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ON FRACTIONAL OPERATORS IN STUMMEL SPACES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. We give boundedness results for the fractional maximal operator and the Riesz potential operator in the framework of Stummel spaces with variable exponents.

1. Introduction

For $0 < p < \infty$ and $0 < \lambda < n$, the Stummel space

$$\mathfrak{S}^{p,\lambda}(\mathbb{R}^n) := \left\{ f \in L^0(\mathbb{R}^n) \mid \|f\|_{\mathfrak{S}^{p,\lambda}} := \sup_{x\in\mathbb{R}^n} \left(\int\limits_{\mathbb{R}^n} \frac{|f(y)|^p}{|x-y|^\lambda} dy \right)^{1/p} < \infty \right\}$$

goes back at least as far as [16] for $p = 2$. In the case $p = 1$, such spaces were also studied in [7, 14] and $\mathfrak{S}^{1,n-2}(\mathbb{R}^n)$ is known as the Stummel–Kato class. Generalized Stummel spaces $\mathfrak{S}^{p,w}(\mathbb{R}^n)$, with the function $|\cdot|^\lambda$ replaced by w, were used in [4, 13, 15] in embedding results for global Morrey spaces. It is worth mentioning that Stummel spaces appear in applications to PDEs (see, e.g., [10–12]).

Very recently, the notion of Stummel spaces with variable exponents was introduced in [1] and, besides obtaining embedding results between Stummel and Morrey spaces with variable exponents, the boundedness of the maximal operator in Stummel and vanishing Stummel spaces with variable exponents was obtained.

The goal of this note is to present the boundedness results for the Riesz potential operator I_α in Stummel spaces with variable exponent, viz.,

$$I_\alpha : \mathfrak{S}^{p(\cdot),\lambda(\cdot)} \hookrightarrow \mathfrak{S}^{q(\cdot),\lambda(\cdot)}, \qquad \frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}.$$

Such result can be obtained by using Welland's inequality and the boundedness of the fractional maximal operator M_α in Stummel spaces with variable exponents, which in turn can be derived from a point-wise inequality and the boundedness of the maximal operator in weighted Lebesgue spaces. This involves uniform bounds for a family of Muckenhoupt weights with non-standard growth (cf., [3]). To the best of the author's knowledge, such results were never studied, even in the constant exponent Stummel spaces.

2. Preliminaries

By $\mathscr{P}$ we denote the class of all bounded measurable functions $p : \mathbb{R}^n \to [1, \infty)$ and define $p^- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$ and $p^+ := \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$.

The function $g : \mathbb{R}^n \to \mathbb{R}$ is said to be *locally* log*-Hölder continuous* if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leqslant \frac{c_{\log}(g)}{\log(\mathrm{e} + 1/|x-y|)}, \quad \text{for all } x, y \in \mathbb{R}^n, \tag{2.1}$$

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and satisfy the log-*Hölder continuity condition at infinity*, also known as the *decay condition*, if there exist $g_\infty \in [1,\infty)$ and $c^*_{\log}(g) > 0$ such that

$$|g(x) - g_\infty| \leqslant \frac{c^*_{\log}(g)}{\log(\mathrm{e} + |x|)}, \quad \text{for all } \ x \in \mathbb{R}^n. \tag{2.2}$$

The function g is said to be log-*Hölder continuous*, denoted by $g \in \mathscr{P}^{\log}$, when it satisfies (2.1) and (2.2).

The *variable exponents Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^n)$, $p \in \mathscr{P}$, is the space of measurable functions f such that

$$\varrho_{p(\cdot)}(f) := \int\limits_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty,$$

and normed by

$$\|f\|_{p(\cdot)} := \inf \left\{ \eta > 0 \mid \varrho_{p(\cdot)}\Big(\frac{f}{\eta}\Big) \leqslant 1 \right\}.$$

We refer to the books [5,6] for the basics on the theory of variable exponent Lebesgue spaces (see also [8,9] for applications of such spaces to integral operators).

