On a fractional Sturm-Liouville problem in higher dimensions*

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Abstract

In this short paper, we consider an n-dimensional fractional Sturm-Liouville eigenvalue problem, by using fractional versions of the gradient operator involving left Caputo and right Riemann-Liouville fractional derivatives. We study the main properties of the eigenfunctions and the eigenvalues of the associated fractional boundary problem.

Keywords: Fractional derivatives; Fractional Sturm-Liouville problem; Fractional variational calculus; Eigenvalue problem; Eigenfunctions; Fractional Clifford analysis.

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1 Introduction

In recent years, many mathematicians directed their attention to some generalizations of the Sturm-Liouville problem in connection with other fields of Mathematics. One of the most important reasons for this emerging interest is the fact that the orthogonal eigenfunctions' system of the fractional Sturm-Liouville problem can be used to solve fractional partial differential equations that are related with anomalous diffusion processes (see [2, 3] and references therein indicated). We consider the n-dimensional fractional Sturm-Liouville eigenvalue problem, by using fractional versions of the gradient operator involving left Caputo and right Riemann-Liouville fractional derivatives. We study the main properties of the eigenfunctions and the eigenvalues of the associated fractional boundary problem. More precisely, we show that the eigenfunctions are orthogonal and the eigenvalues are real and simple.

2 Preliminaries

Let $a, b \in \mathbb{R}$ with $a < b$ and $\alpha > 0$. The left and right Riemann-Liouville fractional integrals $I^{\alpha}_{a^+}$ and $I^{\alpha}_{b^-}$ of order α are given by (see [1])

$$\left(I^{\alpha}_{a^+} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}}\, dt, \qquad x > a \tag{1}$$

$$\left(I^{\alpha}_{b^-} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}}\, dt, \qquad x < b. \tag{2}$$

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By ${}^{RL}D^{\alpha}_{a+}$ and ${}^{RL}D^{\alpha}_{b-}$ we denote the left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ on $[a,b] \subset \mathbb{R}$, which are defined by (see [1])

$$\left({}^{RL}D^{\alpha}_{a+} f\right)(x) \;= \left(D^m I^{m-\alpha}_{a+} f\right)(x) \;= \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_a^x \frac{f(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a \tag{3}$$

$$\left({}^{RL}D^{\alpha}_{b-} f\right)(x) \;= (-1)^m \left(D^m I^{m-\alpha}_{b-} f\right)(x) \;= \frac{(-1)^m}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_x^b \frac{f(t)}{(t-x)^{\alpha-m+1}}\, dt, \qquad x < b. \tag{4}$$

Here, $m = [\alpha] + 1$ and $[\alpha]$ means the integer part of α. Let ${}^{C}D^{\alpha}_{a+}$ be the left Caputo fractional derivative of order $\alpha > 0$ on $[a,b] \subset \mathbb{R}$, which is defined by (see [1])

$$\left({}^{C}D^{\alpha}_{a+} f\right)(x) \;= \left(I^{m-\alpha}_{a+} D^m f\right)(x) \;= \frac{1}{\Gamma(m-\alpha)} \int_a^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}}\, dt, \qquad x > a. \tag{5}$$

We denote by $I^{\alpha}_{a+}(L_p)$, with $p \geq 1$, the class of functions f that are represented by the fractional integral (**??**) of a summable function, that is $f = I^{\alpha}_{a+}\varphi$, with $\varphi \in L_p(a,b)$. A description of the space $I^{\alpha}_{a+}(L_1)$ is given in [4].

Theorem 2.1 *(cf. [4]) A function f belongs to $I^{\alpha}_{a+}(L_1)$, with $\alpha > 0$, if and only if $I^{m-\alpha}_{a+} f$ belongs to $AC^m([a,b])$, $m = [\alpha]+1$ and $\left(I^{m-\alpha}_{a+} f\right)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$.*

In Theorem 2.1, $AC^m([a,b])$ denotes the class of functions f which are continuously differentiable on the segment $[a,b]$ up to the order $m-1$ and $f^{(m-1)}$ is absolutely continuous on $[a,b]$. We note that the conditions $\left(I^{m-\alpha}_{a+} f\right)^{(k)}(a) = 0$, $k = 0, \ldots, m-1$, imply that $f^{(k)}(a) = 0$, for $k = 0, \ldots, m-1$ (see [4]). This conclusion implies (see formula (2.4.1) in [1]) that $\left({}^{C}D^{\alpha}_{a+} f\right)(x) = \left({}^{RL}D^{\alpha}_{a+} f\right)(x)$. Removing the last condition in Theorem 2.1 we obtain the class of functions that admit a summable fractional derivative.

