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Spectral theory for bounded banded matrices with positive bidiagonal factorization and mixed multiple orthogonal polynomials

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Spectral and factorization properties of oscillatory matrices lead to a spectral Favard theorem for bounded banded matrices, that admit a positive bidiagonal factorization, in terms of sequences of mixed multiple orthogonal polynomials with respect to a set positive Lebesgue–Stieltjes measures. A mixed multiple Gauss quadrature formula with corresponding degrees of precision is given.

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Contents

1. Introduction

This work is devoted to spectral theorems beyond self-adjoint or normal operators. We give conditions, the existence of a positive bidiagonal factorization, to be explained later, such we can state a spectral Favard theorem for bounded banded semi-infinite matrices. The study of symmetric tridiagonal operators acting in the Hilbert space ℓ^2 can under an appropriate chosen basis be reduced to an infinite Jacobi matrix enabling a deeper understanding of this spectral theory. Here is where the theory of general orthogonal polynomials comes into place to derive the spectral and the resolvent set for selfadjoint operators (cf. [44]).

Multiple orthogonal polynomials are traditionally linked with the theory of Hermite–Padé and its applications to the constructive function theory. For good introductions to multiple orthogonal polynomials see the book by Nikishin and Sorokin [44] and the chapter by Van Assche in [37, Ch. 23] and for a inspiring basic introduction see [42]. Multiple orthogonal polynomials are a very active research area: for asymptotic of zeros see [7], for a Gauss–Borel perspective see [2], for Christoffel perturbations see [19], for applications to random matrix theory see [16]. Mixed multiple orthogonal polynomials, and the corresponding Riemann–Hilbert problem, have found applications in Brownian bridges, or non-intersecting Brownian motions, that leave from p points and arrive to q points [26], and in the discussion of multicomponent Toda, cf. [1,2]. Mixed multiple orthogonal polynomials also appear in applications to number theory. Apéry, cf. [3], proved in 1979 that $\zeta(3)$ is irrational. The proof is based on an mixed Hermite–Padé approximation to three functions. Also mixed multiple orthogonal polynomials and corresponding mixed Hermite–Padé approximations have been used to show that infinitely many values of the ζ function at odd integers are irrational, cf. [11], and that at least one of the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational, cf. [52].

Another field of application of multiple orthogonal polynomials is the spectral analysis of high order difference operators. Spectral theorems hold beyond self-adjointness for normal operators (the operator commutes with its adjoint). In the case of banded Hessenberg operators, the self-adjointness or normality no more takes place.

A first attempt to tackle these problems has been made by Kalyagin in [38–40,8]. There the author defines a class of operators related with the Hermite–Padé approximants and connects their spectral analysis with the convergence problem for simultaneous Hermite–Padé rational approximants of a system of resolvent functions of the operator (that coincides with the notion of multiple orthogonality). See also [15]. The group comprising Aptekarev, Denisov, and Yattselev has also made significant contributions. Noteworthy among their works are those focused on self-adjoint Jacobi matrices on trees and multiple orthogonal polynomials [4], the asymptotics of coefficients and the essential spectrum of Jacobi matrices on trees generated by Angelescu systems [5], and lastly, their research on the spectral theory of Jacobi matrices on trees whose coefficients are generated by multiple orthogonality [6]. At the same time this connection leads to a new solution of the direct and inverse spectral problems for the operators based on the Jacobi–Perron algorithm and vector continued fractions (cf. [9,49,50]). This approach serves as a base of a new method of the investigation on nonlinear discrete dynamical systems. As an example, global solutions of a hierarchy of discrete KdV equations are obtained (cf. [9,51,12–14]).

Recently, in a series of works (cf. [18,20,23]) we have analyzed the applications of type I and II multiple orthogonal polynomials to certain Markov chains also called non simple random walks (i.e., beyond birth and death). At the end we got a spectral Favard theorem with an application to Markov chains described in terms of a bounded banded $(p+2)$-diagonal (with one superdiagonal and p subdiagonals) oscillatory Hessenberg operators that admit positive bidiagonal factorizations.

The main result we achieved in [20] is that bounded banded Hessenberg matrices that admit a positive bidiagonal factorization have a set of positive Lebesgue–Stieltjes measures, and can be spectrally described by multiple orthogonal polynomials. This extends to the non-normal scenario the spectral Favard theorem for Jacobi matrices (cf. [37]). An important feature of the method applied in [20] to describe the spectrality of a banded Hessenberg operator, is the multiple Gauss quadrature formula that we get with the exact degrees of precision.

In this paper we consider a bounded banded operator T whose semi-infinite matrix is a banded matrix with q superdiagonals and p subdiagonals and such that the leading principal submatrices are given by

$T^{[N]} =$

$$\begin{bmatrix}
T_{0,0} & \cdots & T_{0,q} & 0 & \cdots & 0 \\
\vdots & & & \ddots & & \vdots \\
T_{p,0} & & & & \ddots & 0 \\
0 & \ddots & & & & T_{N-q,N} \\
\vdots & & \ddots & & & \vdots \\
0 & \cdots & 0 & T_{N,N-p} & \cdots & T_{N,N}
\end{bmatrix} \tag{1}$$

where it is assumed that the entries in the extreme diagonals do not vanish

$$T_{n+p,n} \neq 0, \qquad T_{n,n+q} \neq 0, \qquad n \in \mathbb{N}_0. \tag{2}$$

In what follows we will show that when this matrix, after a shift, admits a positive bidiagonal factorization and, consequently, is oscillatory, we can find a spectral Favard theorem. Now, we have mixed multiple orthogonal polynomials with respect to a matrix of positive Lebesgue–Stieltjes measures. As an application we derive the corresponding Gauss quadrature formula for this matrix of measures and determine their degrees of precision.

While the extension given in this paper of the spectral Favard theorem of [20] for banded Hessenberg matrices to arbitrary banded matrices is natural, there were several key issues to resolve before achieving this large extension. The first one was to understand the role of the characteristic polynomial that is no longer an orthogonal polynomial, the second was to find the relationship of the determinants of the two families of mixed multiple orthogonal polynomials to the characteristic polynomial and its zeros, and finally the extension of the positive bidiagonal factorization to this general situation and the application of the oscillation properties of eigenvectors.

Within this introduction we discuss, in the first place, some preliminary material on totally nonnegative matrices, stating (without proof) the results needed later on. Then, we show how the well known bounded tridiagonal Jacobi matrix for which we have the spectral Favard theorem happens to be oscillatory after an adequate shift and have a positive bidiagonal factorization. Finally, we use the Gauss–Borel factorization of a moment matrix to construct mixed multiple orthogonality on the step-line.

1.1. Totally nonnegative and oscillatory matrices

Totally nonnegative (TN) matrices are those with all their minors nonnegative, cf. [30,35], and the set of nonsingular TN matrices is denoted by InTN. Oscillatory matrices, cf. [35], are totally nonnegative, irreducible [36] and nonsingular. Notice that the set of oscillatory matrices is denoted by IITN (irreducible invertible totally nonnegative) in [30]. An oscillatory matrix T is equivalently defined as a totally nonnegative matrix A such that for some n we have that A^n is totally positive (all minors are positive). From Cauchy–Binet Theorem one can deduce the invariance of these sets of matrices under the usual matrix product. Thus, following [30, Theorem 1.1.2] the product of matrices in InTN is again InTN (a similar statement hold for TN or oscillatory matrices).

We have the important result:

Theorem 1.1 *(Gantmacher–Krein Criterion). [35, Chapter 2, Theorem 10]. A totally non negative matrix is oscillatory if and only if it is nonsingular and the elements at the first subdiagonal and first superdiagonal are positive.*

Regarding tridiagonal matrices we have the following classical result:

Theorem 1.2. *[34, Chapter XIII,§9] and [35, Chapter 2, Theorem 11]. A tridiagonal matrix is oscillatory if and only if,*

i) *The matrix entries of the first subdiagonal and first superdiagonal are positive.*
ii) *All leading principal minors are positive.*

Gauss–Borel factorizations are intimately related with these concepts:

Theorem 1.3. *[30, Theorem 2.4.1]* $T \in \mathrm{InTN}$ *if and only if it admits a Gauss–Borel factorization* $T = L^{-1}U^{-1}$ *with* $L, U \in \mathrm{InTN}$*, lower and upper triangular matrices, respectively.*

The following spectral theorems are extracted from [30], see also [35].

Theorem 1.4 *(Eigenvalue). [30, Theorem 5.2.1] Given an oscillatory matrix* $T \in \mathbb{R}^{N\times N}$ *the eigenvalues of* T *are* N *distinct positive numbers.*

Theorem 1.5 *(Interlacing of eigenvalues). [30, Theorem 5.5.2] Given an oscillatory matrix* $T \in \mathbb{R}^{N\times N}$ *the eigenvalues of* T *strictly interlace the eigenvalues of the two principal submatrices of order* $(N-1)$*,* $T(1)$ *or* $T(N)$*, obtained from* T *by deleting the first row and column or the last row and column.*

We need to introduce the following notation. We define the total sign variation of a totally nonzero vector (no entry of the vector u is zero) as $v(u) = \text{cardinal}\{i \in \{1,\dots,n-1\} : u_iu_{i+1} < 0\}$. For a general vector $u \in \mathbb{R}^n$ we define $v_m(u)$ ($v_M(u)$) as the minimum (maximum) value $v(y)$ among all totally nonzero vectors y that coincide with u in its nonzero entries. For $v_m(u) = v_M(u)$ we write $v(u) := v_m(u) = v_M(u)$.

Theorem 1.6 *(Eigenvectors). Let $T \in \mathbb{R}^{N\times N}$ be an oscillatory matrix, and $u^{(k)}$ $(w^{(k)})$ the right (left) eigenvector corresponding to λ_k, the k-th largest eigenvalue of A. Then*

i) *[30, Theorem 5.3.3] We have $v_m(u^{(k)}) = v_M(u^{(k)}) = v(u^{(k)}) = k-1$ $(v_m(w^{(k)}) = v_M(w^{(k)}) = v(w^{(k)}) = k-1)$. Moreover, the first and last entry of $u^{(k)}$ $(w^{(k)})$ are nonzero, and $u^{(1)}$ and $u^{(N)}$ $(w^{(1)}$ and $w^{(N)})$ are totally nonzero; the other vectors may have a zero entry.*

ii) *From Perron–Frobenius theorem we know that $u^{(1)}$ $(w^{(1)})$ can be chosen to be entry-wise positive, and that the other eigenvectors $u^{(k)}$ $(w^{(k)})$, $k = 2,\dots,n$ have at least one entry sign change. In fact, $u^{(N)}$ $(w^{(N)})$ strictly alternates the sign of its entries.*

1.2. Jacobi matrices

Let us consider the tridiagonal semi-infinite real matrix

$$J := \begin{bmatrix} m_0 & 1 & 0 & \cdots & \\ \ell_1 & m_1 & 1 & \ddots & \\ 0 & \ell_2 & m_2 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

and assume that $\ell_k > 0$, $k \in \mathbb{N}$. This matrix is symmetrizable, as the positive diagonal matrix $H = \operatorname{diag}(H_0, H_1, \dots)$, $H_0 = 1$, $H_n := \ell_1 \cdots \ell_n$, is such that $H^{-\frac{1}{2}} J H^{\frac{1}{2}}$ is symmetric.

If the matrix J is bounded, all the possible eigenvalues of the submatrices $J^{[N]}$ belong to the disk $D(0, \|J\|)$. As all the eigenvalues are real, let us consider those that are negative, and let b be the supreme of the absolute values of all negative eigenvalues. Notice that $b \leqslant \|J\|$.

Theorem 1.7. *For $s \geqslant b$, the matrix $J_s = J + sI$ is oscillatory and admits a positive bidiagonal factorization in the form*

$$J_s = \begin{bmatrix} 1 & 0 & \cdots & \\ \alpha_2 & 1 & \ddots & \\ 0 & \alpha_4 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix} \begin{bmatrix} \alpha_1 & 1 & 0 & \cdots & \\ 0 & \alpha_3 & 1 & \ddots & \\ \vdots & \ddots & \alpha_5 & \ddots & \ddots \\ & & \ddots & \ddots & \ddots \end{bmatrix},$$

with $\alpha_n > 0$.

Proof. Take $s \geqslant b$, then J_s has the eigenvalues of its leading principal submatrices $J_s^{[N]} = J^{[N]} + sI_{N+1}$ all positive. The corresponding characteristic polynomials are $P_{N+1}(x-s) = \det\left(xI_{N+1} - J_s^{[N]}\right)$, so that $\det J_s^{[N]} = (-1)^{N+1}P_{N+1}(-s)$ and, as $-s$ is a lower bound for any possible zero of this monic polynomial, we find that $(-1)^{N+1}P_{N+1}(-s) > 0$. Hence, the leading principal minors of J_s are all positive and the entries on the subdiagonal a superdiagonal are positive. Thus, we conclude, attending to Theorem 1.2, that J_s is an oscillatory matrix.

The positive bidiagonal factorization is a consequence of Theorem 1.3 applied to J_s for $s \geq b$. □

The Favard spectral theorem, see [46], ensures for a Jacobi matrix J the existence of a unique probability measure $\mathrm{d}\,\psi$ such that $\int P_n(x)x^m \,\mathrm{d}\,\psi(x) = 0$, for $m \in \{0, \dots, n-1\}$, that is the characteristic polynomials are orthogonal polynomials, and moreover $\int x^n \,\mathrm{d}\,\psi(x) = (J^n)_{0,0}$. Thus, we see that the tridiagonal matrices to which the classical spectral Favard theorem applies are equivalently described as bounded tridiagonal matrices that after a convenient translation admit a positive bidiagonal factorization.

1.3. Mixed multiple orthogonal polynomials on the step-line

Mixed multiple orthogonal polynomials were first introduced in 1994 by Sorokin [48], and further extended in 1997 by him and van Iseghem in [49] when studying matrix orthogonality of vector polynomials. Ten years later, in 2004, it was rediscovered by Daems and Kuijlaars [26] in the context of multiple non-intersecting Brownian motions, where the name mixed multiple orthogonal was coined. It has been discussed also in [1,2,32,33]. Some of the forementioned papers deal $q \times p$ rectangular matrix of weights of rank 1 at each point of the support. However, the most fitted version for the discussion in this paper is the ones in [49] and [33] in where a $q \times p$ rectangular matrix of functionals or measures, respectively, are considered.

Let us present the mixed multiple orthogonal polynomials, on the step line, as they appear from the LU factorizations of a matrix of moments, following the ideas presented in [2].

Definition 1.8 *(Matrix of measures).* Let us consider a matrix of functions, which are right continuous and of bounded variation in a closed interval Δ, $\Psi = \begin{bmatrix} \psi_{1,1} & \cdots & \psi_{1,p} \\ \vdots & & \vdots \\ \psi_{q,1} & \cdots & \psi_{q,p} \end{bmatrix}$ and the associated matrix of Lebesgue–Stieltjes measures $\mathrm{d}\,\Psi = \begin{bmatrix} \mathrm{d}\,\psi_{1,1} & \cdots & \mathrm{d}\,\psi_{1,p} \\ \vdots & & \vdots \\ \mathrm{d}\,\psi_{q,1} & \cdots & \mathrm{d}\,\psi_{q,p} \end{bmatrix}$.

Definition 1.9 *(Monomial matrices).* Given $r \in \mathbb{N}$, we consider the semi-infinite matrices of monomials

$$X_{[r]} := \begin{bmatrix} I_r \\ xI_r \\ x^2 I_r \\ \vdots \end{bmatrix},$$

that, denoting its columns by $X_{[r]}^{(j)}$, we can write $X_{[r]} = \left[X_{[r]}^{(1)} \cdots X_{[r]}^{(r)}\right]$.

Definition 1.10 *(Shift matrices).* The shift matrix is given by $\ell_{[r]} := \begin{bmatrix} 0_r & I_r & 0_r & \cdots \\ 0_r & 0_r & I_r & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$.

Lemma 1.11. *Shift matrices act by left multiplication on monomial matrices according to*

$$\ell_{[r]} X_{[r]} = x X_{[r]}(x), \qquad \ell_{[r]} X_{[r]}^{(j)} = x X_{[r]}^{(j)}(x), \qquad j \in \{1, \dots, r\}.$$

Lemma 1.12. *If we denote by* $\ell := \ell_{[1]} = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$ *we have* $\ell_{[r]} = \ell^r$.

