**Universidade de Aveiro** Departamento de Matemática
**2009**

**Nelson José Rodrigues Faustino**

# Análise de Clifford Discreta

# Discrete Clifford Analysis

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# Discrete Clifford Analysis

Dissertação apresentada à Universidade de Aveiro para cumprimento dos requisitos necessários à obtenção do grau de Doutor em Matemática, realizada sob a orientação científica de Uwe Kähler, Professor Auxiliar com Agregação do Departamento de Matemática da Universidade de Aveiro.

Dissertation submitted to the University of Aveiro in fulfilment of the requirements for the degree of *Doctor of Philosophy* in Mathematics, under the supervision of Uwe Kähler, Assistant Professor with Habilitation at the Department of Mathematics of the University of Aveiro.

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Dedico este trabalho à minha prima Vânia Alexandre (05/09/1985-10/11/2007).

**o júri**

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**palavras-chave**

Fórmula de Cauchy, Chains, Cochains, Mapa De Rham, Funções Monogénicas discretas, Decomposição Fischer, Operator Entrelaçamento, derivada Pincherle, Invariantes segundo Translações, sequências Sheffer, Teorema Stokes, Mapa de Whitney.

**resumo**

Esta tese estuda os fundamentos de uma teoria discreta de funções em dimensões superiores usando a linguagem das Álgebras de Clifford. Esta abordagem combina as ideias do Cálculo Umbral e Formas Diferenciais. O potencial desta abordagem assenta essencialmente da osmose entre ambas as linguagens. Isto permitiu a construção de operadores de entrelaçamento entre estruturas contínuas e discretas, transferindo resultados conhecidos do contínuo para o discreto.

Adicionalmente, isto resultou numa transcrição mimética de bases de polinómios, funções geradoras, Decomposição de Fischer, Lema de Poincaré, Teorema de Stokes, fórmula de Cauchy e fórmula de Borel-Pompeiu.

Esta teoria também inclui a descrição dos homólogos discretos de formas diferenciais, campos vectores e integração discreta. De facto, a construção resultante de formas diferenciais, campos vectores e integração discreta em termos de coordenadas baricêntricas conduz à correspondência entre a teoria de Diferenças Finitas e a teoria de Elementos Finitos, dando um núcleo de aplicações desta abordagem promissora em análise numérica. Algumas ideias preliminares deste ponto de vista foram apresentadas nesta tese.

Também foram apresentados resultados preliminares na teoria discreta de funções em complexos envolvendo simplexes. Algumas ligações com Combinatória e Mecânica Quântica foram também apresentadas ao longo desta tese.

**keywords**

Cauchy’s formula, Chains, Cochains, De Rham map, Discrete Monogenic functions, Fischer Decomposition, Intertwining Operator, Pincherle derivative, Shift-Invariant operators, Sheffer sequences, Stokes theorem, Whitney Map.

**abstract**

This thesis studies the fundamentals of a higher dimensional discrete function theory using the Clifford Algebra setting. This approach combines the ideas of Umbral Calculus and Differential Forms. Its powerful rests mostly on the interplay between both languages. This allowed the construction of intertwining operators between continuous and discrete structures, lifting the well known results from continuum to discrete.

Furthermore, this resulted in a mimetic transcription of basis polynomial, generating functions, Fischer Decomposition, Poincaré and dual-Poincaré lemmata, Stokes theorem and Cauchy’s formula.

This theory also includes the description discrete counterparts of differential forms, vector-fields and discrete integration. Indeed the resulted construction of discrete differential forms, discrete vector-fields and discrete integration in terms of barycentric coordinates leads to the correspondence between the theory of Finite Differences and the theory of Finite Elements, which gives a core of promising applications of this approach in numerical analysis. Some preliminary ideas on this point of view were presented in this thesis.

We also developed some preliminary results in the theory of discrete monogenic functions on simplicial complexes. Some connections with Combinatorics and Quantum Mechanics were also presented along this thesis.

# Contents

# Introduction

*"I have practically never started off with any idea of what I'm going to be doing or where it's going to go. I'm interested in mathematics; I talk, I learn, I discuss and then interesting questions simply emerge. I have never started off with a particular goal, except the goal of understanding mathematics."*

Sir Michael Atiyah

Clifford analysis in its minimal form (see for instance [9, 36, 18, 66]) is based on the study of properties underlying the notion of monogenic function, which intends to be the higher dimensional analogue of an holomorphic function on the complex plane.

To this end, considering the ambient space $\mathbb{R}^n$ endowed with a quadratic form $\mathcal{B}(\cdot,\cdot)$ with signature $(0,n)$ is equivalent to taking an orthonormal basis in $\mathbb{R}^n$ $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ governed by the anti-commutative multiplication rules $\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = -2\delta_{jk}$.

The later relations define a Clifford algebra over $\mathbb{R}^n$, say $\mathbb{R}_{0,n}$, whose elements are called Clifford numbers. Then the Clifford variable is a first order Clifford polynomial $x = \sum_{j=1}^n x_j\mathbf{e}_j$ while the vector derivative in the variable $x$, $D = \sum_{j=1}^n \mathbf{e}_j\partial_{x_j}$ is usually called the Dirac operator.

From the Laplacian splitting $\Delta = -D^2$, Clifford analysis may be regarded as a refinement of harmonic analysis. The fundamental group leaving $D$ invariant is the special orthogonal group $\mathrm{SO}(n)$, doubly covered by the spin group $\mathrm{Spin}(0,n)$ showing that the Dirac operator is invariant under the action of rotations. We will refer several times along this thesis to continuous Clifford analysis, as opposed to the discrete Clifford analysis treated along this thesis.

Currently, there are much interest in finding discrete counterparts of various structures of continuous Clifford analysis since finite difference operators are more suitable to computational physics and numerical computations than continuous ones.

To get a first idea let us take a look at several discrete analogues which were developed in the case of classical complex analysis in the last century. Discrete *monodiffric functions* on square lattices, which shall be understood as an analogue of monogenic functions (in the original sense of the word - *fonction monogène*), were introduced by J. Ferrand in [32] and study extensively in [28]. They are based on following discretizations of the Cauchy-Riemann equations

$$f_{m,n+1} - f_{m+1,n} = i(f_{m+1,n+1} - f_{m,n}).$$

They represent a primer definition of discrete analytic functions.

Extension of the above equations to more general structures can also be find for the case of critical maps [47, 56] or in more advanced cases as circle patterns [67, 6].

In particular, according to [56] discrete analytic functions are defined by means of discrete holomorphic forms acting on oriented maps, i.e. graphs embedded in oriented surfaces. They can also be defined as rotation-free circulations which is the same as requiring that a circulation is also a circulation on the dual graph[1].

In [6] a notion of discrete analytic functions for several variables on quad-graphs is studied. For this quasicristallic rhombic embeddings $\mathcal{D}$ with set of labels $\{\pm\alpha_1, \ldots, \pm\alpha_n\}$ are considered. Extending the labeling $\alpha : \vec{E}(\mathcal{D}) \mapsto \mathbb{C}$ to all edges $\mathbb{Z}^n$, assuming that all edges carrying the labels $\alpha_k$, we say that a function $f : \mathbb{Z}^n \mapsto \mathbb{C}$ is discrete holomorphic, if it satisfies, on each elementary square of $\mathbb{Z}^n$, the equations

$$\frac{f(m + \mathbf{v}_j + \mathbf{v}_k) - f(m)}{f(m + \mathbf{v}_j) - f(m + \mathbf{v}_k)} = \frac{\alpha_j + \alpha_k}{\alpha_j - \alpha_k},$$

where $\mathbf{v}_j$ denotes the j$^{th}$ component of the standard $\mathbb{R}^n$ basis, i.e. the standard shifts of $\mathbb{Z}^n$.

Geometrically speaking, in the above statement the quotient of the diagonals of the $f-$image on each elementary quadrilateral $(m, m+\mathbf{v}_k, m+\mathbf{v}_j+\mathbf{v}_k, m+\mathbf{v}_j) \in \mathcal{F}(\mathcal{D})$ is equal to the quotient of diagonals of the corresponding parallelogram.

We have to remark that this construction is limited to so-called brick structures. If one wants to consider more general situations in higher dimensions, the situations gets even worse. In particular, we would like to stress that Dirac operators which only involve forward/backward- differences do not factorize the *star*-Laplacian,

$$(\Delta_h f)(mh) = \sum_{j=1}^{n} \frac{f(mh + h\mathbf{v}_j) + f(mh - h\mathbf{v}_j) - 2f(mh)}{h^2}.$$

However, in the quaternionic case, by mixing forward and backward differences, it is possible to construct difference Dirac operators which do factorize the *star-Laplacian* [39].

[1] a graph which has a vertex for each plane region of $\mathcal{G}$, and an edge for each edge in $\mathcal{G}$ joining two neighboring regions, for a certain embedding of $\mathcal{G}$.

But this is done by a direct construction via $4 \times 4$-matrices, which do not coincide with the matrix representation of quaternions. The only justification given by the authors is that when the mesh width goes to zero we obtain the quaternions asymptotically. It is not clear from the context how this construction can be carried out in higher dimensions. To do this one needs a more general framework.

One way to overcome this problem was proposed by Wilson in [72], by adding an extra term to the lattice version of the Dirac operator of the order of the cut-off. Another proposal was recently obtained by N. Faustino, U. Kähler, and F. Sommen in [31] using the splitting of the standard Clifford generators $\mathbf{e}_j$ in $\mathbf{e}_j = \mathbf{e}_j^- + \mathbf{e}_j^+$ such that Dirac operators on lattices are constructed by using superpositions of the type $\mathbf{e}_j^+ \partial_h^{+j} + \mathbf{e}_j^- \partial_h^{-j}$.

Although both approaches seem to be very promising tools, however, they have some limitations, particularly in relation to the lattice structure. In a certain sense the first requires the usage of second-order operators to define a Dirac operator while the second at first glance seems to be quite artificial.

As we will see in this thesis, while some of these translations from *continuum* to discrete are straightforward, sometimes it is not so simple to find a right formulation. So in certain sense, there seems to be a rather clear dividing line between the study of continuous and discrete Clifford analysis. While continuous Clifford Analysis is a well established theory with applications in diverse areas like Electromagnetics and Signal Processing, discrete Clifford analysis is a theory under construction driven basically by the recent surge of interest in discrete potential theory, numerical analysis and nowadays in combinatorics.

This interest was further reinforced in [39, 38], where K. Gürlebeck and A. Hommel developed finite difference potential methods in lattice domains based on the concept of discrete fundamental solutions for the difference Dirac operator which generalizes the work developed by Ryabenkij in [63]. More sophisticated numerical applications of this theory were presented recently by N. Faustino, K. Gürlebeck, A. Hommel, and U. Kähler in [29] for the non-linear stationary Navier-Stokes equations and P. Cerejeiras, N. Faustino, and N. Vieira in [14] for the non-linear instationary Schrödinger equation.

Some early results on hypercomplex Bernoulli polynomials were obtained by H. Malonek and G. Tomaz in [55] which could be understood as the discrete monogenic analogues of the standard powers obtained *viz* the Cauchy-Kovaleskaya product (see e.g. [65, 15]).

While both theories encode very different structures and use different methods the motivation behind its study is common if we consider which phenomena they study: Symmetry, dispersion, expansion, and other general phenomena have similar formulations in both worlds. From this point of view, the geometric foundation of Dirac operators on lattices in terms of (discrete) differential forms is, in our opinion, useful on the formulation of well-adapted theoretical approaches on lattices, namely the theory of discrete monogenic functions.

Some of examples for this kind of structure are intimately related with noncommutative geometry [16]. It is well known that lattices in physics are commonly used in the regularization process in field theory as well as in the naive approximation of the topology of space or space-time. However since lattices exhibit some noncommutative geometric nature, the topology of the discrete space or space-time may at higher energies differ from that of the *continuum*.

The approach used by A. Dimakis, F. Müller-Hoissen and its collaborators [23, 24, 25, 26] introduced differential geometry over lattice structures in which coordinates mutually commute but the coordinates no longer commute with $1-$forms. In particular, the continuity is neglected and the space acquires a canonical lattice structure.

According to the last proposal, discrete structures are identified as a certain kind of differential calculi over a discrete set. Indeed, a bijective correspondence between first order differential calculi over discrete sets and graph structures was established in [23], where the vertices of a graph corresponds to the elements of the discrete set and neither multiple edges nor loops are admitted. In particular, this endow some neighborhood relations between elements of the set.

The flexibility of differential calculus rests mostly on its coordinate-free character described by the Grassmann algebra. However Grassmann algebras do not encode some physical structures like spinors and Dirac operators. This is one among many reasons why the correspondence between Clifford algebras and lattice structure should be established *a priori*.

Clifford algebras on lattice structures are defined by using the cup algebra of simplicial homology theory (c.f. [4]). The starting point consists in describing the algebra of endomorphisms associated to the vector space of cochains, and looking for a natural algebraic description in this space just like for Clifford algebras in the *continuum* (see [36], pp. 5-32). This approach, however, has some drawbacks, particularly in relation to lattice gauge theories, where the forward/backward differences $\partial_h^{\pm j}$ are replaced by the symmetric differences $\frac{1}{2}(\partial_h^{-j} + \partial_h^{+j})$. In particular, we want to write a lattice version of the Dirac operator which splits a lattice version of the Laplace operator.

This dissertation is organized in five chapters. In Chapter 1 we will make an overview of the Umbral calculus framework inspired on the ideas of Roman, G.-C. Rota and its collaborators (see e.g. [61, 60]). Special emphasize will given to the Isomorphism theorem, First Expansion Theorem and *Rodrigues formula*.

Some remarks concerning the correspondence between Umbral calculus and the second quantization approach will be pointed out. In this sense operators acting on lattices will appear as the canonical generators of a quantum mechanical system preserving the property that the Bose algebra is the underlying algebra of the system.

In Chapter 2 we will study the interconnection structure beyond the Umbral algebra. The power of the presented approach rests mostly from the fact that differential calculus endowed by a shift-invariant exterior derivative $\mathbf{d}$ makes $O_{\underline{x}} = (O_{x_1}, O_{x_2}, \ldots, O_{x_n})$ appear as a multivariate delta operator. Here, the Pincherle derivatives $O'_{x_j}$ represent the main automorphisms arising in noncommutative differential calculus.

The main novelty of this approach against other approaches is the possibility to unify the continuous and discrete differential calculus as a whole. As a consequence, this allow us to establish a contact with Clifford algebras as a subalgebra of the algebra of endomorphisms. This includes description of Clifford algebras in *continuum* [9, 36, 10] as well as Clifford-like algebras on oriented lattices [5, 68, 34, 49].

We will end this chapter by studying the interplay between discrete differential forms and (geometric) integration theory by taking into account that discrete integration theory can be build up as the natural pairing between chains and cochains. The interconnecting structure between cochains and discrete differential forms given by the Whitney and, De Rham map allow us to derive the discrete Stokes theorem and integration by parts formula by construction and furthermore, the usage of barycentric coordinate functions allows us to explore basic polynomial sequences on discrete domains as a combination between finite difference equations and interpolation theory. It should be stressed that this approach is purely *coordinate-free*.

In Chapter 3 we will introduce the basic structures underlying discrete Clifford analysis. To this end, we will gather some intertwining properties of the exterior differential calculus. In the sequel we restrict ourselves to the algebra of multivariate polynomials $\mathcal{A} = \mathbb{R}[\underline{x}]$.

In terms of the language of Quantum Field Theory (c.f. [58]), the algebra of endomorphisms $\mathrm{End}(\Lambda^*\mathbb{R}[\underline{x}])$ can be realized as the tensor product between the Bose and Fermi algebra $\mathcal{F}^+ \otimes \mathcal{F}^-$. This allows us to sift well known results from *continuum* to discrete by taking into account that the generalization of the Sheffer operator to polynomial differential forms is a mapping between continuous and discrete polynomial forms. Physically speaking, both worlds can be understood as two quantum field systems on which the Sheffer operator acts as gauge transformation preserving the canonical relations between both.

The above scheme is the starting point to define discrete Dirac operators and discrete multiplication operators which play the same role of vector variables. Bear in mind these operators shall be understood as Clifford vectors and due to the fact that the (geometric) Clifford product is non-commutative, we will introduce left and right multiplications as the basic canonical operators belonging to the algebra of endomorphisms induced by the Clifford-valued polynomials, say $\mathrm{End}(\mathcal{P})$. Hereby the construction of left and right actions for the *lowering* operators takes into account the noncommutative relation between coordinates and $1-$forms while the construction of *raising* operators is completely determined by the Bose commutation relations.

At a first glance it follows that the notions of left and right discrete monogenicity only coincide in the case that the *raising* operators are symmetric (e.g. centered finite differences). Furthermore, symmetric extension of the left and right actions of the discrete Dirac operators establish a first contact with the Hermitian Clifford setting [8, 12]. Some comparison with the approaches proposed by Vaz [68], Kanamori and Kawamoto [49], Forgy and Schreiber [34], Bobenko, Mercat and Suris [6] and Faustino, Kähler and Sommen [31] will be pointed out.

For the case of left and right actions of the multiplication operators, say $x', x'| \in \mathrm{End}(\mathcal{P})$ we will also point out some drawbacks regarding the symmetric extension. This is due to the fact the basic endomorphisms $x', x'| \in \mathrm{End}(\mathcal{P})$ does not generate a radial algebra structure in the sense of F.Sommen [66].

Finally we will introduce (orthogonal) discrete Clifford analysis as the algebra of $\mathrm{Spin}(0,n)$-invariant Clifford differential operators $\mathrm{Alg}\{x', D', \mathbf{e}_j \ : \ j = 1, \ldots, n\}$. Umbral counterparts of Euler and Gamma operators, say $E'$ and $\Gamma'$, respectively, will be established as well as basic formulae involving $x', D', E'$ and $\Gamma'$.

In the end of this chapter we will introduce basic operators and relations concerning Clifford differential forms and its interplay with discrete integration. In particular, we will derive discrete versions of the Clifford-Stokes formula and moreover the discrete Cauchy's formula.

In Chapter 4 we will give the key ingredients to develop discrete function theory in the Clifford setting. Starting with the Umbral version of Fischer decomposition, we decompose the space of Umbral homogeneous polynomials in terms of Umbral monogenic polynomials. Furthermore, using the basic (anti-)commutator relations determined on Chapter 3, we determine the Umbral monogenic projectors as the basic operators which map the space of

Umbral homogeneous polynomials onto the space of monogenic polynomials.

The above constructions plays an important role in the development of the theory of discrete spherical monogenics as a refinement of the theory of discrete spherical monogenics. Indeed from the integral representation of the Fischer inner product, the theory of discrete spherical monogenics is equivalent to the theory of discrete monogenic polynomials in $\mathbb{R}^n$. This in combination with the reproducing kernel property for the Umbral Fischer inner product also provides a new way to compute the discrete fundamental solution as an asymptotic expansion.

In the end of this chapter, we will use the Umbral calculus jargon developed along this thesis to determine an alternative proof for the discrete Poincaré lemma. The interplay between polynomial differential forms and the interconnection structure of discrete differential forms will be establish and moreover the correspondence between discrete closed differential forms and discrete monogenic forms.

Finally, in Chapter 5 we explore the physical model of the discrete harmonic oscillator in connection with the theory of discrete Clifford analysis. First we start to split the Hamiltonian operator in terms of *raising* and *lowering* operators. This leads to a representation of the Lie superalgebra $\mathrm{osp}(1|2)$ which is nothing else than a realization of Clifford analysis [18].

The former description in terms of Lie superalgebras allow us to build up in the combinatorial way the discrete counterparts of Clifford-Hermite polynomials as eigenstates of the discrete harmonic oscillator. Correspondence between the spectrum of the Hamiltonian and the MacDonald formula will be explored by means of the integral representation for the Umbral Fischer inner product. Some comparison with the approaches of F. Sommen [65], J. Cnops and Kisil [15] and H. Malonek and G.Tomaz [55] will be pointed out in the end of this chapter.

As it can be seen along this thesis, we do not treat Discrete Clifford Analysis as a *naive* theory established by *brute force* but instead as a theory which emerge due to fruitful interactions between others well-known branches of mathematics like Algebraic Topology, Combinatorics, Non-Commutative Geometry and Quantum Field Theory.

# Chapter 1

# Umbral Calculus Revisited

"*The journey of a thousand miles begins with one step.*"
Lao Tzu

In this chapter we will make a self-contained exposition of umbral calculus to several variables following the approaches of Roman, G.-C. Rota and its collaborators (see e.g. [61, 60]). This includes among other results, proofs of the Isomorphism theorem, First Expansion Theorem and *Rodrigues formula*.

Some remarks concerning the interplay between umbral calculus and the quantization procedure in terms of the Bose algebra setting will be pointed out. We do not claim that the obtained construction of Umbral calculus is new. Most of the results can also be found on the papers [21, 24, 22].