Definition 2.2 *(cf. [4]) A function $f \in L_1(a,b)$ has a summable fractional derivative $\left(D^{\alpha}_{a+} f\right)(x)$ if $\left(I^{m-\alpha}_{a+} f\right)(x)$ belongs to $AC^m([a,b])$, where $m = [\alpha]+1$.*

If a function f admits a summable fractional derivative, then we have the following composition rules (see [4])

$$\left(I^{\alpha}_{a+}\, {}^{C}D^{\alpha}_{a+} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{(x-a)^k}{k!}\, f^{(k)}(a), \tag{6}$$

$$\left(I^{\alpha}_{b-}\, {}^{RL}D^{\alpha}_{b-} f\right)(x) = f(x) - \sum_{k=0}^{m-1} \frac{(b-x)^{\alpha-k-1}}{\Gamma(\alpha-k)}\, \left(I^{m-\alpha}_{a+} f\right)^{(m-k-1)}(b), \tag{7}$$

with $m = [\alpha]+1$. We remark that if $f \in I^{\alpha}_{a+}(L_1)$ then (6) and (7) reduce to $\left(I^{\alpha}_{a+}\, {}^{C}D^{\alpha}_{a+} f\right)(x) = \left(I^{\alpha}_{b-}\, {}^{RL}D^{\alpha}_{b-} f\right)(x) = f(x)$. Nevertheless we note that ${}^{C}D^{\alpha}_{a+}\, I^{\alpha}_{a+} f = {}^{RL}D^{\alpha}_{b-}\, I^{\alpha}_{b-} f = f$ in both cases. Moreover, for $m-1 < \alpha < m$ with $m \in \mathbb{N}$ and $\beta > 0$ we have

$$I^{\alpha}_{b-}\,(b-x)^{\beta-1} \;= \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}\,(b-x)^{\beta+\alpha-1}, \qquad\qquad {}^{C}D^{\alpha}_{a+}\,(x-a)^{\beta-1} \;= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}\,(x-a)^{\beta-1}. \tag{8}$$

3 Fractional Sturm-Liouville problem in higher dimensions

Let us consider the following Riemann-Liouville fractional Sturm-Liouville equation in n-dimensions

$$-\left({}^{RL}\nabla^{\alpha}_{b-} \cdot \left(\mu(x)\; {}^{C}\nabla^{\alpha}_{a+} f\right)\right)(x) = \lambda\, r(x)\, f(x) \tag{9}$$

subject to the following Dirichlet and Neuman boundary conditions

$$\begin{aligned} &\beta_1^{[j]}\, f(x)\,|_{x_j=a_j} + \beta_2^{[j]}\, I^{1-\alpha_j}_{b_j^-}\left(\mu\; {}^{C}_{a_j^+}\partial^{\alpha_j}_{x_j} f\right)(x)\,|_{x_j=a_j} = 0, \\ &\beta_3^{[j]}\, f(x)\,|_{x_j=b_j} + \beta_4^{[j]}\, I^{1-\alpha_j}_{b_j^-}\left(\mu\; {}^{C}_{a_j^+}\partial^{\alpha_j}_{x_j} f\right)(x)\,|_{x_j=b_j} = 0, \end{aligned} \tag{10}$$

where $j = 1, \ldots, n$, $x \in \Omega = \prod_{i=1}^n]a_i, b_i[\subset \mathbb{R}^n$, " $\cdot$ " is the scalar product between two vectors in $\mathbb{R}^n$, and μ, r are positive continuous scalar functions defined on Ω. Moreover

