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IMAGE

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About *IMAGE*

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Acknowledgment: Sebastian Cioabă has completed his term as the *IMAGE* editor in charge of feature articles, having served ILAS in this capacity since *IMAGE* issue 59, way back in Fall 2017. We thank Sebi for his enthusiastic service and many great contributions to these pages!

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LINEAR ALGEBRA EDUCATION

A Library of Lesson Plans for Teaching Linear Algebra

Rachel Quinlan, University of Galway, Galway, Ireland, rachel.quinlan@universityofgalway.ie
Frank Uhlig, Auburn University, Auburn, AL, USA, uhligfd@auburn.edu

1. Introduction. The recent recommendations [1] of the second Linear Algebra Curriculum Study Group (LACSG 2.0) highlight the increasing importance of the teaching of linear algebra amid the current global surge in demand for graduates with skills in data analytics and related fields. They also point out that the opportunities available to us in linear algebra education are evolving rapidly due to research developments in the field, the increased availability and accessibility to students of enabling technologies, and the rapid emergence of new application domains of economic, industrial and scientific significance. In the linear algebra research community, we are fortunate that our subject occupies a privileged and even increasingly prominent position in curricula, and we are challenged to ensure that our practices keep abreast of developments. These include modern matrix tools as well as learning technologies, research advances, and the new combinations of opportunities and demands encountered by graduates in scientific and engineering disciplines as well as in mathematics. The LASCG report also calls for collaborative and creative action from the mathematics community, to avail ourselves of the opportunities afforded by the teaching tools that are now available for linear algebra, with the purpose of equipping students as effectively as we can for modern workplaces and for academic engagement. In this article, we propose a specific community-based collaborative action, with the goal of creating a dynamic forum for sharing practical and applicable teaching ideas.

2. Preserving linear algebra teaching knowledge. In the academic mathematics community, we have many opportunities to share our research insights, interests and advances, through structured events and channels such as conferences, seminars and journals, as well as through informal interactions within our own networks. However, the wealth of practical knowledge and experience that we accumulate in our teaching is not extensively shared in a structured or organised way, and communication of our learning there tends to be more restricted to informal networks and local interactions. Our community of practitioners does not have a written account of what we do in classrooms, how we decide what to do and when to do it, and what considerations inform and constrain those decisions. Some relevant information is available to us via the tools of the trade. Textbooks can organise content and activities for us and for our students, and if we are lucky we are free to choose one that suits our preferences and context. Research advances in mathematics education are documented in the literature of that field, and contributions there may well inform our thinking and enhance our practice. Software to support the learning and teaching of linear algebra is becoming ever more versatile and widely available, and more easily incorporated by students and teachers into educational environments. But none of these resources can contain or convey the knowledge and insights that come from working with our students, observing the consequences of the choices that we make about what they do, adapting our approaches and assessing their effectiveness. To address this situation, we propose a new resource for sharing this kind of knowledge, to be built, curated and shared among the global community of practitioners of linear algebra education, with the purpose of supporting each other, widening discourse, and learning from each other's experiences as broadly as possible.

> "professional knowledge is public, shareable, storable and subject to adaptation"

In [2], Hiebert et al. propose the concept of a *knowledge base* for the teaching profession, and ask what it could look like and how it could be created. They observe that, for example, the medical and legal professions have case literature and records of case law, that can be consulted by practitioners navigating particular scenarios. They comment that the teaching profession has "yet to develop a professional knowledge system."

The challenge of transforming *practitioner knowledge* into professional knowledge is explored at length in [2]. Practitioner knowledge accumulates with experience and resides privately, with individuals or within local contexts. Professional knowledge is public, storable, shareable and subject to scrutiny and to adaptation and improvement. The focus of these authors is not on higher education, but the same questions can be considered there, even specifically in the case of the teaching and learning of linear algebra, a subject on which we all have much to say.

3. Lesson plans. We suggest that a public, shared and stored repository of ideas and experiences that is managed and built by our international community would be a powerful resource both for teaching and for discussion. As a method of organising contributions, we propose the format of *lesson plans*. By a lesson plan, we mean a concise, practically-oriented account of a teaching activity, that addresses either a specific syllabus topic or a general theme in linear algebra. From an author's point of view, a lesson plan would give a practical description, with detailed examples, of something like "How I teach engineering students about linear independence" or "How I teach students about proof in linear algebra courses" or "How I embed the use of software in the learning of linear algebra". A lesson plan contains the necessary information to equip a reader to enter the classroom and deliver a lesson according to the author's script, but that is not its only purpose. It could include some or all of the following elements:

- A description of the envisaged context;
- Key decision points for the teacher;
- A rationale for the approach that is taken;
- Motivation for teaching the topic in the first place;
- Connection to past or future lessons or topics;
- Comments on the experiences of teacher and students with the lesson.

"lesson plans are about writing down the ideas that are not in the textbooks"

A lesson plan is *not* a discussion of different approaches to teaching a topic, and it is not a piece of pedagogical research. It is more like a pitch for a particular approach that is favoured by the author. It might convince some readers, and raise questions for others. It might stimulate reflection, reconsideration, and further contributions to the lesson plan library. Particularly for those topics that are standard items on the curriculum, it is reasonable and desirable to have multiple lesson plans at our disposal, to suit different contexts, curricula and individual student or teacher preferences, and to support critical assessment and development. While an individual lesson plan is not intended as a comparative study of different teaching approaches, a community-generated library of lesson plans could include that, not as an explicit purpose but as a by-product of the independent contributions of multiple authors. While it might also serve some of the purposes of a textbook, it is most interesting for its focus on what is *not* to be found in the textbooks: a practitioner's knowledge that is acquired through interactions with students, and through the process of revisiting ideas and approaches after seeing them in action.

"a record of ideas and experiences"

A single lesson plan is appropriate for presenting a novel teaching or learning activity. It is also appropriate for describing a conventional approach, with a focus for example on the main conceptual ingredients and how they are organised, sequenced and emphasised to prompt an authentic mathematical experience for students. Both novel and traditional methods can be a basis for interesting and effective practice. Every topic in the curriculum, no matter how elementary or advanced, can be the subject of innovative practice in teaching. Creating and preserving a record of such ideas and experiences will benefit the entire international community of linear algebra educators and students. It is envisaged that, over time, the practice of documenting and communicating our deliberate teaching actions and their purposes and effects will lead to a resource that is informative, academically robust, and empowering for authors, readers and all who teach or study linear algebra.

4. How to contribute a lesson plan. The ILAS Education Committee proposes to host and curate a lesson plan library, searchable by topic, author, theme and level (from introductory to advanced courses and topics). We invite contributions on both standard and special interest topics, from all colleagues who teach linear algebra. Authors are welcome to propose single, stand-alone lesson plans, or to assemble collections of plans intended to be used together. Plans that address either specific syllabus topics or themes that arise throughout the curriculum are both welcome. It is expected that works in the library will be shared by default under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License. Individual authors have the option to specify different copyright terms within their documents. Plans will be peer-reviewed before publication, and authors will have an opportunity to make revisions in response to comments from reviewers and editors.

To accompany the present article in *IMAGE*, Frank Uhlig presents an exemplar lesson plan, suitable for the first classes of an introductory linear algebra course. (To view this lesson plan, click **here**.) Plans for subsequent lessons in this series will be among the first contributions to the lesson plan library. Details of the submission process will be publicised in due course. In the meantime, please send your questions and comments regarding lesson plans to Rachel Quinlan (`rachel.quinlan@universityofgalway.ie`).

References.

[1] S. Stewart, S. Axler, R. Beezer, E. Boman, M. Catral, G. Harel, J. McDonald, D. Strong, and M. Wawro. The Linear Algebra Curriculum Study Group (LACSG 2.0) recommendations. *Notices of the American Mathematical Society*, 69(5):813–819, 2022.

[2] J. Hiebert, R. Gallimore, and J. W. Stigler. A Knowledge Base for the Teaching Profession: What Would It Look Like and How Can We Get One? *Educational Researcher*, 31(5):3–15, 2002.

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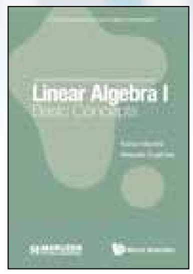

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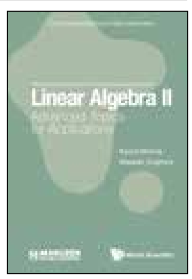

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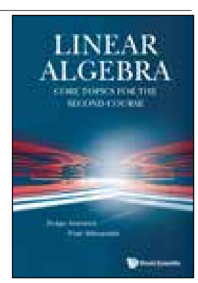

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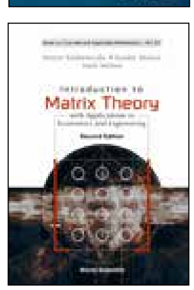

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ILAS NEWS

ILAS Bylaws Amended

Contributed announcement from Daniel B. Szyld, ILAS President

During the October 20th, 2022 meeting of the ILAS Board of Directors, changes to the ILAS bylaws were approved.

The changes refer to the way the Nominating Committee is appointed during the years that the President may run for re-election, and it was prompted by the desire to avoid any appearance of a conflict of interest.

The updated bylaws have been posted on the ILAS website.

We thank Minnie Catral, Froilán Dopico, and Hugo Woerdeman, who brought up the issue and drafted these revisions.

First ILAS Richard A. Brualdi Early Career Prize Awarded

Contributed announcement from Daniel B. Szyld, ILAS President

The inaugural ILAS Richard A. Brualdi Early Career Prize was awarded to Michael Tait of Villanova University.

The citation reads: "For outstanding contributions to spectral theory of graphs, developing new techniques, and settling several long-standing open problems in that area."

Michael will present the prize lecture at the ILAS Conference in June 2023 in Madrid.

The committee consisted of: Marina Arav, Volker Mehrmann, Shmuel Friedland (chair), and Daniel B. Szyld.

2022 Householder Prize Winner

Contributed announcement from Françoise Tisseur

The 2022 Alston S. Householder Prize for the best Ph.D. thesis in numerical linear algebra earned between 1 January 2020 and 31 December 2021 was awarded to Heather Wilber (Cornell University) for the thesis "Computing Numerically with Rational Functions".

Honorable mentions were awarded to:

- Sungwoo Jeong (MIT) for the thesis "Linear Algebra, Random Matrices and Lie Theory";
- Maria del Carmen Quintana Ponce (Universidad Carlos III de Madrid) for the thesis "Linearizations of Rational Matrices".

JOURNAL ANNOUNCEMENTS

LAA Special Issue in honor of Albrecht Böttcher

Contributed announcement from Richard A. Brualdi

Linear Algebra and its Applications (LAA) is pleased to announce a special issue in honor of Albrecht Böttcher on the occasion of his 70th birthday in 2024. Guest editors for this special issue are: Harm Bart (`bart@ese.eur.nl`); Torsten Ehrhardt (`tehrhard@ucsc.edu`); Andre Ran (`a.c.m.ran@vu.nl`); and Ilya Spitkovsky (`spitkovsky@gmail.com`).

Papers intended for this special issue should be submitted by June 1, 2023 via the Editorial Manager (`https://www.editorialmanager.com/laa`) by selecting "SI: Albrecht Böttcher" as the Article Type and Richard A. Brualdi as the handling editor-in-chief.

Authors will have the opportunity to suggest one of the guest editors to handle their submission. Papers will be refereed according to the usual standards of *LAA*.