## 1.1 Shift-Invariant Operators

Let $\underline{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. A monomial over $\underline{x}$ is the product $\underline{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$. Here and elsewhere $\alpha, \beta, \ldots$ denote vectors of nonnegative integers. A *polynomial* is a finite linear combination of monomials. We denote the ring of polynomials over $\underline{x}$ by $\mathbb{R}[\underline{x}]$, $\partial_{x_j} := \frac{\partial}{\partial_{x_j}}$ the partial derivative with respect to $x_j$, $\partial_{\underline{x}} := (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n})$ the gradient operator and by $\mathrm{End}(\mathbb{R}[\underline{x}])$ the algebra of all linear operators acting on $\mathbb{R}[\underline{x}]$.

In the sequel, it will be helpful to use the compact notations

$$\partial_{\underline{x}}^\alpha := \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \ldots \partial_{x_n}^{\alpha_n}, \quad \alpha! = \alpha_1! \alpha_2! \ldots \alpha_n!, \quad \begin{pmatrix} \beta \\ \alpha \end{pmatrix} = \frac{\beta!}{\alpha!(\beta-\alpha)!}, \quad |\alpha| = \sum_{j=1}^n \alpha_j.$$

The differentiation formula $\partial_{\underline{x}}^\alpha \underline{x}^\beta = \frac{\beta!}{\alpha!}\ \underline{x}^{\beta-\alpha}$ naturally leads to the representation of the binomial identity in terms of the gradient operator $\partial_{\underline{x}}$, that is

$$(\underline{x}+\underline{y})^\beta = \sum_{|\alpha|=0}^{|\beta|} \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) \underline{x}^\alpha \underline{y}^{\beta-\alpha} = \sum_{|\alpha|=0}^{|\beta|} \frac{[\partial_{\underline{x}}^\alpha \underline{x}^\beta]_{\underline{x}=\underline{y}}}{\alpha!} \underline{x}^\alpha \tag{1.1}$$

Since $\partial_{\underline{x}}^\alpha \underline{x}^\beta$ vanishes for $|\alpha| > |\beta|$, we get that $\exp(\underline{y}\cdot\partial_{\underline{x}})\underline{x}^\beta$ is finite and coincide with the right-hand side of (1.1), i.e.

$$(\underline{x}+\underline{y})^\beta = \exp(\underline{y}\cdot\partial_{\underline{x}})\underline{x}^\beta. \tag{1.2}$$

Thus we can recast the binomial formula (1.1) by acting the shift-operator $T_{\underline{y}} = \exp(\underline{y}\cdot\partial_{\underline{x}})$ on $\underline{x}^\beta$.

Due to linearity arguments, the above formula is also true for any linear combinations of monomials of the form $\underline{x}^\beta$. Moreover under suitable topological assumptions we get the Taylor expansion

$$f(\underline{x}+\underline{y}) = \sum_{|\alpha|=0}^{\infty} \frac{\underline{y}^\alpha}{\alpha!}[\partial_{\underline{x}} f(\underline{x})]_{\underline{x}=\underline{y}} = \exp(\underline{y}\cdot\partial_{\underline{x}})f(\underline{x}).$$

This motivates the following definition:

**Definition 1.1.1 (Shift-invariant operator)** *An operator $Q \in End(\mathbb{R}[\underline{x}])$ is shift-invariant if it commutes with the shift operator $T_{\underline{y}} = \exp(\underline{y}\cdot\partial_{\underline{x}})$, when acting on $\mathbb{R}[\underline{x}]$, i.e.*

$$[Q, T_{\underline{y}}] = 0 \tag{1.3}$$

*holds for all $P \in \mathbb{R}[\underline{x}]$ and $\underline{y} \in \mathbb{R}^n$, where $[\mathbf{a},\mathbf{b}] = \mathbf{ab} - \mathbf{ba}$ denotes the commutator bracket between $\mathbf{a}$ and $\mathbf{b}$.*

In the co-algebra setting, the shift operator $T_{\underline{y}}$ is nothing else than the co-product in a Hopf algebra over $\mathbb{R}[\underline{x}]$, but we shall not discuss this here- see e.g. [62] (Section V.3) and [50] (Subsection 2.2.) for more details.

We will denote by $\mathcal{U} \subset \mathrm{End}(\mathbb{R}[\underline{x}])$ the set of all shift-invariant operators over $\mathbb{R}[\underline{x}]$. This set, endowed with the standard addition and composition between functions, forms an an $\mathbb{R}-$algebra. In the language of umbral calculus, $\mathcal{U}$ is usually named Umbral Algebra (c.f. [62, 60, 22]). Hence the fundamental result of umbral calculus (see [61], Theorem 2, pp.691) can be stated as follows.

**Theorem 1.1.2 (First Expansion Theorem, c.f.[21])** *A linear operator* $Q : \mathbb{R}[\underline{x}] \to \mathbb{R}[\underline{x}]$ *is shift-invariant if and only if it can be expressed (as a convergent series) in the gradient* $\partial_{\underline{x}}$*, that is*

$$Q = \sum_{|\alpha|=0}^{\infty} \frac{a_\alpha}{\alpha!} \partial_{\underline{x}}^{\alpha}$$

*where* $a_\alpha = \left[Q \underline{x}^\alpha\right]_{\underline{x}=\underline{0}}$.

According to Theorem 1.1.2, shift-invariant operators over polynomials are determined in terms of derivatives. In particular, the multi-index derivatives $\partial_{\underline{x}}^{\alpha}$ and linear combinations of them are shift-invariant. Regardless the last viewpoint, one looks to shift-invariant operators as convergent power series $Q(\underline{x}) = \sum_{|\alpha|=0}^{\infty} \frac{a_\alpha}{\alpha!} \underline{x}^\alpha$, by replacing the vector variable $\underline{x}$ by the gradient $\partial_{\underline{x}}$, namely for the set of formal power series, say $\widehat{\mathbb{R}[\underline{x}]}$, $\iota[Q(\underline{x})] = Q(\partial_{\underline{x}})$, where $\iota : \widehat{\mathbb{R}[\underline{x}]} \to \mathrm{End}(\mathbb{R}[\underline{x}])$ is a mapping between the algebra of formal power series and the algebra $\mathrm{End}(\mathbb{R}[\underline{x}])$.

In the next theorem we will show that this mapping is one-to-one and onto.

**Theorem 1.1.3 (Isomorphism theorem)** *There exists an isomorphism* $\iota : \widehat{\mathbb{R}[\underline{x}]} \to End(\mathbb{R}[\underline{x}])$ *of the algebra* $\widehat{\mathbb{R}[\underline{x}]}$ *onto the algebra* $End(\mathbb{R}[\underline{x}])$

$$\iota : \sum_{|\alpha|\geq 0} \frac{a_\alpha}{\alpha!} \, \underline{x}^\alpha \mapsto \sum_{|\alpha|\geq 0} \frac{a_\alpha}{\alpha!} \, \partial_{\underline{x}}^{\alpha}. \tag{1.4}$$

**Proof:** In the algebra $\widehat{\mathbb{R}[\underline{x}]}$, the product between polynomials is given by the binomial convolution, i.e.

$$\left( \sum_{\alpha} \frac{a_\alpha}{\alpha!} \underline{x}^\alpha \right) \left( \sum_{\beta} \frac{b_\beta}{\beta!} \underline{x}^\beta \right) = \sum_{\gamma} \frac{c_\gamma}{\gamma!} \underline{x}^\gamma$$

with $c_\gamma = \sum_{\alpha+\beta=\gamma} \begin{pmatrix} \gamma \\ \alpha \end{pmatrix} a_\alpha b_\beta$.

Therefore with $Q(\underline{x}) = \sum_\alpha \frac{a_\alpha}{\alpha!} \underline{x}^\alpha \mapsto Q(\partial_{\underline{x}})$ and $S(\underline{x}) = \sum_\beta \frac{b_\beta}{\beta!} \underline{x}^\beta \mapsto S(\partial_{\underline{x}})$ it is enough to show that $[Q(\partial_{\underline{x}}) \; S(\partial_{\underline{x}}) \underline{x}^\gamma]_{\underline{x}=\underline{0}} = c_\gamma$ and this is the case because due to $\underline{x}^{\underline{0}} = 1$ and $(\underline{x}^\gamma)_{|\underline{x}=\underline{0}} = \delta_{\gamma,0}$ we have

$$\left[Q(\partial_{\underline{x}}) \; S(\partial_{\underline{x}}) \underline{x}^\gamma\right]_{\underline{x}=\underline{0}} = \sum_{\alpha+\beta=\gamma} \frac{c_\gamma}{\gamma!} \left[\partial_{\underline{x}}^{\gamma} \underline{x}^\gamma\right]_{\underline{x}=\underline{0}} = c_\gamma.$$

■

Now it arises the question of how to find an inverse for a shift-invariant operator. In the case of the Umbral algebra $\mathcal{U}$ we know the answer due to the Isomorphism theorem (Theorem 1.1.3). Indeed an element $\sum_\alpha \frac{a_\alpha}{\alpha!}\underline{x}^\alpha$ is invertible if and only if $a_{\underline{0}} \neq 0$. This altogether leads to the following corollary:

**Corollary 1.1.4** *An shift-invariant operator $Q \in \mathcal{U}$ has its inverse if and only if $Q1 \neq 0$.*

**Proof:** From the above theorem, we know that the equation

$$\sum_\beta \begin{pmatrix} \gamma \\ \beta \end{pmatrix} c_\beta \underline{x}^{\gamma-\beta} = Q(\partial_{\underline{x}})S(\partial_{\underline{x}})\underline{x}^\gamma = \underline{x}^\gamma$$

fulfils if and only if $\sum_\beta \begin{pmatrix} \gamma \\ \beta \end{pmatrix} [Q(\partial_{\underline{x}})\underline{x}^\beta]_{\underline{x}=\underline{0}}[S(\partial_{\underline{x}})\underline{x}^{\gamma-\beta}]_{\underline{x}=\underline{0}} = c_\gamma = \delta_{\gamma,\underline{0}}$.

Hence $S(\partial_{\underline{x}})$ is the inverse for $Q(\partial_{\underline{x}})$ if and only if $1 = (Q(\partial_{\underline{x}})1)(S(\partial_{\underline{x}})1)$ which is equivalent to $Q(\partial_{\underline{x}})\ 1 \neq 0$.

■

A generalization of the First Expansion Theorem (Theorem 1.1.2) can be obtained by replacing the gradient operator $\partial_{\underline{x}}$ with an arbitrary multivariate lowering operator $O_{\underline{x}} = (O_{x_2}, O_{x_2}, \ldots, O_{x_n})$. To this end, the following definitions will be useful in the sequel:

**Definition 1.1.5 (Multivariate delta operator)** *An operator $O_{\underline{x}} = (O_{x_2}, O_{x_2}, \ldots, O_{x_n}) \in End(\mathbb{R}[\underline{x}])^n$ is a multivariate delta operator if and only if*

1. *$O_{x_j}$ are shift-invariant operators (i.e. $O_{x_j} \in \mathcal{U}$);*
2. *$O_{x_j}(x_j)$ is a non-vanishing constant;*
3. *$O_{x_j}(x_k) = 0$ for $j \neq k$.*

**Remark 1.1.6** *The components of the operator $O_{\underline{x}} = (\exp(t_1\partial_{x_1}), \exp(t_2\partial_{x_2}), \ldots, \exp(t_n\partial_{x_n}))$ are invertible. However $O_{\underline{x}}$ is not a multivariate delta operator. From Corollary 1.1.4 we infer that no one of the components of the multivariate delta operator $O_{\underline{x}}$ is invertible.*

From the stated above and the First Expansion Theorem (Theorem 1.1.2) we have the next corollary:

**Corollary 1.1.7** *The operator $O_{\underline{x}} = (\iota[O_1(\underline{x})], \iota[O_2(\underline{x})], \ldots, \iota[O_n(\underline{x})])$ is a multivariate delta operator if and only if*

$$O_j(\underline{0}) = 0\ , \qquad [\partial_{x_j} O_j(\underline{x})]_{\underline{x}=\underline{0}} \neq 0 \quad \text{and} \quad [\partial_{x_j} O_k(\underline{x})]_{\underline{x}=\underline{0}} = 0 \text{ for } j \neq k. \tag{1.5}$$

*Hereby $\iota : \mathbb{R}[\underline{x}] \to \mathcal{U}$ is the mapping given by the Isomorphism theorem (Theorem 1.1.3).*

There are other useful characterization that follow from direct application of Definition 1.1.5, namely the following lemmata:

**Lemma 1.1.8** *Let $O_{\underline{x}} = (O_{x_1}, O_{x_2}, \ldots, O_{x_n})$ be a multivariate delta operator. Then for every constant polynomial $c \in \mathbb{R}[\underline{x}]$ we have $O_{x_j}(c) = 0$.*

**Proof:** Recall that for all $\underline{y} \in \mathbb{R}^n$ we have $[O_{x_j}, \exp(\underline{y} \cdot \partial_{\underline{x}})] = 0$ and $a = O_{x_j}(x_j)$ is a non-vanishing constant.

In particular we have $O_{x_j}(\exp(c\partial_{x_j})x_j) = O_{x_j}(x_j + c) = a + O_{x_j}(c)$ and at the same time $\exp(c\partial_{x_j})\left(O_{x_j}(x_j)\right) = \exp(c\partial_{x_j})a = a$. Hence $a + O_{x_j}(c) = a$ leads to $O_{x_j}(c) = 0$.

■

**Lemma 1.1.9 (Degree-lowering property)** *If $p(\underline{x}) \in \mathbb{R}[\underline{x}]$ is a polynomial of degree $k$, then $O_{x_j}p(\underline{x})$ is a polynomial of degree $k - 1$.*

**Proof:** Set $q(\underline{x} + \underline{y}) = \exp(\underline{y} \cdot \partial_{\underline{x}})(O_{x_j}(\underline{x}^\alpha))$. By applying the definition and the binomial identity, we have

$$O_{x_j}(\exp(\underline{y} \cdot \partial_{\underline{x}})\underline{x}^\alpha) = O_{x_j}((\underline{x} + \underline{y})^\alpha) = \sum_{|\beta|=0}^{|\alpha|} \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) \underline{y}^\beta O_{x_j}\underline{x}^{\alpha-\beta}.$$

Hence from the shift-invariance property, we end up with $q(\underline{y}) = \sum_{|\beta|=0}^{|\alpha|} \left( \begin{array}{c} \beta \\ \alpha \end{array} \right) \underline{y}^\beta O_{x_j}\underline{x}^{\alpha-\beta}|_{\underline{x}=\underline{0}}$. The coefficient of $\underline{y}^\alpha$ is therefore equal to zero according to Lemma 1.1.8. At the same time the coefficients of $\underline{y}^{\alpha-\mathbf{v}_j}$ are equal to $\left( \begin{array}{c} \alpha \\ \alpha - \mathbf{v}_j \end{array} \right) \neq 0$.

By linearity the above extends to an arbitrary polynomial $p(\underline{x}) \in \mathbb{R}[\underline{x}]$ of degree $|\alpha| = k$, which finishes the proof of the Lemma 1.1.9.

■

From the above lemmata, we show that the multivariate shift-invariant operators acting on the algebra $\mathbb{R}[\underline{x}]$ are degree-lowering operators.

Now we want to generalize the interplay between polynomials of binomial type and differentiation operators given by (1.2). The generalization of the binomial formula (1.1) is given by the following definition:

**Definition 1.1.10 (Polynomial sequence of binomial type)** *A polynomial sequence $V_\alpha(\underline{x})$ is said to be of binomial type if $V_{\underline{0}}(\underline{x}) = 1$ and if it satisfies the binomial identity*

$$V_\beta(\underline{x}+\underline{y}) = \sum_{|\beta|=0}^{|\alpha|} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} V_\alpha(\underline{x}) V_{\beta-\alpha}(\underline{y}) \tag{1.6}$$

We would like to point out that formula (1.6) is a particular case of convolution polynomials. In the Cancellative Semigroups setting, the above definition can also be formulated in terms of the fundamental incidence algebra [61] or more generally as token structures [50, 51] but we shall not explore here.

Now we turn our attention for the construction of a sequence of multivariate polynomials. Motivated from Lemmata 1.1.8 and 1.1.9 we introduce the following definition:

**Definition 1.1.11 (Basic polynomial sequence)** *A polynomial sequence $\{V_\alpha\}_\alpha$, where $V_\alpha$ is a multivariate polynomial of degree $|\alpha|$ such that*

1. $V_{\underline{0}}(\underline{x}) = 1$;

2. $V_\alpha(\underline{0}) = \delta_{\alpha,0}$;

3. $O_{x_j} V_\alpha(\underline{x}) = \alpha_j V_{\alpha-\mathbf{v}_j}(\underline{x})$.

*is called basic polynomial sequence of the multivariate delta operator $O_{\underline{x}} = (O_{x_1}, O_{x_2}, \ldots, O_{x_n})$.*

The next theorem shows that polynomial sequences of binomial type and basic polynomial sequences are indeed equivalent.

**Theorem 1.1.12** *$\{V_\alpha\}_\alpha$ is a basic polynomial sequence of multivariate delta operator if and only if it is a polynomial sequence of binomial type.*

**Proof:** By recursive application of Definition 1.1.11, we obtain $O_{\underline{x}}^\beta V_\alpha(\underline{x}) = \frac{\beta!}{(\beta-\alpha)!} V_{\beta-\alpha}(\underline{x})$, and, therefore, for $|\beta| \leq |\alpha|$, $O_{\underline{x}}^\beta V_\alpha(\underline{x})|_{\underline{x}=\underline{0}} = \beta! \delta_{\beta-\alpha,0}$ while for $|\alpha| < |\beta|$, $O_{\underline{x}}^\beta V_\alpha(\underline{x})|_{\underline{x}=\underline{0}} = 0$.

Hence

$$V_\alpha(\underline{x}) = \sum_{|\beta|=0}^{|\alpha|} \frac{V_\beta(\underline{x})}{\beta!} \left[ O^\beta V_\alpha(\underline{x}) \right]_{\underline{x}=\underline{0}}.$$

Using the shift invariance of $O_{\underline{x}}$, the Taylor series expansion around $\underline{y} \in \mathbb{R}^n$ is given by

$$\begin{aligned} V_\alpha(\underline{x}+\underline{y}) &= \exp(\underline{y} \cdot \partial_{\underline{x}}) V_\alpha(\underline{x}) \\ &= \sum_{|\beta|=0}^{|\alpha|} \frac{V_\beta(\underline{x})}{\beta!} \left[ O_{\underline{x}}^\beta \exp(\underline{y} \cdot \partial_{\underline{x}}) V_\alpha(\underline{x}) \right]_{\underline{x}=\underline{0}} \\ &= \sum_{|\beta|=0}^{|\alpha|} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} V_\alpha(\underline{x}) V_{\beta-\alpha}(\underline{y}). \end{aligned}$$

Conversely, suppose that $\{V_\alpha\}_\alpha$ is a sequence of binomial type. Setting $\underline{y} = \underline{0}$ in (1.6), we infer that $V_{\underline{0}}(\underline{x}) = 1$ and $V_\alpha(\underline{0}) = 0$.

It is sufficient now to define the delta operator $O_{\underline{x}} = (O_{x_1}, O_{x_2}, \ldots, O_{x_n})$ corresponding to $\{V_\alpha\}_\alpha$. We define it according to:

1. $O_{x_j} V_{\underline{0}}(\underline{x}) = 0$;

2. $O_{x_j} V_\alpha(\underline{x}) = \alpha_j V_{\alpha - \mathbf{v}_j}(\underline{x})$;

3. $O_{x_j}$ is a linear operator.

The only thing left is to prove that $O_{\underline{x}}$ is a multivariate shift-invariant operator. For that we use

$$O_{x_j} V_\alpha(\underline{x} + \underline{y}) = \sum_{|\beta|=0}^{|\alpha|} \frac{V_\beta(\underline{y})}{\beta!} \left[ O_{\underline{x}}^{\beta + \mathbf{v}_j} \; V_\alpha(\underline{x} + \underline{y}) \right]_{\underline{y} = \underline{0}} \tag{1.7}$$

Note that the left hand side of (1.7) is equal to $O_{x_j} V_\alpha(\underline{x} + \underline{y}) = \exp(\underline{y} \cdot \partial_{\underline{x}})(O_{x_j} V_\alpha(\underline{x}))$ while the right hand side of (1.7) is equal to

$$\begin{aligned} \sum_{|\beta|=0}^{|\alpha|} \frac{V_\beta(\underline{y})}{\beta!} \left[ O_{\underline{x}}^{\beta + \mathbf{v}_j} \; V_\alpha(\underline{x} + \underline{y}) \right]_{\underline{y} = \underline{0}} &= O_{x_j} \left( \sum_{|\beta|=0}^{|\alpha|} \frac{V_\beta(\underline{y})}{\beta!} \left[ O_{\underline{x}}^{\beta} \; V_\alpha(\underline{x} + \underline{y}) \right]_{\underline{y} = \underline{0}} \right) \\ &= O_{x_j} V_\alpha(\underline{x} + \underline{y}) \\ &= O_{x_j} (\exp(\underline{y} \cdot \partial_{\underline{x}}) V_\alpha(\underline{x})). \end{aligned}$$

This concludes the proof of Theorem (1.1.12).