- ${}^{RL}\nabla^{\alpha}_{b^-}$ and ${}^{C}\nabla^{\alpha}_{a^+}$ are, respectively, the right Riemann-Liouville and left Caputo fractional gradient operators of order $\alpha = (\alpha_1, \dots, \alpha_n)$ given by

$$ {}^{RL}\nabla^{\alpha}_{b^-} = \sum_{i=1}^{n} e_i \; {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \qquad \text{and} \, {}^{C}\nabla^{\alpha}_{a^+} = \sum_{i=1}^{n} e_i \; {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i}, $$

  where for $i = 1, \dots, n$, e_i denotes the standard unit vector in the direction of x_i, and the partial derivatives ${}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}$, ${}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i}$, are the right Riemann-Liouville and left Caputo fractional derivatives of order $\alpha_i \in \left]\frac{1}{2}, 1\right]$ with respect to the variable $x_i \in]a_i, b_i[$;

- $I^{1-\alpha_j}_{b_j^-}$ denotes the right Riemann-Liouville fractional integral of order $1 - \alpha_j$ with respect to the variable $x_j \in]a_j, b_j[$, where $\alpha_j \in \left]\frac{1}{2}, 1\right]$ and $j = 1, \dots, n$;

- the values of $\lambda \in \mathbb{C}$ for which there exists non-trivial solutions $f(x) \in I^{\alpha_j}_{a_j^+}(L_p(\Omega))$, with $p > 1$ and $j = 1, \dots, n$, are called the eigenvalues of the problem.

We remark that $L_p(\Omega) \subset L_1(\Omega)$, for $p > 1$, then since $f(x) \in I^{\alpha_j}_{a_j^+}(L_p(\Omega))$ we have that $f(x) \in I^{\alpha_j}_{a_j^+}(L_1(\Omega))$, for every $j = 1, \dots, n$. Therefore, from Theorem 2.1 we conclude that $f(x)|_{x_j = a_j} = 0$. Let us define the fractional Sturm-Liouville operator ${}^{RLC}L^{\alpha}$ associated to problem (9)-(10) as

$$ {}^{RLC}L^{\alpha} := -{}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu \, {}^{C}\nabla^{\alpha}_{a^+}\right). $$

This operator can be seen as a fractional differential operator of second order since $\alpha_i \in \left]\frac{1}{2}, 1\right]$, for every $i = 1, \dots, n$. Moreover, in the special case of $\alpha = (1, \dots, 1)$ and $\mu(x) = 1$ we recover the Euclidean Laplace operator. Following the same reasoning of the proof of Theorems 3.1, 3.2, and 3.3 in [3] we have, respectively, the following results:

Theorem 3.1 *Let $\alpha^* = \min_{1 \le i \le n}\{\alpha_i\}$, $p \ge 1$, $q \ge 1$ and $\frac{1}{q} + \frac{1}{p} \le 1 + \alpha^*$ ($p \neq 1$ and $q \neq 1$ in the case $\frac{1}{p} + \frac{1}{q} = 1 + \alpha^*$). If $h \in I^{\alpha}_{b^-}(L_p)$ and $\mu(x) \; {}^{C}\nabla^{\alpha}_{a^+} g(x) \in I^{\alpha}_{a^+}(L_p)$, then*

$$ \int_{\Omega} h(x) \; {}^{RLC}L^{\alpha} g(x) \, dx = \int_{\Omega} g(x) \; {}^{RLC}L^{\alpha} h(x) \, dx. $$

Theorem 3.2 *All the eigenvalues of the fractional Sturm-Liouville problem* (9)-(10) *are real.*

Theorem 3.3 *If f and g are two eigenfunctions of the fractional Sturm-Liouville problem* (9)-(10) *corresponding to the eigenvalues λ_1 and λ_2, respectively, with $\lambda_1 \neq \lambda_2$, then the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the weight function r, i.e.,*

$$ \int_{\Omega} r(x) \; f(x) \; g(x) \, dx = 0. $$

Now, as it was done in [3], we prove under which conditions we have that for each eigenvalue corresponds only one linearly independent eigenfunction, up to a constant. Let λ be an eigenvalue of the fractional Sturm-Liouville problem (9)-(10) and f the eigenfunction associated to it. For the equation in (9) we have

$$ -\left({}^{RL}\nabla^{\alpha}_{b^-} \cdot \left(\mu \, {}^{C}\nabla^{\alpha}_{a^+} f\right)\right)(x) = \lambda \, r(x) \; f(x) \; \Leftrightarrow \; \sum_{i=1}^{n} {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x) \; {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right) = -\lambda \, r(x) \; f(x). \tag{11} $$