Call for papers for *PRIMUS* special issue

Contributed announcement from Sepideh Stewart, ILAS Education Committee Chair

The ILAS education committee is pleased to announce a new special issue of *PRIMUS* titled *Teaching Linear Algebra: an International Perspective*.

The call-for-papers can be found here:

`https://primusmath.com/2022/09/12/call-for-papers-teaching-linear-algebra-an-international-perspective`

The issue will be guest-edited by the ILAS Education Committee Members: Anthony Cronin (University College Dublin), Judith McDonald (Washington State University), Rachel Quinlan (National University of Ireland, Galway), Sepideh Stewart (University of Oklahoma), and David Strong (Pepperdine University).

Papers for this special issue should be approximately 10–15 pages long. Submissions will be accepted until February 28th, 2023.

LAMA special issue for ALAMA2022-ALN2gg

Contributed announcement from Fernando De Terán

A special issue of *Linear and Multilinear Algebra* is in progress, devoted to the ALAMA2022-ALN2gg meeting, held at Universidad de Alcalá (in the town of Alcalá de Henares, Madrid region) from June 1 to June 3, 2022:

`https://congresosalcala.fgua.es/alama2022?idioma=en`

The special editors of this issue are:

Steve Kirkland (University of Manitoba, editor-in-chief); Fernando De Terán (Universidad Carlos III de Madrid); Beatrice Meini (Università di Pisa); and Federico Poloni (Università di Pisa).

Papers for this special issue, corresponding to talks given at the meeting, should be submitted using the submission system (`https://www.tandfonline.com/journals/glma20`) and will follow the standard review process of the journal. The corresponding author should explicitly indicate that the submission is intended to be published in a special issue, and select the special issue ALAMA2022-ALN2gg from the drop-down menu. Steve Kirkland should be chosen as editor-in-chief.

The deadline for submissions to this special issue is December 30th, 2022.

OBITUARY NOTICES

Chandler Davis, 1926–2022

Submitted by Daniel B. Szyld

With great sadness I share the news of the passing of Chandler Davis, an ILAS member for many years and an influential figure in Linear Algebra and Operator Theory. He died on Saturday, September 24th, in Toronto.

For more details about his mathematical career, see the review of some of his work by Man-Duen Choi and Peter Rosenthal in *LAA* vol 208/209 (1994), pp. 3–18: `https://www.sciencedirect.com/science/article/pii/002437959490426X` (Special issue dedicated to Chandler Davis edited by Rajendra Bhatia, Shmuel Friedland, and Peter Rosenthal).

Other details about his life can be found on his Wikipedia page: `https://en.wikipedia.org/wiki/Chandler_Davis`

A nice memorial tribute including a statement from Chandler from last July can be found at:

`https://caseazatmiftakhov.org/2022/09/28/chandler-davis-in-memoriam`

We mourn his death, and we honor his rich life and his many contributions.

CONFERENCE REPORTS

Conference Honoring the 65^{th} Birthday of Daniel B. Szyld
Philadelphia, USA, March 24–26, 2022

Report by Kirk M. Soodhalter

On March 24–26, 2022, a conference was held to honor the 65^{th} birthday of our esteemed colleague/friend/mentor, Daniel B. Szyld, on the topic of recent results and new applications of numerical linear algebra and matrix theory. The conference was held at Temple University in Philadelphia, Pennsylvania. The goal was to bring together both researchers working on more classical and those working on newer challenges in numerical linear algebra on the occasion of Daniel B. Szyld's birthday, in order to facilitate discussion among them as well as with computational scientists from cognate disciplines. The scientific committee comprised Chris Beattie, Howard Elman and Kirk M. Soodhalter, and the local organizing committee comprised Isaac Klapper and Benjamin Seibold. The conference had generous financial support from Temple University and from ILAS.

That this conference should bring together researchers held a special meaning beyond intellectual exchange and the celebration of our friend, as the timing of this conference had been delayed due to the scourge of the 2019 novel coronavirus. For nigh two years, many attendees engaged in minimal or no travel at all due to lockdowns and other measures taken to stem the spread of the virus. For most attendees, this was the first conference travel they had engaged in since before the start of the pandemic. The atmosphere was energetic and festive, as the attendees were excited to discuss interesting mathematics and see old friends for the first time in so long.

The attendees were a mixture of colleagues from the numerical linear algebra community, including a few early-career members from nearby East Coast universities. Additionally, close friends of Daniel Szyld from other areas of mathematics and related computational sciences were also in attendance, and his former Ph.D. supervisor, Olof Widlund, was there as well. A number of close friends from the community would have liked to attend but had to send their apologies due to the timing.

Participants of the conference in honor of Daniel Szyld's 65^{th} birthday

The conference took place over a two-and-a-half day period, from Thursday to before lunch on Saturday. Talks were a mixture of invited plenaries, talks from senior colleagues, and those from early-career participants. Topics ranged from classic numerical linear algebra topics such as eigenvalue computations, iterative solvers, and preconditioning, together with topics of more recent interest such as tensor computations, parallel methods in high-performance computing, stochastic methods, and bridging the gap between numerical linear algebra and theoretical computer science. In addition, friends from other fields gave talks on topics such as domain meshing schemes, modeling of swimming microorganisms, animal locomotion on unsteady media, ill-posed problems, and models of brain metabolism. On Thursday at the end of the day, there was a one-hour public lecture given by Volker Mehrmann from TU Berlin on the topic of digital twins: how some recent mathematical and computational advances will have an impact on how we manage large, interconnected infrastructure systems, such as in the energy sector. This talk was open to the wider academic community and public.

On Thursday evening, a banquet was held to celebrate the birthday and career of Daniel Szyld, with family (his wife, children, and grandchildren), friends, and colleagues. The banquet was held at Ocean City Restaurant and consisted of a ten-course meal. After the fifth course, there was a pause for speeches. Kirk M. Soodhalter was honored to have emceed this portion of the evening. This began with a reading of apology messages from those who could not attend. Kirk also had the privilege of speaking in honor of Daniel in his capacity as a Ph.D. supervisor, recounting the central role Daniel's mentoring and support had played in his formative academic years. Benjamin Seibold spoke of having been an early-career colleague whom Daniel supported and gave advice to in his first years as a professor at Temple. Andreas Frommer told of having gotten to know Daniel as they formed a close friendship and multi-year collaboration. Howard Elman spoke about meeting and befriending Daniel when they were early-career researchers. Michele Benzi spoke similarly about becoming friends with Daniel and told of some of their conference travel adventures. Volker Mehrmann gave the final speech, wherein he told of getting to know Daniel, reflecting on key points in Daniel's career that Volker bore witness to, as well as on their differing tastes in music. Daniel then gave a beautiful speech, tying these aspects from different parts of his career together and talking about the people who supported him as well as those who could not be there or are no longer with us. After the speeches, attendees enjoyed the final five courses of the meal. Warm food and warm feelings were not in short supply.

Celebrating Daniel Szyld's birthday and career was a pleasure, and it was a delight to see so many colleagues and friends after being separated for so long.

The full list of attendees was:

- Tamer Aldwairi
- Christopher Beattie
- Michele Benzi
- Marsha Berger
- Matthias Bolten
- Susanne Brenner
- Mingchao Cai
- Daniela Calvetti
- Edmond Chow
- Siobhan Correnty
- Froilán Dopico
- Andrei Draganescu
- Howard Elman
- Mark Embree
- Lisa Fauci
- Andreas Frommer
- Efstratios Gallopoulos
- Jose Garay
- Jonathan Goodman
- Anne Greenbaum
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- Diana Halikias
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- Tonia Hsieh
- Sara Idressi
- Nour Khoudari
- Isaac Klapper
- Pilhwa Lee
- Edward Letzter
- Stephanie Lewkiewicz
- Frederic Magoules
- Vahid Mahzoon
- Volker Mehrmann
- Victor Moll
- Ron Morgan
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- Brian Rider
- Hassane Sadok
- Marcus Sarkis
- Benjamin Seibold
- Tianyi Shi
- Barry Smith
- Erkki Somersalo
- Kirk M. Soodhalter
- Eric de Sturler
- Li-yeng Sung
- Daniel Szyld
- Alex Townsend
- Liz Tucker
- Judith Vogel
- Olof Widlund
- Hugo Woerdeman
- Jacob Woods

Linear Algebra, Matrix Analysis and Applications Meeting and Due giorni di Algebra Lineare Numerica (ALAMA2022-ALN2gg) Alcalá de Henares, Madrid, Spain, June 1–3, 2022

Report by Ana Marco García

The seventh ALAMA (Linear Algebra, Matrix Analysis and Applications) meeting, joint with the eighteenth edition of ALN2gg (Due giorni di Algebra Lineare Numerica), took place in the historical buildings of University of Alcalá (Alcalá de Henares, Madrid, Spain) from June 1 to 3, 2022.

The meeting was celebrated in honour of Ion Zaballa, professor at Universidad del País Vasco (Spain), and of Dario A. Bini, professor at Università di Pisa (Italy).

The main objective of the meeting was to gather researchers whose work is related to linear algebra, matrix analysis, and their applications, so that they could exchange results, experiences and ideas.

The members of the Scientific Committee were Ana Marco, Raymond Honfu Chan, Froilán M. Dopico, Christian Mehl, Juan Manuel Peña, Lothar Reichel, Stefano Serra-Capizzano, Ana M. Urbano and Marc Van Barel. The Organizing Committee consisted of Ana Marco, Fernando De Terán, Fabio Di Benedetto, Bruno Iannazzo, José Javier Martínez, Beatrice Meini, Federico Poloni and Raquel Viaña.

The meeting consisted of four 45-minute invited talks and ninety-two 20-minute contributed talks, delivered in three parallel sessions. There were 105 participants, who, in addition to Spain and Italy, came from Belgium, Croatia, Finland, Germany, Israel, Portugal, Russia, the United Kingdom and the United States.

The plenary speakers were:

- Dario A. Bini (Universitá di Pisa) – Structured matrix analysis: the class of Quasi-Toeplitz matrices
- Françoise Tisseur (ILAS Speaker, University of Manchester) – Structure-preserving transformations for matrix polynomials
- Evgeny Tyrtyshnikov (Moscow State University) – From ill-posedness to correct settings
- Ion Zaballa (Universidad del País Vasco) – Matrix polynomials and friends

The journal *Linear and Multilinear Algebra* will publish a special number of original articles presented in the meeting.

We would like to thank all the sponsors of the event: University of Alcalá (UAH), the Physics and Mathematics Department of UAH, Universidad Carlos III de Madrid (UC3M), the International Linear Algebra Society (ILAS), Foundation Compositio Mathematica, the Spanish Society of Applied Mathematics (SEMA), Comunidad de Madrid, and Ayuntamiento de Alcalá de Henares.

For more information, the conference web site is available at `https://www.alama2022.com`.

XXI Householder Symposium on Numerical Linear Algebra
Selva di Fasano, Italy, June 12–17, 2022

Report by Leonardo Robol and Francesco Tudisco

The 21st Householder Symposium on Numerical Linear Algebra took place in Selva di Fasano, Italy on June 12–17, 2022. The symposium featured an exciting blend of modern numerical linear algebra topics, such as the spectra of infinite-dimensional operators, nonlinear eigenvalue problems, functions of matrices and preconditioning, uncertainty quantification and data assimilation, and low-rank approximation methods. There were 115 lectures, including 23 plenary talks and three addresses by recipients of Householder Prizes.

