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# Approximate Controllability of Delayed Fractional Stochastic Differential Systems with Mixed Noise and Impulsive Effects

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**Abstract:** We herein report a new class of impulsive fractional stochastic differential systems driven by mixed fractional Brownian motions with infinite delay and Hurst parameter $\hat{\mathcal{H}} \in (1/2, 1)$. Using fixed point techniques, a $q$-resolvent family, and fractional calculus, we discuss the existence of a piecewise continuous mild solution for the proposed system. Moreover, under appropriate conditions, we investigate the approximate controllability of the considered system. Finally, the main results are demonstrated with an illustrative example.

**Keywords:** fractional stochastic delay system; impulsive effects; approximate controllability; mixed noise

**MSC:** 34A08; 34K50; 60G22; 93B05

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## 1. Introduction

For a long time, the subject of fractional calculus and its applications has gained a lot of importance, mainly because fractional calculus has become a powerful tool with more accurate and successful results in modeling several complex phenomena in numerous, seemingly diverse and widespread fields of science and engineering. It was found that various, especially interdisciplinary, applications can be elegantly modeled with the help of fractional derivatives [1–4]. See also the recent works of [5–8].

Fractional Brownian motion (fBm for short) is a family of Gaussian random processes that are indexed by the Hurst parameter $\hat{\mathcal{H}} \in (0, 1)$. It is a self-similar stochastic process with long-range dependence and stationary increment properties when $\hat{\mathcal{H}} > 1/2$. For more recent works on fractional Brownian motion, see [9–14] and the references therein.

In order to describe various real-world problems in physical and engineering sciences subject to abrupt changes at certain instants during the evolution process, impulsive fractional differential equations have become important in recent years as mathematical models of many phenomena in both physical and social sciences. Impulsive effects begin at any arbitrary fixed point and continue with a finite time interval, known as non-instantaneous impulses. For more details, we refer the reader to [15–23].

The concept of controllability plays a major role in finite dimensional control theory. However, its generalization to infinite dimensions is too strong and has limited applicability, while approximate controllability is a weaker concept completely adequate in applications [24].

Recently, many authors have established approximate controllability results of (fractional) impulsive systems [25–31]. For example, Kumar et al. [32] investigated the approximate controllability for impulsive semilinear control systems with delay; Anukiruthika et al. [33] analyzed the approximate controllability of semilinear stochastic systems with impulses. Although several works exist in this area, the study of the approximate controllability of impulsive fractional stochastic differential systems driven by mixed noise with infinite delay and Hurst parameter $\hat{\mathcal{H}} \in (1/2, 1)$ is still an understudied topic in the literature. This fact provides the motivation of our current work.

We consider an impulsive fractional stochastic delay differential equation with mixed fractional Brownian motion defined by

$$\begin{cases} {}^cD_t^q z(t) = \mathcal{P}z(t) + \mathcal{F}(t, z_t) + \mathcal{G}(t, z_t)\dfrac{d\hat{\mathcal{W}}(t)}{dt} + \sigma(t)\dfrac{d\mathcal{B}^{\hat{\mathcal{H}}}(t)}{dt}, \quad t \in \cup_{i=0}^m (s_i, t_{i+1}], \\ z(t) = \mathcal{K}_i(t, z_t), \quad t \in \cup_{i=1}^m (t_i, s_i], \\ z(t) = \phi(t), \ \ \phi(t) \in \mathcal{D}_h, \end{cases} \tag{1}$$

where $\mathcal{P} : \mathcal{D}(\mathcal{P}) \subset \mathcal{Z} \to \mathcal{Z}$ is the generator of an $q$-resolvent family $\{\mathcal{S}_q(t) : t \geq 0\}$ on the separable Hilbert space $\mathcal{Z}$, ${}^cD_t^q$ is the Caputo fractional derivative of order $1/2 < q < 1$, and state $z(\cdot)$ takes values in the space $\mathcal{Z}$, and $0 = t_0 = s_0 < t_1 < s_1 < t_2 < \cdots < t_m < s_m < t_{m+1} = T < \infty$. The functions $\mathcal{K}_i(t, z_t)$ represent the non-instantaneous impulses during the intervals $(t_i, s_i]$, $i = 1, 2, \ldots, m$, $\hat{\mathcal{W}} = \{\hat{\mathcal{W}}(t) : t \geq 0\}$ is a $Q$-Wiener process defined on a separable Hilbert space $\mathcal{Y}_1$, and $\mathcal{B}^{\hat{\mathcal{H}}} = \{\mathcal{B}^{\hat{\mathcal{H}}}(t) : t \geq 0\}$ is a $Q$-fBm with the Hurst parameter $\hat{\mathcal{H}} \in (1/2, 1)$, defined on a separable Hilbert space $\mathcal{Y}_2$. The history-valued function $z_t : (-\infty, 0] \to \mathcal{Z}$ is defined as $z_t(\theta) = z(t + \theta), \forall\, \theta \leq 0$, and belongs to an abstract phase space $\mathcal{D}_h$. The initial data $\phi = \{\phi(t), t \in (\infty, 0]\}$ are $\mathcal{F}_0$-measurable, $\mathcal{D}_h$-valued random variable independent of $\hat{\mathcal{W}}$ and $\mathcal{B}^{\hat{\mathcal{H}}}$. The functions $\mathcal{F}$, $\mathcal{G}$, $\sigma$, and $\mathcal{K}_i$ satisfy several suitable hypotheses, which will be specified later.

The work is arranged as follows. In Section 2, relevant preliminaries are given that will be used later. In Section 3, we prove the existence of a piecewise continuous mild solution for the proposed system (1). Then, in Section 4, we study the approximate controllability for problem (1). In Section 5, an example is given to show the application of the obtained results. We end with Section 6, in which we present the conclusion of our results and also suggest directions of possible future research.

## 2. Preliminaries

Let $L(\mathcal{Y}_j, \mathcal{Z})$ denote the space of all linear and bounded operators from $\mathcal{Y}_j$ to $\mathcal{Z}$, $j = 1, 2$. The notation $\|\cdot\|$ represents the norms of $\mathcal{Z}$, $\mathcal{Y}_j$, $L(\mathcal{Y}_j, \mathcal{Z})$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered complete probability space, where $\mathcal{F}_t$ is the $\sigma$-algebra generated by $\{\mathcal{B}^{\hat{\mathcal{H}}}(e), \hat{\mathcal{W}}(e) : e \in [0, t]\}$ and $P$-null sets. Let $\mathcal{Q}_j \in L(\mathcal{Y}_j, \mathcal{Y}_j)$ be the operators defined by $\mathcal{Q}_j e_i^j = \lambda_i^j e_i^j$ with finite trace $Tr(Q_j) = \sum_{i=1}^{\infty} \lambda_i^j < \infty$, where $\{\lambda_i^j\}_{i \geq 1}$ are non-negative real numbers and $\{e_i^j\}_{i \geq 1}$ is a complete orthonormal basis in $\mathcal{Y}_j$. Then, there exists a real independent sequence $\mathscr{B}_i(t)$ of the standard Wiener process such that

$$\hat{\mathcal{W}}(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i^1}\mathscr{B}_i(t) e_i^1.$$

The infinite dimensional $\mathcal{Y}_2$-valued fBm $\mathcal{B}^{\hat{\mathcal{H}}}(t)$ is defined as

$$\mathcal{B}^{\hat{\mathcal{H}}}(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i^2}\mathscr{B}_i^{\hat{\mathcal{H}}}(t) e_i^2,$$

where $\mathscr{B}_i^{\hat{\mathcal{H}}}(t)$ are real, independent fBms.