■

The next theorem shows that each polynomial sequence is uniquely determined by the multivariate delta operator.

**Theorem 1.1.13 (Uniqueness of the basic polynomial sequence)** *Every multivariate delta operator $O_{\underline{x}}$ has a unique sequence of basic polynomials.*

**Proof:** For $\alpha = \underline{0}$ put $V_{\underline{0}}(\underline{x}) = 1$, for $\alpha = \mathbf{v}_j$ put $V_{\mathbf{v}_j}(\underline{x}) = \frac{x_j}{O_{x_j} x_j}$. Then using induction over $|\alpha| = k$, we assume that $V_\beta(\underline{x})$ have been defined for $|\beta| < |\alpha|$.

In order to show that $V_\alpha(\underline{x})$ is uniquely defined, it is enough to show that $V_\alpha(\underline{x})$ has degree $|\alpha| = k$, i.e.

$$V_\alpha(\underline{x}) = \sum_{|\alpha|=k} a_\alpha \underline{x}^\alpha + \sum_{|\beta|=0}^{k-1} b_\beta V_\beta(\underline{x})$$

with $a_\alpha \neq 0$ for some $|\alpha| = k$. Thus we have

$$O_{x_j} V_\alpha(\underline{x}) = \sum_{|\alpha|=k} a_\alpha O_{x_j} \underline{x}^\alpha + \sum_{|\beta|=0}^{k-1} b_\beta O_{x_j} V_\beta(\underline{x}).$$

Notice that the degree of $O_{x_j}\underline{x}^\alpha$ is equal to $k-1$. Hence there exist a unique choice for the constants $b_\beta$, $|\beta| = 0, \ldots, k-1$ for which $O_{x_j} V_\alpha(\underline{x}) = \alpha_j V_{\alpha - \mathbf{v}_j}(\underline{x})$.

This uniquely determines $V_\alpha(\underline{x})$ except for the constant term $b_{\underline{0}}$ which is however determined uniquely by the condition $V_\alpha(\underline{0}) = 0$ for $|\alpha| > 0$.

■

Now we are in conditions to generalize the First Expansion Theorem (Theorem 1.1.2) in terms arbitrary multivariate delta operators $O_{\underline{x}} = (O_{x_1}, O_{x_2}, \ldots, O_{x_n})$

**Theorem 1.1.14** *Let $O_{\underline{x}}$ be a multivariate delta operator with basic polynomial sequence $\{V_\alpha\}_\alpha$ and let $Q$ be a linear operator commuting with all the components of $O_{\underline{x}}$, i.e. $[Q, O_{x_j}] = 0$ for all $j = 1, \ldots, n$.*

*Then we have*

$$Q = \sum_{|\alpha|\geq 0} \frac{a_\alpha}{\alpha!} O_{\underline{x}}^\alpha, \quad \textit{with} \quad a_\alpha = [QV_\alpha(\underline{x})]_{\underline{x}=\underline{0}}.$$

**Proof:** Due to Theorem 1.1.12, it is enough to show that $Q$ is given by the ansatz

$$Q = \sum_{|\alpha|\geq 0} \frac{a_\alpha}{\alpha!} O_{\underline{x}}^\alpha.$$

By letting act a shift-invariant operator $Q$ on the binomial formula

$$V_\alpha(\underline{x} + \underline{y}) = \sum_{|\beta|=0}^{|\alpha|} \frac{V_\beta(\underline{y})}{\beta!} O_{\underline{x}}^\beta V_\alpha(\underline{x})$$

and using the linearity we get $QV_\alpha(\underline{x} + \underline{y}) = \sum_{|\beta|=0}^{|\alpha|} \frac{QV_\beta(\underline{y})}{\beta!} O_{\underline{x}}^\beta V_\alpha(\underline{x})$.

Setting in the above $\underline{y} = \underline{0}$ we arrive at

$$QV_\alpha(\underline{x}) = \sum_{|\beta|=0}^{|\alpha|} \frac{\left[QV_\beta(\underline{y})\right]_{\underline{y}=\underline{0}}}{\beta!} O_{\underline{x}}^\beta V_\alpha(\underline{x}).$$

This proves our theorem for the basic polynomials $V_\alpha(\underline{x})$. The theorem then follows by linearity on the closure of $\{V_\alpha(\underline{x})\}_\alpha$.

■

Now note that every operator $Q$ can be written as the expansion involving linear combinations of multivariate delta operators, i.e. $Q(O_{\underline{x}}) = \sum_{|\alpha|\geq 1} \frac{q_\alpha}{\alpha!} O_{\underline{x}}^\alpha$.

Now we shall derive an expression for the exponential generating function for basic sequences using notation established above.

**Corollary 1.1.15** *The exponential generating function for the basic polynomial sequence $\{V_\alpha(\underline{x})\}_\alpha$ determined by the multivariate delta operator $O_{\underline{x}}$ is given by the following formula*

$$V(\underline{t}, \underline{x}) = \sum_{|\alpha|\geq 0} \frac{V_\alpha(\underline{t})}{\alpha!} \underline{x}^\alpha = \exp(\underline{t} \cdot O^{-1}(\underline{x})),$$

*where $O^{-1}(\underline{t}) := (O_1^{-1}(\underline{t}), O_2^{-1}(\underline{t}), \ldots, O_n^{-1}(\underline{t}))$.*

*Hereby $O_j^{-1}(\underline{t})$ denotes the inverse of the formal power series $O_j(\underline{t})$ such that $O_j(O_j^{-1}(\underline{t})) = t_j$.*

**Proof:** From $\exp(\underline{t} \cdot \partial_{\underline{x}}) = \sum_\alpha \frac{a_\alpha}{\alpha!} O_{\underline{x}}^\alpha$, where $a_\alpha = [\exp(\underline{t} \cdot \partial_{\underline{x}}) V_\alpha(\underline{x})]_{\underline{x}=\underline{0}} = V_\alpha(\underline{t})$ we get

$$\exp(\underline{t} \cdot \partial_{\underline{x}}) = \sum_\alpha \frac{V_\alpha(\underline{t})}{\alpha!} O_{\underline{x}}^\alpha$$

Applying the Isomorphism theorem (Theorem 1.1.3) we have

$$\exp(\underline{t} \cdot \underline{x}) = \sum_\alpha \frac{V_\alpha(\underline{t})}{\alpha!} O(\underline{x})^\alpha$$

Since $O_{\underline{x}}$ is a multivariate delta operator, from Corollary 1.1.7 we have that $a_{\underline{0}} := O_j(\underline{0}) = 0$ and $a_{\mathbf{v}_j} := [\partial_{x_j} O_j(\underline{x})]_{\underline{x}=\underline{0}} \neq 0$ for $j = 1, 2, \ldots, n$ which ensures the existence of $O_j^{-1}$ as a formal power series for each $j = 1, 2, \ldots, n$ such that $O_j(O_j^{-1}(\underline{t})) = t_j$.

After the substitution $O(\underline{x}) = \underline{z}$ we get the thesis.

■

**Remark 1.1.16** *Notice that $\exp(\underline{t} \cdot \underline{x})$ is the generating function for the basic polynomial sequence $\{\underline{x}^\alpha\}_\alpha$ of the $\partial_{\underline{x}}$-operator.*

*In fact behind this proof stands nothing else than the fact that $\exp(\underline{t} \cdot \underline{x})$ acts like a reproducing kernel with respect to the Fischer inner product defined viz (4.1), Chapter 4 (see also Theorem 4.3.1).*

We will see in Chapter 4 that the exponential generating function plays an important role later on construction of discrete spherical monogenics (see Section 4.3).

Using the exponential generating function, we can generalize Theorem 1.1.14 for linear operators. This corresponds to the following:

**Theorem 1.1.17** *Let $O_{\underline{t}}$ a multivariate delta operator and let $Q$ a linear operator. Then*

$$Q = \sum_{\alpha} a_\alpha(\underline{t}) O_{\underline{t}}^{\alpha}$$

*where the polynomials $a_\alpha(\underline{t})$ are obtained viz the generating function $\sum_\alpha a_\alpha(\underline{t})\underline{x}^\alpha = \frac{QV(\underline{t},\underline{x})}{V(\underline{t},\underline{x})}$.*

**Proof:** By acting $O_{\underline{t}}^{\alpha}$ on the exponential generating function $V(\underline{t},\underline{x})$ we obtain from Definition 1.1.11 the relation $O_{\underline{t}}^{\alpha} V(\underline{t},\underline{x}) = \underline{x}^\alpha V(\underline{t},\underline{x})$.

Hence the action of $Q = \sum_\alpha a_\alpha(\underline{t}) O_{\underline{t}}^{\alpha}$ on $V(\underline{t},\underline{x})$ gives

$$QV(\underline{t},\underline{x}) = \sum_{\alpha} a_\alpha(\underline{t}) O_{\underline{t}}^{\alpha} V(\underline{t},\underline{x}) = \left(\sum_{\alpha} a_\alpha(\underline{t})\underline{x}^\alpha\right) V(\underline{t},\underline{x}).$$

This concludes the proof.

■

The resulting theorem corresponds to generalization of the approach obtained by Kurbanov and Maksimov [52] in the one-dimensional case.

According to this approach, the computation of any arbitrary linear operators can be formally expressed in terms of multiplication and differentiation by basic polynomial sequences by first applying the operator $Q$ to the generating function $V(\underline{t},\underline{x})$, divide by $V(\underline{t},\underline{x})$ and afterwards replacing the monomials $\underline{x}^\alpha$ by the multi-index operator $O_{\underline{t}}^{\alpha}$. In particular, this provides a scheme to obtain an asymptotic representation for integration (see [21, 22]).

## 1.2 Pincherle Derivative and Basic Polynomial Sequences

As it was shown in the above section, basic polynomial sequences are naturally handled within the Umbral algebra (i.e. the algebra of shift invariant operators).We will continue to present some more relevant facts about them with special emphasis on their germ structure. A crucial role in this is played by the Pincherle derivative of a multivariate delta operator.

The Pincherle derivative of an operator $Q$ at the coordinate $x_j$ is defined as the commutator

$$Q'_{x_j} = [Q, x_j], \tag{1.8}$$

where on the right hand side of (1.8), $x_j$ acts as a multiplication operator on $\mathbb{R}[\underline{x}]$, i.e. $x_j : P(\underline{x}) \mapsto x_j P(\underline{x})$.

By applying the definition, the following lemma is rather obvious:

**Lemma 1.2.1 (Shift-invariance property)** *The Pincherle derivative of a shift-invariant operator is again shift-invariant.*

**Proof:** Let us assume that $Q$ is a shift-invariant operator. From the First Expansion Theorem (Theorem 1.1.2) $Q$ corresponds to the convergent series $Q = \sum_\alpha \frac{a_\alpha}{\alpha!}\partial_{\underline{x}}^\alpha$.

Since the standard partial derivatives mutually commute and satisfy the Leibniz rule, it immediately follows that

$$\partial_{x_k} Q(F(\underline{x})) = Q\partial_{x_k}(F(\underline{x})), \quad \partial_{x_k}(x_j F(\underline{x})) = \delta_{jk} F(\underline{x}) + x_j \partial_{x_k}(F(\underline{x})).$$

and hence

$$\begin{aligned}
\partial_{x_k}(Q'_{x_j} F(\underline{x})) &= Q(\delta_{jk} F(\underline{x}) + x_j \partial_{x_k} F(\underline{x})) - \delta_{jk} Q F(\underline{x}) - x_j \partial_{x_k}(Q F(\underline{x})) \\
&= Q(x_j \partial_{x_k} F(\underline{x})) - x_j Q(\partial_{x_k} F(\underline{x})) \\
&= Q'_{x_j}(\partial_{x_k} F(\underline{x})).
\end{aligned}$$

Using recursion on $r \in \mathbb{N}_0$, we arrive at $\partial_{x_k}^r (Q'_{x_j} F(\underline{x})) = Q'_{x_j}(\partial_{x_k}^r F(\underline{x}))$ and therefore $\partial_{\underline{x}}^\alpha (Q'_{x_j} F(\underline{x})) = Q'_{x_j}(\partial_{\underline{x}}^\alpha F(\underline{x}))$.

By linearity the above extends to linear combinations of multi-index derivatives $\partial_{\underline{x}}^\alpha$. In particular, this leads to $\exp(\underline{t} \cdot \partial_{\underline{x}})(Q'_{x_j} F(\underline{x})) = Q'_{x_j}(\exp(\underline{t} \cdot \partial_{\underline{x}}) F(\underline{x}))$, which finishes the proof of the lemma.

■

Motivated by the Isomorphism theorem (Theorem 1.1.3) we will call the inverse image under $\iota : \mathbb{R}[\underline{x}] \to \mathcal{U}$ the indicator operator. As a further consequence, the following lemma holds.

**Lemma 1.2.2 (Indicator of the Pincherle derivative)** *Let* $Q(\underline{x}) = \sum_{|\alpha| \geq 0} \frac{a_\alpha}{\alpha!} \underline{x}^\alpha$ *the indicator operator of* $Q = \sum_\alpha \frac{a_\alpha}{\alpha!}\partial_{\underline{x}}^\alpha$. *Then*

$$\partial_{x_j} Q(\underline{x}) = \sum_{|\alpha| \geq 1} \frac{a_\alpha}{(\alpha - \mathbf{v}_j)!} \underline{x}^{\alpha - \mathbf{v}_j}$$

*is the indicator of the Pincherle derivative* $Q'_{x_j} = [Q, x_j]$.

**Proof:** By acting $\partial_{x_j}^r$ on $x_j F(\underline{x})$ we obtain

$$\begin{aligned}
\partial_{x_j}^r (x_j F(\underline{x})) &= \partial^{r-1}(F(\underline{x}) + x_j \partial_{x_j} F(\underline{x})) \\
&= \partial^{r-1}(F(\underline{x}) + \partial^{r-2}(\partial_{x_j} F(\underline{x}) + x_j \partial_{x_j}^2 F(\underline{x})) \\
&= \ldots \\
&= r\partial^{r-1} F(\underline{x}) + x_j \partial_{x_j}^r F(\underline{x})
\end{aligned}$$

Therefore $[\partial_{x_j}^r, x_j] = r\partial_{x_j}^{r-1}$. This leads to $[\partial_{\underline{x}}^\alpha, x_j] = \alpha_j \partial_{\underline{x}}^{\alpha-\mathbf{v}_j}$.

From the First Expansion Theorem (Theorem 1.1.2), $Q'_{x_j}$ corresponds to the convergent series

$$Q'_{x_j} = \sum_{|\alpha|\geq 1} \frac{a_\alpha}{\alpha!}[\partial_{\underline{x}}^\alpha, x_j] = \sum_{|\alpha|\geq 1} \frac{a_\alpha}{(\alpha-\mathbf{v}_j)!}\partial_{\underline{x}}^{\alpha-\mathbf{v}_j}.$$

After the application of the Isomorphism theorem (Theorem 1.1.3) we get the statement.

■

By combining Theorem 1.1.3 with the above lemma, the Pincherle derivative is a commutative derivation operation in the umbral algebra $\mathcal{U}$. In particular, the Pincherle derivative satisfies the Leibniz rule:

**Corollary 1.2.3** *For two shift-invariant operators $Q, R \in \mathcal{U}$, we have*

$$(QR)'_{x_j} = Q'_{x_j}R + QR'_{x_j}.$$

*Furthermore* $(Q^r)'_{x_j} = rQ'_{x_j}Q^{r-1}$.

The next lemma is also a consequence of the Isomorphism theorem (Theorem 1.1.3).

**Lemma 1.2.4** *$O_{\underline{x}}$ is a multivariate delta operator if and only if there exists a multivariate operator $S_{\underline{x}}$ such that $S_{x_j}^{-1}$ exists and $O_{x_j} = \partial_{x_j}S_{x_j}$.*

**Proof:** Assume that $S_{x_j}$ is given by the *ansatz* $S_{x_j} = \sum_{|\alpha|\geq 0} \frac{s_\alpha}{\alpha!}\partial_{\underline{x}}^\alpha$ and $S_{x_j}^{-1}$ exists. Then from Lemma 1.1.7 we get $\partial_{x_j}S_{x_j}$ is a delta operator since $s_{\underline{0}} \neq 0$.

Conversely, assume that $O_{x_j} = \sum_{|\alpha|\geq 1} \frac{a_\alpha}{\alpha!}\partial_{\underline{x}}^\alpha$, with $a_\alpha \neq 0$ for $|\alpha| = 1$ is a delta operator. Then we have $O_{x_j} = \partial_{x_j}S_{x_j}$ with $S_{x_j} = \sum_{|\alpha|\geq 0} \frac{a_{\alpha+\mathbf{v}_j}}{(\alpha+\mathbf{v}_j)!}\partial_{\underline{x}}^\alpha$.

The coefficient $s_{\underline{0}} = a_{\mathbf{v}_j}$ is therefore different from zero. This proves Lemma 1.1.7.

■

We are now in conditions to prove the main theorem in Umbral Calculus:

**Theorem 1.2.5** *Let $\{V_\alpha(\underline{x})\}_\alpha$ a basic polynomial sequence for the multivariate delta operator $O_{\underline{x}}$. Then for $\alpha \in \mathbb{N}^n$, $V_\alpha(\underline{x})$ is uniquely determined by the following equivalent formulae:*

1. $V_\alpha(\underline{x}) = O'_{x_j}(S_{x_j}^{-\alpha_j-1}\underline{x}^\alpha)$;

2. $V_\alpha(\underline{x}) = S_{x_j}^{-\alpha_j}\underline{x}^\alpha - (S_{x_j}^{-\alpha_j})'\underline{x}^{\alpha-\mathbf{v}_j}$;

3. $V_\alpha(\underline{x}) = x_jS_{x_j}^{-\alpha_j}\underline{x}^{\alpha-\mathbf{v}_j}$;

*4.* $V_\alpha(\underline{x}) = x_j(O'_{x_j})^{-1}V_{\alpha-\mathbf{v}_j}(\underline{x})$*;*

**Proof:** First we start prove that the right-hand sides of 1., 2., and 3. coincide.

From Lemma 1.2.4 we have that each component of the multivariate delta operator $O_{\underline{x}}$ is given by $O_{x_j} = \partial_{x_j} S_{x_j}$, where $S_{x_j}$ is an invertible operator. By applying Corollary 1.2.3, we obtain

$$O'_{x_j} S_{x_j}^{-\alpha_j-1} = (S_{x_j} + \partial_{x_j} S'_{x_j}) S_{x_j}^{-\alpha_j-1} = S_{x_j}^{-\alpha_j} - \frac{1}{\alpha_j}(S_{x_j}^{-\alpha_j})'\partial_{x_j}$$

Therefore, $O'_{x_j} S_{x_j}^{-\alpha_j-1}\underline{x}^\alpha = S_{x_j}^{-\alpha_j} x_j^{\alpha_j} - (S_{x_j}^{-\alpha_j})'\underline{x}^{\alpha-\mathbf{v}_j}$, and, hence, the right hand sides of 1. and 2. coincide. Moreover since $(S_{x_j}^{-\alpha_j})'\underline{x}^{\alpha-\mathbf{v}_j} = S_{x_j}^{-\alpha_j}(\underline{x}^\alpha) - x_j S_{x_j}^{-\alpha_j}\underline{x}^{\alpha-\mathbf{v}_j}$, we get that the right-hand sides of 2. and 3. coincide.

Now we show that the polynomial sequence $V_\alpha(\underline{x}) = O'_{x_j}(S_{x_j}^{-\alpha_j-1}\underline{x}^\alpha)$ satisfy the conditions of Definition 1.1.11. Since the assumptions $V_\alpha(\underline{0}) = 0$ for $|\alpha| > 0$ and $V_0(\underline{x}) = 1$ follows by construction, we only need to prove the lowering property $O_{x_j}V_\alpha(\underline{x}) = \alpha_j V_{\alpha-\mathbf{v}_j}(\underline{x})$.