For more information, please refer to the report published in *SIAM News*:

`https://sinews.siam.org/Details-Page/report-of-the-xxi-householder-symposium-on-numerical-linear-algebra`

The 9th International Conference on Matrix Analysis and Applications (ICMAA)
Aveiro, Portugal, June 15–17, 2022

Report by Enide Andrade and Rute Lemos

The 9th International Conference on Matrix Analysis and Applications (ICMAA 2022) was held at the Department of Mathematics of the University of Aveiro in Aveiro, Portugal, June 15–17, 2022. The purpose of this meeting was to stimulate research and interaction among mathematicians working in all aspects of linear and multilinear algebra, matrix analysis, graph theory, and related applications, providing an opportunity for researchers to join together and exchange ideas and developments on these subjects. Forty abstracts, with contributions from both theoretical and applied perspectives, were submitted for presentation, and about fifty-two participants registered for ICMAA 2022, including six Ph.D., master's or bachelor's students.

The keynote speaker was Peter Šemrl from the University of Ljubljana, Ljubljana, Slovenia, who presented a plenary talk entitled *On Wigner's theorem*. The invited speakers were Natália Bebiano from Universidade de Coimbra, Portugal and Chi-Kwong Li from the College of William and Mary, Williamsburg, Virginia, USA. The titles of their plenary talks were *An inverse eigenvalue problem for pseudo-Jacobi matrices* and *Linear maps preserving parallel pairs*, respectively. The titles and abstracts of all talks can be seen at `https://sites.google.com/view/icmaa-2022`.

After a two-year period of pandemic restrictions that forced the previously-scheduled ICMAA 2020 to be postponed, participants had the opportunity to enjoy the fruitful discussions and friendly atmosphere of coffee and lunch breaks. The social program included a welcome reception, *Porto d'Honra*, the conference dinner, and a typical "moliceiro" boat tour on Aveiro's city centre canals.

The conference was organized and supported by the Center for Research & Development in Mathematics and Applications (CIDMA, `https://cidma.ua.pt`), through the Portuguese Foundation for Science and Technology (FCT - *Fundação para a Ciência e a Tecnologia*), within projects UIDB/04106/2020 and UIDP/04106/2020. It was also supported by the International Linear Algebra Society, the University of Aveiro (`https://www.ua.pt`) and by the Department of Mathematics of the University of Aveiro (`https://www.ua.pt/pt/dmat`).

The Scientific Organizing Committee consisted of Enide Andrade (Organizing Committee Chair) and Rute Lemos, University of Aveiro, Aveiro, Portugal; Tin-Yau Tam (Organizing Committee co-Chair), University of Nevada, Reno, USA; Qing-Wen Wang, Shanghai University, Shanghai, China; Fuzhen Zhang, Nova Southeastern University, Fort Lauderdale, USA.

The Local Organizing Committee consisted of Enide Andrade, Rute Lemos, João Pedro Cruz, and Sofia Pinheiro, all from the University of Aveiro, Aveiro, Portugal.

The 24$^{\text{th}}$ ILAS Conference: Classical Connections
Galway, Ireland, June 20–24, 2022

Report by Rachel Quinlan and Helena Šmigoc

The 24$^{\text{th}}$ Conference of the International Linear Algebra Society (ILAS 2022) took place June 20–24, 2022, at the National University of Ireland, Galway.

The conference theme was Classical Connections. This was reflected in the plenary programme and mini-symposia, and participants were encouraged to think about relating talks to their historical roots. The conference attracted over 250 participants, with a rich programme consisting of 10 plenary speakers, 22 mini-symposia, contributed talks, and a poster session. The programme followed a similar format to that of other recent ILAS conferences: lectures and mini-symposia from Monday to Friday, with an excursion on Wednesday afternoon followed by a banquet on Wednesday evening.

The conference proceedings will be published as a special issue of *Linear Algebra and its Applications*.

Plenary lectures were delivered by:

- Shmuel Friedland (University of Illinois Chicago, *LAMA* Lecture) - *Rank of a tensor and quantum entanglement*
- Patrick Farrell (University of Oxford) - *Reynolds-robust preconditioners for the stationary incompressible Navier-Stokes and MHD equations*
- Nicolas Gillis (University of Mons) - *Historical tour on the nonnegative rank*
- Misha Kilmer (Tufts University, SIAG/LA Lecture) - *Bridging the divide: from matrix to tensor algebra for optimal approximation and compression*
- Monique Laurent (Amsterdam, and Tilburg University) - *Graphs, copositive matrices, and sums of squares of polynomials*
- Clément de Seguins Pazzis (Université de Versailles Saint-Quentin-en-Yvelines) - *Decomposing matrices into quadratic ones*
- Christiane Tretter (University of Bern, *LAA* Lecture) - *From finite to infinite dimensions: Chances and challenges in spectral theory*
- Vilmar Trevisan (Universidade Federal do Rio Grande do Sul) - *Eigenvalue Location of Symmetric Matrices*
- Pauline van den Driessche (University of Victoria, Hans Schneider Prize Lecture) - *Linear Algebra is Everywhere: a Duo of Examples from Mathematical Biology*
- Paul van Dooren (Catholic University of Louvain, Israel Gohberg ILAS-IWOTA Lecture) - *Strongly minimal self-conjugate linearizations for polynomial and rational matrices*

Five more junior participants were chosen to be *LAA* Early Career Speakers: Jane Breen, Projesh Nath Choudhury, Jephian Lin, Davide Palitta, and Maria del Carmen Quintana.

A barbecue welcome dinner on Monday was the first social event of the conference. It provided a relaxed setting for catching up with old friends, making new connections, and for the participants to join in the excitement for the week ahead. Participants could choose among three options for the Wednesday excursion: to marvel at the spectacular Cliffs of Moher, to stretch their legs by taking the Marconi Loop walk at the landing site of the first transatlantic flight, or to enjoy the scenery of the South Connemara Coast. Then, that evening, participants had a chance for a culinary experience at the conference banquet, held at the Galmont Hotel.

The Scientific Organizing Committee (Nair Abreu, Peter Cameron, Mirjam Dür, Ernesto Estrada, Vyacheslav Futorny, Stephen Kirkland, Yongdo Lim, Rachel Quinlan, Peter Šemrl, Helena Šmigoc, Françoise Tisseur, and Paul Van Dooren) set up the foundation of the programme, which was fleshed out by the mini-symposium organisers.

The Local Organizing Committee, led by Rachel Quinlan and Niall Madden (both of the University of Galway) and Helena Šmigoc (University College Dublin), included Paul Barry (Waterford Institute of Technology), Jane Breen (Ontario Tech), Anthony Cronin (University College Dublin), Ronan Egan (Dublin City Univeristy), Richard Ellard (Technical University Dublin), Róisín Hill (University of Galway), Kevin Jennings (University of Galway), Thomas Laffey (University College Dublin), Oliver Mason (Maynooth University), Collette McLoughlin (University of Galway), and Kirk Soodhalter (Trinity College Dublin).

The meeting was generously supported by the University of Galway, Fáilte Ireland, the International Linear Algebra Society, the Irish Mathematical Society, and Taylor & Francis.

The ILAS conference banquet

Pauline van den Driessche is presented with the Hans Schneider Prize

Participants of ILAS 2022 in Galway, Ireland

The web site for the meeting remains available at `http://ilas2020.ie`.

The 6th Workshop on Algebraic Designs, Hadamard Matrices & Quanta (Hadamard2020+2) Kraków, Poland, June 27–July 2, 2022

Report by Wojciech Bruzda

The 6th Workshop on Hadamard Matrices and related topics was held from June 27th to July 2nd, 2022 at the Jagiellonian University in Kraków, Poland.

The event, originally announced for 2020, had been postponed twice due to pandemic restrictions. Ultimately organized for 2022, the conference enjoyed the presence of numerous experts in the field who arrived in Kraków from all over the world and delivered high-quality lectures on their recent research. The main subjects of the conference included

- (complex) Hadamard matrices and orthogonal designs,
- mutually unbiased bases and symmetric, informationally complete, positive operator-valued measure(s), and
- applications of combinatorial designs in quantum information.

The meeting was a fruitful collaboration between mathematicians and physicists. The next workshop in this series will take place in 2025 in Seville, Spain.

For more details (including talk slides and videos), please visit the Hadamard2020+2 website:

`https://chaos.if.uj.edu.pl/hadamard2020`

The Advances in Numerical Linear Algebra conference: Celebrating the 60th birthday of Nick Higham Manchester, UK, July 6–8, 2022

Report by Stefan Güttel

An international conference celebrating the 60th birthday of Nick Higham was held at the University of Manchester, July 6–8, 2022. The hybrid meeting brought together more than 90 researchers in numerical linear algebra to discuss current developments, challenges in the light of evolving computer hardware, and the changing needs of applications. More than 30 invited speakers presented their work and emphasized Nick Higham's influence on the field. Nick himself also gave a talk with a historical flavor. A conference dinner in the Manchester Whitworth Art Gallery featured an entertaining speech by Charlie Van Loan and gave all participants ample opportunity to network. The conference was organized by Stefan Güttel, Sven Hammarling, Stephanie Lai, Françoise Tisseur and Marcus Webb and received generous funding from the Royal Society and from MathWorks as well as support from the Manchester Mathematical Sciences (MiMS). Most talks have been recorded and are available on the conference website: `https://nla-group.org/njh60`

Participants of the Advances in Numerical Linear Algebra conference in honor of Nick Higham's 60th birthday

The 33rd International Workshop on Operator Theory and its Applications (IWOTA) Kraków, Poland, September 6–10, 2022

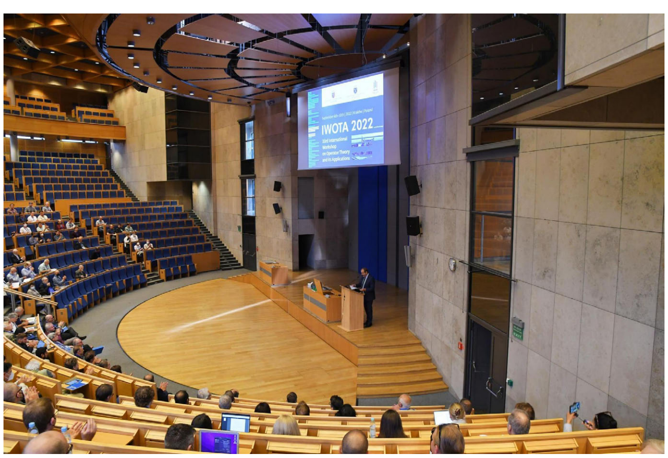
Report by Mark Ptak and Michał Wojtylak

The 33rd International Workshop on Operator Theory and its Applications (IWOTA) took place September 6–10, 2022, at the University of Agriculture, the Jagiellonian University, and the AGH University of Technology in Kraków, Poland. The conference was sponsored by all three universities, the city of Kraków, Stichting Advancement of Mathematics, and ILAS. The organizing committee was chaired by Marek Ptak (University of Agriculture) and Michał Wojtylak (Jagiellonian University), who worked in close collaboration with the IWOTA Steering Committee, in particular with the vice presidents, J. William Helton (University of California San Diego) and Hugo J. Woerdeman (Drexel University).

Participants of IWOTA 2022

Operator Theory is an active research area with many applications in Systems and Control, Quantum Information, Filter Design, etc. This was reflected in both the plenary and session talks. In total there were 366 talks, and the 9 plenary talks included a Hans Schneider ILAS talk by Hugo J. Woerdeman. The titles and abstracts are available at the workshop website: `https://iwota2022.urk.edu.pl`.