Let $\mathscr{B} = \{\mathscr{B}(t), t \in \mathcal{J}\}$, $\mathcal{J} = [0, T]$ be a Wiener process and $\mathscr{B}^{\hat{\mathcal{H}}} = \{\mathscr{B}^{\hat{\mathcal{H}}}(t), t \in \mathcal{J}\}$ be the one-dimensional fBm with Hurst index $\hat{\mathcal{H}} \in (1/2, 1)$. The fBm $\mathscr{B}^{\hat{\mathcal{H}}}(t)$ has the following integral representation:

$$\mathscr{B}^{\hat{\mathcal{H}}}(t) = \int_0^t \mathscr{K}_{\hat{\mathcal{H}}}(t, e)\, d\mathscr{B}(e),$$

where the kernel $\mathscr{K}_{\hat{\mathcal{H}}}(t,e)$ is defined as

$$\mathscr{K}_{\hat{\mathcal{H}}}(t,e) = \mathfrak{X}_{\hat{\mathcal{H}}} e^{1/2-\hat{\mathcal{H}}} \int_e^t (\tau - e)^{\hat{\mathcal{H}}-3/2} \tau^{\hat{\mathcal{H}}-1/2} d\tau \ \text{ for } \ t > e.$$

We apply $\mathscr{K}_{\hat{\mathcal{H}}}(t,e) = 0$ if $t \leq e$. Note that $\frac{\partial \mathscr{K}_{\hat{\mathcal{H}}}}{\partial t}(t,e) = \mathfrak{X}_{\hat{\mathcal{H}}}(t/e)^{\hat{\mathcal{H}}-1/2}(t-e)^{\hat{\mathcal{H}}-3/2}$. Here, $\mathfrak{X}_{\hat{\mathcal{H}}} = [\hat{\mathcal{H}}(2\hat{\mathcal{H}}-1)/\xi(2-2\hat{\mathcal{H}}, \hat{\mathcal{H}}-1/2)]^{1/2}$ and $\xi(\cdot,\cdot)$ is the Beta function. For $\Lambda \in L^2([0,T])$, it follows from [34] that the Wiener-type integral of the function $\Lambda$ w.r.t. fBm $\mathscr{B}^{\hat{\mathcal{H}}}$ is defined by

$$\int_0^T \Lambda(e) d\mathscr{B}^{\hat{\mathcal{H}}}(e) = \int_0^T \mathscr{K}^*_{\hat{\mathcal{H}}} \Lambda(e) d\mathscr{B}(e),$$

where $\mathscr{K}^*_{\hat{\mathcal{H}}}\Lambda(e) = \int_e^T \Lambda(t) \frac{\partial \mathscr{K}_{\hat{\mathcal{H}}}}{\partial t}(t,e)dt$.

Let $\varphi_j \in L(\mathcal{Y}_j, \mathcal{Z})$ and define

$$\|\varphi_j\|_{\mathcal{L}_2^j} = \left[\sum_{i=1}^{\infty} \left\| \sqrt{\lambda_i^j} \varphi_j e_i^j \right\|^2 \right]^{1/2}.$$

If $\|\varphi_j\|_{\mathcal{L}_2^j} < \infty$, then $\varphi_j$ are called $Q_j$-Hilbert–Schmidt operators, and the spaces $\mathcal{L}_2^j(\mathcal{Y}_j, \mathcal{Z})$ are real and separable Hilbert spaces with inner product $\langle \varphi^1, \varphi^2 \rangle_{\mathcal{L}_2^j} = \sum_{i=1}^{\infty} \langle \varphi^1 e_i^j, \varphi^2 e_i^j \rangle$. The stochastic integral of function $\Psi : \mathcal{J} \to \mathcal{L}_2^2(\mathcal{Y}_2, \mathcal{Z})$ w.r.t. fBm $\mathcal{B}^{\hat{\mathcal{H}}}$ is defined by

$$\int_0^t \Psi(e) d\mathcal{B}^{\hat{\mathcal{H}}}(e) = \sum_{i=1}^{\infty} \int_0^t \sqrt{\lambda_i^2} \Psi(e) e_i^2 d\mathscr{B}_i^{\hat{\mathcal{H}}}(e) = \sum_{i=1}^{\infty} \int_0^t \sqrt{\lambda_i^2} \mathscr{K}^*_{\hat{\mathcal{H}}}(\Psi e_i^2) d\mathscr{B}_i(e). \tag{2}$$

**Lemma 1** (See [9]). *If $\Psi : \mathcal{J} \to \mathcal{L}_2^2(\mathcal{Y}_2, \mathcal{Z})$ satisfies $\int_0^T \|\Psi(e)\|^2_{\mathcal{L}_2^2} de < \infty$, then Equation (2) is a well-defined $\mathcal{Z}$-valued random variable such that*

$$\mathbb{E}\left\| \int_0^t \Psi(e) d\mathcal{B}^{\hat{\mathcal{H}}}(e) \right\|^2 \leq 2\hat{\mathcal{H}} t^{2\hat{\mathcal{H}}-1} \int_0^t \|\Psi(e)\|^2_{\mathcal{L}_2^2} de.$$

**Lemma 2** (See [35]). *For any $\alpha \geq 1$ and for an arbitrary $\mathcal{L}_2^1$-valued predictable process $\Upsilon(\cdot)$,*

$$\sup_{e \in [0,t]} \mathbb{E}\left\| \int_0^e \Upsilon(\tau) d\hat{\mathcal{W}}(\tau) \right\|^{2\alpha} \leq (\alpha(2\alpha-1))^{\alpha} \left( \int_0^t \left( \mathbb{E}\|\Upsilon(e)\|^{2\alpha}_{\mathcal{L}_2^1} \right)^{1/\alpha} de \right)^{\alpha}, \ t \in [0,T].$$

*For $\alpha = 1$, we obtain*

$$\sup_{e \in [0,t]} \mathbb{E}\left\| \int_0^e \Upsilon(\tau) d\hat{\mathcal{W}}(\tau) \right\|^2 \leq \int_0^t \mathbb{E}\|\Upsilon(e)\|^2_{\mathcal{L}_2^1} de.$$

Assume that $h : (-\infty, 0] \to (0, \infty)$ with $\varpi = \int_{-\infty}^0 h(t)dt < \infty$ is a continuous function. We define $\mathcal{D}_h$ by

$$\mathcal{D}_h = \Big\{ \phi : (-\infty,0] \to \mathcal{Z}, \ \text{for any } a > 0, \ (\mathbb{E}|\phi(\theta)|^2)^{1/2} \text{ is a measurable and bounded function on } [-a,0] \text{ with } \phi(0) = 0, \text{ and } \int_{-\infty}^0 h(e) \sup_{e \leq \theta \leq 0} (\mathbb{E}|\phi(\theta)|^2)^{1/2} de < \infty \Big\}.$$

If $\mathcal{D}_h$ is endowed with the norm

$$\|\phi\|_{\mathcal{D}_h} = \int_{-\infty}^{0} h(e) \sup_{e\leq\theta\leq 0} (\mathbb{E}\|\phi(\theta)\|^2)^{1/2} de, \ \phi \in \mathcal{D}_h,$$

then $(\mathcal{D}_h, \|\,.\,\|_{\mathcal{D}_h})$ is a Banach space [36].

Define the space $\mathcal{D}_T = \{z : (-\infty, T] \to \mathcal{Z}, z|_{\mathcal{J}_i} \in C(\mathcal{J}_i, \mathcal{Z}), i = 0, 1, \ldots, m$, and there exist $z(t_i^-)$ and $z(t_i^+)$ with $z(t_i^-) = z(t_i)$, and $z_0 = \phi \in \mathcal{D}_h\}$, with the norm

$$\|z\|_{\mathcal{D}_T} = \|\phi\|_{\mathcal{D}_h} + \sup_{t\in[0,T]} (\mathbb{E}\|z(t)\|^2)^{1/2},$$

where $\mathcal{J}_i = (t_i, t_{i+1}]$, $i = 0, 1, \ldots, m$.

**Lemma 3** (see [37])**.** *If for all $t \in [0, T]$, $z_t \in \mathcal{D}_h$, $z_0 \in \mathcal{D}_h$, then*

$$\|z_t\|_{\mathcal{D}_h} \leq \varpi \sup_{t\in[0,T]} (\mathbb{E}\|z(t)\|^2)^{1/2} + \|z_0\|_{\mathcal{D}_h}.$$

**Definition 1** (see [38])**.** *Let $\mathcal{M} > 0$, $\theta \in [\pi/2, \pi]$, and $\omega \in \mathbb{R}$. A closed and linear operator $\mathcal{P}$ is called a sectorial operator if*

1. $\rho(\mathcal{P}) \subset \sum_{(\theta,\omega)} = \{\lambda \in \mathbb{C} : \lambda \neq \omega, |arg(\lambda - \omega)| < \theta\}$,
2. $\|\mathcal{R}(\lambda, \mathcal{P})\| \leq \mathcal{M}/|\lambda - \omega|$, $\lambda \in \sum_{(\theta,\omega)}$.