For this, we notice that from Lemma 1.2.1 it follows that $O_{x_j}$ and $O'_{x_j}$ commute. This leads to

$$\begin{aligned}
O_{x_j}V_\alpha(\underline{x}) &= O'_{x_j} O_{x_j}(S_{x_j}^{-\alpha_j-1}\underline{x}^\alpha) \\
&= O'_{x_j} S_{x_j}(S_{x_j}^{-\alpha_j-1}\partial_{x_j}\underline{x}^\alpha) \\
&= \alpha_j O'_{x_j}(S_{x_j}^{-\alpha_j}\underline{x}^{\alpha-\mathbf{v}_j}) \\
&= \alpha_j V_{\alpha-\mathbf{v}_j}(\underline{x}),
\end{aligned}$$

and therefore $\{V_\alpha\}_\alpha$ is a basic polynomial sequence. We are now able to derive Formula 4. According to Corollary 1.1.4 and Lemma 1.2.2, $(O'_{x_j})^{-1}$ exists. Hence 1. can be rewritten as $\underline{x}^\alpha = S_{x_j}^{\alpha_j+1}(O'_{x_j})^{-1}V_\alpha(\underline{x})$. By replacing $\underline{x}^{\alpha-\mathbf{v}_j}$ by $S_{x_j}^{\alpha_j}(O'_{x_j})^{-1}V_{\alpha-\mathbf{v}_j}(\underline{x})$ on the right-hand side of 3., one gets

$$V_\alpha(\underline{x}) = x_j(O'_{x_j})^{-1}V_{\alpha-\mathbf{v}_j}(\underline{x}).$$

■

Furthermore we are in conditions to derive the Rodrigues formula:

**Corollary 1.2.6 (Rodrigues formula)** *The basic polynomials* $V_\alpha(\underline{x})$ *are obtained through the action of* $(\underline{x}')^\alpha := \prod_{k=1}^n (x'_k)^{\alpha_k}$*, i.e.*

$$V_\alpha(\underline{x}) = (\underline{x}')^\alpha \mathbf{1}, \tag{1.9}$$

*where* $x'_k := x_k(O'_{x_k})^{-1}$.

**Remark 1.2.7** *The properties of basic polynomial sequences are naturally handled within the mapping $\Psi_{\underline{x}}$ defined viz linear extension of $\Psi_{\underline{x}} : \underline{x}^\alpha \mapsto V_\alpha(\underline{x})$.*

*This map is obviously invertible and its inverse $\Psi_{\underline{x}}^{-1}$ corresponds to $\Psi_{\underline{x}}^{-1} : V_\alpha(\underline{x}) \mapsto \underline{x}^\alpha$. This naturally induces a correspondence between the operators $x_j, \partial_{x_j}$ and $O_{x_j}, x_j(O'_{x_j})^{-1}$. Indeed $\Psi_{\underline{x}} : \underline{x}^\alpha \mapsto V_\alpha(\underline{x})$ is an intertwining operator since*

$$O_{x_j}\Psi_{\underline{x}} = \Psi_{\underline{x}}\partial_{x_j}, \quad \text{and} \quad x'_j\Psi_{\underline{x}} = \Psi_{\underline{x}}x_j$$

*fulfils in the ring of multivariate polynomials.*

*This assures that $x_j(O'_{x_j})^{-1}$ and $O_{x_j}$ can be viewed as the generators of the Bose algebra (see formulae (3.10) and Lemma 3.2.4, Subsection 3.2) and moreover Corollary 1.2.6 represents nothing else than the basic lemma in Quantum Field Theory [58].*

**Remark 1.2.8** *We would like to emphasize that there are still many possibilities for constructing operators $x'_j : V_\alpha(\underline{x}) \mapsto V_{\alpha+\mathbf{v}_j}(\underline{x})$ such that $\{V_\alpha(\underline{x})\}_\alpha$ constructed by the Rodrigues formula ( Corollary 1.2.6) are also basic polynomial sequences. In particular, it is also interesting to consider the special choice*

$$x'_j = \frac{1}{2}(x_j(O'_{x_j})^{-1} + (O'_{x_j})^{-1}x_j) \tag{1.10}$$

*Since $(O'_{x_j})^{-1}$ commutes with $O_{x_j}$, one can easily verify that $x'_j$ and $O_{x_j}$ are the canonical generators of the Bose algebra (see formulae (3.10), Section 3.2).*

*One of major reasons to consider (1.10) over $x_j(O'_{x_j})^{-1}$ is that under some special choices of $O_{x_j}$, the operators $O_{x_j}$ and $x'_j$ become symmetric and self-adjoint on a Hilbert space (c.f. [24]).*

The above remarks establish a link between the Umbral Calculus setting and the second quantization approach (c.f. [22]). This in particular is used in Chapter 3 to describe Discrete Clifford Analysis as being the canonical equivalent to the celebrated Wigner Quantum systems introduced by E.P. Wigner in [71] (see also Chapter 5 and reference [53]).

We will finish this chapter with some examples of so obtained representations for the operators $x'_j$ and $O_{x_j}$ involving finite difference operators:

**Example 1.2.9 (Forward/Backward differences)** *Denote by $\partial_h^{\pm j}$ the forward/backward difference operators by*

$$\partial_h^{\pm j} = \pm\frac{\tau_{\pm h\mathbf{v}_j} - \mathbf{id}}{h} = \pm\frac{1}{h}\left(\exp(\pm h\partial_{x_j}) - \mathbf{id}\right).$$

*These operators mutually commute when acting on functions, i.e.*

$$\partial_h^{\pm j}(\partial_h^{\pm k} f(\underline{x})) = \partial_h^{\pm k}(\partial_h^{\pm j} f(\underline{x})). \tag{1.11}$$

*and are interrelated with shifts* $\tau_{\pm h\mathbf{v}_j}$ *by*

$$\tau_{-h\mathbf{v}_j}(\partial_h^{+j} f)(\underline{x}) = (\partial_h^{-j} f)(\underline{x}), \quad \tau_{h\mathbf{v}_j}(\partial_h^{-j} f)(\underline{x}) = (\partial_h^{+j} f)(\underline{x}).$$

*Let us remark that the finite difference* $\partial_h^{\pm j}$ *acting on functions satisfies the product rule*

$$\partial_h^{\pm j}(x_j f(\underline{x})) = \pm\frac{(x_j \pm h) f(\underline{x} + h\mathbf{v}_j) - x_j f(\underline{x})}{h} = x_j \partial_h^{\pm j}(x_j f(\underline{x})) + f(\underline{x} \pm h\mathbf{v}_j). \tag{1.12}$$

*and hence* $(\partial_h^{\pm j})' = \tau_{\pm h\mathbf{v}_j}$.

*Replacing the coordinate function* $x_j$ *by the operators* $x_j\tau_{\mp h\mathbf{v}_j}$ *, we establish the duality between the finite difference operators* $\partial_h^{\pm j}$ *and the "formal" coordinate functions* $x_j\tau_{\mp h\mathbf{v}_j}$.

*We also note that the operators* $x_j\tau_{\pm h\mathbf{v}_j}$ *mutually commute when acting on functions, i.e.*

$$x_j\tau_{\pm h\mathbf{v}_j}(x_k\tau_{\pm h\mathbf{v}_k} f(\underline{x})) = x_k\tau_{\pm h\mathbf{v}_k}(x_j\tau_{\pm h\mathbf{v}_j} f(\underline{x})). \tag{1.13}$$

*Furthermore, the commutative relations (1.11), (1.13) together with the duality relation (1.12) endow an algebraic representation of the Bose algebra, where the "formal" coordinate functions* $x_j\tau_{\mp h\mathbf{v}_j}$ *represent "creation" operators dual to the "annihilation" operators* $\partial_h^{\pm j}$.

*Applying the Rodrigues formula (1.9), we obtain the following multivariate polynomials*

$$(x)_\pm^{(\alpha)} = \prod_{j=1}^n x_j(x_j \pm h)\dots(x_j \pm (\alpha_j - 1)h).$$

*These polynomials form a basic polynomial sequence. It is interesting to see that these multivariate polynomials coincide with the multi-index factorial powers* $(x)_\pm^{(\alpha)}$ *introduced by N. Faustino and U. Kähler in [30].*

*Here we would like to emphasize that the operators* $x_j\tau_{\pm h\mathbf{v}_j}$ *and* $\tau_{\pm h\mathbf{v}_j}$ *do not commute when acting on functions, i.e.*

$$\begin{aligned}
\tau_{\pm h\mathbf{v}_j}(x_k\tau_{\pm h\mathbf{v}_k} f(\underline{x})) &= \pm h\delta_{jk} f(\underline{x} \pm h(\mathbf{v}_j + \mathbf{v}_k)) + x_k\tau_{\pm h\mathbf{v}_k}(\tau_{\pm h\mathbf{v}_j} f(\underline{x})),\\
\tau_{\mp h\mathbf{v}_j}(x_k\tau_{\pm h\mathbf{v}_k} f(\underline{x})) &= \mp h\delta_{jk} f(\underline{x} \pm h(\mathbf{v}_k - \mathbf{v}_j)) + x_k\tau_{\pm h\mathbf{v}_k}(\tau_{\pm h\mathbf{v}_j} f(\underline{x})).
\end{aligned}$$

*We can consider another family of polynomials involving the position operator* $x_j'$ *defined in (1.10). In particular, recursive application of the Rodrigues formula (1.9) yields the following family of polynomials.*

$$\prod_{j=1}^n \frac{1}{2^{\alpha_j}}\left(x_j\tau_{\pm h\mathbf{v}_j} + \tau_{\pm h\mathbf{v}_j} x_j\right)^{\alpha_j}\mathbf{1} = \prod_{j=1}^n \left(x_j \pm \frac{1}{2}h\right)\left(x_j \pm \frac{3}{2}h\right)\dots\left(x_j \pm \frac{2\alpha_j - 1}{2}h\right).$$

*We will come back to the Bose algebra character of the operators* $\partial_h^{\pm j}$ *and* $x'_j$ *in Chapter 4 when establishing the comparison between the Umbral version of Fischer decomposition and the Fischer decomposition for Difference Dirac operators obtained in [30].*

**Example 1.2.10 (Central difference operator)** *In the theory of discrete analytic functions on the complex plane (c.f. [32, 28, 47]) as well as in lattice gauge theories (c.f. [72]) it is common to consider the central finite difference operator (i.e. the average between the forward and backward differences)*

$$O_{x_j} = \frac{1}{2}(\partial_h^{+j} + \partial_h^{-j}) = \frac{1}{h}\sinh(h\partial_{x_j})$$

*instead of the forward/backward differences* $\partial_h^{\pm j}$.

*Two among many reasons behind this choice is that contrary to* $\partial_h^{\pm j}$, $O_{x_j}$ *are symmetric and self-adjoint operators on the Hilbert space* $\ell_2(h\mathbb{Z}^n)$ *and also the converse of discrete integration in discrete Riemann Surfaces (c.f. [56]). The Pincherle derivative for* $O_{x_j}$ *corresponds to*

$$O'_{x_j} = \frac{\tau_{h\mathbf{v}_j} + \tau_{-h\mathbf{v}_j}}{2} = \cosh(h\partial_{x_j}).$$

*It can be easily seen that the multivariate operator* $O_{\underline{x}} = (O_{x_1}, O_{x_2}, \ldots, O_{x_n})$ *is a multivariate delta operator so it admits a basic polynomial sequence. On the other hand* $O'_{x_j}$ *has an inverse since* $O'_{x_j}1 = 1 \neq 0$.

*Now it arises the question how to compute* $(O'_{x_j})^{-1}$. *From the Isomorphism theorem (Theorem 1.1.3), finding an inverse for* $O'_{x_j}$ *is equivalent to the problem of finding an inverse for the series expansion*

$$\cosh(hx_j) = \sum_{k=0}^{\infty} \frac{1}{(2k)!}(hx_j)^{2k}.$$

*We would like to remark that finding an inverse for* $\cosh(hx_j)$ *becomes increasingly cumbersome. Nevertheless, there is a certain systematic in it. For a sake of simplicity, we impose periodic boundary conditions on our lattice such that* $\underline{x} + N\mathbf{v}_j = \underline{x}$, *i.e.* $(\tau_{h\mathbf{v}_j})^N = \mathbf{id}$ *for a certain* $N \in \mathbb{N}$.*(i.e. a periodic lattice). In particular, when* $N/2$ *is odd,* $(O'_{x_j})^{-1}$ *corresponds to (c.f. [24])*

$$(O'_{x_j})^{-1} = \sum_{k=0}^{N/2-1} (-1)^k (\tau_{h\mathbf{v}_j})^k$$

*while for* $N$ *odd one obtain*

$$(O'_{x_j})^{-1} = \sum_{k=0}^{(N-1)/2} (-1)^{k+(N-1)/2} (\tau_{h\mathbf{v}_j})^{2k} + \sum_{k=0}^{(N-1)/2-1} (-1)^k (\tau_{h\mathbf{v}_j})^{2k+1}.$$

In the language of discrete groups, the later inversion formula for the Pincherle derivative can be viewed as a series expansion in terms of left-actions for Cayley graphs (c.f. [25]) while the periodic condition $(\tau_{h\mathbf{v}_j})^N = \mathbf{id}$ ensures that our Cayley graph is a cyclic group. This will be of special interest in Section 3.4 when we introduce the discrete Clifford setting.

We will see afterwards in Subsection 2.1.1 that the exterior differential calculus makes then appear $O_{\underline{x}}$ as a multivariate delta operator and the Pincherle derivatives $O'_{x_j}$, $j = 1, \ldots, n$ as a set of automorphisms (c.f. [26]).

# Chapter 2

# Rediscovering Differential Geometry

*"There is nothing in yesterday mathematics that you can prove with exterior algebra that could not also be proved without it. Exterior algebra is not meant to prove old facts, it is meant to disclose a new world."*

Gian-Carlo Rota

In this Section, we will explore the structure of $(\mathbf{d}, \Lambda^*\mathcal{A})$ beyond the Umbral algebra $\mathcal{U}$.

## 2.1 The Structure of Exterior Differential Calculus

Let $\mathcal{A}$ an associative and Abelian algebra of functions endowed with the standard addition, grading-respecting multiplication and identity $\mathbf{1}$. Denote by $\Lambda^r\mathcal{A}$ the $\mathcal{A}-$modules with $\Lambda^0\mathcal{A} := \mathcal{A}$ and $\Lambda^*\mathcal{A} = \sum_{r=0}^{n} \bigoplus \Lambda^r\mathcal{A}$ denotes the differential envelope of all this forms.

We define the universal differential calculus as the pair $(\mathbf{d}, \Lambda^*\mathcal{A})$, where $\Lambda^*\mathcal{A}$ is a $\mathbb{Z}-$graded associative algebra (over $\mathbb{R}$ or $\mathbb{C}$), $\Lambda^r\mathcal{A}$ (the space of $r-$forms) are $\mathcal{A}-$modules and $\mathbf{d} : \Lambda^r\mathcal{A} \to \Lambda^{r+1}\mathcal{A}$ a $\mathbb{R}-$(or $\mathbb{C}$-) linear map with the following properties,

$$\mathbf{d}(\mathbf{d}\omega) = 0, \tag{2.1}$$

$$\mathbf{d}(\omega^r\omega) = (\mathbf{d}\omega^r)\omega + (-1)^r\omega^r\mathbf{d}\omega, \tag{2.2}$$

where $\omega^r \in \Lambda^r\mathcal{A}$ and $\omega \in \Lambda^*\mathcal{A}$.

The relation (2.1) is known as the nil-potency of $\mathbf{d}$ while the relation (2.2) is known as the (generalized) Leibniz rule. The identity $\mathbf{1} = \mathbf{1}\mathbf{1}$ then implies $\mathbf{d}\mathbf{1} = \mathbf{0}$.

The above treatment is quite general and states that whenever an algebra $\mathcal{A}$ is given as well as an operation which satisfy equations (2.1), (2.2) then we can construct a consistent differential envelope for the algebra.

### 2.1.1 Correspondence between Differential Calculus and Umbral Calculus

In the following, we assume that $\mathcal{A}$ is a set of functions written in $\mathbb{R}^n$ in terms of the $n-$commuting coordinate functions $x_1, x_2, \ldots, x_n$ and $\mathbf{d}$ is constructed inductively as a mapping of coordinate functions $x_j$ onto coordinate differentials $\mathbf{d}x_j$ and $\Lambda^1\mathcal{A}$ is the linear space generated by $\mathbf{d}x_1, \mathbf{d}x_2, \ldots, \mathbf{d}x_n$.

In order to obtain a consistent universal differential calculus $(\mathbf{d}, \Lambda^*\mathcal{A})$ over $\mathcal{A}$, we might want the space of $1-$forms to be generated as a left $\mathcal{A}-$module. We start to define $(\mathbf{d}, \Lambda^*\mathcal{A})$ by imposing the basic relations between coordinate functions and coordinate differentials

$$\mathbf{d}(\mathbf{d}x_j) = 0, \tag{2.3}$$

$$\mathbf{d}x_j = [\mathbf{d}, x_j], \tag{2.4}$$

$$x_j\mathbf{d}x_k = \mathbf{d}x_k x_j, \quad \text{for } j \neq k, \tag{2.5}$$

where on the right hand side of (2.4), the commutator $[\mathbf{d}, x_j]$ acts as a multiplication operator on $\Lambda^*\mathcal{A}$.

These relations are justified by the following assumptions: We want that the operator $\mathbf{d}$ satisfy the nilpotency rule (2.1) (or relation (2.3)) and the Leibniz rule (2.2) (or relation (2.4)). The relation (2.5) might seem a bit strange, but are necessary to ensure that $\mathbf{d}$ is well defined. Indeed, from (2.5) we have that $\mathbf{d}(x_jx_k) = \mathbf{d}(x_kx_j)$.

Now we introduce $\mathcal{T}$ as a complex linear space dual to linear space of $1-$forms $\Lambda^1\mathcal{A}$ with basis $\{O_{x_j} \ : \ j = 1, \ldots, n\}$ dual to $\{\mathbf{d}x_j \ : \ j = 1, \ldots, n\}$, i.e.

$$\langle \mathbf{d}x_k, O_{x_j}\rangle_0 = \delta_{jk}, \tag{2.6}$$

where the left-hand side of (2.6) is nothing else than duality pairing between $\mathbf{d}x_k \in \Lambda^1\mathcal{A}$ and $O_{x_j} \in \mathcal{T}$.

The relation (2.6) can be extended *viz* bi-orthogonal expansion of $\langle \cdot, \cdot \rangle_0$ in terms of the bases $\mathbf{d}x_j$ and $O_{x_j}$, that is

$$\langle \theta, \mathbf{u}\rangle_0 = \sum_{j=1}^n \langle \theta, O_{x_j}\rangle_0 \langle \mathbf{d}x_j, \mathbf{u}\rangle_0$$

The above complex linear space is usually named as the tangent space (c.f. [3, 1]). We are now in conditions to introduce the duality contraction between $1-$forms and vector-fields belonging to the tangent space $\mathcal{T}$.

**Definition 2.1.1 (Duality contraction)** *Let $\theta$ be a linear combination in terms of 1-forms $\mathbf{d}x_j$ and let $\mathbf{u} = \sum_{j=1}^n u_j(\underline{x})O_{x_j}$ be a basic vector-field.*

*We define the duality contraction* $\rfloor$ *as a bilinear form acting on* $\Lambda^1\mathcal{A}$ *as a right* $\mathcal{A}-$*module and on* $\mathcal{T}$ *as a left* $\mathcal{A}-$*module, i.e.*

$$\mathbf{u}\rfloor\left(F(\underline{x})\ \theta\right) \quad = \quad F(\underline{x})\langle\theta,\mathbf{u}\rangle_0.$$

In the language of differential geometry, vector-fields $\mathbf{u}\in\mathcal{T}$ become operators acting on $\mathcal{A}$ *viz*

$$\mathbf{u}(F(\underline{x})) = \mathbf{u}\rfloor(\mathbf{d}F(\underline{x})), \tag{2.7}$$

where the right-hand side of (2.7) can be interpreted as the Lie derivative (see Definition 2.1.15) acting on 0−forms.

Using (2.7), the linear extension of $\mathbf{d}: x_j \mapsto \mathbf{d}x_j$ then corresponds to

$$\mathbf{d}F = \sum_{j=1}^{n} O_{x_j}(F(\underline{x}))\ \mathbf{d}x_j. \tag{2.8}$$

There are some formulae that follow from (2.7) that will be of interest. One more important formula is given by the next lemma.