In order to incorporate (11) and the boundary conditions defined in (10) in a single equation, we need to apply fractional integral operators to (11). Applying firstly $I^{\alpha_j}_{b_j^-}$ and secondly $I^{\alpha_j}_{a_j^+}$, taking into account (7) and (6), and making straightforward calculations, we get that (11) is equivalent to

$$ \begin{aligned} f(x) = {} & (x_j - a_j) \; \xi_1^{[j]}|_{x_j = a_j} + \xi_2^{[j]}|_{x_j = b_j} \; I^{\alpha_j}_{a_j^+} \left(\frac{(b_j - x_j)^{\alpha_j - 1}}{\mu(x) \, \Gamma(\alpha_j)} \right) \\ & + \sum_{\substack{i=1 \\ i \neq j}}^{n} I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)} \, I^{\alpha_j}_{b_j^-} \; {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \left(\mu(x) \; {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right)\right) - \lambda \, I^{\alpha_j}_{a_j^+} \left(\frac{1}{\mu(x)} \, I^{\alpha_j}_{b_j^-} \left(r(x) \; f(x)\right)\right), \end{aligned} \tag{12} $$

where the constants $\xi_1^{[j]}|_{x_j = a_j}$ and $\xi_2^{[j]}|_{x_j = b_j}$ with respect to the variable x_j are given by

$$ \xi_1^{[j]}|_{x_j = a_j} = f'(x)|_{x_j = a_j} \qquad \text{and} \qquad \xi_2^{[j]}|_{x_j = b_j} = I^{1-\alpha_j}_{b_j^-} \left(\mu(x) \; {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i} f(x)\right)|_{x_j = b_j}. $$

Applying the operator $I^{1-\alpha_j}_{b_j^-}\left(\mu\left(x\right)\ {}^{C}_{a_j^+}\partial^{\alpha_j}_{x_j}\right)$ to both sides of (12) and taking into account the relations (8), we obtain that (12) is equivalent to

$$\xi^{[j]}_2|_{x_j=b_j}=I^{1-\alpha_j}_{b_j^-}\left(\mu\left(x\right)\ {}^{C}_{a_j^+}\partial^{\alpha_j}_{x_j}f\left(x\right)\right)-\xi^{[j]}_1|_{x_j=a_j}\,\frac{1}{\Gamma\left(2-\alpha_j\right)}\,I^{1-\alpha_j}_{b_j^-}\left(\mu\left(x\right)\left(x_j-a_j\right)\right)$$
$$-\sum_{\substack{i=1\\i\neq j}}^{n}I^{1}_{b_j^-}\ {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu\left(x\right)\ {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i}f\left(x\right)\right)+\lambda\,I^{1}_{b_j^-}\left(r\left(x\right)\,f\left(x\right)\right). \tag{13}$$

Considering now the first boundary condition (10) and the fact that $f\left(x\right)\Big|_{x_j=a_j}=0$, we have that

$$I^{1-\alpha_j}_{b_j^-}\left(\mu\left(x\right)\ {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i}f\left(x\right)\right)\Big|_{x_j=a_j}=0.$$

Combining the previous conclusion and (13) (with $x_j=a_j$), we get

$$\xi^{[j]}_2|_{x_j=b_j}=\lambda\,I^{1}_{b_j^-}\left(r\left(x\right)\,f\left(x\right)\right)-\sum_{\substack{i=1\\i\neq j}}^{n}I^{1}_{b_j^-}\ {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu\left(x\right)\ {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i}f\left(x\right)\right)$$
$$-\,\xi^{[j]}_1\Big|_{x_j=a_j}\,\frac{1}{\Gamma\left(2-\alpha_j\right)}\,I^{1-\alpha_j}_{b_j^-}\left(\mu\left(x\right)\left(x_j-a_j\right)\right)\Big|_{x_j=a_j}, \tag{14}$$

where $\xi^{[j]}_1\Big|_{x_j=a_j}$ comes from the second boundary condition. Taking into account (14) and (12) with $x_j=b_j$, the second boundary condition in (10) leads to

