{2^*-\gamma-2} U(u - w) - F(u - w).$$

Note that $\frac{\partial}{\partial t}\phi(1,0) = J_M(u) \neq 0$ (by (*i*)) and $\phi(1,0) = Q(u) = 0$. Hence applying the implicit function theorem at the point $(1,0)$, we have that there exists a function $t \equiv t(w)$ with $t(0) = 1$ and $\langle t'(0), w\rangle = -\frac{\partial}{\partial w}\phi(1,0)\left(\frac{\partial}{\partial t}\phi(1,0)\right)^{-1}$. □

Proposition 4. *Suppose hypotheses (H_0) hold and $0 \le \gamma < 1$. We have $c_0 < 0$, there is a critical point $w_0 \in M^+$ of I such that $I(w_0) = c_0$, and w_0 is a local minimizer for I. Moreover, $w_0 > 0$ whenever hypotheses (H_2) hold.*

Proof. Let $u \in M^+ \neq \emptyset$ (see Lemma 2.1). From $J(u) > 0$, we have $U(u) < \frac{1-\gamma}{2^*-\gamma-1}T(u)$ thus $I_M(u) < 0$ and $c_+ < 0$. So, $c_0 \doteq \inf_{u\in M} I(u) \leq \inf_{u\in M^+} I(u) < 0$. From Ekeland's variational principle there exists a bounded minimization sequence $(u_n)_{n\in\mathbb{N}} \subset M$. We need to show that $\|I'(u_n)\| \to 0$ as $n \to \infty$.

Choosing n where $I'(u_n) \neq 0$ and applying Lemma 3.1(iii), for $\delta > 0$ sufficiently small and setting $u \equiv u_n$ and $w \equiv \delta \frac{I'(u_n)}{\|I'(u_n)\|}$, we have that exists $t_n(\delta) \doteq t\left(\delta \frac{I'(u_n)}{\|I'(u_n)\|}\right)$ such that $w_\delta \doteq t_n(\delta)\left(u_n - \delta \frac{I'(u_n)}{\|I'(u_n)\|}\right) \in M$. On the other hand, by (E_b) and the Taylor expansion of I, we have

$$\begin{aligned}\frac{1}{n}\|w_\delta - u_n\| &\geq \langle I'(w_\delta), u_n - w_\delta\rangle + o(\|u_n - w_\delta\|)\\ &= \langle I'(w_\delta), u_n(1-t_n(\delta))\rangle + \left\langle I'(w_\delta), t_n(\delta)\,\delta \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle\\ &\quad + o\left(\left\| u_n - t_n(\delta)\,u_n + \delta \frac{I'(u_n)}{\|I'(u_n)\|} u_n\right\|\right).\end{aligned}$$

Hence

$$\frac{1}{n}\|w_\delta - u_n\| \geq (1-t_n(\delta))\langle I'(w_\delta), u_n\rangle + \delta t_n(\delta)\left\langle I'(w_\delta), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle + o(\delta). \quad (7)$$

Dividing (7) by $\delta > 0$ and passing to the limit as $\delta \to 0$, we have

$$\frac{1}{n}\left(1 + \|u_n\|\,\|t_n'(0)\|\right) \geq \left\langle I'(u_n), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle = \|I'(u_n)\|.$$

Since (u_n) is a bounded sequence, $\|I'(u_n)\| \leq \frac{1}{n}(1 + \|u_n\|\,\|t_n'(0)\|) \leq \frac{C}{n}(1 + \|t_n'(0)\|)$ for a suitable positive constant $C > 0$. Note that $t_n'(0) = \left\langle t'(0), \frac{I'(u_n)}{\|I'(u_n)\|}\right\rangle$. Then by (6), since (u_n) is bounded sequence and $\|w\| = \delta$, we have that

$$|t_n'(0)| \leq \frac{C_1}{|(1-\gamma)T(u_n) - (2^*-\gamma-1)U(u_n)|}$$

for a suitable positive constant C_1. From Lemma 3.1, we have

$$\liminf_{n\to+\infty}\,[(1-\gamma)T(u_n) - (2^*-\gamma-1)U(u_n)] > 0.$$

Thus $|t_n'(0)| \leq K_1$, for a suitable constant $K_1 > 0$ and therefore $\|I'(u_n)\|_{H_1^0(\Omega)} \to 0$ as $n \to \infty$.