With the support and hospitality of the three participating universities and the city of Kraków, participants were able to exchange ideas and experiences on the subject in a friendly atmosphere. There were 375 participants from 41 countries, including participants from war-torn Ukraine. After two years of struggling with pandemic conditions, the meeting was held fully on-site.



Participants gathered for IWOTA 2022

Hugo Woerdeman delivers the Hans Schneider ILAS lecture

UPCOMING CONFERENCES AND WORKSHOPS

Joint Mathematics Meetings in Boston – Invited Lecture – Special Sessions
Boston, USA, January 4–7, 2023

ILAS is again a partner with the American Mathematical Society (AMS) for the annual Joint Mathematics Meetings (JMM) that will take place in Boston, Massachusetts from January 4–7, 2023. This meeting will include an ILAS invited lecture by Apoorva Khare, as well as four ILAS special sessions. All information about the JMM conference itself can be found on the website `https://www.jointmathematicsmeetings.org`. We encourage ILAS members to attend – discounted registration continues until December 20th!

The ILAS Invited Address will be delivered by Apoorva Khare (Indian Institute of Science):

Wednesday, January 4, 2023, 9:00 – 9:50AM.

Analysis applications of Schur polynomials

Abstract: We explain recent results involving Schur polynomials – viewed as functions on the positive orthant – which have led to progress along several classical directions of study:

(a) Understanding functions that preserve positive semidefiniteness when applied entrywise to matrices. This journey takes us from work of Schur, Schoenberg, Rudin, and Loewner (1900s) to modern-day results.

(b) The proofs of these recent results lead to characterizing (weak) majorization of real tuples using Schur polynomials; this has since been generalized to all Weyl groups. Majorization inequalities have been studied since Maclaurin and Newton (1700s).

(c) We also explain how all Schur polynomials are lurking within each smooth function, by completing a 1960s determinantal computation of Loewner. This helps extend determinantal identities by Cauchy and Frobenius (1800s) involving one or two geometric series, to all power series, over all unital commutative rings.

(Partly based on joint works with Alexander Belton, Dominique Guillot, Mihai Putinar, and with Terence Tao.)

The four ILAS Special Sessions will be:

- ILAS Special Session on Matrix Analysis and Applications (10 lectures)

  Organizers: Hugo Woerdeman and Edward Poon

  Wednesday January 4, 2023, 1:00 – 6:00 PM

- ILAS Special Session on Innovative and Effective Ways to Teach Linear Algebra (11 lectures)

  Organizers: David M. Strong, Gil Strang, Sepideh Stewart, and Megan Wawro

  Friday January 6, 2023, 9:00 AM – 12:00 PM

  Saturday January 7, 2023, 1:00 PM – 3:30 PM

- ILAS Special Session on Matrices and Operators (16 lectures)

  Organizers: Mohsen Aliabadi, Tin-Yau Tam, and Pan-Shun Lau

  Saturday January 7, 2023, 8:00 AM – 12:00 PM

  Saturday January 7, 2023, 1:00 PM – 6:00 PM

- ILAS-AIM Special Session on the Inverse Eigenvalue Problem for a Graph and Zero Forcing (12 lectures)

  Organizers: Mary Flagg and Bryan A Curtis

  Thursday January 5, 2023, 8:00 AM – 12:00 PM

  Thursday January 5, 2023, 1:30 PM – 4:30 PM

The Western Canada Linear Algebra Meeting (WCLAM)
Regina, Canada, May 27–28, 2023

The Western Canada Linear Algebra Meeting (WCLAM) provides an opportunity for mathematicians in western Canada working in linear algebra and related fields to meet and present accounts of their recent research, and to have informal discussions. While the meeting has a regional base, it also attracts people from outside the geographical area. Participation is open to anyone who is interested in attending or speaking at the meeting.

Previous WCLAMs were held in Banff (2004, 2010), Brandon (2021), Regina (1993, 2002, 2014), Kananaskis (1996), Lethbridge (1995, 2012), Pullman, WA (2018), Victoria (1998, 2006), and Winnipeg (2000, 2008, 2016).

The 2023 WCLAM will be held at the University of Regina in Regina, Saskatchewan, May 27th – 28th, and the program will include two featured speakers:

- Jane Breen (Ontario Tech University)
- Leslie Hogben (Iowa State University)

The 2023 WCLAM Organising Committee consists of:

- Shaun Fallat (Regina)
- Hadi Kharaghani (Lethbridge)
- Steve Kirkland (Manitoba)
- Peter Lancaster (Calgary)
- Sarah Plosker (Brandon)
- Michael Tsatsomeros (Washington State)
- Pauline van den Driessche (Victoria)

The Local Organizing Committee consists of:

- Shaun Fallat (Regina)
- S. Ahmad Mojallal (Regina)
- Prateek K. Vishwakarma (Regina)

The WCLAM Organising Committee gratefully acknowledges the generous support this meeting has received in the past from PIMS and ILAS, as well current commitments from the Department of Mathematics and Statistics, the Faculty of Science, and the President's Office at the University of Regina.

Registration information and a conference website will be forthcoming.

The 16$^{\text{th}}$ Workshop on Numerical Ranges and Numerical Radii (WONRA 2023)
Coimbra, Portugal, June 7–9, 2023

The 16$^{\text{th}}$ Workshop on Numerical Ranges and Numerical Radii (WONRA 2023) will be held at the Department of Mathematics, University of Coimbra, Coimbra, Portugal, on June 7–9, 2023, close to the 25$^{\text{th}}$ ILAS Conference in Madrid, Spain. The purpose of this workshop is to stimulate research and foster interaction between researchers interested in the subject of numerical ranges and numerical radii. A high level of research activity on the topic has resulted from connections between the subject and many different branches of pure and applied mathematics, such as operator theory, functional analysis, C*-algebras, Banach algebras, matrix norms, inequalities, numerical analysis, perturbation theory, matrix polynomials, systems theory, quantum physics, etc. Moreover, a wide range of tools, including algebra, analysis, geometry, combinatorics and computer programming, are useful in its study. An informal workshop atmosphere will facilitate the exchange of ideas from different scientific areas and, hopefully, the participants will be informed on the latest developments and new ideas. Contributions from both theoretical and applied perspectives are welcome. For further details, please visit `https://sites.google.com/view/wonra-2023`.

The Organizing Committee consists of Natália Bebiano (CMUC, University of Coimbra, `bebiano@mat.uc.pt`); Graça Soares (CMAT-UTAD, University of Trás-os-Montes e Alto Douro, `gsoares@utad.pt`); Rute Lemos (CIDMA, University of Aveiro, `rute@ua.pt`); Ana Nata (CMUC, Polytechnic Institute of Tomar, `anata@ipt.pt`).

The 25th ILAS Conference
Madrid, Spain, June 12–16, 2023

The 25th conference of the International Linear Algebra Society, ILAS 2023, will be held June 12–16, 2023, in Madrid, Spain. The venue is the Escuela Técnica Superior de Ingenierí a de Montes, Forestal y del Medio Natural, of the Universidad Politécnica de Madrid (Polytechnic University of Madrid).

The conference will feature plenary talks from:

- Carlos Beltrán (Universidad de Cantabria)
- Erin C. Carson (Charles University)
- Stefan Guettel (University of Manchester, Taussky Todd Prize)
- Nicholas J. Higham (University of Manchester, Hans Schneider Prize)
- Elias Jarlebring (KHT Stockholm, SIAG-LA speaker)
- Shahla Nasserasr (Rochester Institute of Technology)
- Vanni Noferini (Aalto University, LAMA speaker)
- Rachel Quinlan (NU Galway)
- Michael Tait (Villanova University, Richard A. Brualdi Early Career Prize)
- Cynthia Vinzant (University of Washington)

The organizing committee consists of:

- Roberto Canogar
- Fernando De Terán (chair)
- Froilán M. Dopico
- Ana María Luzón
- Ana Marco
- and José Javier Martínez
- Manuel Alonso Morón and
- Raquel Viaña.

The scientific committee consists of:

- Raymond H. Chan (Hong-Kong)
- Fernando De Terán (Spain)
- Gianna M. Del Corso (Italy)
- Shaun Fallat (Canada)
- Heike Fassbender (Germany)
- Elias Jarlebring (Sweden)
- Linda Patton (USA)
- Jennifer Pestana (UK)
- João Queiro (Portugal)
- Naomi Shaked-Monderer (Israel)
- Daniel Szyld (ILAS President, USA)
- Raf Vandebril (ILAS Vice President for Conferences, Belgium)
- and Zdenek Strakos (Czech Republic).

More information about the conference, including plenary speakers, invited mini-symposia, and relevant dates, is available at `http://www.ilas2023.es`.

The 34$^{\text{th}}$ International Workshop on Operator Theory and its Applications (IWOTA) Helsinki, Finland, July 31–August 4, 2023

The IWOTA conference series is the largest and most important annual event in operator theory and its applications, and it plays a leading role in the advancement of operator theory through the conference activities and proceedings. It contributes to future growth of the field through ample opportunities and partial financial support for early-career researchers and underrepresented groups to present their research and broaden their knowledge.

IWOTA 2023 will contribute to cross-fertilization with other fields of mathematics, such as complex analysis, functional analysis, harmonic analysis, linear algebra, mathematical physics, and inverse problems, as well as their various applications, and provides a medium for an intense exchange of new results, information and opinions, and for international collaboration in operator theory and its applications worldwide. It further sets directions for future research and also makes plans for future IWOTA conferences.

Deadlines:

December 31, 2022 – Special session proposals
Spring 2023 – Submission of abstracts and registration

Invited Plenary Speakers:

- Stephan Ramon Garcia, Pomona College, USA (Hans Schneider ILAS Lecturer)
- Tuomas Hytönen, University of Helsinki, Finland
- Alexander Its, IUPUI, Indianapolis, USA
- Svetlana Jitomirskaya, University of California Irvine, USA
- Stephanie Petermichl, University of Würzburg, Germany
- Gideon Schechtman, Weizmann Institute, Israel
- Gunther Uhlmann, University of Washington, Seattle, USA
- Alexander Volberg, Michigan State University, USA

Organizing committee: Jani Virtanen (chair), University of Helsinki and University of Reading; Hans-Olav Tylli, University of Helsinki; Kari Astala, University of Helsinki. In collaboration with the IWOTA executive steering committee members: J. William Helton (chair), University of California San Diego; Igor Klep, University of Ljubljana, Slovenia; Hugo J. Woerdeman, Drexel University, Pennsylvania.

For further information, visit `https://www.helsinki.fi/en/conferences/iwota2023`.

Algebraic Statistics and Our Changing World: New Methods for New Challenges Chicago, USA, September 18–December 15, 2023

The Institute for Mathematical and Statistical Innovation (IMSI), located in Chicago, will host a long program on Algebraic Statistics and Our Changing World: New Methods for New Challenges. The program will take place from September 18th to December 15th, 2023 and will include four workshops.

Applications for long-term visitors and workshop participants are now open at:

`https://www.imsi.institute/activities/algebraic-statistics-and-our-changing-world`

The focus will be mathematical and statistical challenges arising in applications central to our changing world, such as:

- modeling environmental and ecological systems so that we can better understand the effects of climate change on these systems,
- reimagining urban development and economic systems to address persistent inequity in daily living activities, and
- providing theoretical underpinnings for modern statistical learning techniques to understand the implications of their widespread use and for their easy adaptation to novel applications.