**Lemma 2.1.2** *When acting on functions* $F\in\mathcal{A}$*, the operator* $\mathbf{u}(\cdot) := \mathbf{u}\rfloor(\mathbf{d}\cdot)$ *satisfies*

$$\mathbf{u}\left(\mathbf{d}x_jF(\underline{x})\right) = [\mathbf{u},x_j]F(\underline{x}). \tag{2.9}$$

**Proof:** Starting from the equation (2.7) we obtain

$$[\mathbf{u},x_j]F(\underline{x}) = \mathbf{u}(x_jF(\underline{x})) - x_j\mathbf{u}(F(\underline{x})) = \mathbf{u}\rfloor\left(\mathbf{d}\left(x_jF(\underline{x})\right)\right) - x_j\mathbf{u}\rfloor\left(\mathbf{d}F(\underline{x})\right). \tag{2.10}$$

On the other hand, from the direct application of Definition 2.1.1 and equation (2.4), the right hand side of (2.10) becomes

$$\mathbf{u}\rfloor\left(\mathbf{d}\left(x_jF(\underline{x})\right)\right) - x_j\mathbf{u}\rfloor\left(\mathbf{d}F(\underline{x})\right) = \mathbf{u}\rfloor\left(\mathbf{d}(x_jF(\underline{x})) - x_j\mathbf{d}(F(\underline{x}))\right) = \mathbf{u}\rfloor(\mathbf{d}x_jF(\underline{x})).$$

This altogether proves (2.9).

■

A direct consequence of the above lemma is the generalization of the (possible) noncommutative relations between functions and 1−forms shown in [10, 25, 23]. Indeed, for the particular choice $\mathbf{u} = O_{x_j}$, the left-hand side of (2.7) corresponds to the Pincherle derivative $O'_{x_j}$ acting on $F(\underline{x})$. Moreover, formula (2.8) and the first relation given in (2.4) leads to

$$\mathbf{d}x_jF(\underline{x}) = O'_{x_j}(F(\underline{x}))\mathbf{d}x_j. \tag{2.11}$$

Generalization to higher forms is now straightforward:

For a vector of coordinate differentials $\mathbf{d}\underline{x} = (\mathbf{d}x_1, \mathbf{d}x_2, \ldots, \mathbf{d}x_n)$, we set

$$(\mathbf{d}\underline{x})^\alpha = \prod_{j=1}^{n} (\mathbf{d}x_j)^{\alpha_j}, \quad |\alpha| = r$$

as an ordered basis for the $\mathcal{A}-$module of $r-$forms.

Let us notice that from the property (2.16), the multi-index $\alpha$ belongs to $\{0,1\}^n$ and hence $0 \leq r \leq n$. Therefore, the elements $\omega^r \in \Lambda^r\mathcal{A}$ and $\omega \in \Lambda^*\mathcal{A}$ are given by

$$\begin{aligned} \omega^r &= \textstyle\sum_{|\alpha|=r} F^\alpha(\underline{x})(\mathbf{d}\underline{x})^\alpha, \\ \omega &= \textstyle\sum_{r=0}^{n} \omega^r. \end{aligned} \tag{2.12}$$

Take into account the coordinate expression (2.8) and the relations (2.4), the $\mathbf{d}-$action on $r-$forms (2.12) then corresponds to:

$$\mathbf{d}\omega^r = \sum_{|\alpha|=r} \mathbf{d}(F^\alpha(\underline{x}))(\mathbf{d}\underline{x})^\alpha = \sum_{j=1}^{n} \sum_{|\alpha|=r} O_{x_j}(F^\alpha(\underline{x}))\ \mathbf{d}x_j(\mathbf{d}\underline{x})^\alpha \tag{2.13}$$

which shows that the exterior derivative acting on $r-$forms is completely determined by the exterior derivative acting on 1-forms.

The following result relates the nilpotency character of the exterior derivative $\mathbf{d}$ between the mutually commutativity of basic vector-fields on the tangent space $\mathcal{T}$.

**Lemma 2.1.3** *The nilpotency relation (2.1) is fulfilled in $\Lambda^*\mathcal{A}$ if and only if the basic vector-fields $O_{x_j} \in \mathcal{T}$ mutually commute.*

**Proof:** Starting from the coordinate expression (2.13), it is enough to show Lemma 2.1.3 for the space of $1-$forms.

From the coordinate action (2.8) and (2.16) the coordinate action of $\mathbf{d}^2$ is given by

$$\begin{aligned} \mathbf{d}(\mathbf{d}F) &= \textstyle\sum_{j=1}^{n} \mathbf{d}(O_{x_j}F(\underline{x}))\mathbf{d}x_j \\ &= \textstyle\sum_{j=1}^{n} \sum_{k<j} \left(O_{x_k}(O_{x_j}F(\underline{x})) - O_{x_j}(O_{x_k}F(\underline{x}))\right) \mathbf{d}x_k\mathbf{d}x_j. \end{aligned} \tag{2.14}$$

If the basic vector-fields $O_{x_j} \in \mathcal{T}$ mutually commute, the nilpotency relation (2.1) easily follows.

Conversely, if the nilpotency relation (2.1) is fulfilled, by applying the duality contraction operators $O_{x_j}\rfloor$ and $O_{x_k}\rfloor$ on both sides of (2.14) we obtain

$$0 = O_{x_j}\rfloor\,(O_{x_k}\rfloor\,(\mathbf{d}(\mathbf{d}F(\underline{x})))) = O_{x_k}(O_{x_j}F(\underline{x})) - O_{x_j}(O_{x_k}F(\underline{x})).$$

This completes the proof of Lemma 2.1.3.

■

We are now in conditions to derive the following description of the Umbral algebra in terms of differential forms:

**Proposition 2.1.4** *Assume that when acting on $\Lambda^*\mathcal{A}$, the exterior derivative* $\mathbf{d}$ *satisfies condition* $[\mathbf{d}, \exp(\underline{y} \cdot \partial_{\underline{x}})] = 0$, *i.e.* $\mathbf{d}$ *is shift-invariant* (1.3). *Then the nilpotency relation (2.1) holds on $\Lambda^*\mathcal{A}$.*

**Proof:** As in Lemma 2.1.3, it is enough to show Lemma 2.1.4 for the space of $1-$forms.

From the coordinate expression (2.8), the condition $[\mathbf{d}, \exp(\underline{y} \cdot \partial_{\underline{x}})] = 0$ is equivalent to say that the basic vector-fields $O_{x_j}$ satisfy (1.3) (i.e. $O_{x_j}$ belong to the Umbral algebra $\mathcal{U}$).

Hence the basic vector-fields $O_{x_j} \in \mathcal{T}$ mutually commute since using Theorem 1.1.2, they are expressed in terms of partial derivatives $\partial_{x_j}$. This together with Lemma 2.1.3 concludes the proof of Proposition 2.1.4.

■

With Proposition 2.1.4 we show that differential calculus associated to a shift-invariant exterior derivative $\mathbf{d}$ naturally encodes the multivariate delta operator $O_{\underline{x}}$ (see Definition (1.1.5)) and indeed the multivariate umbral calculus as a whole. Indeed, the relations (2.7) and (2.6) completely determine $O_{x_j}$ as a delta operator (see Definition 1.1.5, Section 1.1). Moreover by combining the Isomorphism theorem (Theorem 1.1.3, Section 1.1) with Lemma 1.2.2 (Section 1.2), the inverse $(O'_{x_j})^{-1}$ always exists and commutes with all $O_{x_k}$, $k = 1, 2, \ldots, n$.

According to (2.11), we obtain an interesting description for the duality contraction given in Definition 2.1.1:

$$\mathbf{u}(F(\underline{x})\mathbf{d}x_k) = \left[\mathbf{u}, x_k(O'_{x_k})^{-1}\right] F(\underline{x}), \tag{2.15}$$

where $\left[\mathbf{u}, x_k(O'_{x_k})^{-1}\right] = \langle \mathbf{d}x_k, \mathbf{u}\rangle_0 \mathbf{id}$.

Now we are in conditions to prove the following consistency relation for the space of 1-forms.

**Proposition 2.1.5** *Under the conditions of equations (2.4)-(2.5) we have*

$$\mathbf{d}x_j \mathbf{d}x_k + \mathbf{d}x_k \mathbf{d}x_j = 0 \tag{2.16}$$

**Proof:** From the equations (2.4) and (2.5) we obtain

$$\mathbf{d}x_j \mathbf{d}x_k + \mathbf{d}x_k \mathbf{d}x_j = \mathbf{d}(x_j \mathbf{d}x_k + x_k \mathbf{d}x_j) = \mathbf{d}(\mathbf{d}(x_j x_k)).$$

From Proposition 2.1.4 $\mathbf{d}(\mathbf{d}(x_j x_k)) = 0$ and hence, the later relation becomes $\mathbf{d}x_j\mathbf{d}x_k + \mathbf{d}x_k\mathbf{d}x_j = 0$. This concludes the proof of (2.16).

■

According to the classical differential calculus literature (see e.g. [3, 1, 10] and references given there), the consistency relation (2.16) is nothing else than the exterior product rule between basic $1-$forms. Geometrically, the product $\mathbf{d}x_j\mathbf{d}x_k$ describes the area of an oriented parallelogram. For $j = k$, the relation (2.16) ensures that parallelogram is not indefinite, while for $j \neq k$ (2.16) ensures that the parallelograms described by $\mathbf{d}x_j\mathbf{d}x_k$ and $\mathbf{d}x_k\mathbf{d}x_j$ has opposite orientations.

There is also an alternative interpretation concerning discrete structures.

**Remark 2.1.6** *According to [23], discrete structures are identified as a certain kind of inner differential calculi over a discrete set, i.e.*

$$\mathbf{d}F = \sum_{j=1}^{n} [\mathbf{d}x_j, F]$$

*where the sum of all $1-$forms $\sum_{j=1}^{n} \mathbf{d}x_j$ assigns the adjacency matrix of an oriented graph.*

*Indeed, the non-vanishing quantities $x_j\mathbf{d}x_k$ can be interpreted as a directed edge from $x_j$ to $x_k$ [23]. Regardless the last viewpoint, the condition (2.5) means that the orientation of the direct edge pointing the vertex $x_j$ to $x_k$ is opposite to the orientation of the direct edge pointing the vertex $x_k$ to $x_j$.*

*The product $x_j\mathbf{d}x_k$ for $j \neq k$ assigns 2-paths on a graph while the relation $\mathbf{d}x_j\mathbf{d}x_k + \mathbf{d}x_k\mathbf{d}x_j = 0$ means that multiple edges and loops are not admitted in our graph structure (see e.g. [34], Section 3.4)*

With the above considerations, we shown that our approach unifies the classical and the discrete differential calculus as a whole.

It's also interesting to make a comparison between our approach and the approach proposed by A. Dimakis/F. Müller-Hoissen in [26], where they construct a differential calculus over associative algebras determined by a set of automorphisms.

**Remark 2.1.7** *On the terminology of [26], the (possible) noncommutative relations between elements $F \in \mathcal{A}$ and $1-$forms $\theta^j \in \Lambda^1\mathcal{A}$ are given by*

$$\theta^j F = \sum_{k\in J} \Psi^j_k(F)\theta^k, \quad \forall_{F\in\mathcal{A}}, \tag{2.17}$$

*where $J$ is some finite set and $F \mapsto \Psi(F)$ is an algebra isomorphism.*

*With the particular choice $\Psi_k^j = [O_{x_j}, x_k]$, the algebra isomorphism is diagonal and it satisfies equation (2.17). Hence, the main automorphisms $\psi_j$ of $\mathcal{A}$ satisfying equation (1.1) correspond to the Pincherle derivatives $O'_{x_j}$ associated to delta operators $O_{x_j}$.*

With the above remark, we shown that our framework gives a quite natural way to describe the generalized differential calculus as a realization of the Umbral algebra. Indeed according to (2.15), the commutator between *raising* and *lowering* operators acting on functions $F(\underline{x}) \in \mathcal{A}$ naturally encodes the duality contraction (see Definition 2.1.1).

We would like to stress that due to the noncommutative character of the formalism, as a consequence of $O'_{x_j} \neq \mathbf{id}$ in general, there is no chance to obtain a 'true' Leibniz rule at the level of vector-fields.

We will come back to the non-Leibniz character of $O_{x_j}$ when we describe the Umbral calculus in terms of *raising* and *lowering* operators.

### 2.1.2 Exterior Product, Interior Product and Lie Derivative

Using the framework developed on the above subsection, we thus have the basic ingredients to define the main operators in exterior differential calculus.

In order to define the exterior and interior product on the algebra of endomorphisms $\mathrm{End}(\Lambda^*\mathcal{A})$, we may think of exterior product as an operator in the space of forms which maps $r$-forms onto $(r+1)$-forms and the interior product as an operator which maps $r$-forms onto $(r-1)$-forms by combining forms with vector-fields.

**Definition 2.1.8 (Exterior product)** *For a $1-$form $\theta \in \Lambda^1\mathcal{A}$, we define the exterior product $\varepsilon : \Lambda^1\mathcal{A} \to \Lambda^*\mathcal{A}$ as a left representation given by the mapping $\varepsilon_\theta : \omega \mapsto \theta\ \omega$.*

From (2.11) and (2.16) we obtain the suggestive action for the exterior product.

**Lemma 2.1.9** *When acting on the space of $\Lambda^*\mathcal{A}$, the exterior product $\varepsilon_{\mathbf{d}x_j} \in End(\Lambda^*\mathcal{A})$ satisfies*

$$\varepsilon_{\mathbf{d}x_j}\omega = (\sigma\ O'_{x_j}\omega)\mathbf{d}x_j, \tag{2.18}$$

*where $\sigma : \Lambda^r\mathcal{A} \to \Lambda^r\mathcal{A}$ denotes the graded involution operator $\sigma : \omega^r \mapsto (-1)^r\omega^r$.*

**Proof:** By acting $\epsilon_{\mathbf{d}x_j}$ on $\omega^r = F(\underline{x})(\mathbf{d}\underline{x})^\alpha$, with $|\alpha| = r$ one obtains from (2.11) the relation $\varepsilon_{\mathbf{d}x_j}\omega = O'_{x_j}F(\underline{x})\mathbf{d}x_j(\mathbf{d}\underline{x})^\alpha$.

First notice that formula (2.18) follows automatically for $r=0$ from (2.11). Otherwise, we can split $(\mathbf{d}\underline{x})^\alpha$, with $|\alpha|=r$ as $(\mathbf{d}\underline{x})^\alpha=\prod_{k=1}^r \mathbf{d}x_{j_k}$, where the indices $j_k$ belong to the set $A_\alpha=\{j\ :\ \alpha_j\neq 0\}$.

Without loss of generality, we assume that $j\not\in A_\alpha$. Using recursion on $r>0$, we obtain

$$\begin{aligned}\mathbf{d}x_j(\mathbf{d}\underline{x})^\alpha &= -\mathbf{d}x_{j_1}\mathbf{d}x_j\prod_{k=2}^r \mathbf{d}x_{j_k}\\ &= (-1)^2\mathbf{d}x_{j_1}\mathbf{d}x_{j_2}\mathbf{d}x_j\prod_{k=3}^r \mathbf{d}x_{j_k}\\ &= \ldots\\ &= (-1)^r(\mathbf{d}\underline{x})^\alpha\mathbf{d}x_j.\end{aligned}$$

Furthermore by acting $\epsilon_{\mathbf{d}x_j}$ on $\omega^r=F(\underline{x})(\mathbf{d}\underline{x})^\alpha$, with $|\alpha|=r$, one obtains from (2.11) the relation $\varepsilon_{\mathbf{d}x_j}\omega=(-1)^r O'_{x_j}F(\underline{x})(\mathbf{d}\underline{x})^\alpha\mathbf{d}x_j$.

By a linearity argument the above extends to arbitrary $r-$forms (given in (2.12)) and moreover to arbitrary forms belonging to the space $\Lambda^*\mathcal{A}$.

■

We would like to point out that the right-hand side of (2.18) can be understood as a right representation operator acting on the algebra of endomorphisms $\mathrm{End}(\Lambda^*\mathcal{A})$. Now we are in conditions to prove the Grassmann algebra structure associated to our exterior calculus.

**Proposition 2.1.10** *Under the conditions of Proposition 2.1.4, the basic endomorphisms $\varepsilon_{\mathbf{d}x_j}\in End(\Lambda^*\mathcal{A})$ satisfies the anti-commutation rule $\{\varepsilon_{\mathbf{d}x_j},\varepsilon_{\mathbf{d}x_k}\}=0$.*

**Proof:** First we would like to point out that the condition $[\mathbf{d},\exp(\underline{y}\cdot\partial_{\underline{x}})]=0$ is equivalent to say that the basic vector-fields $O_{x_j}$ are shift-invariant (see formula (1.3), Chapter 1) and hence from the First Expansion Theorem (Theorem 1.1.2), the basic vector-fields $O_{x_j}$ mutually commute. Hence the Isomorphism theorem (Theorem 1.1.3) ensures that the Pincherle derivatives $O'_{x_j}$ mutually commute.

Finally, by letting act $\{\varepsilon_{\mathbf{d}x_j},\varepsilon_{\mathbf{d}x_k}\}$ on forms $\omega\in\Lambda^*\mathcal{A}$, one obtains from Lemma 2.1.2 and (2.16) the relation

$$\varepsilon_{\mathbf{d}x_j}(\varepsilon_{\mathbf{d}x_k}\omega)+\varepsilon_{\mathbf{d}x_j}(\varepsilon_{\mathbf{d}x_k}\omega)=O'_{x_j}(O'_{x_k}\omega)(\mathbf{d}x_j\mathbf{d}x_k+\mathbf{d}x_k\mathbf{d}x_j)=0.$$

■

Having defined the exterior product operator, our next step is to obtain the extension of the duality contraction action (2.1.1) to the exterior algebra $\Lambda^*\mathcal{A}$. Based on (2.1.1) we start by introducing the following definition:

**Definition 2.1.11 (Duality contractor)** *For a vector-field* $\mathbf{u}$ *belonging to the tangent space* $\mathcal{T}$*, we define the duality contractor* $\mathbf{u}\rfloor \in End(\Lambda^*\mathcal{A})$ *as a right representation given by the mapping*

$$\mathbf{u}\rfloor : \omega\ \theta \mapsto \langle \theta, \mathbf{u}\rangle_0\ \omega, \quad \textit{for all} \ \ \theta \in \Lambda^1\mathcal{A} \tag{2.19}$$

*where* $\langle\cdot,\cdot\rangle_0$ *denotes the dual pairing given by formula (2.6).*

From recursive application of the above Definition on $r-$forms $\Lambda^r\mathcal{A}$ one can derive an explicit expression for $\mathbf{u}\rfloor \in \mathrm{End}(\Lambda^*\mathcal{A})$.

**Lemma 2.1.12** *The action* $\mathbf{u}\rfloor \in End(\Lambda^*\mathcal{A})$ *is given by the linear extension of*

$$\mathbf{u}\rfloor(F(\underline{x})) = 0, \tag{2.20}$$

$$\mathbf{u}\rfloor(F(\underline{x})(\mathbf{d}\underline{x})^\alpha) = (-1)^{|\alpha|+1}\sum_{k=1}^{r}(-1)^{k-1}\mathbf{u}\rfloor(F(\underline{x})\mathbf{d}x_{j_k})\ (\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}}, \tag{2.21}$$

*where the term* $(\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}}$ *means that* $\mathbf{d}x_{j_k}$ *is omitted from the product* $(\mathbf{d}\underline{x})^\alpha$*.*

**Proof:** First notice that formula (2.18) follows automatically for $r = 0$ from (2.6) and (2.1.1). Otherwise notice that for $|\alpha| = r > 0$ we have $(\mathbf{d}\underline{x})^\alpha = \prod_{k=1}^r \mathbf{d}x_{j_k}$, where the indices $j_k$ belong to the set $A_\alpha = \{j \ : \ \alpha_j \neq 0\}$. Using recursion on $r$, we obtain

$$\begin{aligned}
(\mathbf{d}\underline{x})^\alpha &= -\mathbf{d}x_{j_1}\ldots\mathbf{d}x_{j_{k+1}}\mathbf{d}x_{j_k}\mathbf{d}x_{j_{k+2}}\ldots\mathbf{d}x_{j_r}\\
&= (-1)^2\mathbf{d}x_{j_1}\ldots\mathbf{d}x_{j_{k+1}}\mathbf{d}x_{j_{k+2}}\mathbf{d}x_{j_k}\mathbf{d}x_{j_{k+3}}\ldots\mathbf{d}x_{j_r}\\
&= \ldots\\
&= (-1)^{r-k}(\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}}\mathbf{d}x_{j_k}\\
&= (-1)^{r+1}(-1)^{k-1}(\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}}\mathbf{d}x_{j_k}.
\end{aligned}$$

Hence direct application of definition (2.19) leads to

$$\begin{aligned}
O_{x_j}\rfloor(F(\underline{x})(\mathbf{d}\underline{x})^\alpha) &= (-1)^{r+1}(-1)^{k-1}O_{x_j}(F(\underline{x})\mathbf{d}x_{j_k})(\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}}\\
&= (-1)^{r+1}(-1)^{k-1}\delta_{j,j_k}F(\underline{x})\mathbf{d}x_{j_k}(\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}}.
\end{aligned}$$

Using formula (2.15), the extension to arbitrary vector-fields $\mathbf{u}\in\mathcal{T}$ given by

$$\mathbf{u}\rfloor(F(\underline{x})(\mathbf{d}\underline{x})^\alpha) = (-1)^{r+1}\sum_{k=1}^{r}(-1)^{k-1}\left[\mathbf{u}, x_{j_k}(O'_{x_{j_k}})^{-1}\right]F(\underline{x})(\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}}$$

follows by linearity.