Let w_0 be the weak limit in $H_0^1(\Omega)$ of (a subsequence of) the minimizing sequence u_n. Then $w_0 \in M^+$. Indeed, suppose that $w_0 \in M^-$ (since $M^0 = \emptyset$), from Lemma 2.1 there exists $t_+ \equiv t_+(w_0)$ such that $s_f t_+ > 0$ and $t_+ w_0 \in M^-$. But $w_0 \in M^-$ implies $t_+ = 1$. If $s_f = -1$ then $t_+ = 1$ and $s_f t_+ > 0$ are a contradiction (we are done). So, suppose $s_f = 1$. In this case, there exists $t_- \equiv t_-(w_0) > 0$ such that $t_- < t_+ = 1$. Thus, we have $\frac{d}{dt}I(tw_0)\big|_{t=t_-} = \langle I'(t_-w_0), w_0\rangle = (t_-)^{-1}Q(t_-w_0) = 0$, and

$$\begin{aligned}\frac{d^2}{dt^2}I(tw_0)\Big|_{t=t_-} &= \frac{d}{dt}\left(t^\gamma[\phi_u(t) - F(u)]\right)\Big|_{t=t_-} = \frac{d}{dt}\left(t^\gamma[\phi_u(t) - \phi_u(t_-)]\right)\Big|_{t=t_-}\\ &= \gamma\, t_-^{\gamma-1}[\phi_u(t_-) - \phi_u(t_-)] + t_-^\gamma \phi_u'(t_-) = t_-^\gamma \phi_u'(t_-) > 0.\end{aligned}$$

Hence, there exists $\bar{t} > 0$ such that $t_+ > \bar{t} > t_- > t_{\max}$ and $I(\bar{t}w_0) > I(t_-w_0)$. From Lemma 2.1, $I(t_-w_0) < I(\bar{t}w_0) < I(t_+w_0) = I(w_0) = c_0$. This is a contradiction. Therefore $w_0 \in M^+$. This implies that $F(w_0) > \frac{2^*-2}{1-\gamma}U(w_0) > 0$.

We have that w_0 is a weak solution of problem, since $I'(u_n) \to 0$ as $n \to \infty$, we have that $\langle I'(w_0), w\rangle = 0$, for all $w \in H_0^1(\Omega)$. Therefore $c_0 \le I(w_0) \le \lim_{n\to\infty} I(u_n) = c_0$. Then $u_n \to w_0$ (converges strongly) em $H_0^1(\Omega)$ and $I(w_0) = c_0 = \inf_{u\in M} I(u)$.

We now show that w_0 is a local minimum for I. From Lemma 2.1, we have that $I(t_- u) \le I(gu)$, for all $0 < g < h(u) \doteq \left[(1-\gamma)\,T(u)(2^*-\gamma-1)^{-1}U(u)^{-1}\right]^{\frac{1}{2^*-2}}$. From $w_0 \in M^+$, we have $-(2^*-\gamma-1)\int |w_0|^{2^*} + (1-\gamma)\,T(w_0) > 0$ and

$$h(w_0) > t^{-1}. \tag{8}$$

Notice again that, for all $u \in M$, there exist $t_- = t_-(u) > 0$ such that $t_-(u)u \in M^+$. Thus if $w_0 \in M^+$, then $t^{-1} = 1$.

Let $\varepsilon > 0$ sufficiently small such that $1 < h(w_0)$ for $\|w\| < \varepsilon$ and set $t(w) > 0$ a function such that $t(0) = 1$ and $t(w)(w_0 - w) \in M$ for all $\|w\| < \varepsilon$, (see item (iii) of Lemma 3.1). Since $t(w) \to 1$ as $\|w\| \to 0$, we can always assume that $t(w) < h(w_0)$, for all w with $\|w\| < \varepsilon$.

Note that $t(w)(w_0 - w) \in M^+$ and for $0 < g < h(w_0)$ we have,
$I(g(w_0-w)) \ge I(t(w)(w_0-w)) \ge I(w_0)$. From (8) we can take $g = 1$ and conclude that $I(w_0 - w) \ge I(w_0)$, for all $w \in H_0^1(\Omega)$ with $\|w\| < \varepsilon$.
Therefore w_0 is a local minimum for I.

From Lemma 2.1, there exist $t_-(u) \in \mathbb{R}$, such that $s_f\, t_-(|w_0|) > 0$,
$t_-(|w_0|)\,|w_0| \in M^+$, $s_f\, t_-(|w_0|) < t_{\max}(|w_0|) = t_{\max}(w_0)$ and
$I(t_-(|w_0|)\,|w_0|) = \min_{-t_{\max}\le t\le t_{\max}} I(t(|w_0|)\,|w_0|)$. Since $w_0 \in M^+$, then $t_-(w_0) = 1$. Thus $c_0 \le I(t_-(w_0)\,w_0) = \min_{-t_{\max}\le t\le t_{\max}} I(tw_0) \le I(t_-(|w_0|)\,w_0)$.
Note that, since $f > 0$, we have that $I(t_-(|w_0|)\,|w_0|) \le I(t_-(|w_0|)\,w_0) \le c_0$. Therefore $I(t_-(w_0)\,w_0) = c_0$ and we can always take $w_0 > 0$. □