$$\xi^{[j]}_1|_{x_j=a_j}=\frac{\Gamma\left(2-\alpha_j\right)\,\left(b_j-a_j\right)^{\alpha_j-1}}{\Gamma\left(2-\alpha_j\right)\,\left(b_j-a_j\right)^{\alpha_j-1}-\Gamma\left(\alpha_j\right)\,I^{1-\alpha_j}_{b_j^-}\left(\mu\left(x\right)\left(x_j-a_j\right)\right)|_{x_j=a_j}}$$
$$\times\left[-\sum_{\substack{i=1\\i\neq j}}^{n}I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu\left(x\right)}\,I^{\alpha_j}_{b_j^-}\ {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu\left(x\right)\ {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i}f\left(x\right)\right)\right)\Big|_{x_j=b_j}\right.$$
$$+\left(-\lambda\,I^{1}_{b_j^-}\left(r\left(x\right)\,f\left(x\right)\right)+\sum_{\substack{i=1\\i\neq j}}^{n}I^{1}_{b_j^-}\ {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu\left(x\right)\ {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i}f\left(x\right)\right)\right)\,I^{\alpha_j}_{a_j^+}\left(\frac{\left(b_j-x_j\right)^{\alpha_j-1}}{\mu\left(x\right)\,\Gamma\left(\alpha_j\right)}\right)\Big|_{x_j=b_j}$$
$$\left.+\,\lambda\,I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu\left(x\right)}\,I^{\alpha_j}_{b_j^-}\left(r\left(x\right)\,f\left(x\right)\right)\right)\Big|_{x_j=b_j}\right]. \tag{15}$$

Summing up each member of (12) from $j=1,\ldots,n$ we obtain

$$f\left(x\right)=\sum_{j=1}^{n}\left\{\left(x_j-a_j\right)\,\xi^{[j]}_1\Big|_{x_j=a_j}+\xi^{[j]}_2\Big|_{x_j=b_j}\,I^{\alpha_j}_{a_j^+}\left(\frac{\left(b_j-x_j\right)^{\alpha_j-1}}{\mu\left(x\right)\,\Gamma\left(\alpha_j\right)}\right)\right.$$
$$\left.+\sum_{\substack{i=1\\i\neq j}}^{n}I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu\left(x\right)}\,I^{\alpha_j}_{b_j^-}\ {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i}\left(\mu\left(x\right)\ {}^{C}_{a_i^+}\partial^{\alpha_i}_{x_i}f\left(x\right)\right)\right)-\lambda\,I^{\alpha_j}_{a_j^+}\left(\frac{1}{\mu\left(x\right)}\,I^{\alpha_j}_{b_j^-}\left(r\left(x\right)\,f\left(x\right)\right)\right)\right\}, \tag{16}$$

where $\xi^{[j]}_1|_{x_j=a_j}$ and $\xi^{[j]}_2|_{x_j=b_j}$ are given by (15) and (14), respectively. We can consider (16) as a fixed point condition on the function space $C\left(\Omega\right)$ of the form $f=Tf$, where Tf is the right-hand side of (16). Now we calculate the norm of the difference between Tf and Tg for $f,g\in C\left(\Omega\right)$.

$$\|Tf-Tg\|\leq\frac{1}{n}\sum_{j=1}^{n}(\|T_1f-T_1g\|+\|T_2f-T_2g\|+\|T_3f-T_3g\|+\|T_4f-T_4g\|), \tag{17}$$

where each term inside the sum in (17) is associated with a term in (16). In similar way as it was done in [3], we have the following estimated for the four terms inside the sum in (17)

$$\|T_1 f - T_1 g\| \leq \|\phi_7\| \left[\left(\frac{\lambda \|r\| \, \|\phi_3\|}{n^2} + \|\phi_4\| \, M_\mu \right) \left\|\phi_2|_{x_j=b_j}\right\| + \frac{M_\mu}{m_\mu} \left\| \phi_5 \Big|_{x_j=a_j} \right\| + \frac{\lambda \|r\| \, \|\phi_6\|}{n^2 \, m_\mu} \right] \|f-g\| , \tag{18}$$