These are all hard challenges, but by bringing together biologists, social scientists, economists, statisticians, and mathematicians, and viewing the challenges collaboratively through the lens of algebraic statistics and, more generally, nonlinear algebra, we can utilize this new perspective to address these challenges side-by-side.

Combinatorics, algebra, and geometry underlie many of the statistical challenges present in the three focus applications of the program. For example, modeling and inferring biological and social networks rely on statistical models that are fundamentally algebraic, and in many cases estimation is done using combinatorial walks, while understanding modern statistical learning techniques relies on understanding principles rooted in algebraic geometry. Due to such underlying mathematical structures, algebraic and geometric methods have had a long history in statistics, from which the field of algebraic statistics has grown. By taking specific problems and identifying the underlying geometry and algebra, this program will pair domain-specific expertise and recent developments in algebraic statistics to develop interdisciplinary connections aimed at addressing new applications. The goal of the program is not only to see where we can make progress on these applications, but also to identify the mathematical and computational tools that we will need in the future, that we should start developing today.

The 9$^{\text{th}}$ Linear Algebra Workshop (LAW)
Ljubljana, Slovenia, June 10–14, 2024

The central theme of the 9$^{\text{th}}$ Linear Algebra Workshop will be the interplay between operator theory and algebra. Cutting-edge linear algebra topics are welcome. Working groups to solve problems will be encouraged. The workshop will be held at the Department of Mathematics at Ljubljana University. Ljubljana is the capital of Slovenia, a small country in central south-eastern Europe. Slovenia is a member state of the European Union using the European currency. The city is turning into a major European tourist destination. The department is situated in the "Southern Campus," south of the Roman walls, in a nice residential area with many science faculties and institutes. Further developments on the workshop will be posted at `http://www.law05.si/law24`.

Send News for *IMAGE* Issue 70

IMAGE seeks to publish all news of interest to the linear algebra community. Issue 70 of *IMAGE* is due to appear online on June 1, 2023. Send your news for this issue to the appropriate editor by April 15, 2023. Photos are always welcome, as well as suggestions for improving the newsletter. Please send contributions directly to the appropriate editor:

- interviews of senior linear algebraists to Adam Berliner (`berliner@stolaf.edu`)
- problems and solutions to Rajesh Pereira (`pereirar@uoguelph.ca`)
- linear algebra education news and articles to Anthony Cronin (`anthony.cronin@ucd.ie`)
- advertisements to Amy Wehe (`awehe@fitchburgstate.edu`)
- announcements and reports of conferences/workshops/etc. to Jephian C.-H. Lin (`jephianlin@gmail.com`)
- proposals for book reviews and other articles to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`)

Send all other correspondence to the editor-in-chief, Louis Deaett (`louis.deaett@quinnipiac.edu`).

For past issues of *IMAGE*, please visit `http://www.ilasic.org/IMAGE`.

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IMAGE PROBLEM CORNER: OLD PROBLEMS WITH SOLUTIONS

We present solutions to Problems 63-3, 67-1, 68-1, 68-3. Solutions are invited to Problem 63-1; to all of the problems from issue 65; to parts (a) and (b) of Problem 66-2; to Problem 66-4; Problems 68-2 and 68-4 and to all of the new problems from the present issue 69.

Problem 63-3: Products of Rectangular Circulant Matrices

Proposed by Rajesh Pereira, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

An $n \times n$ matrix C is said to be circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \mod n$. Let m, n and q be positive integers. We can define a rectangular circulant matrix as follows: An $m \times n$ matrix C is said to be a rectangular circulant if $c_{ij} = c_{kl}$ whenever $j - i = l - k \mod \gcd(m, n)$. As an example, a 6×9 rectangular circulant looks like

$$\begin{bmatrix} a & b & c & a & b & c & a & b & c \\ c & a & b & c & a & b & c & a & b \\ b & c & a & b & c & a & b & c & a \\ a & b & c & a & b & c & a & b & c \\ c & a & b & c & a & b & c & a & b \\ b & c & a & b & c & a & b & c & a \end{bmatrix}.$$

Let A be an $m \times n$ rectangular circulant and B be an $n \times q$ rectangular circulant.

(a) Show that AB is also a rectangular circulant.

(b) What is the maximum rank that AB can have?

(c) Show that the Moore-Penrose inverse of a rectangular circulant matrix is a rectangular circulant matrix.

Solution 63-3 by M.J. de la Puente, *Universidad Complutense de Madrid, Spain*, mpuente@ucm.es

Let $\mathbb{K}$ be a field and let m and n be positive integers. Let I_m be the identity matrix of size m and $J_{m\times n}$ be the all-ones matrix of size $m \times n$ defined over $\mathbb{K}$. Let $\otimes$ denote the Kronecker product of matrices.

The following characterization of the rectangular circulant as a Kronecker product will be used in all three parts of the solution.

Proposition 1 (Characterization of rectangular circulants). *Let $m, n \in \mathbb{N}$ and let $d = \gcd(m, n)$, $m' = \frac{m}{d}$, and $n' = \frac{n}{d}$. If $C \in M_{m\times n}(\mathbb{K})$ is a rectangular circulant, then $C = J_{m'\times n'} \otimes C'$, with C' a square circulant of size d. Conversely, for any all-ones matrix J and any square circulant matrix S, the matrix $J \otimes S$ is a rectangular circulant.*

Proof. For the first statement, take C' to be the $d \times d$ leading principal submatrix of C. The definition of a rectangular circulant implies that C can expressed as a block matrix

$$C = \begin{array}{c} \\ m' \\ \\ \end{array} \left[\begin{array}{ccc} C' & \overset{n'}{\cdots} & C' \\ \vdots & & \vdots \\ C' & \cdots & C' \end{array} \right].$$

The second statement is straightforward. □

(a) We begin with the following lemma:

Lemma 2 (Refinement of square circulant is block square circulant). *Let S be square circulant of size n, let d divide n, and let $n' = \frac{n}{d}$. Then S is a block circulant square matrix with square blocks of uniform size, i.e.,*

$$S = \begin{bmatrix} S_{11} & \cdots & S_{1n'} \\ \vdots & & \vdots \\ S_{n'1} & \cdots & S_{n'n'} \end{bmatrix},$$

with square blocks S_{ij} of size d such that $S_{ij} = S_{kl}$ whenever $j - i = l - k \mod n'$. Moreover, the sum $\overline{S} := \sum_{i=1}^{n'} S_{ij}$ is the same for all $j \in \{1, 2, \ldots, n'\}$ and $\overline{S}$ is square circulant. Besides, $\overline{S} = \sum_{j=1}^{n'} S_{ij}$ is the same for all $i \in \{1, 2, \ldots, n'\}$.

Proof. The (k,l)-entry of $\overline{S}$ is $\sum_r s_{kr}$, where the sum is extended to all $r \equiv l \mod d$. □

Notice that the blocks S_{ij} are Toeplitz but fail to be square circulant, in general.

Example 3. Let $S = \begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_6 & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_5 & s_6 & s_1 & s_2 & s_3 & s_4 \\ s_4 & s_5 & s_6 & s_1 & s_2 & s_3 \\ s_3 & s_4 & s_5 & s_6 & s_1 & s_2 \\ s_2 & s_3 & s_4 & s_5 & s_6 & s_1 \end{bmatrix}$, so that $n = 6$. Then if we take $d = 2$ in Lemma 2, we have $\overline{S} = \begin{bmatrix} s_1+s_3+s_5 & s_2+s_4+s_6 \\ s_2+s_4+s_6 & s_1+s_3+s_5 \end{bmatrix}$, while taking $d = 3$ gives $\overline{S} = \begin{bmatrix} s_1+s_4 & s_2+s_5 & s_3+s_6 \\ s_3+s_6 & s_1+s_4 & s_2+s_5 \\ s_2+s_5 & s_3+s_6 & s_1+s_4 \end{bmatrix}$.

Let $d_A = \gcd(m,n)$ and $d_B = \gcd(n,q)$ and let A' and B' be as given for A and B, respectively, by Proposition 1. Then, letting $d = \gcd(d_A, d_B)$, we have $d = \gcd(m,n,q)$, and we let $e_A = \frac{d_A}{d}$, $e_B = \frac{d_B}{d}$, $m' = \frac{m}{d}$, $n' = \frac{n}{d}$ and $q' = \frac{q}{d}$. Note that $\gcd(e_A, e_B) = 1$. By Lemma 2, A' and B' can be refined to block square circulants, namely as

$$A' = \begin{bmatrix} A'_{11} & \cdots & A'_{1e_A} \\ \vdots & & \vdots \\ A'_{e_A1} & \cdots & A'_{e_Ae_A} \end{bmatrix} \qquad \text{and} \qquad B' = \begin{bmatrix} B'_{11} & \cdots & B'_{1e_B} \\ \vdots & & \vdots \\ B'_{e_B1} & \cdots & B'_{e_Be_B} \end{bmatrix},$$

each with square blocks A'_{ij} and B'_{ij}, respectively, all of size d. Hence we may define n_A, m_A, n_B and m_B, as well as A_{ij} and B_{ij} for $i, j \in \{1, 2, \ldots, n'\}$, such that

$$A = \begin{bmatrix} A' & \overset{n_A}{\cdots} & A' \\ {}_{m_A}\vdots & & \vdots \\ A' & \cdots & A' \end{bmatrix} = \begin{bmatrix} A'_{11} & \cdots & A'_{1e_A} & \cdots & A'_{11} & \cdots & A'_{1e_A} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A'_{e_A1} & \cdots & A'_{e_Ae_A} & \cdots & A'_{e_A1} & \cdots & A'_{e_Ae_A} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A'_{11} & \cdots & A'_{1e_A} & \cdots & A'_{11} & \cdots & A'_{1e_A} \\ \vdots & & \vdots & & \vdots & & \vdots \\ A'_{e_A1} & \cdots & A'_{e_Ae_A} & \cdots & A'_{e_A1} & \cdots & A'_{e_Ae_A} \end{bmatrix} = [A_{ij}]_{i,j=1,2,\ldots,n'}$$

and

$$B = \begin{bmatrix} B' & \overset{n_B}{\cdots} & B' \\ {}_{m_B}\vdots & & \vdots \\ B' & \cdots & B' \end{bmatrix} = \begin{bmatrix} B'_{11} & \cdots & B'_{1e_B} & \cdots & B'_{11} & \cdots & B'_{1e_B} \\ \vdots & & \vdots & & \vdots & & \vdots \\ B'_{e_B1} & \cdots & B'_{e_Be_B} & \cdots & B'_{e_B1} & \cdots & B'_{e_Be_B} \\ \vdots & & \vdots & & \vdots & & \vdots \\ B'_{11} & \cdots & B'_{1e_B} & \cdots & B'_{11} & \cdots & B'_{1e_B} \\ \vdots & & \vdots & & \vdots & & \vdots \\ B'_{e_B1} & \cdots & B'_{e_Be_B} & \cdots & B'_{e_B1} & \cdots & B'_{e_Be_B} \end{bmatrix} = [B_{ij}]_{i,j=1,2,\ldots,n'}.$$

The right-most expressions in the two equations above give block decompositions of A and B, showing that A and B are block circulant square matrices, with all blocks square of size d. The number of such blocks in any row or column of A or B is $n' = n_Ae_A = n_Be_B$. Now block multiplication of A and B can be performed and we get

$$(AB)_{ij} = \sum_{k=1}^{n'} A_{ik}B_{kj} = \left(\sum_{l=1}^{e_A} A'_{il}\right)\left(\sum_{k=1}^{e_B} B'_{kj}\right), \quad i, j = 1, 2, \ldots, n',$$

where the last equality is due to the fact that e_A and e_B are coprime. Further, take the square circulant matrices $\overline{A'} := \sum_{l=1}^{e_A} A'_{il}$ and $\overline{B'} := \sum_{k=1}^{e_B} B'_{kj} \in M_d(\mathbb{K})$ (which do not depend on i and j) and get $(AB)_{ij} = (AB)_{11} = \overline{A'}\,\overline{B'} \in M_d(\mathbb{K})$. Therefore

$$AB = J_{m' \times q'} \otimes \overline{A'}\,\overline{B'} \tag{1}$$

proving that AB is rectangular circulant, by Proposition 1.