■

As a consequence of the following, the next proposition holds

**Proposition 2.1.13** *For the basic vector-fields* $\mathbf{u}, \mathbf{v} \in \mathcal{T}$*, we have*

$$\{\mathbf{u}\rfloor, \mathbf{v}\rfloor\} = 0. \tag{2.22}$$

**Proof:** Let $O_{x_k}, O_{x_l} \in \mathcal{T}$. Take into account the relation $\mathbf{u}\rfloor(F(\underline{x})(\mathbf{d}\underline{x})^\alpha) = F(\underline{x})\mathbf{u}\rfloor(\mathbf{d}\underline{x})^\alpha$ for $|\alpha| = r$, it is enough to show that $O_{x_k}\rfloor(O_{x_l}\rfloor(\mathbf{d}\underline{x})^\alpha) + O_{x_l}\rfloor(O_{x_k}\rfloor(\mathbf{d}\underline{x})^\alpha)$.

For $r \in \{0, 1\}$, the proof of (2.22) follows by direct application of (2.20) and (2.21) since $O_{x_k}\rfloor(O_{x_l}\rfloor(\mathbf{d}\underline{x})^\alpha) = 0 = O_{x_l}\rfloor(O_{x_k}\rfloor(\mathbf{d}\underline{x})^\alpha)$.

For $r > 1$ we can split $(\mathbf{d}\underline{x})^\alpha$ with $|\alpha| = r$ as $(\mathbf{d}\underline{x})^\alpha = \prod_{k=1}^r \mathbf{d}x_{j_k}$, where the indices $j_k$ belong to the set $A_\alpha = \{j \ : \ \alpha_j \neq 0\}$. Again, without loss of generality, we assume that $k, l \in A_\alpha$, i.e. $k = j_k$ and $l = j_l$. Using recursion on $r$, we obtain

$$(\mathbf{d}\underline{x})^\alpha = (-1)^{r-l}(\mathbf{d}\underline{x})^{\alpha - \mathbf{v}_{j_l}}\mathbf{d}x_{j_l} = (-1)^{r-l}(-1)^{r-1-k}(\mathbf{d}\underline{x})^{\alpha - \mathbf{v}_{j_k} - \mathbf{v}_{j_l}}\mathbf{d}x_{j_k}\mathbf{d}x_{j_l}.$$

Direct application of (2.6) leads to

$$O_{x_l}\rfloor(O_{x_k}\rfloor(\mathbf{d}\underline{x})^\alpha) = (-1)^{r-l}(-1)^{r-1-k}(\mathbf{d}\underline{x})^{\alpha - \mathbf{v}_{j_k} - \mathbf{v}_{j_l}} = -O_{x_k}\rfloor(O_{x_l}\rfloor(\mathbf{d}\underline{x})^\alpha),$$

as required. Thus by linearity arguments the above extends to arbitrary vector-fields $\mathbf{u}, \mathbf{v} \in \mathcal{T}$.

■

The above lemmata gives us a nice motivation to define the interior product as follows:

**Definition 2.1.14 (Interior product)** *We define the interior product* $\mathbf{i}_\mathbf{u} \in End(\Lambda^*\mathcal{A})$ *as the mapping*

$$\mathbf{i}_\mathbf{u} : \omega \mapsto -\mathbf{u}\rfloor(\sigma\omega) \tag{2.23}$$

*where* $\sigma : \Lambda^r\mathcal{A} \to \Lambda^r\mathcal{A}$ *denotes the graded involution operator* $\sigma : \omega^r \mapsto (-1)^r\omega^r$.

Once the interior product has been defined, one can introduce the Lie derivative by means of the interior product combined with the Cartan magic (or homotopy formula) (see e.g. [1]):

**Definition 2.1.15 (Lie derivative)** *For a vector-field $\mathbf{u} \in \mathcal{T}$, we define the Lie derivative in the direction $\mathbf{u}$, $\mathcal{L}_{\mathbf{u}} \in End(\Lambda^* \mathcal{A})$, as the anti-commutator bracket between the exterior derivative $\mathbf{d}$ and the interior product $\mathbf{i}_{\mathbf{u}}$, i.e.*

$$\mathcal{L}_{\mathbf{u}} = \{\mathbf{d}, \mathbf{i}_{\mathbf{u}}\}. \tag{2.24}$$

By assuming that we are in the conditions of Proposition 2.1.4, it is clear that the operators $\mathbf{d}$ and $\mathcal{L}_{\mathbf{u}}$ commute, since

$$\mathbf{d}\mathcal{L}_{\mathbf{u}} = \mathbf{d}(\mathbf{d}\mathbf{i}_{\mathbf{u}} + \mathbf{i}_{\mathbf{u}}\mathbf{d}) = \mathbf{d}\mathbf{i}_{\mathbf{u}}\mathbf{d} = \mathcal{L}_{\mathbf{u}}\mathbf{d}, \tag{2.25}$$

Moreover, from Definition 2.1.14 and formula (2.22) the same is also fulfilled between the operators $\mathbf{i}_{\mathbf{u}}$ and $\mathcal{L}_{\mathbf{u}}$, i.e.

$$\mathbf{i}_{\mathbf{u}}\mathcal{L}_{\mathbf{u}} = \mathbf{i}_{\mathbf{u}}\mathbf{d}\mathbf{i}_{\mathbf{u}} = \mathcal{L}_{\mathbf{u}}\mathbf{i}_{\mathbf{u}}. \tag{2.26}$$

We would like to remark that the Lie derivative extends the notion of directional derivative. In particular, when acting on 0-forms (2.24) is just the operator $\mathbf{u}(\cdot)$ (see formula (2.7)) and when acts on the space of $n-$forms (i.e. volume forms), the relation $\mathcal{L}_{\mathbf{u}}\omega^n = \mathbf{d}(\mathbf{i}_{\mathbf{u}}\omega^n)$ defines implicitly a divergence along the vector-field $\mathbf{u} \in \mathcal{T}$. Indeed $\mathbf{d}(\mathbf{i}_{\mathbf{u}}\omega^n) = \sum_{j=1}^n O_{x_j} u_j(\underline{x})\omega^n$, where the sum $\sum_{j=1}^n O_{x_j} u_j(\underline{x})$ is the umbral counterpart of the divergence operator div.

We will turn to the concept of Lie derivative when we speak about the discrete Poincaré lemma (see Section 4.4) and integration over chains.

The following graded deformed anti-commutation rule between the interior and exterior product will be of special interest afterwards:

**Lemma 2.1.16** *Under the conditions of Proposition 2.1.4, we have*

$$\{\mathbf{i}_{\mathbf{u}}, \varepsilon_{\mathbf{d}x_k}\} = [\mathbf{u}, x_k]. \tag{2.27}$$

**Proof:** From Lemma 2.1.12 is is enough to show that the relation (2.27) holds for $\omega^r = F(\underline{x})(\mathbf{d}\underline{x})^\alpha$, with $|\alpha| = r > 0$.

When $\{\mathbf{i}_{\mathbf{u}}, \varepsilon_{\mathbf{d}x_j}\}$ acts on $0-$forms, the proof of Lemma 2.1.16 follows straightforward from relations (2.20) and (2.9). Otherwise, by combining the relations (2.21), (2.11) and (2.9), we can split $\mathbf{i}_{\mathbf{u}}(\varepsilon_{\mathbf{d}x_k}\omega^r)$ as

$$\mathbf{i}_{\mathbf{u}}(\varepsilon_{\mathbf{d}x_j}\omega^r) = [\mathbf{u}, x_j]\omega^r + \sum_{k=2}^{r+1}(-1)^{k-1}\mathbf{u}\rfloor(O'_{x_{j_j}}F(\underline{x})\mathbf{d}x_{j_k})\mathbf{d}x_j(\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}}.$$

From (2.15) $\mathbf{u}\rfloor(O'_{x_j}F(\underline{x})\mathbf{d}x_{j_k})$ is equal to $[\mathbf{u}, x_{j_k}(O'_{x_{j_k}})^{-1}]O'_{x_{j_k}}F(\underline{x})$ and $[\mathbf{u}, x_{j_k}(O'_{x_{j_k}})^{-1}]$ commutes with $\mathbf{d}x_j$. Furthermore using (2.11) we get

$$\sum_{k=2}^{r+1}(-1)^{k-1}\mathbf{u}\rfloor(O'_{x_j}F(\underline{x})\mathbf{d}x_{j_k})\mathbf{d}x_j(\mathbf{d}\underline{x})^{\alpha-\mathbf{v}_{j_k}} = -\varepsilon_{\mathbf{d}x_j}(\mathbf{i_u}\omega^r).$$

This altogether concludes the proof of Lemma 2.1.16.

■

Having the interior and exterior product at hand, is now interesting to establish a first contact with Clifford algebras (see also Section 3.1). According to [36, 66] they can be defined as a subalgebra of the algebra of the endomorphisms. This consist in two take the following two representations for the algebra of endomorphisms

$$\begin{aligned} \Upsilon^-_{\mathbf{d}x_j} &: \omega \mapsto \varepsilon_{\mathbf{d}x_j}(\omega) - \mathbf{i}_{O_{x_j}}(\omega), \\ \Upsilon^+_{\mathbf{d}x_j} &: \omega \mapsto \varepsilon_{\mathbf{d}x_j}((O'_{x_j})^{-1}\omega) + \mathbf{i}_{O_{x_j}}((O'_{x_j})^{-1}\omega). \end{aligned} \tag{2.28}$$

From Proposition 2.1.10 and Lemmata 2.1.13, 2.1.16, the representations of the algebra of endomorphisms $\Upsilon^\pm_{\mathbf{d}x_j} \in \mathrm{End}(\Lambda^*\mathcal{A})$, satisfy the anti-commutation rules when acting on $\Lambda^*\mathcal{A}$:

$$\begin{aligned} \Upsilon^-_{\mathbf{d}x_j}(\Upsilon^-_{\mathbf{d}x_k}(\omega)) + \Upsilon^-_{\mathbf{d}x_k}(\Upsilon^-_{\mathbf{d}x_j}(\omega)) &= -2\delta_{jk}O'_{x_k}\omega, \\ \Upsilon^+_{\mathbf{d}x_j}(\Upsilon^+_{\mathbf{d}x_k}(\omega)) + \Upsilon^+_{\mathbf{d}x_k}(\Upsilon^+_{\mathbf{d}x_j}(\omega)) &= 2\delta_{jk}(O'_{x_k})^{-1}\omega, \\ \Upsilon^+_{\mathbf{d}x_j}(\Upsilon^-_{\mathbf{d}x_k}(\omega)) + \Upsilon^-_{\mathbf{d}x_k}(\Upsilon^+_{\mathbf{d}x_j}(\omega)) &= \mathbf{0}. \end{aligned} \tag{2.29}$$

The above quantities does not endow the generators of the real Clifford algebra of signature $(n, n)$ but instead a generalization of them. Indeed the ladder structure of the algebra of endomorphisms described above includes a description of Clifford algebras in *continuum* since for $O_{x_k} = \partial_{x_k}$, the Pincherle derivative $O'_{x_k}$ corresponds to the identity operator but also include the description of Clifford-like algebras on oriented lattices introduced by Becher and Joos [5] and later by Vaz [68] and Kanamori,Kawamoto [49].

We will end this section by point out an important remark concerning the described approach:

**Remark 2.1.17** *Most of the ideas developed in this section are intimately related with the approach proposed by M. Barnabei, A. Brini and G. C. Rota in [3].*

*In particular, we note that a priori no problem in relation to the question of associativity and distributivity will appear since the algebra of endomorphisms of a given space equipped* *with the standard sum and a product defined by composition is obviously associative and distributive. On the other hand, the structure of the algebra of endomorphisms lead also to a construction of inner products which are metric-independent in the sense that they only depend on the duality between the tangent and the cotangent space.*

Having obtained the basic features in the language of differential forms, it is now interesting to study the interplay with discrete integration theory.

## 2.2 Discrete Differential Forms and Integration

The integration over a differential form can be viewed as flux of a fluid across a surface or the work of a force along a path. The former definition follows the standard pattern: The surface of integration is partitioned into small pieces in the tangent space and the sum of the values of the form on parallelepipeds approaches the integral as the partition. For further details, we refer the reader to [2], pp. 181.

Geometrically speaking, the value of the integral of the form over parallelepipeds is nothing else than the value of a cochain over a polyhedral chain (c.f. [70, 20, 44]). At this level, we can view discrete differential forms as cochains and integration as a pairing between chains and cochains and the parallelepipeds as the discrete domain of integration.

In this section we will make use of the natural pairing between chains and cochains as the building block to define discrete integration over a discrete domain. Afterwards by introducing the notions of De Rham and Whitney maps, we will build discrete differential forms *viz* interpolation over barycentric coordinates. This naturally lead to the discrete Stokes theorem (Theorem 2.2.23) and to the integration by parts formula (Corollary 2.2.24).

We do not claim that this approach to define discrete differential forms and discrete integration in terms of Whitney maps is new. This also appears e.g. in the works of Bossavit [7], Gradinaru and Hiptmair [37], Hiptmair [42, 43] and more recently in Beauce and Sen [4] and Brezzi, Lipnikov, Shashkov and Simoncini [13].

### 2.2.1 Simplicial Complexes, Chains and Cochains

We will first summarize some basic features of simplices, simplicial complexes, chains and cochains. A detailed framework can be found, for instance, in [57, 44].

**Definition 2.2.1 ($r-$simplex)** *A $r-$simplex is the convex hull of $r+1$ geometrically inde-*

*pendent points*

$$\sigma^r = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r] = \left\{ \sum_{j=0}^{r} \lambda_j \mathbf{p}_j \ : \ \lambda_j \geq 0, \ \sum_{j=1}^{r} \lambda_j = 1 \right\}.$$

The points $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r$ are the vertices of the simplex, $r$ is the dimension of the simplex and the orientation of $[\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$ is endowed by the ordering $\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r$.

We would like to observe that for $r < n$ each $r-$simplex is the intersection of an $n-$simplex with an $r-$dimensional *affine subspace*[1] of dimension $r$. It is rather obvious that each facet of a $r-$simplex is a $(r-1)-$simplex and two $r-$simplexes intersect in a simplex with dimension less or equal to $r$. The above remarks leads to a natural definition for the boundary of a $r-$simplex.

**Definition 2.2.2 (Boundary of a $r$-simplex)** *We call the boundary of $r-$simplex $\sigma^r$ to the union of $(r-1)-$simplexes which are facets of $\sigma^r$.*

**Definition 2.2.3 (Simplicial Complex and Polytope)** *A simplicial complex $\mathcal{K}$ in $\mathbb{R}^n$ is a family of simplexes in $\mathbb{R}^n$ such that,*

1. *Every face of a simplex of $\mathcal{K}$ is in $\mathcal{K}$.*
2. *The intersection of any two simplexes of $\mathcal{K}$ is a face of each of them.*

*A polytope of $\mathcal{K}$, denoted by $|\mathcal{K}|$, is the geometric union of the simplices of $\mathcal{K}$, i.e. $|\mathcal{K}| = \bigcup_{\sigma \in \mathcal{K}} \sigma$.*

The next definition almost corresponds to the notion of a mesh as used in the theory of Finite Element Methods [7, 43, 13].

**Definition 2.2.4 (Simplicial triangulation)** *A simplicial triangulation of a polytope $|\mathcal{K}|$ is the simplicial complex $\mathcal{K}$ such that the union of the simplexes of $\mathcal{K}$ recover the polytope $|\mathcal{K}|$.*

Next we introduce the linear space generated by superpositions of $r-$simplexes. This is known as the linear space of $r-$chains.

**Definition 2.2.5 (r-chains)** *Let $\mathcal{K}$ be a simplicial complex. We denote the free Abelian group generated by a basis consisting of oriented $r-$simplices by $C_r(\mathcal{K}, \mathbb{Z})$.*

*This is the space of finite formal sums of $r-$simplices, with coefficients in $\mathbb{Z}$. Elements of $C_r(\mathcal{K}, \mathbb{Z})$ are called $r-$chains.*

[1] Translation of a $r-$dimensional linear subspace through a fixed vector $\underline{x} \in \mathbb{R}^n$

In other words, $C_r(\mathcal{K}, \mathbb{Z})$ is nothing else than the choice of a finite number of $r-$simplexes, each of them with integer multiplicity. Indeed $k\sigma$ means that we are considering $k-$*superpositions* of $\sigma$ with positive orientation while $(-k)\sigma = k(-\sigma)$ means $k-$superpositions of $\sigma$ with opposite orientation. Taking the boundary of a $r-$simplex, we define the boundary operator as a $(r-1)-$chain with coefficients $a_j = (-1)^j$.

**Definition 2.2.6 (Boundary operator)** *For a $r-$simplex $\sigma = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_r]$, the boundary operator $\partial$ over $\sigma$ is defined as the signed sum of the all the facets of $\sigma$, i.e.*

$$\partial[\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_r] = \sum_{j=0}^{r} (-1)^j [\mathbf{p}_0, \dots, \hat{\mathbf{p}}_j, \dots, \mathbf{p}_r],$$

*where $[\mathbf{p}_0, \dots, \hat{\mathbf{p}}_j, \dots, \mathbf{p}_r]$ is the $(r-1)-$simplex obtained by omitting the vertex $\mathbf{p}_j$.*

According to the above definition, the boundary operator $\partial$ maps an oriented $r-$simplex $\sigma$ to a $(r-1)-$chain containing the facets of $\sigma$.

The proposition below results from the fact that $[\mathbf{p}_0, \dots, \hat{\mathbf{p}}_j, \dots, \mathbf{p}_r]$ and $[\mathbf{p}_0, \dots, \hat{\mathbf{p}}_{j+1}, \dots, \mathbf{p}_r]$ has induced opposing orientations (see also [70, 19]).

**Proposition 2.2.7 (c.f [57])** *For each $r-$simplex $\sigma$ we have $\partial(\partial\sigma) = 0$.*

Moreover, the linear extension of the boundary operator $\partial$ to the space of $r-$chains $C_r(\mathcal{K}, \mathbb{Z})$ produces a chain complex

$$0 \to C_m(\mathcal{K}, \mathbb{Z}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_r(\mathcal{K}, \mathbb{Z}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0(\mathcal{K}, \mathbb{Z}) \to 0.$$

with the property $\partial \circ \partial = 0$.

The following definition corresponds to the simplicial counterpart of a manifold (see also [2, 20]).

**Definition 2.2.8 (Support of a $r-$chain)** *The support of a $r-$simplex $\sigma^r = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_r]$ is denoted by supp$(\sigma^r)$ and corresponds to supp$(\sigma^r) = \{\mathbf{p} \in \mathbb{R}^n \ : \ \mathbf{p} \in \sigma^r\}$.*

*The support of an $r-$chain $\sigma^r = \sum_j a_j \sigma^r_j$ is defined as supp$(\sigma) = \cup_j$supp$(\sigma^r_j)$.*

From the above descriptions, the boundary of the oriented manifold $\partial$ supp$(\sigma)$, which is nothing else than the union of several sub-manifolds $\partial$supp$(\sigma_j)$ (polytopes), can be represented as a chain $\partial\sigma = \sum_j a_j \partial\sigma^j$. However bear in mind that although both concepts produces the same picture when acting on supp$(\sigma)$ and $\sigma$, the meaning of both actions is quite different. By introducing the inner product $\langle \cdot, \cdot \rangle$ between two $r-$chains as $\langle \sigma^r_j, \sigma^r_k \rangle = \delta_{jk}$, it is also possible to define the dual space of $C_r(\mathcal{K}, \mathbb{Z})$ as the space of cochains.

**Definition 2.2.9 (r-cochain)** *We define the space of $r-$cochains $C^r(\mathcal{K},\mathbb{Z})$ as the Abelian group $Hom(C_r(\mathcal{K},\mathbb{Z}),\mathbb{R})$ generated by the linear mappings $\omega_\sigma : \sigma' \mapsto \langle\sigma,\sigma'\rangle$.*

Indeed from the above definition, $\omega_\sigma\left(\sum_j a_j\ \sigma_j^r\right) = \sum_j a_j\omega_{\sigma'}\left(\sigma_j^r\right)$ holds for an $r-$chain $\sum_j a_j\ \sigma_j^r$ (where $a_j \in \mathbb{Z}$) and an $r-$cochain $\omega$ while for two $r-$cochains $\omega_{\sigma'}, \mu_{\sigma'} \in C^r(\mathcal{K},\mathbb{Z})$ and an $r-$chain $\sigma \in C_r(\mathcal{K},\mathbb{Z})$ we have $(\omega_{\sigma'} + \mu_{\sigma'})(\sigma) = \omega_{\sigma'}(\sigma) + \mu_{\sigma'}(\sigma)$.