$$\begin{aligned} \|T_2 f - T_2 g\| \leq \|\phi_2\| & \left[\frac{\lambda \|r\| \, \|\phi_3\|}{n^2} + \|\phi_4\| \, M_\mu \right. \\ & \left. + \|\phi_1\| \left[\left(\frac{\lambda \|r\| \, \|\phi_3\|}{n^2} + \|\phi_4\| \, M_\mu \right) \left\|\phi_2|_{x_j=b_j}\right\| + \frac{M_\mu}{m_\mu} \left\|\phi_5|_{x_j=a_j}\right\| + \frac{\lambda \|r\| \, \|\phi_6\|}{n^2 \, m_\mu} \right] \right] \|f-g\| \end{aligned} \tag{19}$$

$$\|T_3 f - T_3 g\| \leq \frac{\|\phi_5\| \, M_\mu}{m_\mu} \, \|f-g\| , \qquad \|T_4 f - T_4 g\| \leq \frac{\lambda \|r\| \, \|\phi_6\|}{n^2 \, m_\mu} \, \|f-g\| . \tag{20}$$

where

$$\begin{aligned} & \phi_1(x) = \frac{\Gamma(2-\alpha_j)\,(b_j-a_j)^{\alpha_j-1}}{\Gamma(2-\alpha_j)\,(b_j-a_j)^{\alpha_j-1} - \Gamma(\alpha_j)\, I^{1-\alpha_j}_{b_j^-}\left(\mu(x)\,(x_j-a_j)\right)|_{x_j=a_j}}, && \phi_2(x_j) = I^{\alpha_j}_{a_j^+} \frac{(b_j-x_j)^{\alpha_j-1}}{\mu(x)\,\Gamma(\alpha_j)}, \\ & \phi_3(x) = I^1_{b_j^-} 1, && \phi_4(x) = \sum_{\substack{i=1 \\ i\neq j}}^{n} I^1_{b_j^-} \, {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} 1, \\ & \phi_5(x) = I^{\alpha_j}_{a_j^+} I^{\alpha_j}_{b_j^-} \sum_{\substack{i=1 \\ i\neq j}}^{n} {}^{RL}_{b_i^-}\partial^{\alpha_i}_{x_i} \, {}^{RL}_{a_i^+}\partial^{\alpha_i}_{x_i} 1, && \phi_6(x) = I^{\alpha_i}_{a_i^+} I^{\alpha_i}_{b_i^-} 1, \\ & \phi_7(x_j) = (x_j - a_j)\, \phi_1 && M_\mu = \max_{x\in\Omega} |\mu(x)|, \\ & m_\mu = \min_{x\in\Omega} |\mu(x)| . \end{aligned} \tag{21}$$

From (18), (19), and (20), expression (17) becomes $\|Tf - Tg\| \leq \phi_8 \, \|f-g\|$, where

$$\begin{aligned} \phi_8 = \frac{1}{n} \sum_{j=1}^{n} & \left[\|\phi_7\| \left[\left(\frac{\lambda \|r\| \, \|\phi_3\|}{n^2} + \|\phi_4\| \, M_\mu \right) \left\| \phi_2 \Big|_{x_j=b_j} \right\| + \frac{M_\mu}{m_\mu} \left\| \phi_5 \Big|_{x_j=a_j} \right\| + \frac{\lambda \|r\| \, \|\phi_2\|}{n^2 \, m_\mu} \right] \right. \\ & + \|\phi_2\| \left[\frac{\lambda \|r\| \, \|\phi_3\|}{n^2} + \|\phi_4\| \, M_\mu + \|\phi_1\| \left[\left(\frac{\lambda \|r\| \, \|\phi_3\|}{n^2} + \|\phi_4\| \, M_\mu \right) \left\| \phi_2 \Big|_{x_j=b_j} \right\| + \frac{M_\mu}{m_\mu} \left\| \phi_5 \Big|_{x_j=a_j} \right\| + \frac{\lambda \|r\| \, \|\phi_6\|}{n^2 \, m_\mu} \right] \right] \\ & \left. + \frac{\|\phi_5\| \, M_\mu}{m_\mu} + \frac{\lambda \|r\| \, \|\phi_6\|}{n^2 \, m_\mu} \right], \end{aligned}$$

and ϕ_i, $i = 1, \ldots, 7$, M_μ and m_μ are given in (21). Under the assumption that $\phi_8 < 1$ we have that T is a contraction on the space $C(\Omega)$ for a chosen norm. Therefore, the unique fixed point f exists, up to a constant, and solve (9)-(10).

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