Uniformly distributed-order wave equation in higher dimensions*

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Abstract

In this short paper, we obtain the eigenfunctions of the uniformly distributed-order wave equation in $\mathbb{R}^n \times \mathbb{R}^+$, as Laplace integral of Fox H-functions. For the particular case of the first fundamental solution, the fractional moment of second order of the fundamental solution is studied using the Tauberian Theorem.

Keywords: Time-fractional telegraph equation; Distributed order; Hilfer fractional derivative; Laplace transform; Fourier transform; Mellin transform; Mittag-Leffler function; Fox H-function; Fractional moments; Tauberian Theorem.

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1 Introduction

In the last years, fractional partial differential equations with distributed order received increasing attention from researchers on differential equations. One reason of the interest is the relation of these equations with physical processes involving time-scales (see [1, 2] and references therein indicated). More recently, the analysis of fractional differential equations with distributed order has been extended to the case of Hilfer (or composite) fractional derivatives (see [6] for the case of higher dimensions). The Hilfer fractional derivative allows interpolating smoothly between the Riemann-Liouville and the Caputo fractional derivatives. Using Fourier-Laplace transformation techniques, we obtain an integral representation of the eigenfunctions of the uniformly distributed-order wave equation in $\mathbb{R}^n \times \mathbb{R}^+$ with time-fractional Hilfer fractional deriatives. For the particular case of the first fundamental solution, we make use of the Tauberian Theorem to study the second-order moment.

2 Preliminaries

Let $a, b \in \mathbb{R}$ with $a < b$ and $\alpha > 0$. The left Riemann-Liouville fractional integral $I^{\gamma}_{a^+}$ of order $\gamma > 0$ is given by (see [1])

$$\left(I^{\gamma}_{a^+} f\right)(x) = \frac{1}{\Gamma(\gamma)} \int_a^x \frac{f(w)}{(x-w)^{1-\gamma}}\, dw, \qquad x > a.$$

The Hilfer (or composite) fractional derivative ${}_tD^{\gamma,\nu}_{0^+}$ of order $\gamma > 0$ and type $0 \leq \nu \leq 1$ is given by (see [2])

$$\left({}_tD^{\gamma,\nu}_{0^+} f\right)(t) = \left(I^{\nu(m-\gamma)}_{0^+} \frac{d}{dt}\left(I^{(1-\nu)(m-\gamma)}_{0^+} f\right)\right)(t), \tag{1}$$

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where $m=[\gamma]+1$ and $[\gamma]$ means the integer part of γ. We observe that in the case when $\nu=0$ we recover the left Riemann-Liouville fractional derivative and in the case when $\nu=1$ we have the left Caputo fractional derivative. The Fox H-function $H_{p,q}^{m,n}$ is defined, via a Mellin-Barnes type integral, by (see [3])

$$H_{p,q}^{m,n}\left[\, z \;\middle|\; \begin{array}{c} (a_1,\ \alpha_1),\ldots,(a_p,\ \alpha_p) \\ \\ (b_1,\ \beta_1),\ldots,(b_q,\ \beta_q) \end{array} \right] = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{\prod_{j=1}^{m}\Gamma(b_j+\beta_j s)\ \prod_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\ \prod_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}\ z^{-s}\ ds, \tag{2}$$

where $a_i, b_j\in\mathbb{C}$, and $\alpha_i,\beta_j\in\mathbb{R}^+$, for $i=1,\ldots,p$ and $j=1,\ldots,q$, and $\mathcal{C}$ is a suitable contour in the complex plane separating the poles of the two factors in the numerator (see [3]). In [6] it is proved the following result (see Corollary 3.5):

Theorem 2.1 *The solution of the generalized time-fractional wave equation of distributed order in* $\mathbb{R}^n\times\mathbb{R}^+$

$$\int_0^1\int_1^2 b_2(\beta,\nu)\ {}_t\partial_{0^+}^{\beta,\nu}u(x,t)\ d\beta\, d\nu - c^2\,\Delta_x u(x,t) + d^2\,u(x,t)\ = q(x,t) \tag{3}$$

for a given integrable order-density function $b_2(\beta,\nu)$, *subject to the following initial and boundary conditions*

$$\left({}_tI_{0^+}^{(1-\nu)(2-\beta)}u\right)(x,0^+) = g_1(x)\,, \qquad \left[\frac{\partial}{\partial t}\left({}_tI_{0^+}^{(1-\nu)(2-\beta)}u\right)\right](x,0^+) = g_2(x)\,,$$

$$\lim_{|x|\to+\infty} u(x,t)=0, \qquad \int_0^1\int_1^2 b_2(\beta,\nu)\ d\beta\, d\nu = C_1\in\mathbb{R}^+,$$