Definition 1.13 *(Moment matrix).* The matrix of moments is given by

$$\mathcal{M} := \int_\Delta X_{[q]}(x) \,\mathrm{d}\,\Psi(x) \big(X_{[p]}(x)\big)^\top = \int_\Delta \begin{bmatrix} \mathrm{d}\,\Psi(x) & x\,\mathrm{d}\,\Psi(x) & x^2\,\mathrm{d}\,\Psi(x) \cdots \\ x\,\mathrm{d}\,\Psi(x) & x^2\,\mathrm{d}\,\Psi(x) & x^3\,\mathrm{d}\,\Psi(x) \cdots \\ x^2\,\mathrm{d}\,\Psi(x) & x^3\,\mathrm{d}\,\Psi(x) & x^4\,\mathrm{d}\,\Psi(x) \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$= \int_\Delta \left[\begin{array}{ccc|ccc|c} \mathrm{d}\,\psi_{1,1}(x) & \cdots & \mathrm{d}\,\psi_{1,p}(x) & x\,\mathrm{d}\,\psi_{1,1}(x) & \cdots & x\,\mathrm{d}\,\psi_{1,p}(x) & \\ \vdots & & \vdots & \vdots & & \vdots & \cdots \\ \mathrm{d}\,\psi_{q,1}(x) & \cdots & \mathrm{d}\,\psi_{q,p}(x) & x\,\mathrm{d}\,\psi_{q,1}(x) & \cdots & x\,\mathrm{d}\,\psi_{q,p}(x) & \\ \hline x\,\mathrm{d}\,\psi_{1,1}(x) & \cdots & x\,\mathrm{d}\,\psi_{1,p}(x) & x^2\,\mathrm{d}\,\psi_{1,1} & \cdots & x^2\,\mathrm{d}\,\psi_{1,p}(x) & \\ \vdots & & \vdots & \vdots & & \vdots & \cdots \\ x\,\mathrm{d}\,\psi_{q,1}(x) & \cdots & x\,\mathrm{d}\,\psi_{q,p}(x) & x^2\,\mathrm{d}\,\psi_{q,1}(x) & \cdots & x^2\,\mathrm{d}\,\psi_{q,p}(x) & \\ \hline & \vdots & & & \vdots & & \end{array}\right].$$

Lemma 1.14. *The moment matrices are a structured matrix built up with* $q \times p$ *blocks* $\mathcal{M}_{n,m} := \int_\Delta x^{n+m} \,\mathrm{d}\,\Psi(x) \in \mathbb{R}^{q \times p}$. *In fact, is Hankel by blocks, i.e.* $\mathcal{M}_{n+1,m} = \mathcal{M}_{n,m+1}$.

This fact can be reformulated as follows:

Proposition 1.15. *The moment matrix satisfies*

$$\ell_{[q]}\mathcal{M} = \mathcal{M}(\ell_{[p]})^\top. \tag{3}$$

Proof. It follows from

$$\begin{aligned}\ell_{[q]}\mathcal{M} &= \int\limits_{\Delta} \ell_{[q]} X_{[q]}(x)\,\mathrm{d}\,\Psi(x)(X_{[p]}(x))^\top = \int\limits_{\Delta} x X_{[q]}(x)\,\mathrm{d}\,\Psi(x)(X_{[p]}(x))^\top \\ &= \int\limits_{\Delta} X_{[q]}(x)\,\mathrm{d}\,\Psi(x)(X_{[p]}(x))^\top(\ell_{[p]})^\top. \quad \square\end{aligned}$$

Now, let us assume that the moment matrix $\mathcal{M}$ has a Gauss–Borel factorization, i.e. $\mathcal{M} = L^{-1}U^{-1}$, where L is lower triangular and U upper triangular, such that none of the diagonal entries in both triangular matrices are zero, and the respective inverse matrices make sense. It is well known that such LU factorizations do exist whenever the leading principal submatrices are nonsingular.

Definition 1.16 *(Matrix polynomials).* Let us consider the matrices

$$B := LX_{[q]} = \begin{bmatrix} \mathscr{B}_0 \\ \mathscr{B}_1 \\ \vdots \end{bmatrix}, \qquad A := (X_{[p]})^\top U = \begin{bmatrix} \mathscr{A}_0 & \mathscr{A}_1 & \cdots \end{bmatrix},$$

where $\mathscr{B}_n(x)$ is a $q \times q$ matrix polynomial and $\mathscr{A}_n(x)$ is a $p \times p$ matrix polynomial. Observe that

$$\begin{aligned} B &= \begin{bmatrix} B^{(1)} & \cdots & B^{(q)} \end{bmatrix}, & B^{(b)} &= LX^{(b)}_{[q]}, & b &\in \{1,\dots,q\}, \\ A &= \begin{bmatrix} A^{(1)} \\ \vdots \\ A^{(p)} \end{bmatrix}, & A^{(a)} &= (X^{(a)}_{[p]})^\top U, & a &\in \{1,\dots,p\}. \end{aligned}$$

Lemma 1.17. *The semi-infinite vectors*

$$B^{(b)} = \begin{bmatrix} B^{(b)}_0 \\ B^{(b)}_1 \\ \vdots \end{bmatrix}, \qquad A^{(a)} = \begin{bmatrix} A^{(a)}_0 & A^{(a)}_1 & \cdots \end{bmatrix},$$

have as entries polynomials with degrees

$$\deg B^{(b)}_n = \left\lceil \frac{n+2-b}{q} \right\rceil - 1, \qquad \deg A^{(a)}_n = \left\lceil \frac{n+2-a}{p} \right\rceil - 1. \tag{4}$$

For the block polynomials we have

$$\mathscr{B}_n=\begin{bmatrix} B^{(1)}_{nq} & \cdots\cdots\cdots & B^{(q)}_{nq}\\ \vdots & & \vdots\\ B^{(1)}_{nq+q-1} & \cdots\cdots & B^{(q)}_{nq+q-1}\end{bmatrix}, \qquad \mathscr{A}_n=\begin{bmatrix} A^{(1)}_{np} & \cdots\cdots & A^{(1)}_{np+p-1}\\ \vdots & & \vdots\\ A^{(p)}_{np} & \cdots\cdots & A^{(p)}_{np+p-1}\end{bmatrix}.$$

Lemma 1.18.

i) *For $r,n\in\mathbb{N}$, $n\geqslant r$, we have*

$$\sum_{a=1}^{r}\left\lceil\frac{n+1-a}{r}\right\rceil=n. \tag{5}$$

ii) *For the degrees we find $\sum_{b=1}^{q}(\deg B^{(b)}_n+1)=\sum_{a=1}^{p}(\deg A^{(a)}_n+1)=n+1$.*

Proof. i) Let us consider $n=kr+j$, where $j=0,\dots r-1$. For $a=1,\dots,j$, we have that

$$\left\lceil\frac{n+1-a}{r}\right\rceil=\left\lceil\frac{kr+j+1-a}{r}\right\rceil=k+\left\lceil\frac{j+1-a}{r}\right\rceil=k+1.$$

For $a=j+1,\dots,r$, we find

$$\begin{aligned}\left\lceil\frac{n+1-a}{r}\right\rceil=\left\lceil\frac{kr+j+1-a}{r}\right\rceil&=\left\lceil\frac{(k-1)r+r-a+j+1}{r}\right\rceil\\ &=(k-1)+\left\lceil\frac{r-a+j+1}{r}\right\rceil=k.\end{aligned}$$

Therefore, Equation (5) follows.

ii) It follows from the previous result and (4). □

Proposition 1.19 *(Biorthogonality). The following biorthogonality holds*

$$\int_{\Delta} B(x)\,\mathrm{d}\,\Psi(x)A(x)=I,$$

that entrywise can be written

$$\sum_{b=1}^{q}\sum_{a=1}^{p}\int_{\Delta} B^{(b)}_n(x)\,\mathrm{d}\,\psi_{b,a}(x)A^{(a)}_m(x)=\delta_{n,m}, \qquad n,m\in\mathbb{N}_0. \tag{6}$$

Proof. From the definition of the moment matrix $\mathcal{M}$ and the Gauss–Borel factorization $\mathcal{M} = L^{-1}U^{-1}$ we get

$$\int_{\Delta} X_{[q]}(x)\,\mathrm{d}\,\Psi(x)\big(X_{[p]}(x)\big)^{\top} = L^{-1}U^{-1}.$$

By left and right multiplication by the triangular factors L and U, respectively, we get

$$\int_{\Delta} LX_{[q]}(x)\,\mathrm{d}\,\Psi(x)\big(X_{[p]}(x)\big)^{\top}U = I$$

and recalling the definition of B and A we deduce that $\int_{\Delta} B(x)\,\mathrm{d}\,\Psi(x)A(x) = I$, and the result follows immediately. □

Remark 1.20 *(Matrix biorthogonality).*

i) If $p = q$, we recover the well-known matrix biorthogonality, see [27,47,28] for matrix orthogonality,

$$\int_{\Delta} \mathscr{B}_n(x)\,\mathrm{d}\,\Psi(x)\mathscr{A}_m(x) = \delta_{n,m}I_p, \qquad n,m \in \mathbb{N}_0.$$

ii) For $p \neq q$, this matrix orthogonality is lost. However, if we denote for $r \in \mathbb{N}$ by $\mathscr{B}_n^{[r]}$ the $r \times q$ matrix of polynomials obtained from the $\infty \times q$ matrix B by taking consecutive submatrices of size $r \times q$, and similarly for $\mathscr{A}_n^{[r]}$, i.e., obtained from the $p \times \infty$ matrix A consecutive submatrices of size $p \times r$, we get the following generalized matrix biorthogonality

$$\int_{\Delta} \mathscr{B}_n^{[r]}\,\mathrm{d}\,\Psi(x)\mathscr{A}_m^{[r]}(x) = \delta_{n,m}I_r, \qquad n,m \in \mathbb{N}_0.$$

From biorthogonality (6) we get mixed multiple orthogonal relations as follows:

Corollary 1.21 *(Mixed multiple orthogonality). The following orthogonality relations*

$$\sum_{a=1}^{p}\int_{\Delta} A_n^{(a)}(x)\,\mathrm{d}\,\psi_{b,a}(x)x^m = 0, \qquad m \in \Big\{0,\dots,\deg B_{n-1}^{(b)}\Big\}, \qquad b \in \{1,\dots,q\},$$

$$\sum_{b=1}^{q}\int_{\Delta} B_n^{(b)}(x)\,\mathrm{d}\,\psi_{b,a}(x)x^m = 0, \qquad m \in \Big\{0,\dots,\deg A_{n-1}^{(a)}\Big\}, \qquad a \in \{1,\dots,p\},$$

are satisfied.

Remark 1.22. For $q = 1$ the mixed multiple orthogonality is the well known multiple orthogonality, or p-orthogonality, with $A_n^{(a)}$ the type I multiple orthogonal polynomials and the B_n the type II multiple orthogonal polynomials, see [44,37].

We now discuss the connection of the Gauss–Borel factorization of the moment matrix and the Cauchy transforms.

Definition 1.23. Let us consider the formal semi-infinite matrices

$$C(z) := z^{-1}\big(X_{[q]}(z^{-1})\big)^\top L^{-1} = \begin{bmatrix}\mathscr{C}_0 & \mathscr{C}_1 & \cdots\end{bmatrix}, \qquad D(z) := z^{-1}U^{-1}X_{[p]}(z^{-1}) = \begin{bmatrix}\mathscr{D}_0\\ \mathscr{D}_1\\ \vdots\end{bmatrix}. \tag{7}$$

With $\mathscr{C}_n, \mathscr{D}_n$ being $q \times p$ rectangular blocks.

Remark 1.24. The previous definition is formal, as the product of matrices involves series instead of sums, hence to have a meaning we must ensure the convergence of these series.

Remark 1.25. We have that

$$C = \begin{bmatrix}C^{(1)}\\ \vdots\\ C^{(q)}\end{bmatrix}, \qquad D = \begin{bmatrix}D^{(1)} & \cdots & D^{(p)}\end{bmatrix},$$

where $C^{(b)} = \begin{bmatrix}C_0^{(b)} & C_1^{(b)} & \cdots\end{bmatrix}$ are semi-infinite row vectors and $D^{(b)} = \begin{bmatrix}D_0^{(b)}\\ D_1^{(b)}\\ \vdots\end{bmatrix}$ semi-infinite column vectors. The block matrices are

$$\mathscr{C}_n = \begin{bmatrix}C_{np}^{(1)} & \cdots & C_{np+p-1}^{(1)}\\ \vdots & & \vdots\\ C_{np}^{(q)} & \cdots & C_{np+p-1}^{(q)}\end{bmatrix}, \qquad \mathscr{D}_n = \begin{bmatrix}D_{nq}^{(1)} & \cdots & D_{nq}^{(p)}\\ \vdots & & \vdots\\ D_{nq+q-1}^{(1)} & \cdots & D_{nq+q-1}^{(p)}\end{bmatrix}.$$

Proposition 1.26 *(Cauchy transforms). Let us assume that z belongs to the exterior of a disk centered at the origin that includes all the supports* $\operatorname{supp} \mathrm{d}\,\psi_{b,a}$*, for $b \in \{1, \dots, q\}$ and $a \in \{1, \dots, p\}$. Then, the matrices in (7) are the following Cauchy transforms*

$$C(z) = \int \frac{\mathrm{d}\,\Psi(x)}{z - x} A(x), \qquad D(z) = \int B(x) \frac{\mathrm{d}\,\Psi(x)}{z - x}.$$

Proof. We have

$$C(z) = z^{-1}\big(X_{[q]}(z^{-1})\big)^\top \mathcal{M} U = z^{-1} X_{[q]}^\top(z^{-1}) \int X_{[q]}(x)\, \mathrm{d}\,\Psi(x) (X_{[p]}(x))^\top U.$$

Now, notice that for $|x| < |z|$ it holds that

$$z^{-1}\big(X_{[r]}(z^{-1})\big)^{\top}X_{[r]}(x) = \frac{1}{z}\sum_{n=0}^{\infty}\frac{x^n}{z^n}I_r = \frac{I_r}{z-x},$$

so that, recalling that $(X_{[p]}(x))^{\top}U = A(x)$, we get $C(z) = \int \frac{\mathrm{d}\,\Psi(x)}{z-x}A(x)$. Analogously,

$$\begin{aligned} D(z) = L\mathcal{M}X_{[p]}(z^{-1})z^{-1} &= \int LX_{[q]}(x)\,\mathrm{d}\,\Psi(x)(X_{[p]}(x))^{\top}X_{[p]}(z^{-1})z^{-1} \\ &= \int B(x)\frac{\mathrm{d}\,\Psi(x)}{z-x}. \quad \square \end{aligned}$$

Remark 1.27. Entrywise, we find

$$C^{(b)}_n(z) = \sum_{a=1}^{p}\int\frac{\mathrm{d}\,\psi_{b,a}(x)}{z-x}A^{(a)}_n(x), \qquad D^{(a)}_n(z) = \sum_{b=1}^{q}\int B^{(b)}_n(x)\frac{\mathrm{d}\,\psi_{b,a}(x)}{z-x},$$

and block entrywise

$$\mathscr{C}_n(z) = \int\frac{\mathrm{d}\,\Psi(x)}{z-x}\mathscr{A}_n(x), \qquad \mathscr{D}_n(z) = \int \mathscr{B}_n(x)\frac{\mathrm{d}\,\Psi(x)}{z-x}.$$

Now, let us discuss how these polynomials connect with the matrix Hermite–Padé problem as considered in [51]. For that aim, we first introduce:

Definition 1.28 *(Stieltjes–Markov functions).* Let us consider the Stieltjes–Markov functions given by

$$\hat{\psi}_{b,a}(z) := \int_{\Delta}\frac{\mathrm{d}\,\psi_{b,a}(x)}{z-x},$$

i.e., the Cauchy transforms of the measures. We also introduce two families of polynomials of the second kind linked to the orthogonal polynomials:

$$\begin{aligned} R^{(a)}_n(z) &:= \sum_{b=1}^{q}\int_{\Delta}\frac{B^{(b)}_n(z)-B^{(b)}_n(x)}{z-x}\,\mathrm{d}\,\psi_{b,a}(x), \\ Q^{(b)}_n(z) &:= \sum_{a=1}^{p}\int_{\Delta}\frac{A^{(a)}_n(z)-A^{(a)}_n(x)}{z-x}\,\mathrm{d}\,\psi_{b,a}(x). \end{aligned}$$

Proposition 1.29 *(Matrix Hermite–Padé).*

i) *The following simultaneous approximation holds*

$$\sum_{b=1}^{q} B_n^{(b)}(z)\hat{\psi}_{b,a}(z) = R_n^{(a)}(z) + O\left(\frac{1}{z^{n_a+1}}\right), \qquad z \to \infty,$$

with

$$n_a := \deg A_{n-1}^{(a)} + 1 = \left\lceil \frac{n+1-a}{p} \right\rceil.$$

We have $\sum_{a=1}^{p} n_a = n$ *and* $\sum_{b=1}^{q}(\deg B_n^{(b)} + 1) = n + 1$.

ii) *Analogously, the simultaneous approximation*

$$\sum_{a=1}^{p} \hat{\psi}_{b,a}(z) A_n^{(a)}(z) = Q_n^{(b)}(z) + O\left(\frac{1}{z^{m_b+1}}\right), \qquad z \to \infty,$$

with

$$m_b := \deg B_{n-1}^{(b)} + 1 = \left\lceil \frac{n+1-b}{q} \right\rceil,$$

is satisfied. We have $\sum_{b=1}^{q} m_b = n$ *and* $\sum_{a=1}^{p}(\deg A_n^{(a)} + 1) = n + 1$.