**Lemma 4** (see [39])**.** *Let $\mathcal{P}$ be a sectorial operator. Then, the unique solution of the linear fractional system*

$$\begin{cases} {}^cD_t^q z(t) = \mathcal{P}z(t) + \mathcal{F}(t), \quad t > t_0 \geq 0, \quad 0 < q < 1, \\ z(t) = \phi(t), \quad t \leq t_0, \end{cases}$$

*is given by*

$$z(t) = \mathcal{T}_q(t - t_0)z(t_0) + \int_{t_0}^{t} \mathcal{S}_q(t - e)\mathcal{F}(e)de,$$

*where*

$$\begin{aligned} \mathcal{T}_q(t) &= \frac{1}{2\pi i} \int_{B_r} e^{\lambda t} \frac{\lambda^{q-1}}{\lambda^q - \mathcal{P}} d\lambda, \\ \mathcal{S}_q(t) &= \frac{1}{2\pi i} \int_{B_r} \frac{e^{\lambda t}}{\lambda^q - \mathcal{P}} d\lambda. \end{aligned}$$

*Here, $B_r$ denotes the Bromwich path.*

### 3. Solvability Results

We assume the following hypotheses.

**Hypothesis 1 (H1).** *If $q \in (0, 1)$ and $\mathcal{P} \in \mathcal{P}^q(\theta_0, \omega_0)$, then, for any $z \in \mathcal{Z}$ and $t > 0$, we have $\|\mathcal{T}_q(t)\| \leq C_1 e^{\omega t}$ and $\|\mathcal{S}_q(t)\| \leq C_2 e^{\omega t}(1 + t^{q-1})$, $\omega > \omega_0$. Thus, we have*

$$\|\mathcal{T}_q(t)\| \leq \mathcal{M}_1 \ \ and \ \ \|\mathcal{S}_q(t)\| \leq \mathcal{M}_2 t^{q-1},$$

*where $\mathcal{M}_1 = \sup_{0\leq t\leq T} C_1 e^{\omega t}$ and $\mathcal{M}_2 = \sup_{0\leq t\leq T} C_2 e^{\omega t}(1 + t^{q-1})$.*

**Hypothesis 2 (H2).** *There exists a constant $N_{\mathcal{F}} > 0$ such that*

$$\mathbb{E}\|\mathcal{F}(t, \psi_1) - \mathcal{F}(t, \psi_2)\|^2 \leq N_{\mathcal{F}} \ \|\psi_1 - \psi_2\|^2_{\mathcal{D}_h}, \quad \forall\, t \in \mathcal{J}, \quad \psi_1, \psi_2 \in \mathcal{D}_h.$$

**Hypothesis 3 (H3).** *Function $\sigma : \mathcal{J} \to \mathcal{L}_2^2(\mathcal{Y}_2, \mathcal{Z})$ satisfies $\int_0^t \|\sigma(e)\|^2_{\mathcal{L}_2^2} de < \infty$, for every $t \in \mathcal{J}$, and there exists a constant $\Lambda_\sigma > 0$ such that $\|\sigma(e)\|^2_{\mathcal{L}_2^2} \leq \Lambda_\sigma$, uniformly in $\mathcal{J}$.*

**Hypothesis 4 (H4).** *There exists a constant $N_{\mathcal{G}} > 0$ such that*

$$\mathbb{E}\|\mathcal{G}(t,\psi_1) - \mathcal{G}(t,\psi_2)\|^2_{\mathcal{L}_2^1} \leq N_{\mathcal{G}} \, \|\psi_1 - \psi_2\|^2_{\mathcal{D}_h}, \quad \forall\, t \in \mathcal{J}, \quad \psi_1, \psi_2 \in \mathcal{D}_h.$$

**Hypothesis 5 (H5).** *There are constants $L_{\mathcal{K}_i} > 0$, $i = 1, 2, \ldots, m$, such that*

$$\mathbb{E}\|\mathcal{K}_i(t,\psi_1) - \mathcal{K}_i(t,\psi_2)\|^2 \leq L_{\mathcal{K}_i} \, \|\psi_1 - \psi_2\|^2_{\mathcal{D}_h}, \ \forall\, t \in \mathcal{J}, \ \psi_1, \psi_2 \in \mathcal{D}_h.$$

**Definition 2.** *An $\mathcal{F}_t$-adapted random process $z : (-\infty, T] \to \mathcal{Z}$ is called the mild solution of (1) if, for every $t \in \mathcal{J}$, $z(t)$ satisfies $z_0 = \phi \in \mathcal{D}_h$, $z(t) = \mathcal{K}_i(t, z_t)$ for all $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$, and*

$$\begin{aligned} z(t) &= \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e,z_e)de \\ &\quad + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e,z_e)d\hat{\mathcal{W}}(e) + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \end{aligned}$$

*for all $t \in [0, t_1]$, and*

$$\begin{aligned} z(t) &= \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, z_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e,z_e)de \\ &\quad + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e,z_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \end{aligned} \tag{3}$$

*for all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$.*

**Theorem 1.** *Assume that conditions (H1)–(H5) are satisfied. Then, problem (1) has a unique mild solution on $(-\infty, T]$, provided that*

$$L_{\mathcal{R}} = \max_{1 \leq i \leq m} \left\{ \eta_0, \varpi^2 L_{\mathcal{K}_i}, \eta_i \right\} < 1,$$

*where*

$$\begin{aligned} \eta_0 &= 2\mathcal{M}_2^2 \varpi^2 \left( \frac{N_{\mathcal{F}} t_1^{2q}}{q^2} + \frac{N_{\mathcal{G}} t_1^{2q-1}}{2q-1} \right), \\ \eta_i &= \left( 3\mathcal{M}_1^2 L_{\mathcal{K}_i} \varpi^2 + 3\mathcal{M}_2^2 \varpi^2 \left\{ \frac{N_{\mathcal{F}} t_{i+1}^{2q}}{q^2} + \frac{N_{\mathcal{G}} t_{i+1}^{2q-1}}{2q-1} \right\} \right). \end{aligned}$$

**Proof.** We define the operator $\Xi$ from $\mathcal{D}_T$ to $\mathcal{D}_T$ as follows:

$$(\Xi z)(t) = \begin{cases} \phi(t), & t \in (-\infty, 0] \\ \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e,z_e)de & \\ + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e,z_e)d\hat{\mathcal{W}}(e) + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in [0, t_1] \\ \mathcal{K}_i(t, z_t), & t \in (t_i, s_i] \\ \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, z_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e,z_e)de & \\ + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e,z_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in (s_i, t_{i+1}]. \end{cases}$$

For $\phi \in \mathcal{D}_h$, define

$$g(t) = \begin{cases} \phi(t), & t \in (-\infty, 0], \\ 0, & t \in \mathcal{J}. \end{cases}$$

Then, $g_0 = \phi$. Next we define

$$\bar{y}(t) = \begin{cases} 0, & t \in (-\infty, 0] \\ y(t), & t \in \mathcal{J} \end{cases}$$

for each $y \in C(\mathcal{J}, R)$ with $z(0) = 0$. If $z(\cdot)$ satisfies (3), then $z(t) = g(t) + \bar{y}(t)$ for $t \in \mathcal{J}$, which implies that $z_t = g_t + \bar{y}_t$ for $t \in \mathcal{J}$, and the function $y(\cdot)$ satisfies

$$y(t) = \begin{cases} \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \bar{y}_e)de + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \bar{y}_e)d\hat{\mathcal{W}}(e) \\ + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in [0, t_1] \\ \mathcal{K}_i(t, g_t + \bar{y}_t), & t \in (t_i, s_i] \\ \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, g_{s_i} + \bar{y}_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \bar{y}_e)de \\ + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \bar{y}_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\hat{\mathcal{B}}^{\mathcal{H}}(e) & t \in (s_i, t_{i+1}]. \end{cases}$$

Set $\mathcal{D}_T^0 = \{y \in \mathcal{D}_T \text{ such that } y_0 = 0\}$. For any $y \in \mathcal{D}_T^0$, we obtain

$$\|y\|_{\mathcal{D}_T^0} = \|y_0\|_{\mathcal{D}_h} + \sup_{t \in \mathcal{J}} (\mathbb{E}\|y(t)\|^2)^{1/2} = \sup_{t \in \mathcal{J}} (\mathbb{E}\|y(t)\|^2)^{1/2}.$$

Thus, $(\mathcal{D}_T^0, \|\cdot\|_{\mathcal{D}_T^0})$ is a Banach space.