Lemma 3.2. *If hypotheses (H) hold, then $c_- < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$.*

Proof. We know that there is $s_0 > 0$ and $\varepsilon > 0$ sufficiently small such that $w_0 + s_0 v_\varepsilon \in M^-$, by using the arguments in [14, Proposition 2.2]. To prove $c_- < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$, we only need to prove that $\sup_{s>0} I(w_0 + sv_\varepsilon) < c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}$, since $c_- = \inf_{u\in M^-} I(u) \le I(w_0 + s_0 v_\varepsilon) \le \sup_{s>0} I(w_0 + sv_\varepsilon)$. Moreover, we only need to consider bounded values for s, since, $I(w_0 + sv_\varepsilon) \to -\infty$ as $s \to +\infty$ implies that there is $s_0 > 0$ such that

$$\sup_{s>0} I(w_0 + sv_\varepsilon) \le \sup_{0<s<s_0} I(w_0 + sv_\varepsilon).$$

Firstly, since w_0 is a solution of Eq.(1), we get from direct computations that

$$\begin{aligned} I(w_0 + sv_\varepsilon) &= \frac{1}{2} T(w_0 + sv_\varepsilon) - \frac{1}{2^*} U(w_0 + sv_\varepsilon) - \frac{1}{\gamma+1} F(w_0 + sv_\varepsilon) \\ &= I(w_0) + I(sv_\varepsilon) + \int |w_0|^{2^*-2} w_0 (sv_\varepsilon) + \int f(x) |w_0|^{\gamma} (sv_\varepsilon) \\ &\quad - \frac{1}{2^*}\left[U(w_0 + sv_\varepsilon) - U(w_0) - U(sv_\varepsilon)\right] \\ &\quad - \frac{1}{\gamma+1}\left[F(w_0 + sv_\varepsilon) - F(w_0) - F(sv_\varepsilon)\right]. \end{aligned}$$

Suppose hypotheses (H_1) hold. Using the elementary inequality

$$\left||a+b|^q - |a|^q - |b|^q\right| \le d_1 \left[|a|^{q-1}|b| + |a|\,|b|^{q-1}\right]$$

for $a, b \in \mathbb{R}$ and $q > 1$, we obtain that

$$\begin{aligned} I(w_0 + sv_\varepsilon) \quad \leq \quad & I(w_0) + I(sv_\varepsilon) + \int |w_0|^{2^*-1}(sv_\varepsilon) + |f|_{L^\infty(\Omega)} \int |w_0|^\gamma (sv_\varepsilon) \\ & + d_2 \int |w_0|^{2^*-1} |sv_\varepsilon| + d_3 \int |w_0| \, |sv_\varepsilon|^{2^*-1} \\ & + d_4 \int |w_0|^\gamma |sv_\varepsilon| + d_5 \int |w_0| \, |sv_\varepsilon|^\gamma , \end{aligned}$$

where, here and below, d_j for $j \in \mathbb{N}$ denote positive constants.

Secondly, since f is continuous at 0 and $f(0) > 0$, there exist $d_6 > 0$ and $\delta_0 > 0$ such that $f(x) \geq d_6$ for any $x \in B_{\delta_0}(0)$, the ball with center at 0 and radius δ_0. Hence, we have

$$\begin{aligned} \sup_{s>0} I(w_0 + sv_\varepsilon) \quad \leq \quad & I(w_0) + \sup_{s>0} \left[\frac{1}{2} T(sv_\varepsilon) - \frac{1}{2^*} U\left(sv_\varepsilon\right) \right] + d_9 \int |w_0|^{2^*-1} v_\varepsilon \\ & + d_{10} \int |w_0| \, |v_\varepsilon|^{2^*-1} + d_{11} \int |w_0|^\gamma v_\varepsilon + d_{12} \int |w_0| v_\varepsilon^\gamma \\ & - d_7 \int_{B_{\delta_0}(0)} v_\varepsilon^{\gamma+1} + d_8 \int_{\Omega \setminus B_{\delta_0}(0)} v_\varepsilon^{\gamma+1}. \end{aligned}$$

Note that for ε small enough, $\int |w_0|^\gamma v_\varepsilon = O(\varepsilon^{\frac{N-2}{4}})$, $\int_{\Omega \setminus B_{\delta_0}(0)} v_\varepsilon^{\gamma+1} = O(\varepsilon^{\frac{N-2}{4}(\gamma+1)})$, $\int |w_0| v_\varepsilon^\gamma = O(\varepsilon^{\frac{N-2}{4}\gamma})$ and $\int_{B_{\delta_0}(0)} v_\varepsilon^{\gamma+1} = O\left(\varepsilon^{\frac{[N-(\gamma+1)\sqrt{\Lambda}]\sqrt{\Lambda}}{2\sqrt{\Lambda-\lambda}}}\right)$. We obtain from the assumption $\frac{N-\sqrt{\Lambda}}{\sqrt{\Lambda}+\sqrt{\Lambda-\lambda}} < \gamma < 1$ and Proposition 2 that

$$\sup_{s>0} I(w_0 + sv_\varepsilon) \quad < \quad I(w_0) + \frac{1}{N} S_\lambda^{\frac{N}{2}} = c_0 + \frac{1}{N} S_\lambda^{\frac{N}{2}}.$$