The Stummel space with a variable exponent $\mathfrak{S}^{p(\cdot),\lambda(\cdot)}_{\Pi}(\mathbb{R}^n)$, with $\Pi \subseteq \mathbb{R}^n$, $p \in \mathscr{P}$ and $0 < \lambda(x) \leq \lambda_+ < n$, was introduced in [1,2], by

$$\mathfrak{S}^{p(\cdot),\lambda(\cdot)}_{\Pi}(\mathbb{R}^n) := \left\{ f \in L^0(\mathbb{R}^n) \mid \|f\|_{\mathfrak{S}^{p(\cdot),\lambda(\cdot)}} := \sup_{x\in\Pi} \left\| \frac{f}{|x-\cdot|^{\lambda(\cdot)}} \right\|_{p(\cdot)} < \infty \right\}.$$

Its vanishing counterpart $V\mathfrak{S}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ was defined in the aforementioned papers as

$$V\mathfrak{S}^{p(\cdot),\lambda(\cdot)}_{\Pi}(\mathbb{R}^n) := \left\{ f \in \mathfrak{S}^{p(\cdot),\lambda(\cdot)}_{\Pi}(\mathbb{R}^n) \mid \lim_{r\to 0} \sup_{x\in\Pi} \left\| \frac{f}{|x-\cdot|^{\lambda(\cdot)}} \mathbf{1}_{B(x,r)} \right\|_{p(\cdot)} < \infty \right\}.$$

3. Main Results

Recall that the *fractional maximal operator* M_α is defined, for $f \in L^1_{\mathrm{loc}}$ and $0 < \alpha < n$, by

$$M_\alpha f(x) = \sup_{r>0} \frac{1}{|B(x,r)|^{1-\frac{\alpha}{n}}} \int\limits_{B(x,r)} |f(y)| dy.$$

We start with a useful point-wise estimate.

Lemma 3.1. *Let $0 < \alpha < n$ and p be an exponent function such that $1 < p_- \leqslant p(x) \leqslant p_+ < n/\alpha$ and the function q is defined pointwise by $1/q(x) = 1/p(x) - \alpha/n$. Then for all $\xi \in \mathbb{R}^n$ and every $x \in \mathbb{R}^n$ such that $M_\alpha(f)(x) < \infty$, we have*

$$M_\alpha(f)(x) \leqslant \left[M\left(|f(\cdot)|^{\frac{p(\cdot)}{q(\cdot)}\frac{n}{n-\alpha}} |\xi-\cdot|^{\frac{\lambda(\cdot)\alpha p(\cdot)}{n-\alpha}} \right)(x) \right]^{\frac{n-\alpha}{n}} \left(\int\limits_{\mathbb{R}^n} \frac{|f(y)|^{p(y)}}{|\xi-y|^{\lambda(y)p(y)}} dy \right)^{\frac{\alpha}{n}}.$$

Now, we formulate the main results of this note:

Theorem 3.1. *Let $0 < \alpha < n$, $p \in \mathscr{P}^{\log}(\mathbb{R}^n)$ with $1 < p_- \leqslant p^+ < n/\alpha$, and q be defined by $1/q(x) = 1/p(x) - \alpha/n$. Moreover, let λ satisfy (2.1), (2.2) and $\lambda^- \geqslant 0$, with*

$$(p(\alpha+\lambda))^+ < n.$$

Then for a bounded set Π, we have

$$M_\alpha : \mathfrak{S}^{p(\cdot),\lambda(\cdot)}_{\Pi}(\mathbb{R}^n) \hookrightarrow \mathfrak{S}^{q(\cdot),\lambda(\cdot)}_{\Pi}(\mathbb{R}^n).$$

If p and λ are constant, then the result holds also for unbounded sets Π.

Let I_α be the *Riesz potential operator* given by

$$I_\alpha f(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n.$$

The boundedness of I_α in Stummel spaces with variable exponents is obtained via *Welland's inequality*

$$|I_\alpha f(x)| \leqslant C(\varepsilon, \alpha, n) \, |M_{\alpha-\varepsilon} f(x)|^{1/2} \, |M_{\alpha+\varepsilon} f(x)|^{1/2},$$

where $0 < \alpha < n$, $0 < \varepsilon < \max(\alpha, n-\alpha)$, and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 3.2. *Under the same assumptions of Theorem* 3.1*, we have*

$$I_\alpha : \mathfrak{S}_\Pi^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n) \hookrightarrow \mathfrak{S}_\Pi^{q(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$$

when the set Π *is bounded or* Π *is unbounded and* p, λ *are constant.*

Similar results of Theorems 3.1 and 3.2, under the additional assumption $p_\infty(\alpha + \lambda^+) < n$, are also valid for the vanishing Stummel spaces with variable exponents.

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