Proof. Let us check only the first case. The other follows by similar arguments. Observe that

$$\begin{aligned}\sum_{b=1}^{q} B_n^{(b)}(z)\hat{\psi}_{b,a}(z) &= \sum_{b=1}^{q} B_n^{(b)}(z) \int_{\Delta} \frac{\mathrm{d}\,\psi_{b,a}(x)}{z-x} \\ &= \sum_{b=1}^{q} \int_{\Delta} \frac{B_n^{(b)}(z) - B_n^{(b)}(x)}{z-x}\,\mathrm{d}\,\psi_{b,a}(x) + \sum_{b=1}^{q} \int_{\Delta} \frac{B_n^{(b)}(x)}{z-x}\,\mathrm{d}\,\psi_{b,a}(x) \\ &= R_n^{(a)}(x) + \sum_{k=0}^{+\infty} \frac{1}{z^{k+1}} \sum_{b=1}^{q} \int_{\Delta} B_n^{(b)}(x) x^k\,\mathrm{d}\,\psi_{b,a}(x).\end{aligned}$$

Using now the mixed multiple orthogonality conditions, see Corollary 1.21, we get the result. The degrees follow from Lemma 1.18. This is exactly the matrix Hermite–Padé problem that appears in [51]. □

Finally, we discuss the recursion relations

Proposition 1.30 *(Banded recursion matrix).*

i) *The following relation is fulfilled*

$$L\ell_{[q]}L^{-1} = U^{-1}(\ell_{[p]})^{\top}U.$$

ii) *The semi-infinite matrix* $T := L\ell_{[q]}L^{-1} = U^{-1}(\ell_{[p]})^{\top}U$ *is a banded matrix with* p *subdiagonals,* q *superdiagonals, where the* p*-th and* q*-th sub and superdiagonal entries are nonzero.*

iii) *The following recursion relations hold*

$$TB = xB, \qquad AT = xA.$$

Proof. i) From the Gauss–Borel factorization and (3) we get

$$\ell_{[q]}L^{-1}U^{-1} = L^{-1}U^{-1}(\ell_{[p]})^{\top},$$

so that

$$L\ell_{[q]}L^{-1} = U^{-1}(\ell_{[p]})^{\top}U.$$

ii) The matrix $L\ell_{[q]}L^{-1}$ has all its superdiagonals above the first q-th superdiagonal with zero entries, while $U^{-1}(\ell_{[p]})^{\top}U$ has all its subdiagonals below the first p-th subdiagonal with zero entries. Hence, T is a general banded matrix with $p+q+1$ diagonals possibly with nonzero entries. The p-th and q-th sub and superdiagonal entries are nonzero, taking into account that L is lower triangular and U an upper triangular such that none of the diagonal entries in both triangular matrices are zero.

iii) By definition $B = LX_{[q]}$ so that $TB = L\ell_{[q]}L^{-1}LX_{[q]} = L\ell_{[q]}X_{[q]} = xB$. Similarly, also by definition, we have $A = (X_{[p]})^{\top}U$ so that $AT = (X_{[p]})^{\top}UU^{-1}(\ell_{[p]})^{\top}U = (X_{[p]})^{\top}(\ell_{[p]})^{\top}U = xA$. □

This banded recursion matrix is the object of study of this paper. It will be the departure point in the next sections. We have considered a matrix of measures and the associated matrix of moments and derived the mixed multiple orthogonality as well as the banded recursion matrix. The aim in this paper is to get conditions on the banded matrix so that we can go back this way, to retrieve the matrix of positive measures and the mixed multiple orthogonal polynomials from the recursion matrix; i.e., to get a spectral Favard theorem.

2. Recursion polynomials and the characteristic polynomial

We begin by introducing the recursion polynomials associated to the banded matrix T, with truncations given in (1), as the entries of semi-infinite left and right eigenvectors:

Definition 2.1 *(Left and right recursion polynomials).* Associated with the semi-infinite banded matrix T we consider the semi-infinite vectors

$$A^{(a)}=\begin{bmatrix}A_0^{(a)} & A_1^{(a)} & \cdots\end{bmatrix}, \qquad a\in\{1,\dots,p\}, \qquad B^{(b)}=\begin{bmatrix}B_0^{(b)}\\ B_1^{(b)}\\ \vdots\end{bmatrix}, \qquad b\in\{1,\dots,q\},$$

that are left and right eigenvectors with eigenvalue x of T, i.e.

$$A^{(a)}T=xA^{(a)}, \qquad a\in\{1,\dots,p\}, \qquad TB^{(b)}=xB^{(b)}, \qquad b\in\{1,\dots,q\}.$$

The entries of these left and right eigenvectors are polynomials in the eigenvalue x, known as left and right recursion polynomials, respectively, determined by the initial conditions

$$\begin{cases}A_0^{(1)}=1,\\ A_1^{(1)}=\nu_1^{(1)},\\ \qquad\vdots\\ A_{p-1}^{(1)}=\nu_{p-1}^{(1)},\end{cases} \qquad \begin{cases}A_0^{(2)}=0,\\ A_1^{(2)}=1,\\ A_2^{(2)}=\nu_2^{(2)},\\ \qquad\vdots\\ A_{p-1}^{(2)}=\nu_{p-1}^{(2)},\end{cases} \qquad \cdots \qquad \begin{cases}A_0^{(p)}=0,\\ \qquad\vdots\\ A_{p-2}^{(p)}=0,\\ A_{p-1}^{(p)}=1,\end{cases} \tag{8}$$

with $\nu_j^{(i)}$ being arbitrary constants, and

$$\begin{cases}B_0^{(1)}=1,\\ B_1^{(1)}=\xi_1^{(1)},\\ \qquad\vdots\\ B_{q-1}^{(1)}=\xi_{q-1}^{(1)},\end{cases} \qquad \begin{cases}B_0^{(2)}=0,\\ B_1^{(2)}=1,\\ B_2^{(2)}=\xi_2^{(2)},\\ \qquad\vdots\\ B_{q-1}^{(2)}=\xi_{q-1}^{(2)},\end{cases} \qquad \cdots \qquad \begin{cases}B_0^{(q)}=0,\\ \qquad\vdots\\ B_{q-2}^{(q)}=0,\\ B_{q-1}^{(q)}=1,\end{cases} \tag{9}$$

with $\xi_j^{(i)}$ also being arbitrary, respectively. We also define the initial condition matrices

$$\nu:=\begin{bmatrix}1 & 0 & \cdots & \cdots & 0\\ \nu_1^{(1)} & 1 & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ \nu_{p-1}^{(1)} & \cdots & \cdots & \nu_{p-1}^{(p-1)} & 1\end{bmatrix}, \qquad \xi:=\begin{bmatrix}1 & 0 & \cdots & \cdots & 0\\ \xi_1^{(1)} & 1 & \ddots & & \vdots\\ \vdots & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & 0\\ \xi_{q-1}^{(1)} & \cdots & \cdots & \xi_{q-1}^{(q-1)} & 1\end{bmatrix}.$$

Once the initial conditions are fixed, the remaining polynomials are found by:

Proposition 2.2 *(General recursion relations). The recursion polynomials are uniquely determined by the initial conditions* (8) *and* (9) *and the recursion relations*

$$\begin{aligned} &A^{(a)}_{n-q}T_{n-q,n}+\cdots+A^{(a)}_{n+p}T_{n+p,n}=xA^{(a)}_n, && n\in\{0,1,\dots\}, \quad a\in\{1,\dots,p\},\\ & && A^{(a)}_{-q}=\cdots=A^{(a)}_{-1}=0,\\ &T_{n,n-p}B^{(b)}_{n-p}+\cdots+T_{n,n+q}B^{(b)}_{n+q}=xB^{(b)}_n, && n\in\{0,1,\dots\}, \quad b\in\{1,\dots,q\},\\ & && B^{(b)}_{-p}=\cdots=B^{(b)}_{-1}=0. \end{aligned} \tag{10}$$

We use the ceiling function $\lceil x\rceil$ that maps x to the least integer greater than or equal to x.

Proposition 2.3. *For the degrees of the left and right recursion polynomials we find*

$$\deg A^{(a)}_n=\left\lceil\frac{n+2-a}{p}\right\rceil-1, \qquad \deg B^{(b)}_n=\left\lceil\frac{n+2-b}{q}\right\rceil-1.$$

Proof. By inspection we can check that, for $j\in\{1,\dots,p\}$ and $k\in\mathbb{N}_0$, it holds that $\deg A^{(a)}_{kp+j}=k$, for $a\in\{1,\dots,j+1\}$ and $\deg A^{(a)}_{kp+j}=k-1$ for $a\in\{j+2,\dots,p\}$ and that, for $j\in\{1,\dots,q\}$ and $k\in\mathbb{N}_0$, $\deg B^{(b)}_{kq+j}=k$, for $b\in\{1,\dots,j+1\}$ and $\deg B^{(b)}_{kp+j}=k-1$ for $b=\{j+2,\dots,q\}$.

However, we notice that

$$\left\lceil\frac{n+2-a}{p}\right\rceil-1=\left\lceil\frac{kp+j+2-a}{p}\right\rceil-1=k-1+\left\lceil\frac{j+2-a}{p}\right\rceil$$

but

$$\left\lceil\frac{j+2-a}{p}\right\rceil=\begin{cases}1, & a\in\{1,\dots,j+1\},\\ 0, & a\in\{j+2,\dots,p\},\end{cases}$$

and the stated result follows. For the recursion polynomials $B^{(b)}_n$ we proceed analogously. □

Definition 2.4 *(Characteristic polynomials).* For the semi-infinite matrix T we consider the polynomials $P_N(x)$ as the characteristic polynomials of the truncated matrices $T^{[N-1]}$, i.e.,

$$P_N(x):=\begin{cases}1, & N=0,\\ \det\left(xI_N-T^{[N-1]}\right), & N\in\mathbb{N}.\end{cases}$$

Obviously, $\deg P_N=N$. For Hessenberg matrices [20] it happens that the characteristic polynomials up to a factor coincide with the right recursion polynomials. However, for the banded situation this does not hold in general. Nevertheless, there is a relation between determinants of the recursion polynomials, right or left, with the characteristic polynomials of the banded matrix T. Let us see this.

Definition 2.5. Let us introduce the following matrices of left and right recursion polynomials

$$A_N := \begin{bmatrix} A_N^{(1)} & \cdots & A_{N+p-1}^{(1)} \\ \vdots & & \vdots \\ A_N^{(p)} & \cdots & A_{N+p-1}^{(p)} \end{bmatrix}, \qquad B_N := \begin{bmatrix} B_N^{(1)} & \cdots & B_N^{(q)} \\ \vdots & & \vdots \\ B_{N+q-1}^{(1)} & \cdots & B_{N+q-1}^{(q)} \end{bmatrix}, \qquad N \in \mathbb{N}_0,$$

and the following products

$$\begin{aligned} \alpha_N &:= (-1)^{(p-1)N} T_{p,0} \cdots T_{N+p-1,N-1}, \\ \beta_N &:= (-1)^{(q-1)N} T_{0,q} \cdots T_{N-1,N+q-1}, \qquad N \in \mathbb{N}, \end{aligned}$$

and $\alpha_0 = \beta_0 = 1$.

Remark 2.6. These are inspired by the matrix polynomials blocks given in the Gauss–Borel construction of mixed multiple orthogonality, see Lemma 1.17. In fact, for $M \in \mathbb{N}_0$, $A_{Mp} = \mathscr{A}_M$ and $B_{Mq} = \mathscr{B}_M$.

Recall that as the entries in the extreme diagonals do not vanish (2) we have that $\alpha_N, \beta_N \neq 0$. In terms of these objects we found the following important result:

Theorem 2.7. *For $N \in \mathbb{N}_0$, the characteristic polynomials and determinants of left and right recursion polynomial blocks satisfy*

$$P_N(x) = \alpha_N \det A_N(x) = \beta_N \det B_N(x).$$

Proof. For $N = 0$ we have that $\det A_0 = \det \nu = 1$. For $N = 1$ we get

$$T_{p,0} \det A_1 = \begin{vmatrix} A_1^{(1)} & \cdots & A_{p-1}^{(1)} & T_{p,0} A_p^{(1)} \\ \vdots & & \vdots & \vdots \\ A_1^{(p)} & \cdots & A_{p-1}^{(p)} & T_{p,0} A_p^{(p)} \end{vmatrix} = \begin{vmatrix} A_1^{(1)} & \cdots & A_{p-1}^{(1)} & (x - T_{0,0}) A_0^{(1)} \\ \vdots & & \vdots & \vdots \\ A_1^{(p)} & \cdots & A_{p-1}^{(p)} & (x - T_{0,0}) A_0^{(p)} \end{vmatrix}$$

where we have used the recursion (10) in the last column of this determinant. Now we express this last determinant as the following product of determinants

$$
\begin{vmatrix} A_1^{(1)} & \cdots & A_{p-1}^{(1)} & (x-T_{0,0})A_0^{(1)} \\ \vdots & & \vdots & \vdots \\ A_1^{(p)} & \cdots & A_{p-1}^{(p)} & (x-T_{0,0})A_0^{(p)} \end{vmatrix} = \begin{vmatrix} A_0^{(1)} & \cdots & A_{p-1}^{(1)} \\ \vdots & & \vdots \\ A_0^{(p)} & \cdots & A_{p-1}^{(p)} \end{vmatrix} \left|\begin{array}{c|c} 0 \cdots\cdots 0 & x-T_{0,0} \\ \hline \multirow{3}{*}{I_{p-1}} & 0 \\ & \vdots \\ & 0 \end{array}\right|
$$

$$
= (-1)^{p+1}(x-T_{0,0}).
$$

We proceed similarly up to $N=p-1$, so for $N \in \{2,\dots,p-1\}$, we get that

$$
T_{p,0}T_{p+1,1}\dots T_{N+p-1,N-1}\det A_N := \begin{vmatrix} A_N^{(1)} & \cdots & A_{p-1}^{(1)} & T_{p,0}A_p^{(1)} & \cdots & T_{N+p-1,N-1}A_{N+p-1}^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_N^{(p)} & \cdots & A_{p-1}^{(p)} & T_{p,0}A_p^{(p)} & \cdots & T_{N+p-1,N-1}A_{N+p-1}^{(p)} \end{vmatrix}
$$

$$
= \begin{vmatrix} A_N^{(1)} & \cdots & A_{p-1}^{(1)} & (x-T_{0,0})A_0^{(1)} - T_{1,0}A_1^{(1)} - \cdots - T_{N-1,0}A_{N-1}^{(1)} & \dots & -T_{0,N-1}A_0^{(1)} - T_{1,N-1}A_1^{(1)} - \cdots + (x-T_{N-1,N-1})A_{N-1}^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_N^{(p)} & \cdots & A_{p-1}^{(p)} & (x-T_{0,0})A_0^{(p)} - T_{1,0}A_1^{(p)} - \cdots - T_{N-1,0}A_{N-1}^{(p)} & \dots & -T_{0,N-1}A_0^{(p)} - T_{1,N-1}A_1^{(p)} - \cdots + (x-T_{N-1,N-1})A_{N-1}^{(p)} \end{vmatrix}
$$

$$
= \begin{vmatrix} A_0^{(1)} & \cdots & A_{p-1}^{(1)} \\ \vdots & & \vdots \\ A_0^{(p)} & \cdots & A_{p-1}^{(p)} \end{vmatrix} \left|\begin{array}{c|c} 0_{N\times(p-N)} & xI_N - T^{[N-1]} \\ \hline I_{p-N} & 0_{(p-N)\times N} \end{array}\right|
$$

$$
= (-1)^{N(p-N)}P_N(x),
$$

where, in the second equality, we have used the recursion relation (10) in the last N columns and cancel the contributions already present in the previous columns.

For $N \geqslant p$, using the recursion relation similarly as above we get

$$
T_{N,N-p}\cdots T_{N+p-1,N-1}\det A_N = \det M,
$$

$$
M := \begin{bmatrix} A_0^{(1)} & \cdots\cdots & A_{N-1}^{(1)} \\ \vdots & & \vdots \\ A_0^{(p)} & \cdots\cdots & A_{N-1}^{(p)} \end{bmatrix} \begin{bmatrix} -T_{0,N-p} & \cdots\cdots\cdots & -T_{0,N-1} \\ \vdots & & \vdots \\ -T_{N-p-1,N-p} & \cdots\cdots & -T_{N-p-1,N-1} \\ x-T_{N-p,N-p} & \cdots\cdots & -T_{N-p,N-1} \\ \vdots & \ddots & \vdots \\ T_{N-1,N-p} & \cdots\cdots & x-T_{N-1,N-1} \end{bmatrix}.
$$

In order to compute this determinant we notice that

$$\left[\begin{array}{ccc|ccc} & 0_{(N-p)\times p} & & & I_{N-p} & \\ \hline A_0^{(1)} & \cdots & A_{p-1}^{(1)} & A_p^{(1)} & \cdots & A_{N-1}^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_0^{(p)} & \cdots & A_{p-1}^{(p)} & A_p^{(p)} & \cdots & A_{N-1}^{(p)} \end{array}\right](xI_N - T^{[N-1]}) =$$

$$\left[\begin{array}{cccc|c} -T_{p,0} & \cdots & \cdots & -T_{p,N-p-1} & \\ 0 & \ddots & & \vdots & \\ \vdots & \ddots & \ddots & \vdots & C_{(N-p)\times p} \\ 0 & \cdots\ 0 & & -T_{N-1,N-p-1} & \\ \hline & 0_{p\times(n-p)} & & & M \end{array}\right]$$

where C is an $(N-p)\times p$ submatrix of $xI_N - T^{[N-1]}$ that is not relevant for the reasoning. Observe that

$$\left|\begin{array}{ccc|ccc} & 0_{(N-p)\times p} & & & I_{N-p} & \\ \hline A_0^{(1)} & \cdots & A_{p-1}^{(1)} & A_p^{(1)} & \cdots & A_{N-1}^{(1)} \\ \vdots & & \vdots & \vdots & & \vdots \\ A_0^{(p)} & \cdots & A_{p-1}^{(p)} & A_p^{(p)} & \cdots & A_{N-1}^{(p)} \end{array}\right| = \left|\begin{array}{c|ccc} 0_{(N-p)\times p} & & I_{N-p} & \\ \hline & A_p^{(1)} & \cdots & A_{N-1}^{(1)} \\ \nu & \vdots & & \vdots \\ & A_p^{(p)} & \cdots & A_{N-1}^{(p)} \end{array}\right|$$

$$= (-1)^{p(N-p)},$$

where the initial conditions of recursion polynomials have been used, and we get

$$(-1)^{p(N-p)} P_N(x) = (-T_{p,0})(-T_{p+1,1})\cdots(-T_{N-1,N-p-1})\det M$$

so that

$$P_N(x) = (-1)^{(p+1)(N-p)} T_{p,0}T_{p+1,1}\cdots T_{N-1,N-p-1}T_{N,N-p}\cdots T_{N+p-1,N-1}\det A_N(x)$$

and observing that $(-1)^{(p+1)(N-p)} = (-1)^{(p-1)N}$ we obtain the stated result. Finally, the second result is proven analogously. $\square$

3. Right and left eigenvectors

We now consider determinantal polynomials constructed in terms of determinants of left and right recursion polynomials that happen to give left and right eigenvectors of $T^{[N]}$.