Define the operator $\Psi$ from $\mathcal{D}_T^0$ to $\mathcal{D}_T^0$ as follows:

$$(\Psi y)(t) = \begin{cases} \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \bar{y}_e)de + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \bar{y}_e)d\hat{\mathcal{W}}(e) \\ + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in [0, t_1] \\ \mathcal{K}_i(t, g_t + \bar{y}_t), & t \in (t_i, s_i] \\ \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i, g_{s_i} + \bar{y}_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \bar{y}_e)de \\ + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \bar{y}_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e) & t \in (s_i, t_{i+1}]. \end{cases}$$

In order to prove the existence result, we need to show that $\Psi$ has a unique fixed point. Let $y, y^* \in \mathcal{D}_T^0$. Then, for all $t \in [0, t_1]$, we have

$$\begin{aligned} \mathbb{E}\|(\Psi y)(t) - (\Psi y^*)(t)\|^2 &\leq 2\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)(\mathcal{F}(e, g_e + \bar{y}_e) - \mathcal{F}(e, g_e + \bar{y}_e^*))de\right\|^2 \\ &\quad + 2\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)(\mathcal{G}(e, g_e + \bar{y}_e) - \mathcal{G}(e, g_e + \bar{y}_e^*))d\hat{\mathcal{W}}(e)\right\|^2 \\ &\leq \frac{2\mathcal{M}_2^2 t_1^q}{q} \int_0^t (t-e)^{q-1} N_{\mathcal{F}} \|\bar{y}_e - \bar{y}_e^*\|_{\mathcal{D}_h}^2 de \\ &\quad + 2\mathcal{M}_2^2 \int_0^t (t-e)^{2q-2} N_{\mathcal{G}} \|\bar{y}_e - \bar{y}_e^*\|_{\mathcal{D}_h}^2 de \\ &\leq \frac{2\mathcal{M}_2^2 t_1^q}{q} \int_0^t (t-e)^{q-1} N_{\mathcal{F}} \varpi^2 \sup_{e \in \mathcal{J}} \mathbb{E}\|y(e) - y^*(e)\|^2 de \\ &\quad + 2\mathcal{M}_2^2 \int_0^t (t-e)^{2q-2} N_{\mathcal{G}} \varpi^2 \sup_{e \in \mathcal{J}} \mathbb{E}\|y(e) - y^*(e)\|^2 de \\ &\leq 2\mathcal{M}_2^2 \varpi^2 \left( \frac{N_{\mathcal{F}} t_1^{2q}}{q^2} + \frac{N_{\mathcal{G}} t_1^{2q-1}}{2q-1} \right) \|y - y^*\|_{\mathcal{D}_T^0}^2. \end{aligned}$$

Hence,

$$\mathbb{E}\|(\Psi y)(t)-(\Psi y^*)(t)\|^2 \leq 2\mathcal{M}_2^2\varpi^2\left(\frac{N_{\mathcal{F}}t_1^{2q}}{q^2}+\frac{N_{\mathcal{G}}t_1^{2q-1}}{2q-1}\right)\|y-y^*\|^2_{\mathcal{D}_T^0}. \tag{4}$$

For $t\in(t_i,s_i]$, $i=1,2,\ldots,m$, we have

$$\begin{aligned}\mathbb{E}\|(\Psi y)(t)-(\Psi y^*)(t)\|^2 &\leq \mathbb{E}\|\mathcal{K}_i(t,g_t+\bar{y}_t)-\mathcal{K}_i(t,g_t+\bar{y}_t^*)\|^2\\ &\leq L_{\mathcal{K}_i}\|\bar{y}_t-\bar{y}_t^*\|^2_{\mathcal{D}_h}\\ &\leq L_{\mathcal{K}_i}\varpi^2\sup_{t\in\mathcal{J}}\mathbb{E}\|y(t)-y^*(t)\|^2\\ &\leq L_{\mathcal{K}_i}\varpi^2\|y-y^*\|^2_{\mathcal{D}_T^0}.\end{aligned}$$

Hence,

$$\mathbb{E}\|(\Psi y)(t)-(\Psi y^*)(t)\|^2 \leq L_{\mathcal{K}_i}\varpi^2\|y-y^*\|^2_{\mathcal{D}_T^0}. \tag{5}$$

Similarly, for $t\in(s_i,t_{i+1}]$, $i=1,2,\ldots,m$, we have

$$\begin{aligned}\mathbb{E}\|(\Psi y)(t)-(\Psi y^*)(t)\|^2 &\leq 3\mathbb{E}\|\mathcal{T}_q(t-s_i)(\mathcal{K}_i(s_i,g_{s_i}+\bar{y}_{s_i})-\mathcal{K}_i(s_i,g_{s_i}+\bar{y}_{s_i}^*))\|^2\\ &\quad+3\mathbb{E}\left\|\int_{s_i}^t\mathcal{S}_q(t-e)(\mathcal{F}(e,g_e+\bar{y}_e)-\mathcal{F}(e,g_e+\bar{y}_e^*))de\right\|^2\\ &\quad+3\mathbb{E}\left\|\int_{s_i}^t\mathcal{S}_q(t-e)(\mathcal{G}(e,g_e+\bar{y}_e)-\mathcal{G}(e,g_e+\bar{y}_e^*))d\hat{\mathcal{W}}(e)\right\|^2\\ &\leq 3\mathcal{M}_1^2L_{\mathcal{K}_i}\varpi^2\|y-y^*\|^2_{\mathcal{D}_T^0}\\ &\quad+\frac{3\mathcal{M}_2^2t_{i+1}^q}{q}\int_{s_i}^t(t-e)^{q-1}N_{\mathcal{F}}\|\bar{y}_e-\bar{y}_e^*\|^2_{\mathcal{D}_h}de\\ &\quad+3\mathcal{M}_2^2\int_{s_i}^t(t-e)^{2q-2}N_{\mathcal{G}}\|\bar{y}_e-\bar{y}_e^*\|^2_{\mathcal{D}_h}de\\ &\leq 3\mathcal{M}_1^2L_{\mathcal{K}_i}\varpi^2\|y-y^*\|^2_{\mathcal{D}_T^0}\\ &\quad+\frac{3\mathcal{M}_2^2t_{i+1}^q}{q}\int_{s_i}^t(t-e)^{q-1}N_{\mathcal{F}}\varpi^2\sup_{e\in\mathcal{J}}\mathbb{E}\|y(e)-y^*(e)\|^2de\\ &\quad+3\mathcal{M}_2^2\int_{s_i}^t(t-e)^{2(q-1)}N_{\mathcal{G}}\varpi^2\sup_{e\in\mathcal{J}}\mathbb{E}\|y(e)-y^*(e)\|^2de\\ &\leq \left(3\mathcal{M}_1^2L_{\mathcal{K}_i}\varpi^2+3\mathcal{M}_2^2\varpi^2\left\{\frac{N_{\mathcal{F}}t_{i+1}^{2q}}{q^2}+\frac{N_{\mathcal{G}}t_{i+1}^{2q-1}}{2q-1}\right\}\right)\|y-y^*\|^2_{\mathcal{D}_T^0}.\end{aligned}$$

Hence,

$$\mathbb{E}\|(\Psi y)(t)-(\Psi y^*)(t)\|^2 \leq \left(3\mathcal{M}_1^2L_{\mathcal{K}_i}\varpi^2+3\mathcal{M}_2^2\varpi^2\left\{\frac{N_{\mathcal{F}}t_{i+1}^{2q}}{q^2}+\frac{N_{\mathcal{G}}t_{i+1}^{2q-1}}{2q-1}\right\}\right)\|y-y^*\|^2_{\mathcal{D}_T^0}. \tag{6}$$

From Equations (4)–(6), we obtain that

$$\mathbb{E}\|\Psi y-\Psi y^*\|^2_{\mathcal{D}_T^0}\leq L_{\mathcal{R}}\|y-y^*\|^2_{\mathcal{D}_T^0},$$

which implies that $\Psi$ is a contraction. Hence, $\Psi$ has a unique fixed point $y\in\mathcal{D}_T^0$, which is a mild solution of problem (1) on $(-\infty,T]$. □

Next, using Krasnoselskii's fixed point theorem, we establish the second existence result. At this stage we make the following assumptions.