(b) In the proof of part (a), we took $d = \gcd(m, n, q)$ and derived the identity (1) where $\overline{A'}\,\overline{B'}$ is a square circulant of size d. There exist invertible matrices P, Q such that $AB = PUQ$, with

$$U = \begin{bmatrix} \overline{A'}\,\overline{B'} & 0_{d\times(n-d)} \\ 0_{(m-d)\times d} & 0_{(m-d)\times(n-d)} \end{bmatrix},$$

whence $\mathrm{rk}(AB) = \mathrm{rk}(\overline{A'}\,\overline{B'}) \le d$. Since one can choose A and B so that $\overline{A'}\,\overline{B'}$ is an invertible circulant the maximum rank that AB can be is $d = \gcd(m, n, q)$.

(c) Let $A \in M_{m\times n}(\mathbb{K})$ be rectangular circulant. Write $d = \gcd(m, n)$, $m' = \frac{m}{d}$, $n' = \frac{n}{d}$ and express $A = J_{m'\times n'} \otimes A'$, with A' square circulant of size d, as in proposition 1. We first state some easily proved statements about the Moore-Penrose inverse of $J_{m\times n}$, Kronecker products and circulant matrices respectively. It is easy to see that $J^+_{m\times n} = \frac{1}{mn} J_{n\times m}$. In section 2.6 in [2], it is shown that $(A \otimes B)^+ = A^+ \otimes B^+$. In p. 90 in [1] it is shown that the Moore-Penrose inverse of a square circulant is square circulant. Hence we have

$$A^+ = (J_{m'\times n'} \otimes A')^+ = \left(\frac{1}{m'n'} J_{n'\times m'}\right) \otimes (A')^+ = J_{n'\times m'} \otimes \frac{1}{m'n'}(A')^+$$

with $\frac{1}{m'n'}(A')^+$ square circulant, whence A^+ is rectangular circulant, by Proposition 1.

References

[1] P. J. Davis. *Circulant Matrices*. Chelsea Publishing, New York, 2nd edition, 1994.

[2] A. N. Langville and W. J. Stewart. The Kronecker product and stochastic automata networks. *J. Comput. Appl. Math.*, 167(2):429–447, 2004.

Problem 67-1: Integer Solutions of a Matrix Equation

Proposed by Gérald Bourgeois, *Université de la Polynésie française, FAA'A, Tahiti, Polynésie française*, `bourgeois.gerald@gmail.com`

Let n be a positive integer, and let J_n be the $n \times n$ matrix all of whose entries are equal to 1.

(a) Show that there exists a matrix $X \in M_n(\mathbb{Z})$ such that $X^2 + X = J_n$ if and only if $n = m^2 + m$ for some $m \in \mathbb{Z}$.

(b) When n is of the form $m^2 + m$, find the number of $n \times n$ zero-one matrices X which solve $X^2 + X = J_n$.

Solution 67-1 by Gary Greaves, *Nanyang Technological University, Singapore*, `gary@ntu.edu.sg`
Ferdinand Ihringer, *Ghent University, Belgium*, `Ferdinand.Ihringer@ugent.be`
Franklin Kenter, *U.S. Naval Academy, Annapolis, MD*, `kenter@usna.edu`
and Brendan Rooney, *Rochester Institute of Technology, Rochester, NY*, `brsma@rit.edu`

For each positive integer m, let $\mathcal{A}_m = \{0, 1, \ldots, m\}$. The Kautz digraph of degree m and dimension n is the directed graph whose vertex set consists of all n-tuples $(a_1, a_2, \ldots, a_n)$ of elements in $\mathcal{A}_m$ that satisfy the condition $a_i \neq a_{i+1}$ for all i in the range $1 \le i \le n-1$ and where there is a directed edge from $(a_1, a_2, \ldots, a_n)$ to $(b_1, b_2, \ldots, b_n)$ if and only if $b_i = a_{i+1}$, for all i in the range $1 \le i \le n-1$.

A key elementary property of the Kautz digraph of degree m and dimension 2 is that, given any two not necessarily distinct vertices (a_1, a_2) and (b_1, b_2), there is exactly one directed path of length less than or equal to two from (a_1, a_2) to (b_1, b_2). If $b_1 = a_2$, this path is of length one and if $b_1 \neq a_2$, this path is of length two and has (a_2, b_1) as its intermediate vertex.

(a) Suppose that $n = m^2 + m$ for some $m \in \mathbb{Z}$. Since $m^2 + m = (-m-1)^2 + (-m-1)$, we can assume without loss of generality that $m \geq 1$. Let X be the adjacency matrix of the Kautz digraph of degree m and dimension 2. Then it follows from the elementary property noted above that $X^2 + X = J_n$, as required.

Conversely, suppose that $X^2 + X = J_n$ for some matrix $X \in M_n(\mathbb{Z})$. Let Y be the Jordan normal form of X. Then we can write $X = PYP^{-1}$ for some matrix P. It follows that $Y^2 + Y = P^{-1}J_nP$. Since the Jordan normal form is unique, and J_n is diagonalisable, we find that Y must be a diagonal matrix. Thus, X is diagonalisable. Now it is straightforward to show that the all-ones vector $\mathbf{1}$ is an eigenvector for X. Whence,

$$\begin{aligned} X^2\mathbf{1} + X\mathbf{1} &= J_n\mathbf{1} \\ m^2\mathbf{1} + m\mathbf{1} &= n\mathbf{1} \end{aligned}$$

for some integer m, as required.

(b) First, observe that the diagonal of X must be 0. It was shown by Gimbert in 1999 [1, Theorem 1] that the Kautz digraphs of dimension 2 are (up to isomorphism) unique with respect to being graphs whose adjacency matrix X satisfies $X^2 + X = J_n$. Let Γ be the Kautz digraph of degree m, and let X be its adjacency matrix. Then the automorphism group $\mathrm{Aut}(\Gamma)$ is the symmetric group S_{m+1} (see [2]). Hence, the number of $n \times n$ zero-one matrices X which solve $X^2 + X = J_n$ is $(m^2 + m)!/(m+1)!$.

References

[1] J. Gimbert. On digraphs with unique walks of closed lengths between vertices. *Australas. J. Combin.*, 20:77–90, 1999.

[2] J. L. Villar. The underlying graph of a line digraph. *Discrete Appl. Math.*, 37/38:525–538, 1992.

Problem 68-1: A Product of Projections

Proposed by Johanns de Andrade Bezerra, *Federal University of Paraíba, João Pessoa, Brazil*, `veganismo.direitoanimal@gmail.com`

A complex square matrix M which satisfies $M^2 = M$ is said to be a projection. Let A be an $n \times n$ Hermitian projection and B be a $n \times n$ projection. Show that if AB is a normal matrix, then AB must be a Hermitian projection.

Solution 68-1 by Roger A. Horn, *University of Utah, Salt Lake City, Utah, USA*, `rhorn@math.utah.edu`

Observe that: (1) The eigenvalues of a projection can only be 0 and 1; (2) The hypotheses are invariant under a simultaneous unitary similarity of A and B, so we may assume that $A = I_k \oplus 0_{n-k}$ for some $k \in \{0, 1, \ldots, n\}$; (3) A normal matrix is Hermitian if and only if its eigenvalues are real; and (4) The off-diagonal blocks of a block triangular normal matrix are zero blocks and each diagonal block is normal.

Partition the projection $B = [B_{ij}]$ conformally with A. Then

$$AB = \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \end{bmatrix}$$

is normal, so $B_{12} = 0$ and B_{11} is normal. Then

$$B = \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11}^2 & 0 \\ * & * \end{bmatrix} = B^2,$$

so B_{11} is a normal projection and hence it is Hermitian. Therefore,

$$AB = \begin{bmatrix} B_{11} & 0 \\ 0 & 0 \end{bmatrix}$$

is a Hermitian projection.

Also solved by Oskar Maria Baksalary, *Adam Mickiewicz University, Poznań, Poland*, `obaksalary@gmail.com`
and Götz Trenkler, *Dortmund University of Technology, Dortmund, Germany*, `trenkler@statistik.tu-dortmund.de`
and Eugene A. Herman, *Grinnell College, Grinnell, Iowa, USA*, `eaherman@gmail.com`

Problem 68-3: Real Orthogonal Roots of Real Orthogonal Matrices

Proposed by Richard William FAREBROTHER, *Bayston Hill, Shrewsbury, England*, R.W.Farebrother@hotmail.com

A real square matrix Q is said to be real orthogonal if $QQ^T = Q^TQ = I$. Let Q be an $n \times n$ real orthogonal matrix and let k be a positive integer. Show that the equation $X^k = Q$ has a solution where X is a real orthogonal matrix if and only if the equation $x^k = \det(Q)$ has a real solution.

Solution 68-3 by Eugene A. HERMAN, *Grinnell College, Grinnell, Iowa, USA*, eaherman@gmail.com

The determinant of an orthogonal matrix is always ± 1. So the equation $x^k = \det(Q)$ has a real solution if and only if $\det(Q) = 1$ or $\det(Q) = -1$ and k is odd. Suppose k is even and $\det(Q) = -1$ and suppose $X^k = Q$ has a solution where X is orthogonal. Then $-1 = \det(Q) = \det(X^k) = (\det(X))^k = (\pm 1)^k = 1$. This contradiction shows that no such matrix X exists.

For the converse, we use the fact that Q is orthogonally similar to a block diagonal matrix whose blocks are either 2×2 rotation matrices or 1×1 with entry ± 1. We may assume the blocks are arranged as we wish: We assume the 1×1 blocks with entry -1 are consecutive. If $\det(Q) = 1$, the number of occurrences of such blocks is even, and so we may group them in pairs. Then each such pair constitutes the 2×2 diagonal block $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$, which is a rotation matrix. For a 2×2 rotation matrix through an angle θ, the rotation through the angle θ/k is an orthogonal matrix and a kth root of the given matrix. Of course 1 is a kth root of 1, and so Q has an orthogonal kth root when $\det(Q) = 1$. If $\det(Q) = -1$ and k is odd, each ± 1 is a kth root of itself and each 2×2 rotation matrix has an orthogonal kth root as before. Thus Q again has an orthogonal kth root.

Also solved by Johanns DE ANDRADE BEZERRA, *Federal University of Paraíba, João Pessoa, Brazil*, veganismo.direitoanimal@gmail.com

IMAGE PROBLEM CORNER: NEW PROBLEMS

Problems: We introduce three new problems in this issue and invite readers to submit solutions for publication in *IMAGE*.
Submissions: Please submit proposed problems and solutions in macro-free LaTeX along with the PDF file by e-mail to *IMAGE* Problem Corner editor Rajesh Pereira (pereirar@uoguelph.ca).