We would like to point out that each cochain also corresponds to one value per simplex, since all the $r-$simplexes form a basis to the linear space $C_r(\mathcal{K},\mathbb{Z})$. From the structure of $C^r(\mathcal{K},\mathbb{Z})$, the definition for the dual of the boundary operator $\partial$ is rather obvious:

**Definition 2.2.10 (Coboundary operator)** *For $\omega^r \in C^r(\mathcal{K},\mathbb{R})$ and $\sigma^{r+1} \in C_r(\mathcal{K},\mathbb{Z})$ we define $\delta$ by*

$$(\delta\omega^r)(\sigma^{r+1}) = \omega^r(\partial\sigma^{r+1}). \tag{2.30}$$

This definition of coboundary operator induces the cochain complex

$$0 \leftarrow C^m(\mathcal{K},\mathbb{R}) \stackrel{\delta}{\leftarrow} \ldots \stackrel{\delta}{\leftarrow} C^r(\mathcal{K},\mathbb{R}) \stackrel{\delta}{\leftarrow} \ldots \stackrel{\delta}{\leftarrow} C^0(\mathcal{K},\mathbb{R}) \leftarrow 0.$$

where it is easy to see that $\delta \circ \delta = 0$ follows from definition.

The next definition will be useful in the sequel.

**Definition 2.2.11** *Given an $r-$simplex $\sigma^r = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$, the cone with vertex $\mathbf{q}$ and base $\sigma^r$ is defined as $\mathbf{q} \diamond \sigma^r = [\mathbf{q}, \mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$.*

Direct application of Definition 2.2.6 combined with standard arguments leads to the following lemma (see also [19]):

**Lemma 2.2.12 (c.f [57])** *The operator $\diamond$ satisfies $\partial(\mathbf{q} \diamond \sigma^r) + \mathbf{q} \diamond \partial(\sigma^r) = \sigma^r$.*

The standard cone construction can be straightforward extended to the space of chains $C_r(\mathcal{K},\mathbb{Z})$ by introducing the notion of *trivially star-shaped* complex (c.f. [19])

**Definition 2.2.13 (Trivially-shaped complex)** *A simplicial complex $\mathcal{K}$ is trivially star-shaped if there exists a vertex $\mathbf{q}$ such that for all $\sigma^r \in \mathcal{K}$, the cone $\mathbf{q} \diamond \sigma^r$ is a $(r+1)-$chain, i.e. $\mathbf{q} \diamond \sigma^r \in C_{r+1}(\mathcal{K},\mathbb{Z})$.*

The cone operation with respect to $\mathbf{q}$ is defined as $\kappa_{\mathbf{q}} : \sigma^r \mapsto \mathbf{q} \diamond \sigma^r$.

Furthermore the following lemma follows by construction (c.f. [19]):

**Lemma 2.2.14 (Homotopy mapping property)** *In trivially star-shaped simplicial complexes, the cone operator* $\kappa_{\mathbf{q}} : C_r(\mathcal{K}, \mathbb{Z}) \to C_{r+1}(\mathcal{K}, \mathbb{Z})$ *satisfies*

$$\kappa_{\mathbf{q}}\, \partial + \partial\, \kappa_{\mathbf{q}} = \mathbf{id}. \tag{2.31}$$

Using the duality between chains and cochains, the cocone operator can be defined as follows:

**Definition 2.2.15 (Cocone operator)** *The cocone operator* $\chi_{\mathbf{q}} : C^r(\mathcal{K}, \mathbb{R}) \to C^{r-1}(\mathcal{K}, \mathbb{R})$ *is defined as* $(\chi_{\mathbf{q}}\omega^r)(\sigma^{r-1}) = \omega^r(\kappa_{\mathbf{q}}\sigma^{r-1})$.

Using duality arguments between chains and cochains, the following identity is fulfilled at the level of cochains:

**Lemma 2.2.16 (c.f [19])** *The cocone operator* $\chi_{\mathbf{q}} : C^r(\mathcal{K}, \mathbb{R}) \to C^{r-1}(\mathcal{K}, \mathbb{R})$ *at the level of cochains satisfies the following identity*

$$\chi_{\mathbf{q}}\delta + \delta\chi_{\mathbf{q}} = \mathbf{id}.$$

With the later result the operator $\chi_{\mathbf{q}} : C^r(\mathcal{K}, \mathbb{R}) \to C^{r-1}(\mathcal{K}, \mathbb{R})$ corresponds to the mimetic transcription of the homotopy mapping at the level of cochains. It is also clear that $\chi_{\mathbf{q}}$ is a local operator and in particular if the chain is closed, i.e. $\partial\sigma^r = 0$ we obtain the discrete Poincaré lemma at the level of cochains (c.f. [19]). We will come back to this when we prove in Section 4.4 the discrete Poincaré lemma by means of umbral homogeneous polynomial forms. In fact, it was shown in [27] that the operator $\chi_{\mathbf{q}}$ can be replaced by a tangent vector-field with positive eigenvalues.

Next we explore the correspondence between cochains and discrete differential forms.

### 2.2.2 Whitney, De Rham Maps and Discrete Integration

Let us restrict ourselves to simplicial triangulations $\mathcal{K}$ representing a finite partition of a certain compact set $\mathcal{M}$ of dimension $n$. In the following we will use the notation $\Lambda^r\mathcal{A}(\mathcal{M})$ instead of $\Lambda^r\mathcal{A}$ to emphasize that $\Lambda^r\mathcal{A}(\mathcal{M})$ is a set of $r-$forms acting on functions supported on $\mathcal{M}$.

In order to encode the structure of $C^r(\mathcal{K}, \mathbb{R})$ in the space $\Lambda^r\mathcal{A}(\mathcal{M})$, we must introduce two basic operators: A reduction operator which maps differential forms in cochains and a reconstruction operator which translates cochains to differential forms. Since the natural pairing between chains and cochains give rise to scalars and the integration over forms can be obtained by measuring polytope quantities, we choose to take the De Rham map as reduction operator. This map is defined as follows (c.f. [70]).

**Definition 2.2.17 (De Rham Map)** *For an $r-$form $\omega^r \in \Lambda^r\mathcal{A}(\mathcal{M})$ and an $r-$chain, we define the De Rham map $R : \Lambda^r\mathcal{A}(\mathcal{M}) \to C^r(\mathcal{K}, \mathbb{R})$ as*

$$R[\omega^r](\sigma^r) = \int_{supp\ (\sigma^r)} \omega^r, \quad \textit{for all } \sigma^r \in C_r(\mathcal{K}, \mathbb{Z}) \tag{2.32}$$

Having chosen the reduction operator, it raises the question how to choose the right reconstruction operator $W$. In contrast to $R$, where the De Rham map appears as a natural choice, the choice for $W$ has some degrees of freedom. Indeed there are many possibilities to split $\sigma^r \in C_r(\mathcal{K}, \mathbb{Z})$ in terms of $r-$simplexes.

In order to define $W$, we must impose the conditions:

1. $W$ is a left-inverse for $R$: $RW = \mathbf{id}$;

2. $WR$ approximates the identity: $WR \approx \mathbf{id}$.

A possible construction of $W$ satisfying the above conditions is the Whitney map [70]. In order to define this map, we need to introduce barycentric coordinates associated to a given $r-$simplex $\sigma^r$. More concretely, we pick up functions $\mathbf{b}_{\mathbf{p}_0}, \mathbf{b}_{\mathbf{p}_1}, \ldots, \mathbf{b}_{\mathbf{p}_n} : \mathcal{M} \mapsto \mathbb{R}$ with the property $\sum_{j=0}^{r} \mathbf{b}_{\mathbf{p}_j} = \mathbf{1}$.

**Definition 2.2.18 (Whitney Map)** *The Whitney map $W : C^r(\mathcal{K}) \to \Lambda^r(\mathcal{K})$ is the linear extension of the mapping defined by*

$$W(\sigma) = r! \sum_{j=0}^{r} (-1)^j \mathbf{b}_{\mathbf{p}_j} \mathbf{d}\mathbf{b}_{\mathbf{p}_0} \ldots \mathbf{d}\mathbf{b}_{\mathbf{p}_{j-1}} \mathbf{d}\mathbf{b}_{\mathbf{p}_{j+1}} \ldots \mathbf{d}\mathbf{b}_{\mathbf{p}_r}, \quad \textit{with} \quad \sigma = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r] \tag{2.33}$$

*Hereby $\mathbf{b}_{\mathbf{p}_0}, \mathbf{b}_{\mathbf{p}_1}, \ldots, \mathbf{b}_{\mathbf{p}_n} : \mathcal{M} \mapsto \mathbb{R}$ are $0-$forms satisfying $\sum_{j=0}^{r} \mathbf{b}_{\mathbf{p}_j} = \mathbf{1}$.*

**Remark 2.2.19** *According to the above construction, the Whitney map builds piecewise continuous polynomial $r-$forms viz interpolation on $r-$simplexes $[\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$. This map is a particular example of conforming reconstruction operators used in the theory of Finite Elements [37, 43, 13].*

In this thesis we will take the Whitney map as reconstruction operator. However we would like to point out that there are other (physically relevant) algebraic realizations of reconstruction operators available (see e.g. [7, 42, 44] and the references given there).

The following properties will be important on the sequel to encode the cochain structure on the space of differential forms.

**Proposition 2.2.20 (see [70])** *The De Rham map and the Whitney map intertwines the exterior derivative* $\mathbf{d}$ *and the coboundary operator* $\delta$*, i.e.*

$$R\mathbf{d} = \delta R, \quad W\delta = \mathbf{d}W. \tag{2.34}$$

Now we have the full machinery required to introduce the discrete counterparts of differential forms and integration at the level of cochains.

**Definition 2.2.21 (Discrete Differential Form)** *Given* $\omega \in \Lambda^*\mathcal{A}(\mathcal{M})$*, we define a discrete differential form* $\omega_\sharp$ *as* $\omega_\sharp = WR\omega$*.*

**Definition 2.2.22 (Discrete Integration)** *For a discrete* $r-$*form* $\omega_\sharp^r \in \Lambda^r\mathcal{A}(\mathcal{M})$ *and a* $r-$*chain* $\sigma^r \in C_r(\mathcal{K}, \mathbb{Z})$ *with support* $supp(\sigma^r)$ *contained in* $\mathcal{M}$*, we define the integral of* $\omega_\sharp^r$ *along* $supp(\sigma^r)$ *as*

$$\int_{supp(\sigma^r)} \omega_\sharp^r = \omega^r(\sigma^r).$$

From the above considerations, discrete differential forms are defined by restricting the differential form $\omega^r$ to the polytope $|\mathcal{K}|$ by means of linear interpolation. On the other hand discrete integration then makes $\mathbf{d}$ appear as the dual of $\partial$. Moreover, the discrete Stokes theorem holds by construction:

**Theorem 2.2.23 (Discrete Stokes theorem)** *For a discrete* $r-$*form* $\omega_\sharp^r \in \Lambda^r\mathcal{A}(\mathcal{M})$ *and a* $r-$*chain* $\sigma^r$ *with support* $supp(\sigma^r)$ *contained in* $\mathcal{M}$*, it holds*

$$\int_{supp(\sigma^r)} \mathbf{d}\omega_\sharp^r = \int_{\partial \ supp(\sigma^r)} \omega_\sharp^r.$$

This corresponds to the discrete counterpart of the classical Gauss divergence and Stokes circulation theorems (see also [2, 20]). As a corollary of the discrete Stokes theorem, we get from (2.2) a formula for the integration by parts.

**Corollary 2.2.24 (Integration by parts)** *For a discrete* $r-$*form* $\omega_\sharp^r \in \Lambda^r\mathcal{A}(\mathcal{M})$ *and a* $s-$*form* $\omega_\sharp^s \in \Lambda^s\mathcal{A}(\mathcal{K})$*, and for an* $(r+s)-$*chain* $\sigma^{r+s}$ *with support* $supp(\sigma^{r+s})$ *contained in* $\mathcal{M}$*, we have*

$$\int_{supp(\sigma^{r+s})} \mathbf{d}(\omega_\sharp^r)\omega_\sharp^s + (-1)^r\omega_\sharp^r\mathbf{d}\omega_\sharp^s = \int_{\partial \ supp(\sigma^{r+s})} \omega_\sharp^r\omega_\sharp^s.$$

### 2.2.3 Discrete Differential Forms and Discrete Vector-Fields

Now we turn our attention to the world of differential forms again. In what follows we retain the notation adopted in Section 2.1. Additionally, we take a set of $(n+1)-$geometrically independent points $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{R}^n$. To motivate the construction of discrete differential forms, we will use linear interpolation on $n-$simplexes of the form $\sigma^n = [\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n]$:

Given the the nodal values of $F(\underline{x})$ on the vertices of $\sigma^n$, i.e. $\{F(\underline{x}(\mathbf{p}_j)) \ : \ j = 0, \dots, n\}$, we construct $F_\sharp$ on $\mathrm{supp}(\sigma^n)$ by means of

$$F_\sharp = \sum_{k=0}^{n} F(\underline{x}(\mathbf{p}_k))\mathbf{b}_{\mathbf{p}_k}, \tag{2.35}$$

where $\mathbf{b}_{\mathbf{p}_0}, \mathbf{b}_{\mathbf{p}_1}, \dots, \mathbf{b}_{\mathbf{p}_n}$ are barycentric coordinates satisfying the constraint

$$\sum_{j=0}^{n} \mathbf{b}_{\mathbf{p}_j} = \mathbf{1}. \tag{2.36}$$

By applying $\mathbf{d}$ on both sides of (2.36) and taking into account the identity $\mathbf{d1} = 0$ we obtain

$$\mathbf{d}\mathbf{b}_{\mathbf{p}_0} = -\sum_{j=1}^{n} \mathbf{d}\mathbf{b}_{\mathbf{p}_j}. \tag{2.37}$$

This leads to

$$\mathbf{d}F_\sharp = -\sum_{k=1}^{n} F(\underline{x}(\mathbf{p}_0))\mathbf{d}\mathbf{b}_{\mathbf{p}_k} + \sum_{k=1}^{n} F(\underline{x}(\mathbf{p}_k))\mathbf{d}\mathbf{b}_{\mathbf{p}_k} = \sum_{k=1}^{n} (F(\underline{x}(\mathbf{p}_k)) - F(\underline{x}(\mathbf{p}_0)))\mathbf{d}\mathbf{b}_{\mathbf{p}_k}. \tag{2.38}$$

In order to get the interconnecting structure of discrete vector-fields in terms of $1-$chains, we start to observe that the constraint (2.36) and Definition 2.2.18 assures that the $\mathbf{d}-$action on $\mathbf{b}_{\mathbf{p}_j}$ is given by

$$\mathbf{d}\mathbf{b}_{\mathbf{p}_j} = \sum_{k=0}^{n} W[\mathbf{p}_k, \mathbf{p}_j]. \tag{2.39}$$

By applying the De Rham map $R$ on both sides, we end up with $\delta R \mathbf{b}_{\mathbf{p}_j} = \sum_{k=0}^{n} [\mathbf{p}_k, \mathbf{p}_j]$ hence $\langle \delta R \mathbf{b}_{\mathbf{p}_j}, [\mathbf{p}_0, \mathbf{p}_l] \rangle = \delta_{jl}$ follows by taking into account the inner product $\langle \cdot, \cdot \rangle$ between chains. This leads to the orthogonal constraint $\int_{\partial \mathrm{supp}[\mathbf{p}_0, \mathbf{p}_l]} \mathbf{b}_{\mathbf{p}_j} = \delta_{jl}$.

Hence for functions $F_\sharp = \sum_{j=0}^{n} F(\underline{x}(\mathbf{p}_j))\mathbf{b}_{\mathbf{p}_j}$ with support $\mathrm{supp}[\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_n]$, direct application of the discrete Stokes theorem (Theorem 2.2.23) leads to

$$\int_{\partial \ \mathrm{supp} \ [\mathbf{p}_0, \mathbf{p}_l]} F_\sharp = \sum_{j=1}^{n} \left(F(\underline{x}(\mathbf{p}_j)) - F(\underline{x}(\mathbf{p}_0))\right) \int_{\partial \ \mathrm{supp} \ [\mathbf{p}_0, \mathbf{p}_l]} \mathbf{b}_{\mathbf{p}_j} = F(\underline{x}(\mathbf{p}_l)) - F(\underline{x}(\mathbf{p}_0)).$$

**Remark 2.2.25** *Note that here the coefficients $F(\underline{x}(\mathbf{p}_k)) - F(\underline{x}(\mathbf{p}_0))$ are nothing else than the flux of $F_\sharp$ along the boundary of the $1-$chain $[\mathbf{p}_0, \mathbf{p}_k]$.*

According to (2.8), the $\mathbf{d}-$action is written in terms of the $n$-basic vector-fields $O_{x_j}$ belonging to the tangent space $\mathcal{T}$. This tangent space can be identified with the ambient space $\mathbb{R}^n$, so we must consider the fluxes $x_j(\mathbf{p}_k) - x_j(\mathbf{p}_0)$ of the discrete differential forms $(\mathbf{d}x_j)_\sharp$ to be vectors on $\mathbb{R}^n$. Furthermore we obtain the following coordinate-free expression for the finite differences $O_{x_j}(\cdot)$ determined *viz* the contraction formula (2.7)

$$O_{x_j}F(\underline{x}(\mathbf{p}_0)) = \frac{F(\underline{x}(\mathbf{p}_k)) - F(\underline{x}(\mathbf{p}_0))}{x_j(\mathbf{p}_k) - x_j(\mathbf{p}_0)} = \frac{\int_{\partial \text{supp}[\mathbf{p}_0,\mathbf{p}_k]} F_\sharp}{\int_{\text{supp}[\mathbf{p}_0,\mathbf{p}_k]} (\mathbf{d}x_j)_\sharp}. \tag{2.40}$$

This suggests an identification for discrete vector-fields in terms of 1-cochains. The coordinate-free expression for the Pincherle derivative $O'_{x_j}(\cdot)$ (see formula 1.8, Section 1.2) acting on $F(\underline{x}(\mathbf{p}_0))$ corresponds to

$$O'_{x_j}F(\underline{x}(\mathbf{p}_0)) = \frac{1}{\int_{\text{supp}[\mathbf{p}_0,\mathbf{p}_k]} (\mathbf{d}x_j)_\sharp} \int_{\partial \text{supp}[\mathbf{p}_0,\mathbf{p}_k]} ((x_jF)_\sharp - x_j(\mathbf{p}_0)F_\sharp) = F(\underline{x}(\mathbf{p}_j)).$$

**Remark 2.2.26** *According to formulae (2.9) and (2.38), $\int_{\partial supp[\mathbf{p}_0,\mathbf{p}_k]} ((x_jF)_\sharp - x_j(\mathbf{p}_0)F_\sharp)$ can be recasted by applying the discrete Integration by parts formula (Corollary 2.2.24) to the functions $x_j$ and $F$.*

With the former construction, the discrete operators (2.40) and (2.41) are local operators which assigns the connectivity of the discrete domain. In particular, every 1-path $[\mathbf{p}, \mathbf{q}]$, described in terms of the Whitney 1-forms $W[\mathbf{p}, \mathbf{q}] = \mathbf{b_p}\mathbf{d}\mathbf{b_q} - \mathbf{b_q}\mathbf{d}\mathbf{b_p}$ corresponds to an edge connecting nodes $\mathbf{p}$ and $\mathbf{q}$. This induces a graph structure on the set of points $\text{supp}(\sigma^0)$, where $\sigma^0$ denotes the sum of all $0-$simplices of $\mathcal{K}$.

Hereby we assume that the set $\text{supp}(\sigma^0)$ is equipped with a group addition $\mathbf{p} + \mathbf{v} = \mathbf{q}$, which allows to *go from* $\mathbf{p}$ *in the direction* $\mathbf{v}$ *and arrive at* $\mathbf{q}$ on the discrete space and a group inverse $-\mathbf{v}$ defined by $\mathbf{q} + (-\mathbf{v}) = \mathbf{p}$ which corresponds to the direction $\mathbf{v}$ oriented in the opposite side. This particular class of graphs are Cayley graphs, i.e. discrete groups [25]. The left action of a Cayley graph on $\text{supp}(\sigma^0)$ corresponds to the Pincherle derivative $O'_{x_j}$ acting on $F(\underline{x})$

$$O'_{x_j}F(\underline{x}) = \sum_{[\mathbf{p}_0,\mathbf{p}_0+\mathbf{v}_j]\in\mathcal{K}} F(\underline{x}(\mathbf{p}_0 + \mathbf{v}_j))\mathbf{b}_{\mathbf{p}_0}. \tag{2.41}$$

Notice that $O'_{x_j}$ is obviously a shift-invariant operator and its inverse given by

$$(O'_{x_j})^{-1}F(\underline{x}) = \sum_{[\mathbf{p}_0,\mathbf{p}_0+(-\mathbf{v}_j)]\in\mathcal{K}} F\left(\underline{x}\left(\mathbf{p}_0 + (-\mathbf{v}_j)\right)\right)\mathbf{b}_{\mathbf{p}_0}. \tag{2.42}$$

corresponds to the right action of a Cayley graph. This in particular generalize the classical notions of forward and backward shifts acting on $\mathbb{Z}^n$ (c.f. [23]).