**Hypothesis 6 (H6).** *The map $\mathcal{F} : \mathcal{J} \times \mathcal{D}_h \to \mathcal{Z}$ is a continuous function, and there exists a continuous function $\xi_1 : \mathcal{J} \to (0, \infty)$ such that*

$$\mathbb{E}\|\mathcal{F}(t,\psi)\|^2 \leq \xi_1(t)\ \|\psi\|^2_{\mathcal{D}_h},$$

*for all $t \in \mathcal{J}$, and $\xi_1^* = \sup_{t\in\mathcal{J}} \xi_1(t)$.*

**Hypothesis 7 (H7).** *The map $\mathcal{G} : \mathcal{J} \times \mathcal{D}_h \to \mathcal{L}_2^1(\mathcal{Y}_1, \mathcal{Z})$ is a continuous function, and there exists a continuous function $\xi_2 : \mathcal{J} \to (0, \infty)$ such that*

$$\mathbb{E}\|\mathcal{G}(t,\psi)\|^2_{\mathcal{L}_2^1} \leq \xi_2(t)\ \|\psi\|^2_{\mathcal{D}_h},$$

*for all $t \in \mathcal{J}$ and $\xi_2^* = \sup_{t\in\mathcal{J}} \xi_2(t)$.*

**Hypothesis 8 (H8).** *The inequality*

$$L_{\mathcal{HR}} = 2\mathcal{M}_2^2\varpi^2\left(\frac{N_{\mathcal{F}}T^{2q}}{q^2} + \frac{N_{\mathcal{G}}T^{2q-1}}{2q-1}\right) < 1$$

*holds and*

$$\max_{1\leq i\leq m}\{\kappa_0, \upsilon_i\lambda_3, \kappa_i\} < \pi,$$

*where*

$$\begin{aligned}
\kappa_0 &= 3\mathcal{M}_2^2 t_1^{2q}\left(\frac{\lambda_1}{q^2} + \frac{\lambda_2}{t_1(2q-1)} + \frac{2\hat{\mathcal{H}}\Lambda_\sigma t_1^{2\hat{\mathcal{H}}-2}}{2q-1}\right),\\
\kappa_i &= 4\mathcal{M}_1^2\upsilon_i\lambda_3 + 4\mathcal{M}_2^2 t_{i+1}^{2q}\left(\frac{\lambda_1}{q^2} + \frac{\lambda_2}{t_{i+1}(2q-1)} + \frac{2\hat{\mathcal{H}}\Lambda_\sigma t_{i+1}^{2\hat{\mathcal{H}}-2}}{2q-1}\right).
\end{aligned}$$

**Hypothesis 9 (H9).** *The maps $\mathcal{K}_i : (t_i, s_i] \times \mathcal{D}_h \to \mathcal{Z}$, $i = 1, 2, \ldots, m$, are continuous functions and*

*i. there exist constants $\upsilon_i > 0$, $i = 1, 2, \ldots, m$, such that $\mathbb{E}\|\mathcal{K}_i(t,\psi)\|^2 \leq \upsilon_i\ \|\psi\|^2_{\mathcal{D}_h}$ for all $t \in \mathcal{J}$;*

*ii. the set $\{b_i : b_i \in V(\pi, \mathcal{K}_i)\}$ is an equicontinuous subset of $C((t_i, s_i], \mathcal{Z})$, $i = 1, 2, \ldots, m$, where $V(\pi, \mathcal{K}_i) = \{t \to \mathcal{K}_i(t, y_t) : y \in \mathcal{D}_\pi\}$.*

The set $\mathcal{D}_r = \{y \in \mathcal{D}_T^0 : \|y\|^2_{\mathcal{D}_T^0} \leq r, r > 0\}$ is clearly a convex closed bounded set in $\mathcal{D}_T^0$ for each $y \in \mathcal{D}_r$. By Lemma 3, we obtain

$$\begin{aligned}
\|x_t + \bar{y}_t\|^2_{\mathcal{D}_h} &\leq 2(\|x_t\|^2_{\mathcal{D}_h} + \|\bar{y}_t\|^2_{\mathcal{D}_h})\\
&\leq 4\Big(\varpi^2 \sup_{\nu\in[0,t]} \mathbb{E}\|x(\nu)\|^2 + \|x_0\|^2_{\mathcal{D}_h}\Big) + 4\Big(\varpi^2 \sup_{\nu\in[0,t]} \mathbb{E}\|\bar{y}(\nu)\|^2 + \|\bar{y}_0\|^2_{\mathcal{D}_h}\Big)\\
&\leq 8(\|\phi\|^2_{\mathcal{D}_h} + \varpi^2 r).
\end{aligned}$$

Let

$$\lambda_1 = 8\xi_1^*(\|\phi\|^2_{\mathcal{D}_h} + \varpi^2 r),\ \ \lambda_2 = 8\xi_2^*(\|\phi\|^2_{\mathcal{D}_h} + \varpi^2 r),\ \ \lambda_3 = 8(\|\phi\|^2_{\mathcal{D}_h} + \varpi^2 r).$$

**Theorem 2.** *Assume conditions (H1)–(H9) are satisfied. Then, problem (1) has at least one mild solution on $(-\infty, T]$.*

**Proof.** Let $\mathcal{E}_1 : \mathcal{D}_r \to \mathcal{D}_r$ and $\mathcal{E}_2 : \mathcal{D}_r \to \mathcal{D}_r$ be defined as

$$\mathcal{E}_1(y)(t) = \begin{cases} 0 & t \in [0, t_1] \\ \mathcal{K}_i(t, g_t + \bar{y}_t), & t \in (t_i, s_i] \\ \mathcal{T}_q(t - s_i)\mathcal{K}_i(s_i, g_{s_i} + \bar{y}_{s_i}) & t \in (s_i, t_{i+1}] \end{cases}$$

and

$$\mathcal{E}_2(y)(t) = \begin{cases} \int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \bar{y}_e)de \\ + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \bar{y}_e)d\hat{\mathcal{W}}(e) + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), & t \in [0, t_1] \\ 0, & t \in (t_i, s_i] \\ \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \bar{y}_e)de \\ + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \bar{y}_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e) & t \in (s_i, t_{i+1}]. \end{cases}$$

For convenience, we divide the proof into various steps.

Step 1. We show that $\mathcal{E}_1 y + \mathcal{E}_2 y^* \in \mathcal{D}_r$. For $y, y^* \in \mathcal{D}_r$ and for $t \in [0, t_1]$, we obtain

$$\begin{aligned} \mathbb{E}\|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 \leq\ & 3\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \bar{y}^*_e)de\right\|^2 \\ & + 3\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e, g_e + \bar{y}^*_e)d\hat{\mathcal{W}}(e)\right\|^2 \\ & + 3\mathbb{E}\left\|\int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e)\right\|^2 \\ \leq\ & 3\mathcal{M}_2^2\left(\int_0^t (t-e)^{q-1}de\right)\left(\int_0^t (t-e)^{q-1}\xi_1(e)\|g_e + \bar{y}^*_e\|^2_{\mathcal{D}_h}de\right) \\ & + 3\mathcal{M}_2^2 \int_0^t (t-e)^{2q-2}\xi_2(e)\|g_e + \bar{y}^*_e\|^2_{\mathcal{D}_h}de \\ & + 6\hat{\mathcal{H}}\Lambda_\sigma \mathcal{M}_2^2 t_1^{2\hat{\mathcal{H}}-1}\int_0^t (t-e)^{2q-2}de \\ \leq\ & 3\mathcal{M}_2^2 t_1^{2q}\left(\frac{\lambda_1}{q^2} + \frac{\lambda_2}{t_1(2q-1)} + \frac{2\hat{\mathcal{H}}\Lambda_\sigma t_1^{2\hat{\mathcal{H}}-2}}{2q-1}\right). \end{aligned}$$

Hence,

$$\mathbb{E}\|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 \leq 3\mathcal{M}_2^2 t_1^{2q}\left(\frac{\lambda_1}{q^2} + \frac{\lambda_2}{t_1(2q-1)} + \frac{2\hat{\mathcal{H}}\Lambda_\sigma t_1^{2\hat{\mathcal{H}}-2}}{2q-1}\right). \quad (7)$$

For $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \mathbb{E}\|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 &\leq \mathbb{E}\|\mathcal{K}_i(t, g_t + \bar{y}_t)\|^2 \\ &\leq v_i \|g_t + \bar{y}_t\|^2_{\mathcal{D}_h} \\ &\leq v_i \lambda_3. \end{aligned}$$

Hence,

$$\mathbb{E}\|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 \leq v_i \lambda_3. \quad (8)$$