When hypotheses (H_2) hold, instead of (H_1), the proof is similar so we omit the details. □

Proposition 5. *If hypotheses (H) hold, then there is a critical point $w_1 \in M^-$ of I such that $I(w_1) = c_-$. Moreover, if hypotheses (H_2) hold then $w_1 > 0$.*

Proof. First for to show that there is $w_1 \in M^-$ such that $I(w_1) = c_-$ and w_1 is a solution of Eq.(1), we use the item (1) and (2) of Proposition 3, and the same idea of the proof the Proposition 3.7 in Chen-Rocha [5]. We omit the details here.
Next we will show that $w_1 > 0$, if $f > 0$. From Lemma 2.1, there exist $t_+(u) \in \mathbb{R}$, such that $s_f \, t_+(|w_1|) > 0$, $t_+(|w_1|)|w_1| \in M^-$, $s_f \, t_+(|w_1|) > t_{\max}(|w_1|) = t_{\max}(w_1)$ and $I(t_+(|w_1|)|w_1|) = \max_{s_f t \geq 0} I(t(|w_1|)|w_1|)$. Since $w_1 \in M^-$, then $t_+(w_1) = 1$. Thus $I(t_+(w_1)w_1) = I(w_1) = \max_{s_f t \geq 0} I(tw_1) \geq I(t_+(|w_1|)w_1)$.
Note that, since $f > 0$, we have that $I(t_+(|w_1|)w_1) \geq I(t_+(|w_1|)|w_1|) \geq c_-$.
Therefore $I(t_+(w_1)w_1) = c_-$ and we can always take $w_1 > 0$. □

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REFERENCES

[1] F. Catrina and Z.Q. Wang, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence) and symmetry of extremal functions*, Comm. Pure. Appl. Math, **56** (2001), 229–258.

[2] N. Chaudhuri and M. Ramaswamy, *Existence of positive solutions of some semilinear elliptic equations with singular coefficients*, Proc. Roy. Soc. Edinburgh, **131A** (2001), 1275–1295.

[3] J. Chen, *On a semilinear elliptic equation with singular term and Hardy-Sobolev critical growth*, Math. Nachr. **280** (2007), 838–850.

[4] J. Chen, S.J. Li and Y.Q. Li, *Multiple solutions for a semilinear equation involving singular potential and critical exponent*, Z. angew. Math. Phys. **56** (2005), 453–474.

[5] J. Chen, E. M. Rocha, *Four solutions of an inhomogeneous elliptic equation with critical exponent and singular term*, Nonlinear Analysis, Elsevier, **71** (2009), 4739–4750.

[6] D. G. Costa and E. A. Silva, *A note on problems involving critical Sobolev exponents*, Diff. Integ. Eqns, **8** (1995), 673–679.

[7] A. Ferrero and F. Gazzola, *Existence of solutions for singular critical growth semilinear elliptic equations*, J. Differential Equations, **177** (2001), 494–522.

[8] X. He and W. Zou, *Infinitely many arbitrarily small solutions for sigular elliptic problems with critical Sobolev–Hardy exponents*, Proc. Edinb. Math. Soc. **52** (2009), 97–108.

[9] D. Kang and Y. Deng, *Multiple solutions for inhomogeneous elliptic problems involving critical Sobolev-Hardy exponents*, J. Math. Anal. Appl., **60** (2005), 729–753.

[10] S. Li and W. Zou, *Remarks on a class of elliptic problems with critical exponents*, Nonlinear Anal, **32** (1998), 769–774.

[11] D. Ruiz and M. Willem, *Elliptic problems with critical exponents and Hardy potentials*, J. Differential Equations, **190** (2003), 524–538.

[12] J. Sang, *Multiplicity results and bifurcation for nonlinear elliptic problems involving critical Sobolev exponents*, Nonlin. Analysis **23** (1994), 1493–1498.

[13] G. Tarantello, *On nonhomogeneous elliptic equations involving critical Sobolev exponent*, Ann. Inst. H.Poincaré Anal. Non Linéaire, **9** (1992), 281–309.

[14] G. Tarantello, *Multiplicity results for an inhomogeneous Neumann problem with critical exponent*, Manuscripta Math. **81** (1993), 51–78.

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