NEW PROBLEMS:

Problem 69-1: Block Triangular k-Potent Matrices

Proposed by Jeffrey STUART, *Pacific Lutheran University, Tacoma, Washington, USA*, jeffrey.stuart@plu.edu

A complex matrix N is called *k-potent* if there exists a positive integer k such that $N^{k+1} = N$. Let A, B and C be complex matrices with A and B square. Let M be the complex matrix

$$M = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}.$$

(a) Suppose M is k-potent for some positive integer k. Show that A and B must be k-potent for the same k.

(b) Suppose A and B are k-potent matrices that have no eigenvalue in common. Show that M is k-potent for all choices of the matrix C.

(c) Suppose A and B are k-potent matrices that have at least one eigenvalue in common. Show that there exists a matrix C for which M is not k-potent.

Problem 69-2: Matrix Units

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let n be a positive integer and let S_{n^2} denote the symmetric group on n^2 elements. A *matrix unit* is any real matrix which has a single entry equal to 1 and all of its other entries equal to 0. Let $\{A_i\}_{i=1}^{n^2}$ be the set of all distinct $n \times n$ matrix units listed in any order. Express the sum $\sum_{\sigma \in S_{n^2}} \prod_{i=1}^{n^2} A_{\sigma(i)}$ as a single matrix.

Problem 69-3: Sums of Roots of Unity

Proposed by Rajesh PEREIRA, *University of Guelph, Guelph, Canada*, pereirar@uoguelph.ca

Let m and n be integers with $1 \leq m \leq n$. Let $\omega_1, \omega_2, \ldots, \omega_m$ be any m distinct nth roots of unity. Show that $\sum \prod_{j=1}^{m} \omega_j^{p_j} = 0$, where the sum is taken over all m-tuples of nonnegative integers $(p_1, p_2, \ldots, p_m)$ that satisfy $\sum_{j=1}^{m} p_j = n - m + 1$.

Solutions to Problems 63-3, 67-1, 68-1 and 68-3 are on pages 23–27.

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Lesson Plan 1

Introduction to

"Linear Algebra" **and** *"Matrix Theory"*

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Concepts, Notions and Definitions in Lesson Plan 1

Vectors

in $\mathbb{R}^n$ or $\mathbb{C}^n$

Matrices (Rows before Columns)

Vector Algebra, Matrix Algebra

Linear Transformations

Dot Product of Vectors

Linear Combinations of Vectors

Unit Vectors e_i

Standard Basis $\mathcal{E} = \{e_i\}$

Standard Matrix Representation of Linear Transformations

Matrix $\times$ Vector Product

The subject matter of this first Lesson Plan will take about 3 to 4 class hours. It is important to encourage student questions and deal with them extensively, beginning on the very first day of class.

A Global Preamble for this Lesson Plan Series as a Whole

Our overall aim is to rejuvenate our teachings of elementary Linear Algebra to become congruent with our current uses and knowledge of this area. Linear Algebra is most successfully approached today via modern Matrix Theory. Matrices and matrix problems are the life fluid of much of engineering research, of the internet and search engines, of controls, optimizations, AI, large data problems, et cetera and most of today's numerical computations.

In this series of lesson plans we immediately equate Linear Transformations with Matrix $\times$ Vector Products and subsequently rework and rethink all of 'classical Linear Algebra' matricially, in contrast to how most of us and our teachers and their teachers and on down have been taught from an antiquated and obsolete knowledge base for decades, if not centuries.

Matrices, their behavior, their intrinsic structure and our computational prowess with them - if taught coherently in an alive manner - will benefit all of current and future science and engineering research and give our students a solid modern knowledge base that will benefit us all.

In this matrix based course formal proofs are few and far apart in Lesson Plans 1, 4, 6, and 7. We prefer constructive proofs and use computer validation to illustrate important matrix results.

Our Goals in the First Lesson:

Contents-wise *:*

We introduce n-vectors and m by n matrices lightly; study their interaction in light of linear transformations; introduce unit vectors, the vector dot-product, component functions, Riesz Theorem and the standard matrix representation $A_{m,n}$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$.

We formally prove that linear transformations and standard matrix representations are equivalent in Riesz's Representation Theorem (1907).

We explain the first two formal (and simple) proofs of the course slowly and deliberately. There will be many student question as to why, how , ... here. **Practice !**

Pedagogy-wise :

We aim to engage the students and with the students on day 1 and in every class session; to ask questions and mold their answers, to discuss their ideas and guide them; to ask for examples and simple computations in class and to deal with misunderstandings in a respectful, open, even a 'loving' ('That's ok ...') way.

And try to elicit student answers to student questions.
('How could you, would you explain this to another student down your hall residence...?').
We build the course on student input, be they understandings or misunderstandings. And we share our knowledge freely and openly and do not ever rush through the syllabus.

Our Path through *"Linear Algebra"* *and* *"Matrix Theory"*

Linear Algebra deals with vector spaces in the abstract, while
Matrix Theory deals with vector spaces concretely.

This is how I would start my first class meeting *of a first Linear Algebra class:*

"Welcome to your first class on Linear Algebra and Matrices.

"Here we will study many concepts and methods that may be new for you, so please bear with me if and when not every notion is completely clear at its first mention; talk with your fellow class mates, listen and ask ... . In class we will go over these notions and concepts many times in the future. And in a few weeks of studies you will become experts, too.
Welcome and let me begin with some words of explanation."

Abstract Linear Algebra is popular as a first introduction into formal Algebra.

In such an abstract course one would introduce finite dimensional vector spaces through 11 axioms for vector addition, scalar vector multiplication and the distributive properties linking vector addition and scalar vector stretching and so forth.

And then study linear transformations and the structure of various sets of vectors.

We would argue through several algebraic and logic based notions for linear vector (in)dependence when we study bases and subspaces and learn how to construct indirect proofs and proofs by induction.

Such an approach would introduce students into mathematicians' lingo and prepare them to think like (abstract) mathematicians.

This would be a worthy endeavor and worth the effort for maths, science and engineering majors.

Matrix Theory instead deals with finite dimensional vector spaces and linear transformations concretely.

A matrix is a rectangular array of numbers, real or complex. This array interacts with vectors as any linear transformation does, but concretely by matrix $\times$ vector multiplication.

Sets of vectors, their linear (in)dependence and possible spanning and basis structures can be read off the row echelon form of their column vector matrix as we will see in Lesson Plan 2.

Likewise, the ranges and nullspaces of linear transformations are displayed by the row echelon form of their standard matrix representation.

Eigenvalues and eigenvectors of linear transformations are best understood and computed by subspace iterations and matrix eigensolving software.

Our first, the immediate task, however, is to link Linear Algebra and Matrix Theory through the standard matrix representation of linear transformations in his Lesson Plan 1.

The basic notions and tools are vectors and their accumulation in rectangular or square matrices, as well as the geometry and algebra behind the interaction of vectors and matrices.

The preceding two pages have been directed to instructors who need to know what lies ahead. But they also serve as a reminder to students of what this set of Lessons has addressed, once the semester ends and exams loom.

Before starting with math proper, it is good to just stop for a while, contemplate and let students and things settle. Ask them about their professional plans, their majors and study plans. Maybe scroll back and mention how these 'new matrix and vector words' and notions will become familiar quickly and help their personal growth.

Invite your students to relax and watch how vectors and matrices and related notions are introduced and handled in this modern day and age.

And then the technical, the mathematical part of our lessons begins concretely.

The actual Start of Lesson 1

Vectors contain a number of entries that may be real numbers, complex numbers, named variables; such as the column vector

$$x = \begin{pmatrix} 2.35 \\ \pi \\ 5\sqrt{-1} \\ x_4 \\ -70 \\ x_6 \\ -19 \cdot s \end{pmatrix} \text{ or the row vector } y = \left(1, -2, 3i, 2^2, t^3, 6 \right).$$

The vector x with seven entries lies in 7-dimensional space and y belongs to a 6-dimensional space.

The students might want to think up some examples of vectors and of non-vectors; play with them: try to add them, multiply them, stretch them by a scalar factor, reverse their direction, find the zero vector ... . Let everyone have fun and the students get chalk on their hands!

To create a matrix we bunch equally dimensioned vectors together, stacking column vectors left to right and row vectors top to bottom:

If $u = \begin{pmatrix} 2.2 & -4 & 56 & 2\sqrt{7} \end{pmatrix}$ and $v = \begin{pmatrix} 2 & 0 & 17 & -88i \end{pmatrix}$ are two 4-dimensional row vectors, then

$$A = \begin{pmatrix} \cdots & u & \cdots \\ \cdots & v & \cdots \end{pmatrix} = \begin{pmatrix} 2.2 & -4 & 56 & 2\sqrt{7} \\ 2 & 0 & 17 & -88i \end{pmatrix}$$

is a 2 by 4 matrix.

Often we write a matrix B with its column and row numbers subscripted as $B_{m,n}$. Thus $A = A_{2,4}$ above.

Always remember *"rows before columns"*, or the number m of rows of a matrix B precedes the number n of columns in **double subscript notation** $B_{m,n}$.

With $m = 2$ and $n = 4$ for A above we have $A_{m,n} = A_{2,4}$. Note that neither $x_{7,1}$ nor $y_{1,6}$ from two pages back can appear in a matrix that has either $u_{1,6}$ or $v_{1,6}$ as a row or column since the vector dimensions (rows before column indices) do not conform.

To study and comprehend vector spaces and their linear transformations and their structures is the task of abstract Linear Algebra.

Matrix Theory **instead** *studies the concrete effects of matrices when they map vectors from one space to itself or to another.*

Let this distinction set in.

Then ask and listen to the students' future uses of Linear Algebra and/or of Matrix Theory ... . Mention Linearization ... .

We have to answer two questions now:

Question 1: What are linear transformations of vector spaces?

Question 2: How do matrices map vectors, concretely and practically?

DEFINITION

A function f between two vector spaces $\mathcal{U}$ and $\mathcal{V}$ is a linear transformation if

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) \qquad (*)$$

for all vectors $x, y \in \mathcal{U}$ and all scalars α and β.

In equation $(*)$ the expression $\alpha x + \beta y$ is the diagonal of the parallelogram with sides αx and βy.

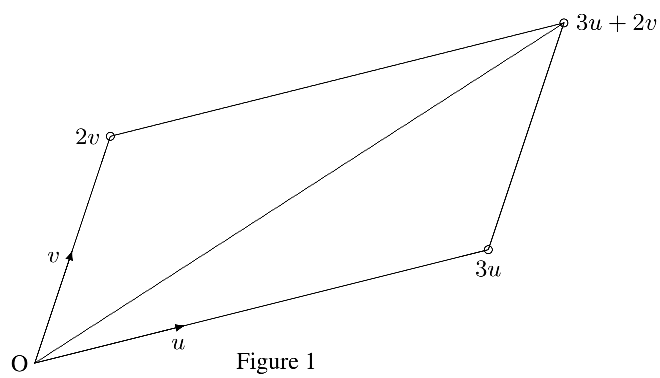

Figure 1

In equation $(*)$, the right hand expression $\alpha f(x) + \beta f(y)$ is the diagonal of the parallelogram in the range space with sides $\alpha f(x)$ and $\beta f(y)$.