Similarly, for $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} \mathbb{E}\|(\mathcal{E}_1 y)(t) + (\mathcal{E}_2 y^*)(t)\|^2 \leq\ & 4\mathbb{E}\|\mathcal{T}_q(t - s_i)\mathcal{K}_i(s_i, g_{s_i} + \bar{y}_{s_i})\|^2 \\ & + 4\mathbb{E}\left\|\int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{F}(e, g_e + \bar{y}^*_e)de\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&+4\mathbb{E}\left\|\int_{s_i}^{t}\mathcal{S}_q(t-e)\mathcal{G}(e,g_e+\bar{y}_e^*)d\hat{\mathcal{W}}(e)\right\|^2\\
&+4\mathbb{E}\left\|\int_{s_i}^{t}\mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e)\right\|^2\\
\leq\;& 4\mathcal{M}_1^2 v_i\lambda_3\\
&+4\mathcal{M}_2^2\left(\int_{s_i}^{t}(t-e)^{q-1}de\right)\left(\int_{s_i}^{t}(t-e)^{q-1}\xi_1(e)\|g_e+\bar{y}_e^*\|^2_{\mathcal{D}_h}de\right)\\
&+4\mathcal{M}_2^2\int_{s_i}^{t}(t-e)^{2q-2}\xi_2(e)\|g_e+\bar{y}_e^*\|^2_{\mathcal{D}_h}de\\
&+8\hat{\mathcal{H}}\Lambda_\sigma\mathcal{M}_2^2 t_{i+1}^{2\hat{\mathcal{H}}-1}\int_{s_i}^{t}(t-e)^{2q-2}de\\
\leq\;& 4\mathcal{M}_1^2 v_i\lambda_3+4\mathcal{M}_2^2 t_{i+1}^{2q}\left(\frac{\lambda_1}{q^2}+\frac{\lambda_2}{t_{i+1}(2q-1)}+\frac{2\hat{\mathcal{H}}\Lambda_\sigma t_{i+1}^{2\hat{\mathcal{H}}-2}}{2q-1}\right).
\end{aligned}
$$

Therefore,

$$
\mathbb{E}\|(\mathcal{E}_1 y)(t)+(\mathcal{E}_2 y^*)(t)\|^2 \leq 4\mathcal{M}_1^2 v_i\lambda_3+4\mathcal{M}_2^2 t_{i+1}^{2q}\left(\frac{\lambda_1}{q^2}+\frac{\lambda_2}{t_{i+1}(2q-1)}+\frac{2\hat{\mathcal{H}}\Lambda_\sigma t_{i+1}^{2\hat{\mathcal{H}}-2}}{2q-1}\right). \tag{9}
$$

Equations (7)–(9) imply that

$$
\|\mathcal{E}_1 y+\mathcal{E}_2 y^*\|^2_{\mathcal{D}_T^0} \leq r.
$$

Thus, $\mathcal{E}_1 y+\mathcal{E}_2 y^* \in \mathcal{D}_r$.

Step 2. We show that the operator $\mathcal{E}_1$ is continuous on $\mathcal{D}_r$. Let $\{y^n\}_{n=1}^{\infty}$ be a sequence such that $y^n \to y$ in $\mathcal{D}_r$. For all $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$, we have

$$
\mathbb{E}\|(\mathcal{E}_1 y^n)(t)-(\mathcal{E}_1 y)(t)\|^2 \leq \mathbb{E}\|\mathcal{K}_i(t,g_t+\bar{y^n}_t)-\mathcal{K}_i(t,g_t+\bar{y}_t)\|^2.
$$

Since the maps $\mathcal{K}_i$, $i = 1, 2, \ldots, m$, are continuous functions, one has

$$
\lim_{n\to\infty}\|\mathcal{E}_1 y^n-\mathcal{E}_1 y\|^2_{\mathcal{D}_T^0}=0. \tag{10}
$$

For all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$
\mathbb{E}\|(\mathcal{E}_1 y^n)(t)-(\mathcal{E}_1 y)(t)\|^2 \leq \mathbb{E}\|\mathcal{T}_q(t-s_i)(\mathcal{K}_i(s_i,g_{s_i}+\bar{y^n}_{s_i})-\mathcal{K}_i(s_i,g_{s_i}+\bar{y}_{s_i}))\|^2.
$$

Therefore,

$$
\lim_{n\to\infty}\|\mathcal{E}_1 y^n-\mathcal{E}_1 y\|^2_{\mathcal{D}_T^0}=0. \tag{11}
$$

Equations (10) and (11) imply that the operator $\mathcal{E}_1$ is continuous on $\mathcal{D}_r$.

Step 3. The operator $\mathcal{E}_1$ maps bounded sets into bounded sets in $\mathcal{D}_r$. Let us show that for $r > 0$ there exists a $r > 0$ such that, for each $y \in \mathcal{D}_r$, we obtain $\mathbb{E}\|\mathcal{E}_1(y)(t)\|^2 \leq r$, for all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$. For all $t \in (s_i, t_{i+1}]$, $i = 1, 2, \ldots, m$, we have

$$
\mathbb{E}\|(\mathcal{E}_1 y)(t)\|^2 \leq \mathbb{E}\|\mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i,g_{s_i}+\bar{y}_{s_i})\|^2 \leq \mathcal{M}_1^2 v_i\lambda_3.
$$

For all $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$, we have

$$
\mathbb{E}\|(\mathcal{E}_1 y)(t)\|^2 \leq \mathbb{E}\|\mathcal{K}_i(t,g_t+\bar{y}_t)\|^2 \leq v_i\lambda_3.
$$

From the above equations, we obtain

$$\|\mathcal{E}_1 y\|^2_{\mathcal{D}_T^0} \quad \leq \quad r,$$

where $r = \max\{\mathcal{M}_1^2 v_i \lambda_3, v_i \lambda_3\}$. Hence, the operator $\mathcal{E}_1$ maps bounded sets into bounded sets in $\mathcal{D}_r$.

Step 4. The operator $\mathcal{E}_1$ is equicontinuous. For all $\Delta_1, \Delta_2 \in (t_i, s_i]$, $\Delta_1 < \Delta_2$, and $y \in \mathcal{D}_r$, we obtain

$$\mathbb{E}\|(\mathcal{E}_1 y)(\Delta_2) - (\mathcal{E}_1 y)(\Delta_1)\|^2 \leq \mathbb{E}\|\mathcal{K}_i(\Delta_2, g_{\Delta_2} + \bar{y}_{\Delta_2}) - \mathcal{K}_i(\Delta_1, g_{\Delta_1} + \bar{y}_{\Delta_1})\|^2. \tag{12}$$

For all $\Delta_1, \Delta_2 \in (s_i, t_{i+1}]$, $\Delta_1 < \Delta_2$, and $y \in \mathcal{D}_r$, we obtain

$$\mathbb{E}\|(\mathcal{E}_1 y)(\Delta_2) - (\mathcal{E}_1 y)(\Delta_1)\|^2 \quad \leq \quad \mathbb{E}\|(\mathcal{T}_q(\Delta_2 - s_i) - \mathcal{T}_q(\Delta_1 - s_i))\mathcal{K}_i(s_i, g_{s_i} + \bar{y}_{s_i})\|^2.$$

Since $\mathcal{T}_q$ is strongly continuous, it allows us to conclude that

$$\lim_{n\to\infty} \|\mathcal{T}_q(\Delta_2 - s_i) - \mathcal{T}_q(\Delta_1 - s_i)\|^2 = 0. \tag{13}$$

Equations (12) and (13) with (H9)(ii) imply that the operator $\mathcal{E}_1$ is equicontinuous on $\mathcal{D}_r$. Finally, combining steps 1–4 together with Ascoli's theorem, we conclude that the operator $\mathcal{E}_1$ is completely continuous.