Definition 3.1. Let us introduce the determinantal polynomials

$$Q_{n,N} := \begin{vmatrix} A_n^{(1)} & \cdots\cdots\cdots & A_n^{(p)} \\ A_{N+1}^{(1)} & \cdots\cdots & A_{N+1}^{(p)} \\ \vdots & & \vdots \\ A_{N+p-1}^{(1)} & \cdots\cdots & A_{N+p-1}^{(p)} \end{vmatrix}, \qquad R_{n,N} := \begin{vmatrix} B_n^{(1)} & \cdots\cdots\cdots & B_n^{(q)} \\ B_{N+1}^{(1)} & \cdots\cdots & B_{N+1}^{(q)} \\ \vdots & & \vdots \\ B_{N+q-1}^{(1)} & \cdots\cdots & B_{N+q-1}^{(q)} \end{vmatrix}, \tag{11}$$

the semi-infinite row and column vectors

$$Q_N := \begin{bmatrix} Q_{0,N} & Q_{1,N} & \cdots \end{bmatrix}, \qquad R_N := \begin{bmatrix} R_{0,N} \\ R_{1,N} \\ \vdots \end{bmatrix},$$

and corresponding truncations

$$Q^{\langle N\rangle} := \begin{bmatrix} Q_{0,N} & Q_{1,N} & \cdots & Q_{N,N} \end{bmatrix}, \qquad R^{\langle N\rangle} := \begin{bmatrix} R_{0,N} \\ R_{1,N} \\ \vdots \\ R_{N,N} \end{bmatrix}.$$

Proposition 3.2. *The following properties for polynomials $Q_{n,N}, R_{n,N}$ are satisfied*

i) $Q_{N+1,N} = \cdots = Q_{N+p-1,N} = R_{N+1,N} = \cdots = R_{N+q-1,N} = 0.$
ii) $\alpha_N Q_{N,N} = \beta_N R_{N,N} = P_N$ *and* $(-1)^{p-1}\alpha_{N+1} Q_{N+p,N} = (-1)^{q-1}\beta_{N+1} R_{N+q,N} = P_{N+1}$.
iii) $Q_N T = x Q_N$ *and* $T R_N = x R_N$.
iv)

$$Q^{\langle N\rangle} T^{[N]} + \begin{bmatrix} 0 & \cdots & 0 & T_{N+p,N} Q_{N+p,N} \end{bmatrix} = x Q^{\langle N\rangle},$$

$$T^{[N]}R^{\langle N\rangle}+\begin{bmatrix}0\\ \vdots\\ 0\\ T_{N,N+q}R_{N+q,N}\end{bmatrix}=xR^{\langle N\rangle}. \tag{12}$$

Proof. i) As $Q_{n,N}$ and $R_{n,N}$ are the determinants in (11) we see that they vanish whenever two rows are equal, which happens precisely in the indicated cases.

ii) It follows from Theorem 2.7.

iii) It is a direct consequence of the fact that all appropriate rows/columns in the determinants in (11) satisfy corresponding recurrences.

iv) It follows from the previous points i) and iii). □

Now, we are ready to give a set of left and right eigenvectors of the banded finite matrix $T^{[N]}$. Let us assume that its eigenvalues $\lambda_k^{[N]}$, $k\in\{1,\dots,N+1\}$ are simple (which happens for example for oscillatory matrices). These eigenvalues are the zeros of the characteristic polynomials $P_{N+1}(x)$. We also assume that $\lambda_1^{[N]}>\lambda_2^{[N]}>\cdots>\lambda_{N+1}^{[N]}$.

Proposition 3.3. *For $k\in\{1,\dots,N+1\}$, the vectors $Q^{\langle N\rangle}\big|_{x=\lambda_k^{[N]}}$ and $R^{\langle N\rangle}\big|_{x=\lambda_k^{[N]}}$ are left and right eigenvectors of $T^{[N]}$, respectively.*

Proof. Properties ii) and iv) in Proposition 3.2 and an evaluation at $\lambda_k^{[N]}$ leads to the result. □

4. Christoffel–Darboux formula

We present now a generalized Christoffel–Darboux formula for the determinantal polynomials and the characteristic polynomial of a banded matrix. These results are an extension of the formulas found in [25] for the non-mixed case, see also [22]. Christoffel–Darboux formulas, not of the type described here, for the mixed multiple orthogonality were discussed in [26] and also in [2,10].

Proposition 4.1 *(Christoffel–Darboux type formulas).*

i) *For the determinantal polynomials $Q_{n,N}$ and $R_{n,N}$ introduced in (11) we get the following generalized Christoffel–Darboux formula*

$$\sum_{n=0}^{N}Q_{n,N}(x)R_{n,N}(y)=\frac{1}{\alpha_N\beta_N}\frac{P_{N+1}(x)P_N(y)-P_N(x)P_{N+1}(y)}{x-y}. \tag{13}$$

ii) *The following generalized confluent Christoffel–Darboux relation is fulfilled*

$$\sum_{n=0}^{N}Q_{n,N}R_{n,N}=\frac{1}{\alpha_N\beta_N}\big(P'_{N+1}P_N-P'_NP_{N+1}\big). \tag{14}$$

Proof. We use (12) to get

$$-Q^{\langle N\rangle}(x)\begin{bmatrix}0\\ \vdots\\ 0\\ T_{N,N+q}R_{N+q,N}(y)\end{bmatrix}+\begin{bmatrix}0\cdots 0 & T_{N+p,N}Q_{N+p,N}(x)\end{bmatrix}R^{\langle N\rangle}(y)$$
$$=(x-y)Q^{\langle N\rangle}(x)R^{\langle N\rangle}(y).$$

Now, recalling $Q_{N,N}=\alpha_N^{-1}P_N$, $Q_{N+p,N}=(-1)^{p-1}\alpha_{N+1}^{-1}P_{N+1}$, $\alpha_{N+1}=(-1)^{p-1}T_{N+p,N}\alpha_N$, $R_{N,N}=\beta_N^{-1}P_N$, $R_{N+q,N}=(-1)^{q-1}\beta_{N+1}^{-1}P_{N+1}$, $\beta_{N+1}=(-1)^{q-1}T_{N,N+q}\beta_N$, we obtain (13). Finally, (14) appears as a limit in (13). □

5. Biorthogonality and Christoffel numbers

We now discuss, for the truncated situation, how to construct biorthogonal families of left and right eigenvectors and introduce the Christoffel numbers in this setting.

Definition 5.1 *(Christoffel numbers).* The Christoffel numbers or coefficients are defined as

$$\mu_{k,1}^{[N]}:=\frac{\begin{vmatrix}A^{(2)}_{N+1}(\lambda_k^{[N]})\cdots\cdots A^{(p)}_{N+1}(\lambda_k^{[N]})\\ \vdots\qquad\qquad\vdots\\ A^{(2)}_{N+p-1}(\lambda_k^{[N]})\cdots A^{(p)}_{N+p-1}(\lambda_k^{[N]})\end{vmatrix}}{\beta_N\sum_{l=0}^N Q_{l,N}(\lambda_k^{[N]})R_{l,N}(\lambda_k^{[N]})},$$

$$\mu_{k,2}^{[N]}:=-\frac{\begin{vmatrix}A^{(1)}_{N+1}(\lambda_k^{[N]}) & A^{(3)}_{N+1}(\lambda_k^{[N]})\cdots\cdots A^{(p)}_{N+1}(\lambda_k^{[N]})\\ \vdots & \vdots\qquad\qquad\vdots\\ A^{(1)}_{N+p-1}(\lambda_k^{[N]}) & A^{(3)}_{N+p-1}(\lambda_k^{[N]})\cdots A^{(p)}_{N+p-1}(\lambda_k^{[N]})\end{vmatrix}}{\beta_N\sum_{l=0}^N Q_{l,N}(\lambda_k^{[N]})R_{l,N}(\lambda_k^{[N]})},$$

$$\vdots$$

$$\mu_{k,p}^{[N]}:=(-1)^{p-1}\frac{\begin{vmatrix}A^{(1)}_{N+1}(\lambda_k^{[N]})\cdots\cdots A^{(p-1)}_{N+1}(\lambda_k^{[N]})\\ \vdots\qquad\qquad\vdots\\ A^{(1)}_{N+p-1}(\lambda_k^{[N]})\cdots A^{(p-1)}_{N+p-1}(\lambda_k^{[N]})\end{vmatrix}}{\beta_N\sum_{l=0}^N Q_{l,N}(\lambda_k^{[N]})R_{l,N}(\lambda_k^{[N]})},$$

$$\rho_{k,1}^{[N]}:=\beta_N\begin{vmatrix}B^{(2)}_{N+1}(\lambda_k^{[N]})\cdots\cdots B^{(q)}_{N+1}(\lambda_k^{[N]})\\ \vdots\qquad\qquad\vdots\\ B^{(2)}_{N+q-1}(\lambda_k^{[N]})\cdots B^{(p)}_{N+q-1}(\lambda_k^{[N]})\end{vmatrix},$$

$$\rho_{k,2}^{[N]} := -\beta_N \begin{vmatrix} B_{N+1}^{(1)}\big(\lambda_k^{[N]}\big) & B_{N+1}^{(3)}\big(\lambda_k^{[N]}\big) \cdots\cdots & B_{N+1}^{(q)}\big(\lambda_k^{[N]}\big) \\ \vdots & \vdots & \vdots \\ B_{N+q-1}^{(1)}\big(\lambda_k^{[N]}\big) & B_{n+q-1}^{(3)}\big(\lambda_k^{[N]}\big) \cdots & B_{N+q-1}^{(p)}\big(\lambda_k^{[N]}\big) \end{vmatrix},$$

$$\vdots$$

$$\rho_{k,q}^{[N]} := (-1)^{q-1}\beta_N \begin{vmatrix} B_{N+1}^{(1)}\big(\lambda_k^{[N]}\big) \cdots\cdots & B_{N+1}^{(q-1)}\big(\lambda_k^{[N]}\big) \\ \vdots & \vdots \\ B_{N+q-1}^{(1)}\big(\lambda_k^{[N]}\big) \cdots & B_{N+q-1}^{(q-1)}\big(\lambda_k^{[N]}\big) \end{vmatrix}.$$

Proposition 5.2 *(Spectral properties). Assume that P_{N+1} has simple zeros at the set $\big\{\lambda_k^{[N]}\big\}_{k=1}^{N+1}$, so that the vectors $u_k^{\langle N\rangle} := R^{\langle N\rangle}\big(\lambda_k^{[N]}\big)$ and $\tilde{w}_k^{\langle N\rangle} := Q^{\langle N\rangle}\big(\lambda_k^{[N]}\big)$ are right and left eigenvectors of $T^{[N]}$, respectively, $k = 1, \dots, N+1$. Then:*

i) *Biorthogonal families left and right eigenvectors $\big\{w_k^{\langle N\rangle}\big\}_{k=1}^{N+1}$ and $\big\{u_k^{\langle N\rangle}\big\}_{k=1}^{N+1}$, are*

$$w_k^{\langle N\rangle} = \frac{Q^{\langle N\rangle}\big(\lambda_k^{[N]}\big)}{\beta_N \sum_{l=0}^{N} Q_{l,N}\big(\lambda_k^{[N]}\big) R_{l,N}\big(\lambda_k^{[N]}\big)}, \qquad u_k^{\langle N\rangle} = \beta_N R^{\langle N\rangle}\big(\lambda_k^{[N]}\big).$$

ii) *The following expression holds*

$$w_{k,n}^{\langle N\rangle} = \frac{\alpha_N Q_{n-1,N}\big(\lambda_k^{[N]}\big)}{P_N\big(\lambda_k^{[N]}\big) P'_{N+1}\big(\lambda_k^{[N]}\big)}, \qquad u_{k,n}^{\langle N\rangle} = \beta_N R_{n-1,N}\big(\lambda_k^{[N]}\big). \tag{15}$$

iii) *In terms of the Christoffel numbers we can write*

$$w_{k,n}^{\langle N\rangle} = A_{n-1}^{(1)}\big(\lambda_k^{[N]}\big)\mu_{k,1}^{[N]} + \cdots + A_{n-1}^{(p)}\big(\lambda_k^{[N]}\big)\mu_{k,p}^{[N]}, \tag{16}$$

$$u_{k,n}^{\langle N\rangle} = B_{n-1}^{(1)}\big(\lambda_k^{[N]}\big)\rho_{k,1}^{[N]} + \cdots + B_{n-1}^{(q)}\big(\lambda_k^{[N]}\big)\rho_{k,q}^{[N]}. \tag{17}$$

iv) *For the Christoffel numbers it holds that*

$$\begin{bmatrix} \mu_{k,1}^{[N]} \\ \mu_{k,2}^{[N]} \\ \vdots \\ \mu_{k,p}^{[N]} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \nu_1^{(1)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \nu_{p-1}^{(1)} & \cdots & \nu_{p-1}^{(p-1)} & 1 \end{bmatrix}^{-1} \begin{bmatrix} w_{k,1}^{\langle N\rangle} \\ w_{k,2}^{\langle N\rangle} \\ \vdots \\ w_{k,p}^{\langle N\rangle} \end{bmatrix},$$

$$\begin{bmatrix} \rho^{[N]}_{k,1} \\ \rho^{[N]}_{k,2} \\ \vdots \\ \rho^{[N]}_{k,q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \xi^{(1)}_1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \xi^{(1)}_{q-1} & \cdots & \cdots & \xi^{(q-1)}_{q-1} & 1 \end{bmatrix}^{-1} \begin{bmatrix} u^{\langle N \rangle}_{k,1} \\ u^{\langle N \rangle}_{k,2} \\ \vdots \\ u^{\langle N \rangle}_{k,q} \end{bmatrix}. \tag{18}$$

v) *The corresponding matrices $\mathscr{U}$ (with columns the right eigenvectors u_k arranged in the standard order) and $\mathscr{W}$ (with rows the left eigenvectors w_k arranged in the standard order) satisfy*

$$\mathscr{U}\mathscr{W} = \mathscr{W}\mathscr{U} = I_{N+1}.$$

vi) *In terms of the eigenvalues diagonal matrix $D = \operatorname{diag}\left(\lambda^{[N]}_1, \dots, \lambda^{[N]}_{N+1}\right)$ we have*

$$\mathscr{U} D^n \mathscr{W} = \left(T^{[N]}\right)^n, \qquad n \in \mathbb{N}_0.$$

Proof. i) As the zeros are simple we have that left and right eigenvectors are orthogonal, i.e., $\tilde{w}^{\langle N \rangle}_k u^{\langle N \rangle}_l = \delta_{k,l} \sum_{r=0}^N Q_{r,N}\left(\lambda^{[N]}_k\right) R_{r,N}\left(\lambda^{[N]}_k\right)$. Hence, we divide by $\sum_{r=0}^N Q_{r,N}\left(\lambda^{[N]}_k\right) R_{r,N}\left(\lambda^{[N]}_k\right)$ to get normalized left eigenvectors.

ii) It follows from the previous result and Equation (14).

iii) In Equation (11) expand the determinant in $Q_{n-1,N}$ along its first row.

iv) Use (16) for the first p entries

$$\begin{bmatrix} w^{\langle N \rangle}_{k,1} \\ \vdots \\ w^{\langle N \rangle}_{k,p} \end{bmatrix} = \begin{bmatrix} A^{(1)}_0\left(\lambda^{[N]}_k\right) & \cdots & A^{(p)}_0\left(\lambda^{[N]}_k\right) \\ \vdots & & \vdots \\ A^{(1)}_{p-1}\left(\lambda^{[N]}_k\right) & \cdots & A^{(p)}_{p-1}\left(\lambda^{[N]}_k\right) \end{bmatrix} \begin{bmatrix} \mu^{[N]}_{k,1} \\ \vdots \\ \mu^{[N]}_{k,p} \end{bmatrix}$$

and the initial conditions (8)

$$\begin{bmatrix} A^{(1)}_0\left(\lambda^{[N]}_k\right) & \cdots & A^{(p)}_0\left(\lambda^{[N]}_k\right) \\ A^{(1)}_1\left(\lambda^{[N]}_k\right) & \cdots & A^{(p)}_1\left(\lambda^{[N]}_k\right) \\ \vdots & & \vdots \\ A^{(1)}_{p-1}\left(\lambda^{[N]}_k\right) & \cdots & A^{(p)}_{p-1}\left(\lambda^{[N]}_k\right) \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \nu^{(1)}_1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \nu^{(1)}_{p-1} & \cdots & \cdots & \nu^{(p-1)}_{p-1} & 1 \end{bmatrix}$$

to obtain the result. For the right vectors proceed similarly.

v) It follows from the biorthogonality of the left and right eigenvectors.

vi) Notice that $\mathscr{U} D^n = \left(T^{[N]}\right)^n \mathscr{U}$ and use $\mathscr{U}^{-1} = \mathscr{W}$ to get $\mathscr{U} D^n \mathscr{W} = \left(T^{[N]}\right)^n$ as desired. □

6. Mixed multiple discrete orthogonality

We reformulate the previous discussed biorthogonality in terms of a set of discrete measures and corresponding mixed multiple discrete orthogonality. We remind that $\lambda_1^{[N]} > \lambda_2^{[N]} > \cdots > \lambda_{N+1}^{[N]}$.