where $(x,\,t)\in\mathbb{R}^n\times\mathbb{R}^+$, $c,d\in\mathbb{R}^+$, Δ_x *is the classical Laplace operator in* $\mathbb{R}^n$, *and the partial time-fractional derivative of order* $\beta\in]1,2]$ *and type* $\nu\in]0,1[$ *is in the Hilfer sense given by* (1), *is given, in terms of convolution integrals, by*

$$u(x,t) = \int_{\mathbb{R}^n} g_1(z)\ G_2(x-z,\,t)\ dz + \int_{\mathbb{R}^n} g_2(z)\ G_3(x-z,\,t)\ dz + \int_{\mathbb{R}^n}\int_0^t q(z,\,w)\ G_4(x-z,\,t-w)\ dw\, dz,$$

where G_2, G_3, *and* G_4 *are given by*

$$G_2(x,t) = \frac{-1}{\pi^{\frac{n-1}{2}}\,(2|x|)^n}\int_0^{+\infty}\frac{re^{-rt}}{\rho\sin(\gamma\pi)}\left[\rho^*\sin(\gamma^*\pi)\,\mathcal{H}\left(\frac{1}{|x|\,\sqrt{\rho}}\right) + \frac{d^2}{c^2}\,\sin(\gamma\pi)\,\mathcal{H}^*\left(\frac{1}{|x|\,\sqrt{\rho}}\right)\right] dr,$$

$$G_3(x,t) = \frac{1}{\pi^{\frac{n-1}{2}}\,(2|x|)^n}\int_0^{+\infty}\frac{e^{-rt}}{\rho\sin(\gamma\pi)}\left[\rho^*\sin(\gamma^*\pi)\,\mathcal{H}\left(\frac{1}{|x|\,\sqrt{\rho}}\right) + \frac{d^2}{c^2}\,\sin(\gamma\pi)\,\mathcal{H}^*\left(\frac{1}{|x|\,\sqrt{\rho}}\right)\right] dr,$$

$$G_4(x,t) = \frac{-1}{c^2\,\pi^{\frac{n-1}{2}}\,(2|x|)^n}\int_0^{+\infty}\frac{e^{-rt}}{\rho}\,\mathcal{H}^*\left(\frac{1}{|x|\,\sqrt{\rho}}\right) dr$$

with

$$\begin{cases}\rho\ = \left|B_2\left(re^{i\pi}\right)\right| \\ \\ \gamma\ = \dfrac{1}{\pi}\,\arg\left(B_2\left(re^{i\pi}\right)\right)\end{cases}, \qquad \begin{cases}\rho^*\ = \left|B_2^*\left(re^{i\pi}\right)\right| \\ \\ \gamma^*\ = \dfrac{1}{\pi}\,\arg\left(B_2\left(re^{i\pi}\right)\right)\end{cases}, \tag{4}$$

and

$$\begin{aligned} B_2(\mathbf{s})\ &= \frac{1}{c^2}\left(\int_0^1\int_1^2 b_2(\beta,\nu)\ \mathbf{s}^{\beta}\ d\beta\, d\nu + d^2\right) \\ B_2^*(\mathbf{s})\ &= \frac{1}{c^2}\left(\int_0^1\int_1^2 b_2(\beta,\nu)\ \mathbf{s}^{-\nu(2-\beta)}\ d\beta\, d\nu + d^2\right)\end{aligned} \tag{5}$$

and the functions $\mathcal{H}$ *and* $\mathcal{H}^*$ *are expressed in terms of the following Fox H-functions*

$$\mathcal{H}\left(\frac{1}{|x|\,\sqrt{\rho}}\right) = H_{3,2}^{0,2}\left[\ \frac{1}{|x|\,\sqrt{\rho}}\ \middle|\ \begin{array}{c}(1-n,\,1)\,,\,\left(1,\,\dfrac{1}{2}\right),\,\left(0,\,\dfrac{\gamma}{2}\right) \\ \\ \left(\dfrac{1-n}{2},\,\dfrac{1}{2}\right),\,\left(0,\,\dfrac{\gamma}{2}\right)\end{array}\right],$$

$$\mathcal{H}^*\left(\frac{1}{|x|\,\sqrt{\rho}}\right) = H_{3,2}^{0,2}\left[\ \frac{1}{|x|\,\sqrt{\rho}}\ \middle|\ \begin{array}{c}(1-n,\,1)\,,\,\left(0,\,\dfrac{1}{2}\right),\,\left(-\gamma,\,\dfrac{\gamma}{2}\right) \\ \\ \left(\dfrac{1-n}{2},\,\dfrac{1}{2}\right),\,\left(-\gamma,\,\dfrac{\gamma}{2}\right)\end{array}\right]. \tag{6}$$