Step 5. The operator $\mathcal{E}_2$ is a contraction map. For $y, y^* \in \mathcal{D}_r$ and for $t \in (t_i, s_i]$, $i = 1, 2, \ldots, m$, we have

$$\mathbb{E}\|(\mathcal{E}_2 y)(t) - (\mathcal{E}_2 y^*)(t)\|^2 = 0. \tag{14}$$

Similarly, for $y, y^* \in \mathcal{D}_r$ and for $t \in (s_i, t_{i+1}]$, $i = 0, 1, \ldots, m$, we have

$$\begin{aligned}
\mathbb{E}\|(\mathcal{E}_2 y)(t) - (\mathcal{E}_2 y^*)(t)\|^2 \quad &\leq \quad 2\mathbb{E}\left\|\int_{s_i}^t \mathcal{S}_q(t-e)(\mathcal{F}(e, g_e + \bar{y}_e) - \mathcal{F}(e, g_e + \bar{y}^*_e))de\right\|^2 \\
&\quad +2\mathbb{E}\left\|\int_{s_i}^t \mathcal{S}_q(t-e)(\mathcal{G}(e, g_e + \bar{y}_e) - \mathcal{G}(e, g_e + \bar{y}^*_e))d\hat{\mathcal{W}}(e)\right\|^2 \\
&\leq \quad 2\mathcal{M}_2^2 \varpi^2 \left(\frac{N_{\mathcal{F}} T^{2q}}{q^2} + \frac{N_{\mathcal{G}} T^{2q-1}}{2q-1}\right)\|y - y^*\|^2_{\mathcal{D}_T^0}.
\end{aligned}$$

Hence,

$$\mathbb{E}\|(\mathcal{E}_2 y)(t) - (\mathcal{E}_2 y^*)(t)\|^2 \leq 2\mathcal{M}_2^2 \varpi^2 \left(\frac{N_{\mathcal{F}} T^{2q}}{q^2} + \frac{N_{\mathcal{G}} T^{2q-1}}{2q-1}\right)\|y - y^*\|^2_{\mathcal{D}_T^0}. \tag{15}$$

From above, we obtain

$$\|\mathcal{E}_2 y - \mathcal{E}_2 y^*\|^2_{\mathcal{D}_T^0} \leq L_{\mathcal{HR}}\|y - y^*\|^2_{\mathcal{D}_T^0}.$$

Thus, $\mathcal{E}_2$ is a contraction map. By Krasnoselskii's fixed point theorem, we obtain that problem (1) has at least one solution on $(-\infty, T]$. □

## 4. Approximate Controllability

We consider the following control system:

$$\begin{cases}
{}^c D_t^q z(t) = \mathcal{P} z(t) + \mathcal{A}\hat{u}(t) + \mathcal{F}(t, z_t) + \mathcal{G}(t, z_t)\dfrac{d\hat{\mathcal{W}}(t)}{dt} + \sigma(t)\dfrac{d\mathcal{B}^{\hat{\mathcal{H}}}(t)}{dt}, \quad t \in \cup_{i=0}^m (s_i, t_{i+1}], \\
z(t) = \mathcal{K}_i(t, z_t), \quad t \in \cup_{i=1}^m (t_i, s_i], \\
z(t) = \phi(t), \quad \phi(t) \in \mathcal{D}_h.
\end{cases} \tag{16}$$

The control $\hat{u}(\cdot) \in L^2(\mathcal{J},\mathcal{U})$, where $L^2(\mathcal{J},\mathcal{U})$ is the Hilbert space of all admissible control functions. The operator $\mathcal{A}$ is linear and bounded from the separable Hilbert space $\mathcal{U}$ into $\mathcal{Z}$. Assume that the linear system

$$\begin{cases} {}^cD_t^q z(t) = \mathcal{P}z(t) + \mathcal{A}\hat{u}(t), & t \in [0,T], \\ z(t) = \phi(t), & \phi(t) \in \mathcal{D}_h. \end{cases} \tag{17}$$

Define the operator $t_{s_i}^{t_{i+1}}$ associated with system of (17) as

$$t_{s_i}^{t_{i+1}} = \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\mathcal{A}\mathcal{A}^*\mathcal{S}_q^*(t_{i+1}-e)de.$$

Here, $\mathcal{A}^*$ and $\mathcal{S}_q^*(t)$ are the adjoint of $\mathcal{A}$ and $\mathcal{S}_q(t)$, respectively. The operator $t_{s_i}^{t_{i+1}}$ is a bounded and linear operator.

**Definition 3.** *System (16) is approximately controllable on $[0,T]$ if $\overline{\mathcal{R}(T,\phi,\hat{u})} = L^2(\mathcal{F}_T,\mathcal{Z})$, where $\mathcal{R}(T,\phi,\hat{u}) = \{z(\phi,\hat{u})(T) : z$ is the solution of problem (16) and $\hat{u} \in L^2(\mathcal{J},\mathcal{U})\}$.*

The following assumption is needed.

[AC]: System (17) is approximate controllability on $\mathcal{J}$.

Note that system (17) is approximately controllable on $\mathcal{J}$ only if

$$\Delta(\Lambda, t_{s_i}^{t_{i+1}}) = (\Lambda I + t_{s_i}^{t_{i+1}})^{-1} \to 0 \text{ as } \Lambda \to 0. \tag{18}$$

**Definition 4.** *An $\mathcal{F}_t$-adapted random process $z : (-\infty, T] \to \mathcal{Z}$ is called the mild solution of (16) if for every $t \in \mathcal{J}$, $z(t)$ satisfies $z_0 = \phi \in \mathcal{D}_h$, $z(t) = \mathcal{K}_i(t,z_t)$ for all $t \in (t_i,s_i]$, $i = 1,2,\ldots,m$, and*

$$\begin{aligned} z(t) &= \int_0^t \mathcal{S}_q(t-e)[\mathcal{F}(e,z_e) + \mathcal{A}\hat{u}(e)]de \\ &\quad + \int_0^t \mathcal{S}_q(t-e)\mathcal{G}(e,z_e)d\hat{\mathcal{W}}(e) + \int_0^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \end{aligned}$$

*for all $t \in [0,t_1]$, and*

$$\begin{aligned} z(t) &= \mathcal{T}_q(t-s_i)\mathcal{K}_i(s_i,z_{s_i}) + \int_{s_i}^t \mathcal{S}_q(t-e)[\mathcal{F}(e,z_e) + \mathcal{A}\hat{u}(e)]de \\ &\quad + \int_{s_i}^t \mathcal{S}_q(t-e)\mathcal{G}(e,z_e)d\hat{\mathcal{W}}(e) + \int_{s_i}^t \mathcal{S}_q(t-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \end{aligned} \tag{19}$$

*for all $t \in (s_i,t_{i+1}]$, $i = 1,2,\ldots,m$.*

**Lemma 5.** *For any $z_{t_{i+1}} \in L^2(\mathcal{F}_T,\mathcal{Z})$, there exist $\phi_1 \in L^2(\Omega, L^2([s_i,t_{i+1}], \mathcal{L}_2^1(\mathcal{Y}_1,\mathcal{Z}))$ and $\phi_2 \in L^2([s_i,t_{i+1}], \mathcal{L}_2^2(\mathcal{Y}_2,\mathcal{Z}))$ such that*

$$z_{t_{i+1}} = \mathbb{E}z_{t_{i+1}} + \int_{s_i}^{t_{i+1}} \phi_1(e)d\hat{\mathcal{W}}(e) + \int_{s_i}^{t_{i+1}} \phi_2(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e).$$

Next, we choose the control $\hat{u}^\Lambda(t)$ as follows:

$$\hat{u}^\Lambda(t) = \mathcal{A}^*\mathcal{S}_q^*(t_{i+1}-t)\Delta(\Lambda, t_{s_i}^{t_{i+1}})p(z(\cdot)), \tag{20}$$

where

$$p(z(\cdot)) = z_{t_{i+1}} - \mathcal{T}_q(t_{i+1}-s_i)\mathcal{K}_i(s_i,z_{s_i}) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\mathcal{F}(e,z_e)de$$

$$-\int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\mathcal{G}(e,z_e)d\hat{\mathcal{W}}(e) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e),$$
$$\forall\, t \in (s_i,t_{i+1}], \quad i=0,1,\ldots,m,$$

and $\mathcal{K}_0(0,\cdot)=0$, $z(t_{m+1})=z_{t_{m+1}}=z_T$.

**Theorem 3.** *Assume the hypotheses (H1)–(H9) are satisfied. Then, the problem (16) has at least one mild solution on* $(-\infty,T]$.

**Proof.** The proof is a consequence of Theorem 2. □

**Theorem 4.** *Assume that the hypotheses (H1)–(H9) and [AC] are satisfied. Then functions* $\mathcal{F}$ *and* $\mathcal{G}$ *are uniformly bounded on their respective domains. Moreover, the system (16) is approximately controllable on* $[0,T]$.