Definition 6.1 *(Step functions).* Let us consider the following step functions

$$\psi_{b,a}^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{k,b}^{[N]}\mu_{k,a}^{[N]}, & \lambda_{k+1}^{[N]} \leqslant x < \lambda_k^{[N]}, \quad k \in \{1,\dots,N\}, \\ \rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]}, & x \geqslant \lambda_1^{[N]}. \end{cases}$$

We now show that last step of these step functions is bounded. This implies in the case of positive Christoffel coefficients that these step functions are uniformly bounded in N. For that aim we need to introduce the matrix $I_{q,p} \in \mathbb{R}^{q\times p}$, with $(I_{q,p})_{k,l} = \delta_{k,l}$. Thus, if $p = q$ we are dealing with the identity matrix, however if $p \neq q$ is a rectangular matrix with a square block with the identity $I_{\min(p,q)}$ completed with a zero block.

Proposition 6.2. *For $a \in \{1,\dots,p\}$ and $b \in \{1,\dots,q\}$, we have*

$$\rho_{1,b}^{[N]}\mu_{1,a}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}\mu_{N+1,a}^{[N]} = (\xi^{-1} I_{q,p} \nu^{-\top})_{b,a}.$$

Proof. Let us write (18) in the alternative form

$$\begin{bmatrix} \mu_{1,1}^{[N]} & \cdots & \mu_{1,p}^{[N]} \\ \vdots & & \vdots \\ \mu_{N+1,1}^{[N]} & \cdots & \mu_{N+1,p}^{[N]} \end{bmatrix} = \begin{bmatrix} w_{1,1}^{\langle N\rangle} & \cdots & w_{1,p}^{\langle N\rangle} \\ \vdots & & \vdots \\ w_{N+1,1}^{\langle N\rangle} & \cdots & w_{N+1,p}^{\langle N\rangle} \end{bmatrix} \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \nu_1^{(1)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \nu_{p-1}^{(1)} & \cdots & \nu_{p-1}^{(p-1)} & 1 \end{bmatrix}^{-\top},$$

$$\begin{bmatrix} \rho_{1,1}^{[N]} & \cdots & \rho_{N+1,1}^{[N]} \\ \vdots & & \vdots \\ \rho_{1,q}^{[N]} & \cdots & \rho_{N+1,q}^{[N]} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ \xi_1^{(1)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \xi_{q-1}^{(1)} & \cdots & \xi_{q-1}^{(q-1)} & 1 \end{bmatrix}^{-1} \begin{bmatrix} u_{1,1}^{\langle N\rangle} & \cdots & u_{N+1,1}^{\langle N\rangle} \\ \vdots & & \vdots \\ u_{1,q}^{\langle N\rangle} & \cdots & u_{N+1,q}^{\langle N\rangle} \end{bmatrix}.$$

From $\mathscr{U}\mathscr{W} = I$, we obtain

$$\begin{bmatrix} u^{\langle N\rangle}_{1,1} & \cdots & u^{\langle N\rangle}_{N+1,1}\\ \vdots & & \vdots\\ u^{\langle N\rangle}_{1,q} & \cdots & u^{\langle N\rangle}_{N+1,q}\end{bmatrix}\begin{bmatrix} w^{\langle N\rangle}_{1,1} & \cdots & w^{\langle N\rangle}_{1,p}\\ \vdots & & \vdots\\ w^{\langle N\rangle}_{N+1,1} & \cdots & w^{\langle N\rangle}_{N+1,p}\end{bmatrix}=I_{q,p}.$$

Hence,

$$\begin{bmatrix} \rho^{[N]}_{1,1} & \cdots & \rho^{[N]}_{N+1,1}\\ \vdots & & \vdots\\ \rho^{[N]}_{1,q} & \cdots & \rho^{[N]}_{N+1,q}\end{bmatrix}\begin{bmatrix} \mu^{[N]}_{1,1} & \cdots & \mu^{[N]}_{1,p}\\ \vdots & & \vdots\\ \mu^{[N]}_{N+1,1} & \cdots & \mu^{[N]}_{N+1,p}\end{bmatrix}=\xi^{-1}I_{q,p}\nu^{-\top} \tag{19}$$

and we get $\mu^{[N]}_{1,a}\rho^{[N]}_{1,b}+\cdots+\mu^{[N]}_{N+1,a}\rho^{[N]}_{N+1,b}=(\xi^{-1}I_{q,p}\nu^{-\top})_{b,a}$. □

Notice that these functions have bounded variation and are right continuous, so it makes sense to consider the associated Lebesgue–Stieltjes measures.

Definition 6.3 *(Matrix of discrete measures).* Let us introduce a $q\times p$ matrix $\Psi^{[N]}:=\begin{bmatrix} \psi^{[N]}_{1,1} & \cdots & \psi^{[N]}_{1,p}\\ \vdots & & \vdots\\ \psi^{[N]}_{q,1} & \cdots & \psi^{[N]}_{q,p}\end{bmatrix}$ and the corresponding $q\times p$ matrix of discrete Lebesgue–Stieltjes measures supported at the zeros of P_{N+1},

$$\mathrm{d}\,\Psi^{[N]}=\begin{bmatrix} \mathrm{d}\,\psi^{[N]}_{1,1} & \cdots & \mathrm{d}\,\psi^{[N]}_{1,p}\\ \vdots & & \vdots\\ \mathrm{d}\,\psi^{[N]}_{q,1} & \cdots & \mathrm{d}\,\psi^{[N]}_{q,p}\end{bmatrix}=\sum_{k=1}^{N+1}\begin{bmatrix}\rho^{[N]}_{k,1}\\ \vdots\\ \rho^{[N]}_{k,q}\end{bmatrix}\begin{bmatrix}\mu^{[N]}_{k,1} & \cdots & \mu^{[N]}_{k,p}\end{bmatrix}\delta\big(x-\lambda^{[N]}_k\big). \tag{20}$$

Remark 6.4. This matrix of discrete measures is rank 1 at each point of the support.

Theorem 6.5 *(Mixed multiple discrete biorthogonality). Assume that the recursion polynomials P_{N+1} have simple zeros $\{\lambda^{[N]}_k\}_{k=1}^{N+1}$. The following biorthogonal relations hold*

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int B^{(b)}_n(x)\,\mathrm{d}\,\psi^{[N]}_{b,a}(x)A^{(a)}_m(x)=\delta_{n,m},\qquad n,m\in\{0,\dots,N\}.$$

Proof. It follows from Equations (16), (17) and $\mathscr{U}\mathscr{W}=I$. □

From this biorthogonality we get the following:

Corollary 6.6 *(Mixed multiple discrete orthogonality). Assume that the polynomial P_{N+1} has simple zeros $\{\lambda^{[N]}_k\}_{k=1}^{N+1}$. Then, the following discrete type mixed multiple orthogonality for $m\in\{1,\dots,N\}$ is satisfied:*

$$\sum_{a=1}^{p}\int x^n\,\mathrm{d}\,\psi_{b,a}^{[N]}A_m^{(a)}=0,\qquad n\in\big\{0,\dots,\deg B_{m-1}^{(b)}\big\},\qquad b\in\{1,\dots,q\},$$

$$\sum_{b=1}^{q}\int B_m^{(b)}\,\mathrm{d}\,\psi_{b,a}^{[N]}x^n=0,\qquad n\in\big\{0,\dots,\deg A_{m-1}^{(a)}\big\},\qquad a\in\{1,\dots,p\}.$$

7. Positive bidiagonal factorization and Christoffel numbers positivity

Positive bidiagonal factorization (PBF) accommodates naturally to TN banded matrices as all the subdiagonals may be constructed in terms of simpler bidiagonal matrices.

Definition 7.1 *(Positive bidiagonal factorization).* We say that a banded matrix T as in (1) admits a PBF if

$$T=L_1\cdots L_p\Delta U_q\cdots U_1,$$

with $\Delta=\operatorname{diag}(\Delta_0,\Delta_1,\dots)$ and bidiagonal matrices given respectively by

$$L_k:=\begin{bmatrix} 1 & 0 & \cdots & \\ L_{k|0} & 1 & \ddots & \\ 0 & L_{k|1} & 1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix},\qquad U_k:=\begin{bmatrix} 1 & U_{k|0} & 0 & \cdots \\ 0 & 1 & U_{k|1} & \ddots \\ \vdots & \ddots & 1 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

and such that the positivity constraints $L_{k|i},U_{k|i},\Delta_i>0$, for $i\in\mathbb{N}_0$, are satisfied.

Remark 7.2. Notice that $L_1^{[N]},\dots,L_p^{[N]},\Delta^{[N]},U_q^{[N]},\dots,U_1^{[N]}\in\text{InTN}$.

Proposition 7.3. *The above positive bidiagonal factorization of T induces the following positive bidiagonal factorization for the leading principal submatrix $T^{[N]}$*

$$T^{[N]}=L_1^{[N]}\cdots L_p^{[N]}\Delta^{[N]}U_q^{[N]}\cdots U_1^{[N]}. \tag{21}$$

Proposition 7.4. *If T has a PBF then its leading principal submatrices $T^{[N]}$ are oscillatory.*

Proof. As all factors are InTN the product matrix is InTN. Moreover, as all parameters in the bidiagonal factors are positive then using Gantmacher–Krein Criterion we get that the matrix is oscillatory. □

Proposition 7.5 *(Interlacing). Let us assume that T is oscillatory. Then:*

i) *The polynomial P_{N+1} interlaces P_N.*

ii) *For $x \in \mathbb{R}$, for the corresponding Wronskian we find $P'_{N+1}P_N - P'_N P_{N+1} > 0$. In particular,*

$$(P'_{N+1}P_N)\big|_{x=\lambda^{[N]}_k} > 0, \qquad (P_{N+1}P'_N)\big|_{x=\lambda^{[N-1]}_k} < 0.$$

iii) *The confluent kernel is a positive function; i.e., $\alpha_N\beta_N\sum_{n=0}^{N} Q_{n,N}(x)R_{n,N}(x) > 0$ for $x \in \mathbb{R}$.*

Proof. i) Given that $T^{[N]}$ is oscillatory the polynomial P_{N+1} interlaces P_N, see Theorem 1.5.

ii) As the polynomials interlace its Wronskian $P'_{N+1}P_N - P'_N P_{N+1}$ has constant sign for $x \in \mathbb{R}$[1] and, as the characteristic polynomials are monic, we have that $P'_{N+1}P_N - P'_N P_{N+1} = x^{2N} + O(x^{2N-1})$ for $|x| \to \infty$. Hence, the Wronskian is positive and $P_N\big(\lambda^{[N]}_k\big)P'_N\big(\lambda^{[N]}_k\big) > 0$ and $P'_N\big(\lambda^{[N-1]}_k\big)P_N\big(\lambda^{[N-1]}_k\big) > 0$.

iii) From (14) we get $\sum_{n=0}^{N} Q_{n,N}R_{n,N} = \frac{1}{\alpha_N\beta_N}\big(P'_{N+1}P_N - P'_N P_{N+1}\big)$ and the result follows immediately. □

We now explore some consequences that a positive bidiagonal factorization has. For that aim we introduce the idea of Darboux transformation of a banded Hessenberg matrix. Darboux transformations for banded Hessenberg matrices (beyond the tridiagonal situation) were discussed in [14]. In [21] for the tetradiagonal case, and corresponding multiple orthogonal polynomials in the step-line with two weights, the PBF factorization is given in terms of the values of the orthogonal polynomials of type I and II at 0 and, consequently, an spectral interpretation of the Darboux transformation is given.

Definition 7.6 *(Darboux transformations of banded matrices).* Let us assume that T admits a bidiagonal factorization (not necessarily positive). For each of its truncations $T^{[N]}$ we consider a chain of new auxiliary matrices, called Darboux transformations, given by the consecutive permutation of the unitriangular matrices in the factorization (21),

$$\begin{aligned}
\hat{T}^{[N,+1]} &= L^{[N]}_2 \cdots L^{[N]}_p \Delta^{[N]} U^{[N]}_q \cdots U^{[N]}_1 L^{[N]}_1,\\
\hat{T}^{[N,+2]} &= L^{[N]}_3 \cdots L^{[N]}_p \Delta^{[N]} U^{[N]}_q \cdots U^{[N]}_1 L^{[N]}_1 L^{[N]}_2,\\
&\vdots\\
\hat{T}^{[N,+(p-1)]} &= L^{[N]}_p \Delta^{[N]} U^{[N]}_q \cdots U^{[N]}_1 L^{[N]}_1 L^{[N]}_2 \cdots L^{[N]}_{p-1},
\end{aligned}$$

[1] In terms of $\pi^{[N]}_k := \frac{P_{N+1}}{x-\lambda^{[N]}_k} = \prod_{l\neq k}\big(x-\lambda^{[N]}_l\big)$ we have $P_N = \sum_{k=1}^{N+1} b_k\pi^{[N]}_k$ with $b_l = \frac{P_N(\lambda^{[N]}_l)}{\pi^{[N]}_l(\lambda^{[N]}_l)} \neq 0$ and that, as these polynomials interlaces, all the b_k have the same sign; indeed, $\operatorname{sgn} P_N(\lambda^{[N]}_l) = -\operatorname{sgn} P_N(\lambda^{[N]}_{l+1})$ by interlacing and $\operatorname{sgn}\pi^{[N]}_l(\lambda^{[N]}_l) = -\operatorname{sgn}\pi^{[N]}_{l+1}(\lambda^{[N]}_{l+1})$ by definition. Consequently, $\frac{P_N}{P_{N+1}} = \sum_{k=1}^{N+1}\frac{b_k}{x-\lambda^{[N]}_k}$, so that $P'_{N+1}P_N - P'_N P_{N+1} = P^2_{N+1}\Big(\frac{P_N}{P_{N+1}}\Big)' = -P^2_{N+1}\sum_{k=1}^{N+1}\frac{b_k}{(x-\lambda^{[N]}_k)^2} = -\sum_{k=1}^{N+1} b_k(\pi^{[N]}_k)^2$, and the result follows.

$$\hat{T}^{[N,+p]} = \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]},$$

and

$$\begin{aligned}
\hat{T}^{[N,-1]} &= U_1^{[N]} L_1^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_2^{[N]},\\
\hat{T}^{[N,-2]} &= U_2^{[N]} U_1^{[N]} L_1^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_3^{[N]},\\
&\vdots\\
\hat{T}^{[N,-(q-1)]} &= U_{q-1}^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]},\\
\hat{T}^{[N,-q]} &= U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} L_2^{[N]} \cdots L_p^{[N]} \Delta^{[N]}.
\end{aligned}$$

Lemma 7.7. *Darboux transformations are banded matrices with only its first p subdiagonals, main diagonal and q superdiagonals possibly different from zero. If T admits a PBF then entries in these diagonals are positive.*

Proof. It is a simple computation recalling the positivity of the nonzero entries. □

Lemma 7.8. *Let us assume that T has a PBF. Then, for $k \in \{1, \dots, p\}$, we find:*

i) *The Darboux transformations $\hat{T}^{[N,+a]}$, $a \in \{1, \dots, p\}$, $\hat{T}^{[N,-b]}$, $b \in \{1, \dots, q\}$ are oscillatory.*
ii) *The characteristic polynomial of the Darboux transformations $\hat{T}^{[N,+a]}$, $a \in \{1, \dots, p\}$, $\hat{T}^{[N,-b]}$, $b \in \{1, \dots, q\}$ is P_{N+1}.*
iii) *If w, u are left and right eigenvectors of $T^{[N]}$, respectively, then $\hat{w} = wL_1^{[N]} \cdots L_a^{[N]}$ is a left eigenvector of $\hat{T}^{[N,+a]}$ and $\hat{u} = U_b^{[N]} \cdots U_1^{[N]} u$ is a right eigenvector of $\hat{T}^{[N,-b]}$.*

Proof. i) Each bidiagonal factor belongs to InTN. Then, the Darboux transformation $\hat{T}^{[N,k]}$ is a product of matrices in InTN and, consequently, belongs to InTN. Moreover, the entries in the first subdiagonal and first superdiagonal are sums of products of elements coming from the entries of the positive subdiagonal or superdiagonal of the matrices $L_j^{[N]}$ and $U_m^{[N]}$, for $j = 1, \dots, p$, and $m = 1, \dots, q$. According to Gantmacher–Krein Criterion is an oscillatory matrix.

ii) As $\hat{T}^{[N,+a]} = (L_1^{[N]} \cdots L_a^{[N]})^{-1} T^{[N]} L_1^{[N]} \cdots L_a^{[N]}$ its characteristic polynomial is P_{N+1}. Similarly, as $\hat{T}^{[N,-b]} = U_b^{[N]} \cdots U_1^{[N]} T^{[N]} (U_b^{[N]} \cdots U_1^{[N]})^{-1}$ the corresponding characteristic polynomial is again P_{N+1}.

iii) We see that

$$\begin{aligned}
\lambda \hat{w} = \lambda w L_1^{[N]} \cdots L_a^{[N]} &= w L_1^{[N]} \cdots L_a^{[N]} L_{a+1}^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_1^{[N]} L_1^{[N]} \cdots L_a^{[N]}\\
&= \hat{w} \hat{T}^{[N]},\\
\lambda \hat{u} = \lambda U_b^{[N]} \cdots U_1^{[N]} u &= U_b^{[N]} \cdots U_1^{[N]} L_1^{[N]} \cdots L_p^{[N]} \Delta^{[N]} U_q^{[N]} \cdots U_{b+1}^{[N]} U_b^{[N]} \cdots U_1^{[N]} u
\end{aligned}$$

$$= \hat{T}^{[N]}\hat{u}. \quad \square$$

In order to show the positivity of the Christoffel coefficients we require of some preliminary notation.