Linear functions preserve the parallelogram law between their domain and range spaces.

Think of f as a delivery service that sends α items x and β items y in one package or it sends α packages with x inside and β packages with y inside. The effect will be the same if f is linear. never mind the extra packaging material ...

Any two real or complex entry vectors $x = (x_i)$ and $y = (y_i)$ in n-space can be added by adding their components become $x + y = (x_i + y_i)_n$.

Likewise vectors can be multiplied by any scalar α to become $\alpha x = (\alpha x_i)_n$.

This describes the linear algebraic vector structure of n-space.

Likewise compatibly sized real or complex matrices $A_{m,n} = (a_{i,j})$ and $B_{m,n} = (b_{i,j})$ are added entry by entry to become $A + B = (a_{i,j} + b_{i,j})_{m,n}$.

Single matrices $A_{m,n}$ are multiplied by scalars to become $\lambda A = (\lambda a_{i,j})_{m,n}$.

Thus matrices of size m by n also have a linear algebraic structure.

Both compatibly sized vectors and matrices are closed under addition and scalar multiplication.

Our final tool is the dot product of n-vectors $x = (x_i)_n$ and $y = (y_i)_n$.

The dot product of two same n-dimensional vectors x and y is the scalar $x \cdot y = \sum_{i=1}^{n} x_i y_i$.

This may be the end of the very first hour of Linear Algebra class.

Now is the time to play with vectors and matrices. Form matrices by stacking equal dimensioned row vectors horizontally or equal dimensioned column vectors vertically.
It may also be advisable to introduce the class to Matlab or Python or other matrix capable software now and learn how to set up vectors and matrices inside the chosen software.

How to create a random entry matrix $A_{4,4}$*, and then the block matrix* $B_{8,8}$ *that contains* $7 \cdot A$ *in its* $(1,1)$ *block,* $-A$ *in its* $(1,2)$ *block,* $-2 \cdot A + 5 \cdot eye(n)$ *in its* $(2,1)$ *block for the identity matrix* $eye(4)$*, as well as the zero matrix* $zeros(4,4)$ *in its* $(2,2)$ *diagonal block.*
Maybe set up a computer help session or two with the TAs or yourself some afternoon in a campus computer lab and let students and you have fun and games ...

For simplicity, we will assume from now on and for a while that our vector spaces and matrices are all formed over the reals.

[Complex numbers only occur in the second half of the course when we study eigenvalues and eigenvectors of matrices.]

Any function f that maps $\mathbb{R}^n$ into $\mathbb{R}^m$ is comprised of m individual component functions f_j that each map $\mathbb{R}^n$ into $\mathbb{R}$.

Students should contemplate this fact for a while until comfortable with this component function decomposition.

We will now study linear functions $f : \mathbb{R}^n \to \mathbb{R}^m$ more closely.

Indeed, we want to understand the following fact and prove it.

(1) *If $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear, then each of its m component functions f_i is linear*.

<u>*Proof*</u>:

We have $f(\alpha x + \beta y) = \begin{pmatrix} \vdots \\ f_j(\alpha x + \beta y) \\ \vdots \end{pmatrix}$ and

$$\alpha f(x) + \beta f(y) = \alpha \begin{pmatrix} \vdots \\ f_j(x) \\ \vdots \end{pmatrix} + \beta \begin{pmatrix} \vdots \\ f_j(y) \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ \alpha f_j(x) + \beta f_j(y) \\ \vdots \end{pmatrix},$$

when expressed for the ith component function f_j of f.

Assuming that f is linear, both left hand sides are equal.
And therefore the top and bottom right hand sides are also equal so that any component function f_j of f must be linear, too. $\square$

Now for the converse.

(2) *If each component function $f_j : \mathbb{R}^n \mapsto \mathbb{R}$ of a function $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ is linear, then f is linear.*

Proof:

$$f(\alpha x + \beta y) = \begin{pmatrix} \vdots \\ f_j(\alpha x + \beta y) \\ \vdots \end{pmatrix} = \begin{pmatrix} \vdots \\ \alpha f_j(x) + \beta f_j(y) \\ \vdots \end{pmatrix}$$

$$= \begin{pmatrix} \vdots \\ \alpha f_j(x) \\ \vdots \end{pmatrix} + \begin{pmatrix} \vdots \\ \beta f_j(y) \\ \vdots \end{pmatrix}$$

$$= \alpha \begin{pmatrix} \vdots \\ f_j(x) \\ \vdots \end{pmatrix} + \beta \begin{pmatrix} \vdots \\ f_j(y) \\ \vdots \end{pmatrix} = \alpha f(x) + \beta f(y)$$

for each $1 \leq j \leq m$ and f is linear. $\square$

Compatible n-vector sums of the kind $\alpha u + \beta v + \gamma w + ...$ are called linear combinations of the vectors $u, v, w, ...$ with scalars $\alpha, \beta, \gamma, ...$.

Each n-vector $x = (x_i)_n$ can be written as the linear combination of the standard unit vectors $e_i \in \mathbb{R}^n$.

Here each standard unit vectors e_i has $n - 1$ zero entries and a single entry of 1 in position $1 \leq i \leq n$.

Thus $x = (x_i)_n = x_1 e_1 + ... + x_n e_n$.

And due to the linearity of component functions $f_j : \mathbb{R}^n \to \mathbb{R}$ of linear functions f we have

$$f_j(x) = x_1 f_j(e_1) + ... + x_n f_j(e_n) = x \cdot (f_j(e_1), ..., f_j(e_n)) \in \mathbb{R}.$$

Hence all linear component functions f_j of a linear transformation f act as the dot product of the variable vector x and the constant vector $(f_j(e_1), ..., f_j(e_n))$ that defines $f_j : \mathbb{R}^n \to \mathbb{R}$.

This is Riesz's Representation Theorem (1907) in functional analysis and for infinite dimensional spaces, developed here for finite dimensions n.

In finite dimensions it helps us to understand matrix multiplications and matrix $\times$ vector mappings.

How do we incorporate Riesz's Theorem into concrete matrix theory and matrix operations?

We stack the defining vectors $(f_i(e_1), ..., f_i(e_n))_n$ of each component function f_i of a linear function $f : \mathbb{R}^m \to \mathbb{R}^n$ in an m, n matrix

$$A = \begin{pmatrix} f_1(e_1) & ... & f_1(e_n) \\ \vdots & & \vdots \\ f_m(e_1) & ... & f_m(e_n) \end{pmatrix} = \begin{pmatrix} \vdots & ... & \vdots \\ f(e_1) & ... & f(e_n) \\ \vdots & ... & \vdots \end{pmatrix}_{m,n} .$$

$A_{m,n}$ is called the standard matrix representation of the linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$.

$$A = \begin{pmatrix} f_1(e_1) & ... & f_1(e_n) \\ \vdots & & \vdots \\ f_m(e_1) & ... & f_m(e_n) \end{pmatrix} = \begin{pmatrix} \vdots & ... & \vdots \\ f(e_1) & ... & f(e_n) \\ \vdots & ... & \vdots \end{pmatrix}_{m,n} .$$

The standard matrix representation A contains the images $\begin{pmatrix} \vdots \\ f(e_i) \\ \vdots \end{pmatrix}$ of the standard unit vectors $e_i \in \mathbb{R}^n$ under f in its n columns and - alternately - the defining vectors $\begin{pmatrix} f_j(e_1) & \cdots & f_j(e_n) \end{pmatrix}$ of each component function f_j of f in its m rows.

Remember : '<u>rows before columns</u>'.

How to mate the standard matrix representations $A_{m,n}$ of a linear transformation $f : \mathbb{R}^n \to \mathbb{R}^m$ with a vector $x \in \mathbb{R}^n$ to evaluate $f(x) = A \cdot x$?

To evaluate the necessary dot products of $(f_j(e_1), ..., f_j(e_n))_n$ and $x \in \mathbb{R}^n$ by using $A_{m,n}$, we write x as a column vector, place it to A's right and evaluate $Ax = A \cdot x$ $(= f(x))$ as

$$Ax = \begin{pmatrix} f_1(e_1) & \ldots & f_1(e_n) \\ \vdots & & \vdots \\ f_m(e_1) & \ldots & f_m(e_n) \end{pmatrix}_{m,n} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}_n$$

$$= \begin{pmatrix} \vdots \\ \begin{pmatrix} f_j(e_1) & \ldots & f_j(e_n) \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ \vdots \end{pmatrix}_m$$

as m repeated matrix row $\times$ column vector dot products.

Thus we have achieved a concrete way to express general linear transformations between finite dimensional vector spaces by using matrix $\times$ vector products.

The Linear Transformation Theorem

Every linear transformation $f : \mathbb{R}^n \mapsto \mathbb{R}^m$ can be expressed as a constant matrix $A \times$ vector product $A_{m,n}\, x_n$.

This establishes the equivalence of classical finite dimensional Linear Algebra and Matrix Theory.

Matrix Theory is concrete and codeable. It can answer and solve all questions of abstract Linear Algebra.

Matrix Theory is modern and the language for computing.

It, rather than abstract reasoning, is used in all matrix applications nowadays.

And that we shall teach.

This ends the introductory first lesson plan that covers 2 or 3 class hours of fundamentals in Linear Algebra and in the concrete matrix/vector setting.

Some ideas for homework (and classwork on an added class day):
Think of a matrix $A_{m,n}$ as a stack of n column vectors as one would stack books | next to each other on a shelf, left to right | | | ....

What does the matrix $\times$ vector product $A \cdot x$ do with the columns $\begin{pmatrix} \vdots \\ c_i \\ \vdots \end{pmatrix}$ *in A?*

$$A_{m,n} \cdot x = \begin{pmatrix} \vdots & \cdots & \vdots \\ c_1 & & c_n \\ \vdots & \cdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} =$$

$$= x_1 \begin{pmatrix} \vdots \\ c_1 \\ \vdots \end{pmatrix} + x_2 \begin{pmatrix} \vdots \\ c_2 \\ \vdots \end{pmatrix} + ... + x_n \begin{pmatrix} \vdots \\ c_n \\ \vdots \end{pmatrix}.$$

Hint: multiplying $A_{m,n}$ by an n-vector x on the right creates a linear combination of the column vectors in A.

How would the students multiply an m by n matrix $A_{m,n}$ by an m-vector y from the left? What should $y \cdot A_{m,n} =$ $\begin{pmatrix} y_1 & \cdots & y_m \end{pmatrix} \begin{pmatrix} \cdots & r_1 & \cdots \\ & \vdots & \\ \cdots & r_m & \cdots \end{pmatrix}$ *mean?*

A horizontal row stack ... of books.

Ask students to create matrices and vectors in either column or row form and try to multiply them in class. Some such products will work and others will fail.
Why, and how is that recognized?

Our first lesson on linear transformations and matrix times vector multiplications might take two to four actual class hours and this leads us to simple but tedious applications in the second lesson.

The second lesson plan will again contain outlines for two to four class sessions.

Sometimes full of concrete number crunching, sometimes dealing with abstractions and a few small proofs, too.

The subjects of Lesson Plan 2 are :
row reduction of rectangular matrices to row echelon form,
interpreting row reduced forms in terms of spanning sets of matrix columns or rows,
minimal spanning sets or bases for vector subspaces, and
to learn how to solve solve systems of linear equations via row reduction.

The second lesson plan will take a class to near the halfway point of a modern first Linear Algebra and Matrix Theory course.

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