**Proof.** Let $z^{\Lambda}$ be a fixed point of $\mathcal{E}_1+\mathcal{E}_2$. Using Fubini's theorem, we get

$$z^{\Lambda}(t_{i+1}) = z_{t_{i+1}} - \Lambda\Delta(\Lambda, t_{s_i}^{t_{i+1}})p(z^{\Lambda}(\cdot)), \tag{21}$$

where

$$\begin{aligned} p(z^{\Lambda}(\cdot)) &= z_{t_{i+1}} - \mathcal{T}_q(t_{i+1}-s_i)\mathcal{K}_i(s_i,z_{s_i}^{\Lambda}) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\mathcal{F}(e,z_e^{\Lambda})de \\ &\quad - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\mathcal{G}(e,z_e^{\Lambda})d\hat{\mathcal{W}}(e) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \\ &\quad \forall\, t \in (s_i,t_{i+1}],\ i=0,1,\ldots,m. \end{aligned}$$

The functions $\mathcal{F}$ and $\mathcal{G}$ are uniformly bounded. Hence, there exists a subsequence, still represented by $\mathcal{F}(e,z_e^{\Lambda})$ and $\mathcal{G}(e,z_e^{\Lambda})$, that weakly converge to, say, $\mathcal{F}(e)$ and $\mathcal{G}(e)$ in $\mathcal{Z}$ and $\mathcal{L}_2^1(\mathcal{Y}_1,\mathcal{Z})$, respectively. Let us define

$$\begin{aligned} \eta &= z_{t_{i+1}} - \mathcal{T}_q(t_{i+1}-s_i)\mathcal{K}_i(s_i,z_{s_i}) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\mathcal{F}(e)de \\ &\quad - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\mathcal{G}(e)d\hat{\mathcal{W}}(e) - \int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)\sigma(e)d\mathcal{B}^{\hat{\mathcal{H}}}(e), \\ &\quad \forall\, t \in (s_i,t_{i+1}],\ i=0,1,\ldots,m. \end{aligned}$$

For $t\in(s_i,t_{i+1}]$, $i=0,1,\ldots,m$, we have

$$\begin{aligned} \mathbb{E}\|p(z^{\Lambda})-\eta\|^2 &\leq 3\mathbb{E}\|\mathcal{T}_q(t_{i+1}-s_i)(\mathcal{K}_i(s_i,z_{s_i}^{\Lambda})-\mathcal{K}_i(s_i,z_{s_i}))\|^2 \\ &\quad +3\mathbb{E}\left\|\int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)(\mathcal{F}(e,z_e^{\Lambda})-\mathcal{F}(e))de\right\|^2 \\ &\quad +3\mathbb{E}\left\|\int_{s_i}^{t_{i+1}} \mathcal{S}_q(t_{i+1}-e)(\mathcal{G}(e,z_e^{\Lambda})-\mathcal{G}(e))d\hat{\mathcal{W}}(e)\right\|^2. \end{aligned}$$

By the infinite dimensional version of the Arzela–Ascoli theorem, we obtain that

$$\bar{k}(\cdot) \to \int_{\cdot}^{\cdot} \mathcal{S}_q(\cdot-e)\bar{k}(e)de$$

is a compact operator. For all $t\in[0,T]$,

$$\mathbb{E}\|p(z^{\Lambda})-\eta\|^2 \to 0 \text{ as } \Lambda\to 0^+. \tag{22}$$

By Equation (21), we get

$$\mathbb{E}\|z^{\Lambda}(t_{i+1})-z_{t_{i+1}}\|^2 \leq \mathbb{E}\|\Lambda\Delta(\Lambda,t_{s_i}^{t_{i+1}})(\eta)\|^2+\mathbb{E}\|\Lambda\Delta(\Lambda,t_{s_i}^{t_{i+1}})\|^2\mathbb{E}\|p(z^{\Lambda})-\eta\|^2.$$

By (18) and (22), we get

$$\mathbb{E}\|z^{\Lambda}(t_{i+1})-z_{t_{i+1}}\|^2 \to 0 \text{ as } \theta\to 0^+.$$

Thus, the system (16) is approximate controllable on the interval $[0,T]$. □

## 5. Example

We consider the following fractional stochastic control system:

$$\begin{cases} {}^cD_t^q y(t,z) = \dfrac{\partial^2}{\partial z^2}y(t,z)+\Theta(t,z)+\int_{-\infty}^t e^{4(r-t)}y(r,z)dr \\ \qquad +\int_{-\infty}^t e^{6(r-t)}y(r,z)dr\dfrac{d\hat{\mathcal{W}}(t)}{dt}+P(t)\dfrac{d\mathcal{B}^{\hat{\mathcal{H}}}(t)}{dt}, \\ \qquad y\in(0,\pi),\ t\in[2i,2i+1],\ i=0,1,\dots,m, \\ y(t,z)=\int_{-\infty}^t G_i(r-t)y(r,z)dr,\ t\in(2i-1,2i],\ i=1,2,\dots,m, \\ y(t,0)=\ 0\ =y(t,\pi), \\ y(t,z)=\phi(t,z),\ t\in(-\infty,0], \end{cases} \tag{23}$$

where ${}^cD_t^q$ is the Caputo derivative of order $1/2<q<1$, $0=s_0=t_0<t_1<s_1<t_2<\dots<t_m<s_m<t_{m+1}=T<\infty$ with $s_i=2i,\ t_i=2i-1$.

Let $\mathcal{Z}=L^2([0,\pi])$ and the operator $\mathcal{P}$ be defined by

$$\mathcal{P}w=w''\ ,\quad \mathcal{D}(\mathcal{P})=H^2(0,\pi)\cap H_0^1(0,\pi).$$

Clearly, $\mathcal{P}$ is the generator of an analytic semigroup $\{\mathcal{S}(t):t\geqslant 0\}$. The spectral representation of $\mathcal{S}(t)$ is given by

$$\mathcal{S}(t)w=\sum_{n\in\mathbb{N}}e^{-n^2t}\langle w,w_n\rangle w_n,$$

where

$$w_n(y)=\sqrt{2/\pi}\sin(ny),\ n\in\mathbb{N},$$

is the orthogonal set of eigenvectors corresponding to the eigenvalue $\lambda_n=-n^2$ of $\mathcal{P}$. The semigroup $\{\mathcal{S}(t):t\geq 0\}$ is compact and uniformly bounded, so that $\mathcal{R}(\lambda,\mathcal{P})=(\lambda I-\mathcal{P})^{-1}$ is a compact operator for all $\lambda\in\rho(\mathcal{P})$, i.e., $\mathcal{P}\in\mathcal{P}^q(\theta_0,\omega_0)$. Let $h(e)=e^{2e}$, $e<0$. Then $\varpi=\int_{-\infty}^0 h(e)de=1/2$ and we define

$$\|\phi\|_{\mathcal{D}_h}=\int_{-\infty}^0 h(e)\sup_{e\leq\theta\leq 0}(\mathbb{E}|\phi(\theta)|^2)^{1/2}de,\quad \phi\in\mathcal{D}_h.$$

Hence, $(t,\phi)\in[0,T]\times\mathcal{D}_h$. The bounded linear operator $\mathcal{A}$ is defined by $\mathcal{A}\hat{u}(t)(z)=\Theta(t,z)$.

Define the functions $\mathcal{F}:\mathcal{J}\times\mathcal{D}_h\to\mathcal{Z}$, $\mathcal{G}:\mathcal{J}\times\mathcal{D}_h\to L_2(\mathcal{Y}_1,\mathcal{Z})$, and $\mathcal{K}_i:(t_i,s_i]\times\mathcal{D}_h\to\mathcal{Z}$ as

$$\begin{aligned} \mathcal{F}(t,\phi)(z) &= \int_{-\infty}^0 e^{4\theta}(\phi(\theta)(z))d\theta, \\ \mathcal{G}(t,\phi)(z) &= \int_{-\infty}^0 e^{6\theta}(\phi(\theta)(z))d\theta, \\ \mathcal{K}_i(t,\phi)(z) &= \int_{-\infty}^0 G(\theta)(\phi(\theta)(z))d\theta. \end{aligned}$$

Assume that

$$\int_0^T \|\sigma(e)\|^2_{\mathcal{L}_2^2} de < \infty.$$

The system (23) can be written as an abstract formulation of (1), and thus previous theorems can be applied to guarantee both existence and approximate controllability results.

## 6. Conclusions

We have investigated impulsive fractional stochastic control systems defined on separable Hilbert spaces. The proposed problem is driven by mixed noise, i.e., it involves both a $Q$-Wiener process and a $Q$-fractional Brownian motion with the Hurst parameter $\hat{\mathcal{H}} \in (1/2, 1)$. For our results, we have mainly applied fixed point techniques, a $q$-resolvent family, and fractional calculus. The obtained results are supported by an illustrative example. As further directions of investigation and continuation to this work, it would be interesting to investigate the sensitivity on the noise range and develop numerical and computational methods to approximate the solution. We also intend to extend our results via discrete fractional calculus.

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