Definition 7.9. Let us define the matrices

$$\Lambda := \begin{bmatrix}\Lambda^{(1)} \cdots \cdot \Lambda^{(p)}\end{bmatrix} \in \mathbb{R}^{p\times p}, \quad \Upsilon := \begin{bmatrix}\Upsilon^{(1)} \\ \vdots \\ \Upsilon^{(q)}\end{bmatrix} \in \mathbb{R}^{q\times q},$$

with

$$\Lambda^{(1)} := \begin{bmatrix}1\\0\\ \vdots \\ \vdots \\ 0\end{bmatrix}, \qquad \Lambda^{(k)} := \frac{1}{r_k} L_1^{[p-1]} \cdots L_{k-1}^{[p-1]} \begin{bmatrix}1\\0\\ \vdots \\ \vdots \\ 0\end{bmatrix},$$
$$r_k := L_{k|0} L_{k-1|1} \cdots L_{1|k-1}, \qquad k \in \{2, \dots, p\},$$

and

$$\Upsilon^{(1)} := \begin{bmatrix}1 & 0 \cdots \cdot 0\end{bmatrix}, \qquad \Upsilon^{(k)} := \frac{1}{s_k} \begin{bmatrix}1 & 0 \cdots \cdot 0\end{bmatrix} U_1^{[q-1]} \cdots U_{k-1}^{[q-1]},$$
$$s_k := U_{k|0} U_{k-1|1} \cdots U_{1|k-1}, \qquad k \in \{2, \dots, q\}.$$

Lemma 7.10. *The matrices Λ and Υ are positive upper and lower unitriangular matrices, respectively.*

Theorem 7.11 *(Christoffel numbers positivity). Let us assume that T has a PBF and choose the matrices of initial conditions as*

$$\nu^{-\top} = \Lambda \mathcal{A}, \qquad \xi^{-1} = \mathcal{B}\Upsilon, \tag{22}$$

for some upper and lower unitriangular nonnegative matrices $\mathcal{A} \in \mathbb{R}^{p\times p}$ and $\mathcal{B} \in \mathbb{R}^{q\times q}$, respectively. Then,

$$\rho_{k,b}^{[N]} > 0, \qquad \mu_{k,a}^{[N]} > 0, \qquad k \in 1, \dots, N+1, \qquad a \in \{1, \dots, p\}, \qquad b \in \{1, \dots, q\}.$$

Proof. Recall that the Christoffel numbers can be expressed, in terms of the initial condition matrices ξ and ν, through the formulas

$$\begin{bmatrix}\mu^{[N]}_{k,1} & \cdots & \mu^{[N]}_{k,p}\end{bmatrix} = \begin{bmatrix}w^{\langle N\rangle}_{k,1} & \cdots & w^{\langle N\rangle}_{k,p}\end{bmatrix}\nu^{-\top}, \qquad \begin{bmatrix}\rho^{[N]}_{k,1}\\ \vdots\\ \rho^{[N]}_{k,q}\end{bmatrix} = \xi^{-1}\begin{bmatrix}u^{\langle N\rangle}_{k,1}\\ \vdots\\ u^{\langle N\rangle}_{k,q}\end{bmatrix},$$

that relates these Christoffel numbers with the corresponding biorthogonal families of right and left eigenvectors. Notice that the entries of these biorthogonal right and left eigenvectors can be written as $w^{\langle N\rangle}_{k,a} = \alpha_N \left.\frac{Q_{a-1,N}}{P'_{N+1}P_N}\right|_{x=\lambda^{[N]}_k}$ and $u^{\langle N\rangle}_{k,b} = \beta_N R_{b-1,N}(\lambda^{[N]}_k)$, see (15). Hence, recall iii) in Proposition 7.5, the Christoffel numbers are positive if and only if

$$\beta_N\xi^{-1}\begin{bmatrix}R_{0,N}\\ \vdots\\ R_{q-1,N}\end{bmatrix}, \qquad \frac{1}{\beta_N}\begin{bmatrix}Q_{0,N} & \cdots & Q_{p-1,N}\end{bmatrix}\nu^{-\top}$$

are positive vectors at the points $x = \lambda^{[N]}_k$, $k \in \{1,\dots,N+1\}$. We will show now that is possible to choose the initial condition matrices ν, ξ such that this holds true.

Now, we consider left and right eigenvectors with last entry normalized to 1

$$\begin{bmatrix}\left.\frac{Q_{0,N}}{Q_{N,N}}\right|_{x=\lambda^{[N]}_k} & \left.\frac{Q_{1,N}}{Q_{N,N}}\right|_{x=\lambda^{[N]}_k} & \cdots & 1\end{bmatrix}, \qquad \begin{bmatrix}\left.\frac{R_{0,N}}{R_{N,N}}\right|_{x=\lambda^{[N]}_k}\\ \left.\frac{R_{1,N}}{R_{N,N}}\right|_{x=\lambda^{[N]}_k}\\ \vdots\\ 1\end{bmatrix}.$$

It is important to recall that according to Theorem 1.6 the last entry of any eigenvector is nonzero, i.e. so that we can normalize the last entry to 1. Despite, this is not the biorthogonal normalization is interesting for our purposes. Recall that $Q_{N,N} = \alpha_N^{-1}P_N$, $R_{N,N} = \beta_N^{-1}P_N$ and that, according to Theorem 1.6, the first eigenvector entries are not zero; i.e., $\alpha_N \left.\frac{Q_{0,N}}{P_N}\right|_{x=\lambda^{[N]}_k}, \beta_N \left.\frac{R_{0,N}}{P_N}\right|_{x=\lambda^{[N]}_k} \neq 0$. As the last entry is positive the change sign properties described in Theorem 1.6 leads to

$$\begin{aligned}
&\alpha_N \left.\frac{Q_{0,N}}{P_N}\right|_{x=\lambda^{[N]}_1} > 0, && \alpha_N \left.\frac{Q_{0,N}}{P_N}\right|_{x=\lambda^{[N]}_2} < 0,\\
&\alpha_N \left.\frac{Q_{0,N}}{P_N}\right|_{x=\lambda^{[N]}_3} > 0,\\
&\beta_N \left.\frac{R_{0,N}}{P_N}\right|_{x=\lambda^{[N]}_1} > 0, && \beta_N \left.\frac{R_{0,N}}{P_N}\right|_{x=\lambda^{[N]}_2} < 0,\\
&\beta_N \left.\frac{R_{0,N}}{P_N}\right|_{x=\lambda^{[N]}_3} > 0,
\end{aligned}$$

and so on, alternating the sign. Now, as T is oscillatory and the characteristic polynomial P_{N+1} interlaces P_N we have that $\operatorname{sgn} P_N\big(\lambda^{[N]}_k\big) = (-1)^{k-1}$ so that

$$\alpha_N Q_{0,N}\big(\lambda^{[N]}_k\big), \beta_N R_{0,N}\big(\lambda^{[N]}_k\big) > 0, \qquad k \in \{1, \dots, N+1\}.$$

Now, we start using the Darboux transformations. Recall that $\hat{T}^{[N,\pm 1]}$ is an oscillatory matrix with characteristic polynomial P_{N+1}. Then, a left eigenvector of $T^{[N,+1]}$ for the eigenvalue $\lambda^{[N]}_k$ can be chosen as

$$\begin{bmatrix} \alpha_N \frac{Q_{0,N}}{P_N}\Big|_{x=\lambda^{[N]}_k} & \alpha_N \frac{Q_{1,N}}{P_N}\Big|_{x=\lambda^{[N]}_k} & \cdots & 1 \end{bmatrix} L^{[N]}_1 = \begin{bmatrix} \alpha_N \frac{(Q_{0,N}+L_{1|0}Q_{1,N})}{P_N}\Big|_{x=\lambda^{[N]}_k} & \cdots & 1 \end{bmatrix},$$

and a right eigenvector of $T^{[N,-1]}$ for the eigenvalue $\lambda^{[N]}_k$ can be taken as

$$U^{[N]}_1 \begin{bmatrix} \beta_N \frac{R_{0,N}}{P_N}\Big|_{x=\lambda^{[N]}_k} \\ \beta_N \frac{R_{1,N}}{P_N}\Big|_{x=\lambda^{[N]}_k} \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \beta_N \frac{(R_{0,N}+U_{1|0}R_{1,N})}{P_N}\Big|_{x=\lambda^{[N]}_k} \\ \vdots \\ 1 \end{bmatrix}.$$

Using again the sign properties of the eigenvectors associated to an oscillatory matrix we get

$$\begin{aligned}
&\alpha_N \left.\frac{\frac{1}{L_{1|0}}Q_{0,N} + Q_{1,N}}{P_N}\right|_{x=\lambda^{[N]}_1} > 0, && \alpha_N \left.\frac{\frac{1}{L_{1|0}}Q_{0,N} + Q_{1,N}}{P_N}\right|_{x=\lambda^{[N]}_2} < 0,\\
&\alpha_N \left.\frac{\frac{1}{L_{1|0}}Q_{0,N} + Q_{1,N}}{P_N}\right|_{x=\lambda^{[N]}_3} > 0,\\
&\beta_N \left.\frac{\frac{1}{U_{1|0}}R_{0,N} + R_{1,N}}{P_N}\right|_{x=\lambda^{[N]}_1} > 0, && \beta_N \left.\frac{\frac{1}{U_{1|0}}R_{0,N} + R_{1,N}}{P_N}\right|_{x=\lambda^{[N]}_2} < 0,\\
&\beta_N \left.\frac{\frac{1}{U_{1|0}}R_{0,N} + R_{1,N}}{P_N}\right|_{x=\lambda^{[N]}_3} > 0,
\end{aligned}$$

and so on alternating the sign. Recalling the sign of P_N at the zeros of P_{N+1} we get

$$\alpha_N \left(\frac{1}{L_{1|0}}Q_{0,N} + Q_{1,N}\right)\Big|_{x=\lambda^{[N]}_k}, \beta_N \left(\frac{1}{U_{1|0}}R_{0,N} + R_{1,N}\right)\Big|_{x=\lambda^{[N]}_k} > 0, \quad k \in \{1, \dots, N+1\}.$$

Now we consider the matrices $\hat{T}^{[N,\pm 2]}$, both oscillatory matrices with characteristic polynomial P_{N+1}. Then, for $T^{[N,+2]}$, a corresponding left eigenvector for the eigenvalue $\lambda^{[N]}_k$ can be chosen as

$$\left[\alpha_N \frac{Q_{0,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} \ \alpha_N \ \frac{Q_{1,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} \cdots 1\right] L_1^{[N]} L_2^{[N]}$$
$$= \left[\alpha_N \ \frac{Q_{0,N}+(L_{1|0}+L_{2|0})Q_{1,N}+L_{1|1}L_{2|0}Q_{2,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} \cdots 1\right],$$

and for $T^{[N,-2]}$ a corresponding right eigenvector for the eigenvalue $\lambda_k^{[N]}$ can be taken as

$$U_2U_1 \begin{bmatrix} \beta_N \frac{R_{0,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} \\ \beta_N \ \frac{R_{1,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} \beta_N \ \frac{R_{0,N}+(U_{1|0}+U_{2|0})R_{1,N}+U_{1|1}U_{2|0}R_{2,N}}{P_N}\Big|_{x=\lambda_k^{[N]}} \\ \vdots \\ 1 \end{bmatrix}.$$

Hence, using the sign properties of the eigenvectors associated to an oscillatory matrix we get

$$\alpha_N \ \frac{\frac{1}{L_{1|1}L_{2|0}}Q_{0,N} + \frac{L_{1|0}+L_{2|0}}{L_{1|1}L_{2|0}}Q_{1,N} + Q_{2,N}}{P_N}\Bigg|_{x=\lambda_1^{[N]}} > 0,$$
$$\alpha_N \ \frac{\frac{1}{L_{1|1}L_{2|0}}Q_{0,N} + \frac{L_{1|0}+L_{2|0}}{L_{1|1}L_{2|0}}Q_{1,N} + Q_{2,N}}{P_N}\Bigg|_{x=\lambda_2^{[N]}} < 0,$$
$$\beta_N \ \frac{\frac{1}{U_{1|1}U_{2|0}}R_{0,N} + \frac{U_{1|0}+U_{2|0}}{U_{1|1}U_{2|0}}R_{1,N} + R_{2,N}}{P_N}\Bigg|_{x=\lambda_1^{[N]}} > 0,$$
$$\beta_N \ \frac{\frac{1}{U_{1|1}U_{2|0}}R_{0,N} + \frac{U_{1|0}+U_{2|0}}{U_{1|1}U_{2|0}}R_{1,N} + R_{2,N}}{P_N}\Bigg|_{x=\lambda_2^{[N]}} < 0,$$

and so on, alternating the sign. Recalling again the sign of P_N at the zeros of P_{N+1}, we get we obtain

$$\alpha_N \left(\frac{1}{L_{1|1}L_{2|0}}Q_{0,N} + \frac{L_{1|0}+L_{2|0}}{L_{1|1}L_{2|0}}Q_{1,N} + Q_{2,N}\right)\Big|_{x=\lambda_k^{[N]}} > 0,$$
$$\beta_N \left(\frac{1}{U_{1|1}U_{2|0}}R_{0,N} + \frac{U_{1|0}+U_{2|0}}{U_{1|1}U_{2|0}}R_{1,N} + R_{2,N}\right)\Big|_{x=\lambda_k^{[N]}} > 0,$$

for $k \in \{1, \ldots, N+1\}$.

Consequently, after repeating this process up to $T^{[N,+(p-1)]}$ and $T^{[N,-(q-1)]}$ we find that

$$\beta_N \Upsilon \begin{bmatrix} R_{0,N} \\ \vdots \\ R_{q-1,N} \end{bmatrix}, \qquad \alpha_N \begin{bmatrix} Q_{0,N} \cdots Q_{p-1,N} \end{bmatrix} \Lambda,$$

are positive vectors at the points $x = \lambda_k^{[N]}$, $k \in \{1, \dots, N\}$. Therefore, if the initial condition matrices are chosen as indicated in (22) we get the result. □

8. Resolvent, second kind polynomials and Weyl functions

From here on we assume that $N \geqslant \max(p, q)$.

Definition 8.1. Given $r \in \mathbb{N}$, we write $\{e_1^{[r]}, \dots, e_r^{[r]}\}$ for the canonical basis of $\mathbb{R}^r$ and consider the $r \times (N+1)$ matrix $E_{[r,N+1]} := \begin{bmatrix} I_r & 0_{r\times(N+1-r)} \end{bmatrix}$. Then, we introduce the vectors $e_a^\nu, e_b^\xi \in \mathbb{R}^{N+1}$ with

$$e_a^\nu := E_{[p,N+1]}^\top \nu^{-\top} e_a^{[p]}, \qquad (e_b^\xi)^\top := (e_b^{[q]})^\top \xi^{-1} E_{[q,N+1]}.$$

Lemma 8.2. *For the matrices $\mathscr{U}$ and $\mathscr{W}$ (introduced in v) of Proposition 5.2) we find*

$$(e_b^\xi)^\top \mathscr{U} = \begin{bmatrix} \rho_{1,b}^{[N]} \cdots \rho_{N+1,b}^{[N]} \end{bmatrix}, \qquad \mathscr{W} e_a^\nu = \begin{bmatrix} \mu_{1,a}^{[N]} \\ \vdots \\ \mu_{N+1,a}^{[N]} \end{bmatrix}, \qquad a \in \{1, \dots, p\}, \qquad b \in \{1, \dots, q\}.$$

Remark 8.3. In matrix form, the above Lemma 8.2 reads

$$\begin{bmatrix} \rho_{1,1}^{[N]} & \cdots & \rho_{N+1,1}^{[N]} \\ \vdots & & \vdots \\ \rho_{1,q}^{[N]} & \cdots & \rho_{N+1,q}^{[N]} \end{bmatrix} = \xi^{-1} E_{[q,N+1]} \mathscr{U}, \qquad \begin{bmatrix} \mu_{1,1}^{[N]} & \cdots & \mu_{1,p}^{[N]} \\ \vdots & & \vdots \\ \mu_{N+1,1}^{[N]} & \cdots & \mu_{N+1,p}^{[N]} \end{bmatrix} = \mathscr{W} E_{[p,N+1]}^\top \nu^{-\top}.$$

Also observe that (19) follows from these relations and two facts: $\mathscr{U}\mathscr{W} = I_{N+1}$ and $E_{[q,N+1]} E_{[p,N+1]}^\top = I_{q,p}$.

For the following we need of the adjugate matrix $\operatorname{adj} A$ of a matrix A; i.e. of the transpose of the matrix of cofactors (cf. [36]).