Moreover, from the expression (86) in [6], we have that the second-order moment in the Laplace domain for the first fundamental solution of (3) ($g_1(x) = \delta(x)$, $g_2(x) = 0 = q(x,t)$, and $d = 0$) is given by

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}}\,\mathbf{s}\,B_2^*(\mathbf{s})\,\left(B_2(\mathbf{s})\right)^{\frac{n-5}{2}}, \qquad n \neq 5+2k,\ k \in \mathbb{N}_0. \tag{7}$$

In order to perform the Tauberian analysis in the last section, let us recall some necessary Laplace inversion formulas that can be found in [4]:

- Formula (2.1.1.1):

$$\mathcal{L}^{-1}\left\{\frac{1}{\mathbf{s}^\nu}\right\}(t) = \frac{t^{\nu-1}}{\Gamma(\nu)}, \qquad \nu > 0, \tag{8}$$

- Formula (2.5.1.12):

$$\mathcal{L}^{-1}\left\{\frac{1}{\mathbf{s}^\nu}\,\ln^n(\mathbf{s})\right\}(t) = \left(-\frac{d}{d\mu}\right)^n\left[\frac{t^{\mu-1}}{\Gamma(\mu)}\right]\Bigg|_{\mu=\nu}, \qquad n \in \mathbb{N}. \tag{9}$$

- Formula (2.5.6.5):

$$\mathcal{L}^{-1}\left\{\frac{1}{\mathbf{s}^\nu}\,\ln^\mu(a\mathbf{s})\right\}(t) = \frac{a^{\nu-1}}{\Gamma(-\mu)}\int_0^{+\infty}\frac{w^{-\mu-1}}{\Gamma(\nu+w)}\left(\frac{t}{a}\right)^{w+\nu-1}dw, \qquad \mathrm{Re}(\mu) < 0,\ a > 0,\ \mathrm{Re}(\mathbf{s}) > 0. \tag{10}$$

3 Uniformly distributed-order operator

The aim of this section is to obtain a representation in terms of Laplace integrals of Fox H-functions of the eigenfunctions of the uniformly distributed-order operator. In this sense let us consider the following eigenfunction equation

$$\int_1^2 {}_t\partial_{0^+}^{\beta,\eta}u(x,t)\,d\beta - c^2\,\Delta_x u(x,t) = \lambda\,u(x,t), \tag{11}$$

subject to the following initial and boundary conditions

$$\left({}_tI_{0^+}^{(1-\eta)(2-\beta)}u\right)(x,0^+) = \delta(x) = \prod_{i=1}^n \delta(x_i), \quad \left[\frac{\partial}{\partial t}\left({}_tI_{0^+}^{(1-\eta)(2-\beta)}u\right)\right](x,0^+) = 0, \quad \lim_{|x|\to+\infty} u(x,t) = 0. \tag{12}$$

The boundary value problem (11)-(12) is a particular case of Theorem 2.1, where b_2 is given by

$$b_2(\beta,\nu) = \delta(\nu-\eta)\,p(\beta), \quad \text{with } p(\beta) = 1 \quad \text{and } 0 < \eta < 1, \tag{13}$$

$d = i\sqrt{\lambda}$, $g_1(x) = \delta(x)$, and $g_2(x) = 0 = q(x,t)$. In these conditions, the integral representation of the solution of (11)-(12) is given by

$$u(x,t) = \frac{-1}{\pi^{\frac{n-1}{2}}(2|x|)^n}\int_0^{+\infty}\frac{re^{-rt}}{\rho\sin(\gamma\pi)}\left[\rho^*\sin(\gamma^*\pi)\,\mathcal{H}\left(\frac{1}{|x|\,\sqrt{\rho}}\right) - \frac{\lambda}{c^2}\,\sin(\gamma\pi)\,\mathcal{H}^*\left(\frac{1}{|x|\,\sqrt{\rho}}\right)\right]dr, \tag{14}$$

where ρ, ρ^*, $\mathcal{H}$, and $\mathcal{H}^*$, are given, respectively by (4), and (6). Moreover, from (13), we have from (5)

$$B_2(\mathbf{s}) = \frac{1}{c^2}\,\frac{\mathbf{s}(\mathbf{s}-1)}{\ln(\mathbf{s})}, \qquad \text{and} \qquad B_2(\mathbf{s}) = \frac{1}{c^2\,\eta}\,\frac{1-\mathbf{s}^{-\eta}}{\ln(\mathbf{s})}. \tag{15}$$

Remark 3.1 *If we consider $\lambda = 0$ in (11) the solution $u(x,t)$ corresponds to the first fundamental solution.*