Definition 8.4 *(Second kind polynomials).* The second kind characteristic polynomials are given by

$$P_{N+1}^{(b,a)}(x) := (e_b^\xi)^\top \operatorname{adj}(x I_{N+1} - T^{[N]}) e_a^\nu, \qquad a \in \{1, \dots, p\}, \qquad b \in \{1, \dots, q\}.$$

Proposition 8.5. *For the second kind characteristic polynomials we find*

$$P^{(b,a)}_{N+1}(x)=\sum_{k=1}^{N+1}\rho^{[N]}_{k,b}\mu^{[N]}_{k,a}\pi^{[N]}_k(x),\qquad \pi^{[N]}_k(x):=\prod_{\substack{l\in\{1,\dots,N+1\}\\ l\neq k}}\big(x-\lambda^{[N]}_l\big).$$

Proof. It follows from $\operatorname{adj}(xI_{N+1}-T^{[N]})=\operatorname{adj}(\mathscr{U}(xI_{N+1}-D)\mathscr{W})=\mathscr{U}\operatorname{adj}(xI_{N+1}-D)\mathscr{W}$. □

Proposition 8.6. *The second kind characteristic polynomials are the second kind polynomials of the characteristic polynomial; i.e.,*

$$\begin{aligned}P^{(b,a)}_{N+1}(z)&=\int\frac{P_{N+1}(z)-P_{N+1}(x)}{z-x}\,\mathrm{d}\,\psi^{[N]}_{b,a}(x)\\&=\alpha_{N+1}\int\frac{\det(A_{N+1}(z))-\det(A_{N+1}(x))}{z-x}\,\mathrm{d}\,\psi^{[N]}_{b,a}(x)\\&=\beta_{N+1}\int\frac{\det(B_{N+1}(z))-\det(B_{N+1}(x))}{z-x}\,\mathrm{d}\,\psi^{[N]}_{b,a}(x).\end{aligned}$$

Proof. We have

$$\int\frac{P_{N+1}(z)-P_{N+1}(x)}{z-x}\,\mathrm{d}\,\psi^{[N]}_{b,a}(x)=\sum_{k=1}^{N+1}\rho^{[N]}_{k,b}\mu^{[N]}_{k,a}\int\delta\big(x-\lambda^{[N]}_k\big)\frac{P_{N+1}(z)-P_{N+1}(x)}{z-x},$$

but

$$\int\delta\big(x-\lambda^{[N]}_k\big)\frac{P_{N+1}(z)-P_{N+1}(x)}{z-x}=\frac{P_{N+1}(z)-P_{N+1}\big(\lambda^{[N]}_k\big)}{z-\lambda^{[N]}_k}=\frac{P_{N+1}(z)}{z-\lambda^{[N]}_k}=\pi^{[N]}_k(z),$$

and using Proposition 8.5 we obtain the first result. For the second we use Theorem 2.7. □

Remark 8.7. The second kind characteristic polynomial matrix is

$$\begin{aligned}P^{(1)}_{N+1}&:=\xi^{-1}E_{[q,N+1]}\operatorname{adj}(xI_{N+1}-T^{[N]})E^{\top}_{[p,N+1]}\nu^{-\top}\\&=\sum_{k=1}^{N+1}\pi^{[N]}_k(x)\begin{bmatrix}\rho^{[N]}_{k,1}\\ \vdots\\ \rho^{[N]}_{k,q}\end{bmatrix}\begin{bmatrix}\mu^{[N]}_{k,1}&\cdots&\mu^{[N]}_{k,p}\end{bmatrix}\\&=\int\frac{P_{N+1}(z)-P_{N+1}(x)}{z-x}\,\mathrm{d}\,\Psi^{[N]}(x),\end{aligned}$$

is a $q\times p$ matrix of polynomials whose entries are the polynomials of the second kind: $(P^{(1)}_{N+1})_{b,a}=P^{(b,a)}_{N+1}$.

Proposition 8.8. *If T has a PBF and (22) is satisfied then* $\deg P_{N+1}^{(b,a)} = N$.

Proof. The choice (22) ensures that the entries of the vectors e_a^ν and e_b^ξ are positive. The PBF of T also ensures that all the Christoffel numbers are positive. Then, the definition of the second kind polynomials through the adjugate matrix leads to the degree N of these polynomials. □

The moments of the pq discrete measures $\mathrm{d}\,\psi_{b,a}^{[N]}$ are linked to the components of the powers of $T^{[N]}$:

Proposition 8.9 *(Discrete moments). For the discrete moments we have*

$$\begin{aligned}\int x^n \,\mathrm{d}\,\psi_{b,a}^{[N]}(x) &= \sum_{k=1}^{N+1} \rho_{k,b}^{[N]} \mu_{k,a}^{[N]} \big(\lambda_k^{[N]}\big)^n \\ &= (e_b^\xi)^\top \big(T^{[N]}\big)^n e_a^\nu, \qquad a \in \{1,\dots,p\}, \qquad b \in \{1,\dots,q\}.\end{aligned}$$

Proof. We have that $(e_b^\xi)^\top \big(T^{[N]}\big)^n e_a^\nu = (e_b^\xi)^\top \mathscr{U} D^n \mathscr{W} e_a^\nu$ so that

$$(e_b^\xi)^\top \big(T^{[N]}\big)^n e_a^\nu = \begin{bmatrix}\rho_{1,b}^{[N]} & \cdots & \rho_{N+1,b}^{[N]}\end{bmatrix} D^n \begin{bmatrix}\mu_{1,a}^{[N]} \\ \vdots \\ \mu_{N+1,a}^{[N]}\end{bmatrix},$$

and the result follows. □

Remark 8.10. In matrix form we can write

$$\int x^n \,\mathrm{d}\,\Psi^{[N]}(x) = \xi^{-1} E_{[q,N+1]} \big(T^{[N]}\big)^n E_{[p,N+1]}^\top \nu^{-\top}.$$

Definition 8.11 *(Resolvent).* The resolvent matrix $R^{[N]}(z)$ of the leading principal submatrix $T^{[N]}$ is

$$R^{[N]}(z) := \big(zI_{N+1} - T^{[N]}\big)^{-1} = \frac{\operatorname{adj}\big(zI_{N+1} - T^{[N]}\big)}{\det(zI_{N+1} - T^{[N]})}.$$

Lemma 8.12. *We have*

$$R^{[N]}(z) = \mathscr{U}(zI_{N+1} - D)^{-1} \mathscr{W}. \tag{23}$$

Proof. It follows immediately from the spectral decomposition of the matrix $T^{[N]}$. □

Definition 8.13 *(Weyl's functions).* The Weyl functions are

$$S_{b,a}^{[N]} := (e_b^\xi)^\top R^{[N]} e_a^\nu, \qquad a \in \{1,\dots,p\}, \qquad b \in \{1,\dots,q\}.$$

Proposition 8.14. *The Weyl functions can be expressed as follows*

$$\begin{aligned} S^{[N]}_{b,a}(z) &= \frac{P^{(b,a)}_{N+1}(z)}{P_{N+1}(z)} \\ &= \sum_{k=1}^{N+1} \frac{\rho^{[N]}_{k,b}\mu^{[N]}_{k,a}}{z-\lambda^{[N]}_k} = \int \frac{\mathrm{d}\,\psi^{[N]}_{b,a}(x)}{z-x}, \qquad a \in \{1,\dots,p\}, \qquad b \in \{1,\dots,q\}. \end{aligned}$$

Proof. The first equalities follow from adjugate expressions. The second expressions can be deduced from (23). Indeed, recalling Lemma 8.2 we get that the Weyl functions are

$$S^{[N]}_{b,a}(z) = \begin{bmatrix} \rho^{[N]}_{1,b} & \cdots & \rho^{[N]}_{N+1,b} \end{bmatrix} (zI_{N+1} - D)^{-1} \begin{bmatrix} \mu^{[N]}_{1,a} \\ \vdots \\ \mu^{[N]}_{N+1,a} \end{bmatrix} = \sum_{k=1}^{N+1} \frac{\rho^{[N]}_{k,b}\mu^{[N]}_{k,a}}{z-\lambda^{[N]}_k}. \qquad \square$$

Remark 8.15. For the $q \times p$ matrix of Weyl functions $S^{[N]} = \begin{bmatrix} S^{[N]}_{1,1} & \cdots & S^{[N]}_{1,p} \\ \vdots & & \vdots \\ S^{[N]}_{q,1} & \cdots & S^{[N]}_{q,p} \end{bmatrix} :=$ $\xi^{-1}E_{[q,N+1]}R^{[N]}_z E^{\top}_{[p,N+1]}\nu^{-1}$, we can write

$$S^{[N]}(z) = \frac{P^{(1)}_{N+1}(z)}{P_{N+1}(z)} = \sum_{k=1}^{N+1} \frac{1}{z-\lambda^{[N]}_k} \begin{bmatrix} \rho^{[N]}_{k,1} \\ \vdots \\ \rho^{[N]}_{k,q} \end{bmatrix} \begin{bmatrix} \mu^{[N]}_{k,1} & \cdots & \mu^{[N]}_{k,p} \end{bmatrix} = \int \frac{\mathrm{d}\,\Psi^{[N]}(x)}{z-x}.$$

Proposition 8.16. *If T has a PBF and (22) is satisfied then $P^{(b,a)}_{N+1}$ is interlaced by P_{N+1}.*

Proof. Notice that if T has a PBF all the singularities of the Weyl functions are simple poles with positive residues. Consequently, each of the second kind polynomials $P^{(b,a)}_{N+1}$ is interlaced by the characteristic polynomial P_{N+1}. $\square$

We now connect these constructions with the polynomials used in discussion of the Hermite–Padé problem in Proposition 1.29. Let us remind that $\{e^{[N+1]}_1, \dots, e^{[N+1]}_{N+1}\}$ the canonical basis of R^{N+1}.

Proposition 8.17 *(Vectorial polynomials of the second type). For $n \in \{1, \dots, N+1\}$ we find*

$$\int \frac{\mathrm{d}\,\Psi^{[N]}(x)}{z-x} \begin{bmatrix} A^{(1)}_{n-1}(x) \\ \vdots \\ A^{(p)}_{n-1}(x) \end{bmatrix} = \xi^{-1}E_{[q,N+1]}R^{[N]}(z)e^{[N+1]}_n,$$

$$\int \left[B^{(1)}_{n-1}(x)\cdots B^{(q)}_{n-1}(x)\right] \frac{\mathrm{d}\,\Psi^{[N]}(x)}{z-x} = e_n^{[N+1]\top} R^{[N]}(z) E^{\top}_{[p,N+1]}\nu^{-\top},$$

and entrywise

$$\sum_{a=1}^{p}\int \frac{\mathrm{d}\,\psi^{[N]}_{b,a}(x)}{z-x} A^{(a)}_{n-1}(x) = \left(e^{\xi}_b\right)^{\top} R^{[N]}(z) e_n^{[N+1]},$$

$$\sum_{b=1}^{q}\int B^{(b)}_{n-1}(x) \frac{\mathrm{d}\,\psi^{[N]}_{b,a}(x)}{z-x} = e_n^{[N+1]\top} R^{[N]}(z) e^{\nu}_a.$$

Proof. From (16) and (20) we get

$$\int \frac{\mathrm{d}\,\Psi^{[N]}(x)}{z-x}\begin{bmatrix} A^{(1)}_{n-1}(x)\\ \vdots \\ A^{(p)}_{n-1}(x)\end{bmatrix} = \sum_{k=1}^{N+1}\begin{bmatrix}\rho_{k,1}\\ \vdots \\ \rho_{k,q}\end{bmatrix}\frac{w^{\langle N\rangle}_{k,n}}{z-\lambda^{[N]}_k}$$

and (18) implies

$$\begin{aligned}\int \frac{\mathrm{d}\,\Psi^{[N]}(x)}{z-x}\begin{bmatrix} A^{(1)}_{n-1}(x)\\ \vdots \\ A^{(p)}_{n-1}(x)\end{bmatrix} &= \xi^{-1}\sum_{k=1}^{N+1}\begin{bmatrix}u^{\langle N\rangle}_{k,1}\\ \vdots \\ u^{\langle N\rangle}_{k,q}\end{bmatrix}\frac{w^{\langle N\rangle}_{k,n}}{z-\lambda^{[N]}_k}\\ &= \xi^{-1}E_{[q,N+1]}\sum_{k=1}^{N+1}\begin{bmatrix}u^{\langle N\rangle}_{k,1}\\ \vdots \\ \vdots \\ u^{\langle N\rangle}_{k,N+1}\end{bmatrix}\frac{1}{z-\lambda^{[N]}_k}\left[w^{\langle N\rangle}_{k,1}\cdots w^{\langle N\rangle}_{k,N+1}\right] e_n^{[N+1]},\end{aligned}$$

and the result follows. Now, proceeding similarly and using (17) and (18) we obtain the second relation. □

9. Spectral Favard theorem

As the submatrices $T^{[N]}$ are oscillatory, we know that $P_{N+1}(x)$ strictly interlaces $P_N(x)$ so that the positive sequence $\{\lambda^{[N]}_1\}_{N=1}^{\infty}$ is a strictly increasing sequence and $\{\lambda^{[N]}_{N+1}\}_{N=1}^{\infty}$ is a strictly decreasing sequence. As well, for bounded operators, $\|T\|_\infty<\infty$, we have $\|T^{[N]}\|_\infty<\|T\|_\infty<\infty$. Therefore, there exists the limits $\zeta := \lim_{N\to\infty}\lambda^{[N]}_{N+1}\geqslant 0$ and $\eta := \lim_{N\to\infty}\lambda^{[N]}_1 \leqslant \|T\|_\infty$. Following [24,37] we call $\Delta := [\zeta,\eta]\subseteq[0,\|T\|_\infty]$ the true interval of orthogonality, that is the smallest interval containing all zeros of the characteristic polynomials P_n, i.e. the eigenvalues of the leading principal submatrices of T.

Theorem 9.1 *(Favard spectral representation). Let us assume that*

i) *The banded matrix T is bounded and there exist $s \geqslant 0$ such that $T + sI$ has a PBF.*
ii) *The sequences $\{A_n^{(1)}, \dots, A_n^{(p)}\}_{n=0}^{\infty}$, $\{B_n^{(1)}, \dots, B_n^{(q)}\}_{n=0}^{\infty}$ of recursion polynomials are determined by the initial condition matrices ν and ξ, respectively, such that $\nu^{-\top} = \Lambda\mathcal{A}$, $\xi^{-1} = \mathcal{B}\Upsilon$, and $\mathcal{A} \in \mathbb{R}^{p\times p}$ is a nonnegative upper unitriangular matrices and $\mathcal{B} \in \mathbb{R}^{q\times q}$ is a nonnegative lower unitriangular matrix.*

Then, there exists pq non decreasing positive functions $\psi_{b,a}$, $a \in \{1, \dots, p\}$ and $b \in \{1, \dots, q\}$ and corresponding positive Lebesgue–Stieltjes measures $\mathrm{d}\,\psi_{b,a}$ with compact support Δ such that the following biorthogonality holds

$$\sum_{a=1}^{p}\sum_{b=1}^{q}\int_{\Delta} B_l^{(b)}(x)\,\mathrm{d}\,\psi_{b,a}(x) A_k^{(a)}(x) = \delta_{k,l}, \qquad k, l \in \mathbb{N}_0.$$

Proof. The shift in the matrix $T \to T + sI$ only shifts by s the eigenvalues of the truncations $T^{[N]}$, so that they are positive, and the dependent variable of the recursion polynomials, but do not alter the interlacing properties of the polynomials and the positivity of the corresponding Christoffel numbers. From Theorem 7.11 we know that the sequences $\{\psi_{a,b}^{[N]}\}_{N=0}^{\infty}$, $a \in \{1, \dots, p\}$, $b \in \{1, \dots, q\}$ given in Definition 6.1 are positive. Moreover, Proposition 6.2 implies that they are uniformly bounded and nondecreasing. Consequently, following Helly's results, see [24, §II] there exist subsequences that converge when $N \to \infty$ to positive nondecreasing functions $\psi_{b,a}$ with support on Δ and that the discrete biorthogonal relations lead to the stated biorthogonal properties. □

Corollary 9.2 *(Mixed multiple orthogonal relations). In the conditions of Theorem 9.1, the mixed multiple orthogonal relations are fulfilled*

$$\begin{aligned}
&\sum_{a=1}^{p}\int_{\Delta} x^n\,\mathrm{d}\,\psi_{b,a}(x) A_m^{(a)}(x) = 0, && n \in \{0, \dots, \deg B_{m-1}^{(b)}\}, && b \in \{1, \dots, q\},\\
&\sum_{b=1}^{q}\int_{\Delta} B_m^{(b)}(x)\,\mathrm{d}\,\psi_{b,a}(x) x^n = 0, && n \in \{0, \dots, \deg A_{m-1}^{(a)}\}, && a \in \{1, \dots, p\}.
\end{aligned}$$

Definition 9.3. Let us consider the semi-infinite matrix

$$E_{[r]} := \left[\begin{array}{cccc|c}
1 & 0 & \cdots & 0 & 0 \cdots \\
 & \ddots & \overset{r \text{ times}}{\ddots} & \vdots & \vdots \\
0 & \ddots & \ddots & 0 & r \text{ times} \\
\vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \cdots
\end{array}\right],$$

and the infinite vectors

$$u_a^\nu = E_{[p]}^\top \nu^{-\top} e_a^{[p]} \qquad u_b^\xi = \left(e_b^{[q]}\right)^\top \xi^{-1} E_{[q]}$$

Proposition 9.4 *(Spectral representation of moments and Stieltjes–Markov functions). In the conditions of Theorem 9.1 and in terms of the spectral functions $\psi_{b,a}$, $a \in \{1, \dots, p\}$, $b \in \{1, \dots, q\}$ we find the following relations between entries of powers or the resolvent of the banded matrix and moments or the Cauchy transform of the measures, respectively:*

$$(u_b^\xi)^\top T^n u_a^\nu = \int_\Delta x^n \,\mathrm{d}\,\psi_{b,a}(x), \qquad (u_b^\xi)^\top (zI - T)^{-1} u_a^\nu = \int_\Delta \frac{\mathrm{d}\,\psi_{b,a}(x)}{z - x} =: \hat{\psi}_{b,a}(z).$$

Proof. Propositions 8.9 and 8.14 and Helly's second theorem lead to the spectral representation for the moments and Stieltjes–Markov functions $\hat{\psi}_{b,a}(z)$ of T. □

Remark 9.5. In terms of $\Psi = \begin{bmatrix} \psi_{1,1} & \cdots & \psi_{1,p} \\ \vdots & & \vdots \\ \psi_{q,1} & \cdots & \psi_{q,p} \end{bmatrix}$ we have

$$\xi^{-1} E_{[q]} T^n E_{[p]}^\top \nu^{-\top} = \int_\Delta x^n \,\mathrm{d}\,\Psi(x), \qquad \xi^{-1} E_{[q]} (zI - T)^{-1} E_{[p]}^\top \nu^{-\top} = \int_\Delta \frac{\mathrm{d}\,\Psi(x)}{z - x}.$$

Proposition 9.6 *(Normal convergence of Weyl functions). Given the conditions of Theorem 9.1, the Weyl functions in Proposition 8.14 converge uniformly in compact subsets of $\bar{\mathbb{C}} \setminus \Delta$ to the Stieltjes–Markov functions, i.e.,*

$$S_{b,a}^{[N]}(z) = \frac{P_{N+1}^{(b,a)}(z)}{P_{N+1}(z)} \underset{N \to \infty}{\Longrightarrow} \hat{\psi}_{b,a}(z), \qquad a \in \{1, \dots, p\}, \qquad b \in \{1, \dots, q\}.$$

Proof. Notice the uniform boundedness in N in compact subsets of $\bar{\mathbb{C}} \setminus \Delta$ of the Weyl functions $S_{b,a}^{[N]}$ for each pair a, b. Then, Vitali convergence theorem see [45, Theorem 6.2.8] leads to the result. □

Remark 9.7. Despite the positivity of Christoffel numbers described in Theorem 7.11 we only have the bound proved in Proposition 6.2. Therefore, we know that the functions $\psi_{b,a}^{[N]}$ given in Definition 6.1 that are right continuous, of bounded variation, increasing and positive are also uniformly bounded. Therefore, Helly's theorem can be applied to the large N limit.

However, this is not applicable to each family of Christoffel numbers separately. That is,

$$\varphi_b^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \rho_{1,b}^{[N]} + \cdots + \rho_{k,b}^{[N]}, & \lambda_{k+1}^{[N]} \leqslant x < \lambda_k^{[N]}, \quad k \in \{1,\dots,N\}, \\ \rho_{1,b}^{[N]} + \cdots + \rho_{N+1,b}^{[N]}, & x \geqslant \lambda_1^{[N]}, \end{cases}$$

$$\tilde\varphi_a^{[N]} := \begin{cases} 0, & x < \lambda_{N+1}^{[N]}, \\ \mu_{1,a}^{[N]} + \cdots + \mu_{k,a}^{[N]}, & \lambda_{k+1}^{[N]} \leqslant x < \lambda_k^{[N]}, \quad k \in \{1,\dots,N\}, \\ \mu_{1,a}^{[N]} + \cdots + \mu_{N+1,a}^{[N]}, & x \geqslant \lambda_1^{[N]}, \end{cases}$$

are right continuous, of bounded variation, increasing and positive. But, in principle, they might be not bounded and therefore Helly's result may not be applicable.

Thus, to get measures from these functions we need to ensure the existence of bounds as follows $\rho_{1,b}^{[N]} + \cdots + \rho_{N+1,b}^{[N]} \leqslant R_b$ and $\mu_{1,a}^{[N]} + \cdots + \mu_{N+1,a}^{[N]} \leqslant M_a$. For such situation, the large limit will lead to the existence of spectral measures $\mathrm{d}\,\varphi_a$, $a \in 1,\dots,p$ and $\mathrm{d}\,\tilde\varphi_b$, $b \in \{1,\dots,q\}$. If these measures are absolutely continuous w.r.t. the measure $\mathrm{d}\,\mu$, with Radon–Nikodym derivatives the weights w_a and $\tilde w_b$, respectively, we could write $\mathrm{d}\,\varphi_a = w_a \,\mathrm{d}\,\mu$ and $\mathrm{d}\,\tilde\varphi_b = \tilde w_b \,\mathrm{d}\,\mu$. A natural conjecture, that we have not yet proven, is that in this situation $\mathrm{d}\,\psi_{b,a} = w_a \tilde w_b \,\mathrm{d}\,\mu$. This rank one simplification is assumed in a large number of papers dealing with mixed multiple orthogonality.

10. Mixed multiple Gaussian quadrature and degrees of precision

Gaussian quadrature formulas are an important tool in the theory of orthogonal polynomials and its applications to approximation theory, see for example [24,37]. Its extension to non-mixed multiple orthogonal polynomials was discussed in [17,31,25,20], degrees of precision were presented in [20]. Now, we give its extension to the mixed multiple orthogonal situation. Notice that for $p = q$ we are dealing with standard matrix orthogonal polynomials and such quadrature formulas have been discussed for this situation, see the excellent review [28] and references therein cited.

Let us assume that T has a PBF, and that the conditions of Theorem 9.1 hold, and introduce:

Definition 10.1. The degrees of precision or orders $d_{b,a}(N)$, $a \in \{1,\dots,p\}$, $b \in \{1,\dots,q\}$, are the largest natural numbers such that

$$\big(u_b^{\xi}\big)^{\top} T^n u_a^{\nu} = (e_b^{\xi})^{\top} \big(T^{[N]}\big)^n e_a^{\nu}, \qquad 0 \leqslant n \leqslant d_{b,a}(N), \qquad a \in \{1,\dots,p\}, \qquad b \in \{1,\dots,q\}.$$

Proposition 10.2. *In terms of the recursion polynomial degrees, see Proposition 2.3, the degrees of precision are*

$$d_{b,a}(N) = \deg A_N^{(a)} + \deg B_N^{(b)} + 1 = \left\lceil \frac{N+2-a}{p} \right\rceil + \left\lceil \frac{N+2-b}{q} \right\rceil - 1,$$

$$a \in \{1, \dots, p\}, \qquad b \in \{1, \dots, q\}.$$

Proof. The vectors e_b^{ξ}, u_b^{ξ}, e_a^{ν}, and u_a^{ν} are nonnegative, with the first q or p entries being positive numbers, and the remaining entries being zero, respectively. In the computation of $\left(u_b^{\xi}\right)^{\top} T^n u_a^{\nu}$, our focus is on determining whether $(T^n)_{j,i}$, where $j \in \{0, \dots, b-1\}$ and $i \in \{0, \dots, a-1\}$, involves nonzero factors $T_{k,l}$ with $k > N$ or $l > N$.

We observe that $(T^n)_{j,i}$ can be expressed as sums of products of the form

$$T_{j,i_1} T_{i_1,i_2} \cdots T_{i_{n-2},i_{n-1}} T_{i_{n-1},i},$$

where each factor is a positive entry from the banded matrix T. Our objective is to analyze those products that might lead to the appearance of undesired nonzero factors $T_{k,l}$ with $k > N$ or $l > N$ at an earlier stage.

When considering a specific row k, the entry $T_{k,k+q}$ is the last nonzero entry as we move to the right. Similarly, $T_{k,k-p}$ (for $k \geq p$) is the last positive entry as we move to the left. These "optimal ascending jumps" of q units provide the fastest upward movement in a column, while the "optimal descending jumps" of p units offer the quickest downward movement.

Examining products of $s + 1$ factors involving optimal ascending jumps, designed to go from $b - 1$ to $N + 1$ as rapidly as possible, we have:

$$T_{j,j+q} T_{j+q,j+2q} T_{j+2q,j+3q} \cdots T_{j+(s-1)q,j+sq} T_{j+sq,N+1}.$$

Here, s is a nonnegative integer ensuring that $j + (s + 1)q \geq N + 1$ for the first time, which can be expressed as:

$$s \geq \frac{N + 1 - j}{q} - 1.$$

Hence, $s = \left\lceil \frac{N + 1 - j}{q} \right\rceil - 1$.

Moving downward to the i position using optimal descending jumps involves a product of r factors:

$$T_{N+1,N+1-p} T_{N+1-p,N+1-2p} T_{N+1-2p,N+1-3p} \cdots T_{N+1-(r-1)p,i}.$$

The condition $N + 1 - rp \leq i$ dictates $r = \left\lceil \frac{N + 1 - i}{p} \right\rceil$.

Combining these insights, the product:

$$\begin{aligned} &T_{j,j+q} T_{j+q,j+2q} T_{j+2q,j+3q} \cdots T_{j+(s-1)q,j+sq} T_{j+sq,N+1} \\ &\qquad \times T_{N+1,N+1-p} T_{N+1-p,N+1-2p} \cdots T_{N+1-(r-2)p,N+1-(r-1)p} T_{N+1-(r-1)p,i} \end{aligned}$$

illustrates the quickest path to reach an element $T_{k,l}$ where $k > N$ or $l > N$. This product involves $r+s+1$ factors.

Thus, to avoid such scenarios, we must consider, for the entry $(T^n)_{j,i}$, a power of at most $n < r+s+1$, i.e.,

$$n = r+s = \left\lceil \frac{N+1-j}{q} \right\rceil - 1 + \left\lceil \frac{N+1-i}{p} \right\rceil.$$

Finally, we will determine the smallest value among these powers for $j \in \{0,\dots,b-1\}$ and $i \in \{0,\dots,a-1\}$. This will give us the expression for the n-th power as follows:

$$n = \left\lceil \frac{N+2-b}{q} \right\rceil + \left\lceil \frac{N+2-a}{p} \right\rceil - 1,$$

which leads us to the desired result. □

Theorem 10.3 *(Mixed multiple Gaussian quadrature formulas). The following Gauss quadrature formulas hold*

$$\begin{aligned}&\int_\Delta x^n \,\mathrm{d}\,\psi_{b,a}(x) = \sum_{k=1}^{N+1} \rho^{[N]}_{k,b}\mu^{[N]}_{k,a}\big(\lambda^{[N]}_k\big)^n,\\ &0 \leqslant n \leqslant d_{b,a}(N), \qquad a \in \{1,\dots,p\}, \qquad b \in \{1,\dots,q\}.\end{aligned} \tag{24}$$

Here the degrees of precision $d_{b,a}$ are optimal (for any power largest than n a positive remainder appears, an exactness is lost).

Proof. On the one hand, from Proposition 9.4 we have that $\big(e^\xi_b\big)^\top T^n e^\nu_a = \int_\Delta x^n \,\mathrm{d}\,\psi_{b,a}(x)$. On the other hand, from Proposition 8.9, we know that $\big(e^\xi_b\big)^\top \big(T^{[N]}\big)^n e^\nu_a = \sum_{k=1}^{N+1} \rho^{[N]}_{k,b}\mu^{[N]}_{k,a}\big(\lambda^{[N]}_k\big)^n$. Hence, as we have

$$\big(e^\xi_b\big)^\top T^n e^\nu_a = \big(e^\xi_b\big)^\top \big(T^{[N]}\big)^n e^\nu_a, \quad 0 \leqslant n \leqslant d_{b,a}(N), \quad a \in \{1,\dots,p\}, \quad b \in \{1,\dots,q\},$$

we get (24). Notice that for $n > d_{b,a}(N)$ a positive remainder will appear and exactness will be lost. Indeed, observe that T^n is oscillatory and that $e^\nu_a = \Lambda\mathcal{A}e_a$ and $e^\xi_b = \Upsilon\mathcal{B}e_a$ are positive vectors, so all the objects involved imply positive contributions. □

Remark 10.4. In terms of the number of nodes or interpolation points $\mathcal{N} = N+1$, the zeros of the characteristic polynomial P_{N+1} or, equivalently the eigenvalues of $T^{[N]}$, we have that in the non multiple case for which $a=b=p=q=1$ the degree of precision is $2\mathcal{N}-1$ and we recover the well known Gauss quadrature formula, see for example [24,37]. For the non mixed multiple situation we recover the result we got and discussed in [20, Theorem 7], that is, degrees of precision $d_a = \mathcal{N} - 1 + \deg A^{(a)}_{\mathcal{N}-1}$.

Remark 10.5. Notice that for the standard orthogonality, i.e. $p = q = 1$, the nodes are the zeros of an orthogonal polynomial of certain degree. This also happens for the non mixed multiple situation as the characteristic polynomials and one of the families of recursion polynomials, say B_n, coincide. However, for mixed multiple orthogonality the nodes are the zeros of the characteristic polynomial of the corresponding truncation, which is not an orthogonal polynomial. Consequently, the nodes are not, in general, the zeros of the left or right recursion polynomials, that are the ones satisfying the mixed multiple orthogonal relations.

Remark 10.6. A quadrature is said to be interpolatory if there is a polynomial that interpolates the function for which a weighted integral is supposed to be approximated by a quadrature. In the non mixing multiple orthogonal quadrature the interpolation polynomial is $P_N = B_N$ for all the measures $\mathrm{d}\,\psi_a$, $a \in \{1, \dots, p\}$. Now, for the mixed multiple orthogonality for each $b \in \{1, \dots, q\}$, we use the interpolation polynomials $B_N^{(b)}$ for the measures $\mathrm{d}\,\psi_{b,a}$, $a \in \{1, \dots, p\}$, so that in order to have an interpolatory quadrature we need the degrees of precision to be at least $\deg B_N^{(b)} - 1$, which in fact is the case.

Remark 10.7. For the case $p = q$, i.e. when we are dealing with the usual matrix orthogonality, as we are working with $p \times p$ blocks we take $\mathcal{N} = Mp$, with $M \in \mathbb{N}$, the degree of precision given in Proposition 10.2 must be the smaller degree of precision in $p \times p$ block, i.e.

$$\begin{aligned} d(N) = 2\deg B_N^{(p)} + 1 = 2\left\lceil \frac{\mathcal{N}+1-p}{p} \right\rceil - 1 = 2\left\lceil \frac{(M-1)p+1}{p} \right\rceil - 1 \\ = 2(M-1) + 2 - 1 = 2M - 1. \end{aligned}$$

This is the optimal degree of precision according to Durán and Polo [29].

11. Conclusions and outlook

In this paper, we introduce an extension of the spectral Favard theorem, which establishes the presence of positive measures for bounded Jacobi matrices. This extension is formulated to encompass situations featuring a band structure with p subdiagonals and q superdiagonals.

The foundation for this extension arose from our observation that shifting a Jacobi matrix yields an oscillatory matrix, which can be factorized into a positive bidiagonal configuration. Our pivotal discovery lies in the fact that this positive bidiagonal factorization, indicative of the matrix's oscillatory nature, allows for a spectral interpretation. Consequently, we effectively validate the existence of mixed multiple orthogonal polynomials tailored to bounded banded matrices that adhere to this specific pattern.

An additional outcome of our work is the derivation of a multiple Gauss quadrature approach, incorporating explicit degrees of precision, for scenarios involving mixed multiple orthogonality.

Looking ahead, our first objective is to extend the Karlin–MacGregor spectral interpretation of birth and death Markov chains [41] to encompass multiple potential transitions, spanning up to p backwards and q forwards. We have already accomplished this extension within the Hessenberg framework, as demonstrated in [20]. In that context, either $q = 1$ and p can be arbitrary or $p = 1$ and q can be arbitrary.

An intriguing avenue for exploration involves the functional analysis interpretation of our findings. In the tridiagonal Jacobi scenario, the Favard spectral theorem serves as a crucial element in establishing the spectral theorem for bounded self-adjoint operators A [46, Sections 5.2 and 5.3]. Central to this proof is the role of cyclic vectors φ, which facilitate the construction of $\psi_n = A^n\varphi$ and vectors φ_n using the Gram–Schmidt method. In this basis, A takes on the form of a Jacobi matrix, thus allowing the classical spectral Favard theorem to be applied. As a result, the spectral theorem for A becomes readily demonstrable [46, Theorem 5.3.1]. Moreover, in [43, Section 2] (where the Jacobi matrix is denoted as $\mathscr{L}$ and the cyclic vector for $\mathscr{L}$ is e_0), it is shown that if Q_n represents the corresponding orthonormal polynomials, then $e_n = Q_n(\mathscr{L})e_0$, thereby implying $\delta_{k,l} = (e_k, e_l) = (Q_k(\mathscr{L})e_0, Q_l(\mathscr{L})e_0) = \int Q_k(x)Q_l(x)\,\mathrm{d}\mu$.

The question that arises is whether a similar construction exists for the banded operators T discussed in this paper. Preliminary observations suggest that, rather than a cyclic vector, we might expect cyclic subspaces, which could potentially encompass both left and right cyclic subspaces. Understanding the construction delineated in [43] and the formula $e_n = Q_n(\mathscr{L})e_0$ within the framework of banded scenarios is of paramount importance. We hold the belief that the spectral outcomes presented in Proposition 9.4 and Remark 9.5 hold significance within this particular context.

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