4 Second order moment of the fundamental solution

In this section, we obtain the expression for second-order moment of the first fundamental solution of (11) in the Laplace domain, and we apply the Tauberian Theorem to study the asymptotic behaviour in the time domain for $t \to 0^+$ and $t \to +\infty$. Considering (15) in (7) and making straightforward calculations, we arrive to

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\,c^{n-3}\eta}\,\frac{\mathbf{s}^{\frac{n-3}{2}}\,(1-\mathbf{s}^{-\eta})\,(\mathbf{s}-1)^{\frac{n-5}{2}}}{(\ln(\mathbf{s}))^{\frac{n-3}{2}}}, \qquad n \neq 5+2k,\ k \in \mathbb{N}_0. \tag{16}$$

When $\mathbf{s} \to 0^+$ we have the following asymptotic behaviour

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) = \frac{(-1)^{\frac{n-3}{2}}\, 2^{1-n}\, \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}\eta} \frac{\mathbf{s}^{\frac{n-3}{2}-\eta}\left(1-\mathbf{s}^{\eta}\right)\left(1-\mathbf{s}\right)^{\frac{n-5}{2}}}{\left(\ln(\mathbf{s})\right)^{\frac{n-3}{2}}} \sim \frac{(-1)^{\frac{n-3}{2}}\, 2^{1-n}\, \Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}\eta} \frac{\left(\ln(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{\eta+\frac{3-n}{2}}}.$$

Making use of (9) (for $n=1$), (8) (for $n=3$), and (10) (for $n=4+2k,\, k\in\mathbb{N}_0$) to invert the Laplace transform, and applying the Tauberian Theorem, we obtain for $t\to+\infty$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{2c^2}{\eta}\, \dfrac{t^{\eta}\ln(t)}{\Gamma(1+\eta)}, & n=1 \\[2ex] \dfrac{c}{4\eta}\,\mathcal{L}^{-1}\left\{\dfrac{\left(-\ln(\mathbf{s})\right)^{\frac{1}{2}}}{\mathbf{s}^{\eta+\frac{1}{2}}}\right\}(t), & n=2 \\[2ex] \dfrac{1}{4\pi\eta}\, \dfrac{t^{\eta-1}}{\Gamma(\eta)}, & n=3 \\[2ex] \dfrac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}\,\eta}\, \dfrac{(-1)^{\frac{n-3}{2}}}{\Gamma\left(\frac{n-3}{2}\right)} \displaystyle\int_0^{+\infty} \dfrac{w^{\frac{n-5}{2}}\, t^{w+\eta+\frac{1-n}{2}}}{\Gamma\left(w+\eta+\frac{3-n}{2}\right)}\, dw, & n=4+2k,\, k\in\mathbb{N}_0 \end{cases}.$$

When $\mathbf{s}\to+\infty$ we have the following asymptotic behaviour

$$\widetilde{\mathbf{M}}^2(\mathbf{s}) \sim \frac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}\eta} \frac{\left(\ln(\mathbf{s})\right)^{\frac{3-n}{2}}}{\mathbf{s}^{4-n}}.$$

Making use of (8) (for $n=1$), (9) (for $n=3$), and (10) (for $n=4+2k,\, k\in\mathbb{N}_0$) to invert the Laplace transform, and applying the Tauberian Theorem, we obtain for $t\to0^+$

$$\mathbf{M}^2(t) \sim \begin{cases} \dfrac{c^2}{\eta}\, t^2\, \ln\left(\dfrac{1}{t}\right), & n=1 \\[2ex] \dfrac{c}{4\eta}\,\mathcal{L}^{-1}\left\{\dfrac{\left(\ln(\mathbf{s})\right)^{\frac{1}{2}}}{\mathbf{s}^{2}}\right\}(t), & n=2 \\[2ex] \dfrac{1}{4\pi\eta}, & n=3 \\[2ex] \dfrac{2^{1-n}\,\Gamma\left(\frac{5-n}{2}\right)}{\pi^{\frac{n-1}{2}}\, c^{n-3}\,\eta}\, \dfrac{1}{\Gamma\left(\frac{n-3}{2}\right)} \displaystyle\int_0^{+\infty} \dfrac{w^{\frac{n-5}{2}}\, t^{w+3-n}}{\Gamma\left(w+4-n\right)}\, dw, & n=4+2k,\, k\in\mathbb{N}_0 \end{cases}.$$

Remark 4.1 *When $n=1$, all the results presented in this section correspond to the correspondent ones obtained in [7] (Section 3). Moreover, when $n=3$ if we consider the limit case of $\eta=1$ (i.e., the case where the time-fractional partial derivatives are in the Caputo sense), we obtain the results presented in Section 5.1 of [5].*

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