Now we move from fluxes defined over $1-$simplexes to fluxes defined over $r-$simplexes. It turns out that there will be two dimensions which will be relevant in the construction: The dimension $n$ of the ambient space $\mathbb{R}^n$ and the dimension $r$ from the chain. By combining linear interpolation with the action of the exterior derivative, the mimetic transcription from differential forms to discrete differential forms consists in represent every discrete $r-$form on the r-simplex $\sigma^r = [\mathbf{p}_0, \mathbf{p}_{j_1}, \ldots, \mathbf{p}_{j_r}]$, with $j_1 < \ldots < j_r$, as a linear combination of elements of the form $\mathbf{b}_{\sigma^r} = \mathbf{b}_{\mathbf{p}_0}\mathbf{d}\mathbf{b}_{\mathbf{p}_{j_1}}\ldots\mathbf{d}\mathbf{b}_{\mathbf{p}_{j_r}}$ for which $\mathbf{d}\mathbf{b}_{\sigma^r} = \mathbf{d}\mathbf{b}_{\mathbf{p}_0}\mathbf{d}\mathbf{b}_{\mathbf{p}_{j_1}}\ldots\mathbf{d}\mathbf{b}_{\mathbf{p}_{j_r}}$.

The following lemma gives the interconnection structure for the space of discrete $r-$form

**Lemma 2.2.27** *Let $\sigma^r = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$ an $r-$simplex and $\mathbf{q}\diamond\sigma^r = [\mathbf{q}, \mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$ a cone with vertex $\mathbf{q}$ and base $\sigma^r$. Then we have*

$$\mathbf{d}\mathbf{b}_{\sigma^r} = \frac{1}{(r+1)!}\sum_{\mathbf{q}} W[\mathbf{q}\diamond\sigma^r]. \tag{2.43}$$

**Proof:** In order to prove (2.43), we use induction over the degree of the differential form. If $r = 0$ and $r = 1$, the proof of (2.43) follows from the definition of $W$ and from the relation (2.39), respectively. Otherwise, by taking the $r-$simplex $\sigma^r = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$, we pick up cones of the form $\mathbf{q}\diamond\sigma^r = [\mathbf{q}, \mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$.

Note first that for each cone $\mathbf{q}\diamond\sigma^r$, the term $\frac{1}{(r+1)!}W[\mathbf{q}\diamond\sigma^r]$ can be splitted into the sum

$$\mathbf{b}_{\mathbf{q}}\mathbf{d}\mathbf{b}_{\mathbf{p}_0}\mathbf{d}\mathbf{b}_{\mathbf{p}_1}\ldots\mathbf{d}\mathbf{b}_{\mathbf{p}_r} - \mathbf{b}_{\mathbf{p}^0}\mathbf{d}\mathbf{b}_{\mathbf{q}}\mathbf{d}\mathbf{b}_{\mathbf{p}_1}\ldots\mathbf{d}\mathbf{b}_{\mathbf{p}_r} + \ldots + (-1)^{r+1}\mathbf{b}_{\mathbf{p}^0}\mathbf{d}\mathbf{b}_{\mathbf{p}_1}\ldots\mathbf{d}\mathbf{b}_{\mathbf{p}_r}\mathbf{d}\mathbf{b}_{\mathbf{q}}.$$

Summing up for all $\mathbf{q}$, we obtain from the constraints (2.36) and (2.37), that the term $\sum_{\mathbf{q}}\mathbf{b}_{\mathbf{q}}\mathbf{d}\mathbf{b}_{\mathbf{p}_0}\mathbf{d}\mathbf{b}_{\mathbf{p}_1}\ldots\mathbf{d}\mathbf{b}_{\mathbf{p}_r}$ is equal to $\mathbf{d}\mathbf{b}_{\mathbf{p}_0}\mathbf{d}\mathbf{b}_{\mathbf{p}_1}\ldots\mathbf{d}\mathbf{b}_{\mathbf{p}_r} = \mathbf{d}\mathbf{b}_{\sigma^r}$ while the remainder terms are equal to zero.

This results in $\frac{1}{(r+1)!}\sum_{\mathbf{q}} W[\mathbf{q}\diamond\sigma^r] = \mathbf{d}\mathbf{b}_{\sigma^r}$, which concludes the proof of Lemma 2.2.27.

■

Notice that for an $r-$simplex $\sigma^r$, the support $\mathrm{supp}(\sigma^r)$ corresponds to the $r-$parallelepiped spanned by $(\mathbf{p}_{j_1} - \mathbf{p}_0)\wedge\ldots\wedge(\mathbf{p}_{j_r} - \mathbf{p}_0)$, where $\wedge$ stands the wedge product on the ambient space $\mathbb{R}^n$, the associated discrete $r-$form $(\mathbf{d}\underline{x})^\alpha_\sharp$ restricted to $\sigma^r = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_r]$ is given by

$$\sum_k \frac{(x_{j_1}(\mathbf{p}_k) - x_{j_1}(\mathbf{p}_0))}{(x_{j_k}(\mathbf{p}_1) - x_{j_k}(\mathbf{p}_0))} c^k_{\mathbf{q}\diamond\sigma^r} \mathbf{b}_{\sigma^r_k}, \quad \text{with } \sigma^r_k = [\mathbf{p}_0, \mathbf{p}_1, \ldots, \mathbf{p}_{k-1}, \mathbf{q}, \mathbf{p}_k, \ldots, \mathbf{p}_r].$$

Hereby the constants $c^k_{\mathbf{q}\diamond\sigma^r}$ are given by products involving the displacements $(x_{j_l}(\mathbf{p}_l) - x_{j_l}(\mathbf{p}_0))$ for $l = 1, \ldots, k$, $(x_{j_k}(\mathbf{q}) - x_{j_k}(\mathbf{p}_0))$ and $(x_{j_{l+1}}(\mathbf{p}_l) - x_{j_{l+1}}(\mathbf{p}_0))$ for $l = k, \ldots, r$.

From the constraint (2.2.3) we obtain that the coefficients $x_{j_r}(\mathbf{q}) - x_{j_r}(\mathbf{p})$ vanish for $\mathbf{p} \neq \mathbf{p}^0$ and $\mathbf{q} \neq \mathbf{p}^r$. This gives rise to $\int_{\text{supp}(\sigma^r)} (\mathbf{d}\underline{x})^\alpha_\sharp = \prod_{k=1}^n \left( x_{j_k}(\mathbf{p}^k) - x_{j_k}(\mathbf{p}^0) \right)$.

Hence when acting on the discrete $r-$form, $\omega^r_\sharp = \sum_{|\alpha|=r} F^\alpha_\sharp (\mathbf{d}\underline{x})^\alpha_\sharp$, $\mathbf{d}\omega^r_\sharp(\mathbf{p}_0)$ is the principal multi-linear part of the flux of $\omega^r_\sharp$ over $\partial\text{supp}(\sigma^r)$, i.e.

$$\mathbf{d}\omega^r_\sharp(\mathbf{p}_0) = \frac{\int_{\partial\ \text{supp}(\sigma^r)} \omega^r_\sharp}{\int_{\text{supp}(\sigma^r)} (\mathbf{d}\underline{x})^\alpha_\sharp}.$$

Furthermore, for $\sigma^r_{\pm j} = [\mathbf{p}_0, \mathbf{p}_0 + (\pm\mathbf{v}_j), \mathbf{p}_2, \ldots, \mathbf{p}_r]$ the transcription of (2.41) and (2.42) to the space of $r-$forms then corresponds to

$$\begin{aligned} O'_{x_j}\omega^r(\underline{x}(\mathbf{p}_0)) &= \tfrac{1}{\int_{\text{supp}(\sigma^r_{+j})} (\mathbf{d}x_j (\mathbf{d}\underline{x})^\alpha)_\sharp} \int_{\partial\ \text{supp}(\sigma^r_{+j})} \left( (x_j\omega^r)_\sharp - (x_j \mathbf{d}\omega^r)_\sharp \right), \\ (O'_{x_j})^{-1}\omega^r(\underline{x}(\mathbf{p}_0)) &= \tfrac{1}{\int_{\text{supp}(\sigma^r_{-j})} (\mathbf{d}x_j (\mathbf{d}\underline{x})^\alpha)_\sharp} \int_{\partial\ \text{supp}(\sigma^r_{-j})} \left( (x_j\omega^r)_\sharp - (x_j \mathbf{d}\omega^r)_\sharp \right). \end{aligned} \tag{2.44}$$

At this level it becomes interesting to make the bridge between basic polynomial sequences with discrete integration formulae and discrete vector-fields:

**Remark 2.2.28** *Recall that from Remark 2.2.19, $\omega^r_\sharp$ (linear combination of Whitney $r-$forms) is a piecewise continuous polynomial $r-$form built viz linear interpolation. On the other hand, from Section 1.2, the basic polynomial sequences are computed in terms raising operators $x'_j$ which involve the explicit computation of the inverse for the Pincherle derivative.*

*In terms of discrete integration theory, the action of $O'_{x_j}$ and $(O'_{x_j})^{-1}$ restricted to $r-$simplexes $\sigma^r \in |\mathcal{K}|$ have integral representation given by formulae (2.44) (see also Remark 2.2.26). In particular $\omega^r\left(\underline{x}\left(\mathbf{p}_0 + (-\mathbf{v}_j)\right)\right) = (O'_{x_j})^{-1}\omega^r(\underline{x}(\mathbf{p}_0))$ is obtained from $O'_{x_j}\omega^r(\underline{x}(\mathbf{p}_0))$ by reversing the orientation of the 1-path $[\mathbf{p}_0, \mathbf{p}_j]$ on $\sigma^r$.*

*This clearly generalizes Duffin's approach to generate discrete powers. In fact, the discrete powers obtained in [28] are discrete $0-$forms which satisfy the axioms of basic polynomial sequences (see Definition 1.1.11, Chapter 1).*

With the former approach, we show that the *basic polynomial sequences* on discrete domains results from the interplay between finite difference operators with linear interpolation theory. On the other hand, the interior product $\mathbf{i}_{\mathbf{u}}$ (see Lemma 2.1.12) and afterwards the Lie derivative $\mathcal{L}_{\mathbf{u}}$ (see Definition 2.1.15) and the Clifford-like basis (see formulae (2.28)) can be determined in terms of linear interpolation along $r-$chains. Indeed the insertion of (2.44) on the right-hand side of (2.20)-(2.21) fix the interior product of an $r-$form in terms of Whitney $r-$forms.

According to the arguments explored along this chapter, we are able to conclude that the theory of discrete differential forms result from the correspondence between non-commutative geometry (c.f. [26]), the action of discrete groups (c.f. [23, 25]) and Whitney forms coming from Algebraic Topology (c.f. [70, 7, 42, 4]). It may also turn out a connection with the extrusion approach proposed in the Ph.D thesis of A. Hirani [44] (see in particular pages 81-86) but we shall not explore it here.

# Chapter 3

# Ladder Structures in Discrete Clifford Analysis

"*The modern evolution... has on the whole been marked by a trend of algebraization.*"
Herman Weyl

## 3.1 Clifford Algebras

In this section we just recollect some basic features of Clifford algebras; we may refer [9, 36, 18] for further details.

### 3.1.1 Real Clifford Algebras

Let $\mathbb{R}^n$ be endowed with the non-degenerate bilinear symmetric form $\mathcal{B}(\cdot,\cdot)$ of signature $(p,q)$, with $p+q=n$ and let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be an orthogonal basis of $\mathbb{R}^n$.

We assume that the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ satisfy the relations

$$\begin{array}{rcll} \mathcal{B}(\mathbf{e}_j, \mathbf{e}_j) & = & -1, & j = 1, \ldots, p \\ \mathcal{B}(\mathbf{e}_j, \mathbf{e}_j) & = & 1, & j = p+1, \ldots, n \\ \mathcal{B}(\mathbf{e}_j, \mathbf{e}_k) & = & 0, & j \neq k. \end{array}$$

We define $\mathbb{R}_{p,q}$ as the free algebra generated by the identity $\mathbf{1}$ and the $e_j$'s modulo the relations

$$\mathbf{e}_j \mathbf{e}_k + \mathbf{e}_k \mathbf{e}_j = -2\mathcal{B}(\mathbf{e}_j, \mathbf{e}_k). \tag{3.1}$$

This corresponds to the real Clifford algebra over $\mathbb{R}^n$. Elements of $\mathbb{R}_{p,q}$ are called Clifford numbers while relation (3.1) is the so-called Kronecker factorization.

For $\underline{\mathbf{e}} = (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n)$, we set $\underline{\mathbf{e}}^\alpha = \mathbf{e}_1^{\alpha_1}\mathbf{e}_2^{\alpha_2}\ldots\mathbf{e}_n^{\alpha_n}$. A canonical basis for $\mathbb{R}_{p,q}$ is obtained by considering for any multi-index $\alpha \in \{0,1\}^n$, the element $\underline{\mathbf{e}}^\alpha$. Moreover for $\alpha = \underline{0}$, we put $\underline{\mathbf{e}}^{\underline{0}} = \mathbf{1}$.

An element $\mathbf{a} \in \mathbb{R}_{p,q}$ is called a $r-$vector if $\mathbf{a}$ may be written as a sum of elements of the form $a_\alpha \underline{\mathbf{e}}^\alpha$, with with $|\alpha| = r$ (i.e. $\alpha$ has $r$ non-vanishing indices). The space of all $r-$vectors is denoted by $\mathbb{R}^r_{p,q}$ and $[\cdot]_r : \mathbb{R}_{p,q} \to \mathbb{R}^r_{p,q}$ denotes the projection operator of $\mathbb{R}_{p,q}$ onto $\mathbb{R}^r_{p,q}$ defined as

$$[\mathbf{a}]_r = \sum_{|\alpha|=r} a_\alpha \underline{\mathbf{e}}^\alpha, \tag{3.2}$$

leading to the identification of $\mathbb{R}$ with the subspace $\mathbb{R}^0_{p,q}$, $\mathbb{R}^n$ with the subspace of real Clifford vectors of signature $(p,q)$, $\mathbb{R}^1_{p,q}$, and of the space of volume-forms with the subspace $\mathbb{R}^n_{p,q}$ generated by the single $n-$vector $\mathbf{e}_1 \ldots \mathbf{e}_n$, called the pseudoscalar.

Moreover, every element $\mathbf{a} \in \mathbb{R}_{p,q}$ may be decomposed in a unique way as a finite sum of the form $\mathbf{a} = \sum_{r=0}^n [\mathbf{a}]_r$ and hence

$$\mathbb{R}_{p,q} = \sum_{r=0}^{n} \oplus \mathbb{R}^r_{p,q}.$$

This algebra is universal and its dimension over $\mathbb{R}$ is equal to $2^n$. The Clifford algebra $\mathbb{R}_{p,q}$ is in fact an algebra of radial-type $R(\mathcal{S})$ generated by $\mathcal{S} = \mathbb{R}^1_{p,q}$, that is

$$[\{x,y\},z] = 0 \quad \text{for any } x,y,z \in \mathcal{S}. \tag{3.3}$$

We would like to stress that there is indeed no a priori defined linear space to which the vector variables $x = \sum_{j=1}^n x_j \mathbf{e}_j \in \mathcal{S}$ belong. Nevertheless, by only using (3.3) one can already deduce many properties [66]. In particular, the space $\mathbb{R}^0_{p,q}$ is the center of $\mathbb{R}_{p,q}$ and it is generated by the anti-commuting products $\{x,y\}$, between vectors $x,y \in \mathbb{R}^1_{p,q}$. We also note hereby that $\mathbb{R}_{p,q}$ is in fact equivalent to the quotient algebra $R(\mathcal{S})/I(\mathcal{S})$, where $I(\mathcal{S})$ is its two sided ideal of $R(\mathcal{S})$ generated by all the products of the form $[\ \{x,y\}, z\ ]$. In fact, an epimorphism from $R(\mathcal{S})$ to $\mathbb{R}_{p,q}$ which maps $I(\mathcal{S})$ onto zero already tacitly exists ( [36], page 17).

The product of two vectors is usually called geometric Clifford product and it is decomposed in symmetric and antisymmetric parts, according to

$$xy = \frac{1}{2}(xy + yx) + \frac{1}{2}(xy - yx). \tag{3.4}$$

From the above splitting, we define a *dot product* $(\bullet)$ and a *wedge product* $(\wedge)$ by

$$x \bullet y = \frac{1}{2}(xy + yx) = -\mathcal{B}(x,y) = \sum_{j=1}^{p} x_j y_j - \sum_{j=p+1}^{p+q} x_j y_j$$

and

$$x \wedge y = \frac{1}{2}(xy - yx) = \sum_{j<k}(x_j y_k - x_k y_j)\mathbf{e}_j\mathbf{e}_k.$$

Geometrically speaking, the vector $x$ is orthogonal (respectively, parallel) to the vector $y$ if $x \bullet y = 0$ (respectively, $x \wedge y = 0$). Therefore, the orthogonality between $x$ and $y$ then leads to $xy = -yx$ while the commutativity between $x$ and $y$ (i.e. $xy = yx$) occurs if $x$ is parallel to $y$. By (3.4), $x^2 = x \bullet x$ is real for any vector $x$ and hence the vector $x$, is invertible if and only if $x^2 \neq 0$. In this case the inverse $x^{-1}$ is given by $x^{-1} = \frac{x}{x^2}$ while if $x^2 = 0$ then either $x$ is zero or it is a zero divisor and hence not invertible.

For $x \in \mathbb{R}^1_{p,q}$ and $\mathbf{a}^r \in \mathbb{R}^r_{p,q}$, the *inner* and *wedge* products can be extended to the whole algebra $\mathbb{R}_{p,q}$ by the formulae

$$\begin{aligned} x \bullet \mathbf{a}^r &= [x\mathbf{a}^r]_{r-1} = \tfrac{1}{2}\left(x\mathbf{a}^r - (-1)^r \mathbf{a}^r x\right), \\ x \wedge \mathbf{a}^r &= [x\mathbf{a}^r]_{r+1} = \tfrac{1}{2}\left(x\mathbf{a}^r + (-1)^r \mathbf{a}^r x\right), \end{aligned} \tag{3.5}$$

and finally $x\mathbf{a}^r = x \bullet \mathbf{a}^r + x \wedge \mathbf{a}^r$ corresponds to the extension of the geometric Clifford product (3.4).

There are essentially two linear anti-automorphisms (reversion and conjugation) and a linear automorphism (main involution) acting on $\mathbb{R}_{p,q}$. They are defined as:

- the *main involution* is defined by

$$\mathbf{e}_j{}^{\text{‘}} = -\mathbf{e}_j, \quad 1^{\text{‘}} = 1, \quad (j = 1, \ldots, n), \quad (ab)^{\text{‘}} = a^{\text{‘}}b^{\text{‘}}, \quad \forall\, a, b \in \mathbb{R}_{p,q};$$

- the *reversion* is defined by

$$\mathbf{e}_j^* = \mathbf{e}_j, \quad 1^* = 1, \quad (i = 1, \ldots, n), \quad (ab)^* = b^* a^*, \quad \forall\, a, b \in \mathbb{R}_{p,q};$$

- the *conjugation* is defined by

$$\mathbf{e}_j^\dagger = -\mathbf{e}_j, \quad \mathbf{1}^\dagger = \mathbf{1}, \quad (j = 1, \ldots, n), \quad (ab)^\dagger = b^\dagger a^\dagger, \quad \forall\, a, b \in \mathbb{R}_{p,q}.$$

We stress that the *conjugation* can be obtained as a composition between *main involution* and *reversion* i.e. $x^\dagger = (x^{\text{‘}})^* = (x^*)^{\text{‘}}$, $\forall x \in \mathbb{R}_{p,q}$. From the definition we can derive the action on the basis elements $\underline{\mathbf{e}}^\alpha$ by the rules:

$$(\underline{\mathbf{e}}^\alpha)^{\text{‘}} = (-1)^{|\alpha|}\underline{\mathbf{e}}^\alpha, \qquad (\underline{\mathbf{e}}^\alpha)^* = (-1)^{\frac{|\alpha|(|\alpha|-1)}{2}}\underline{\mathbf{e}}^\alpha, \qquad (\underline{\mathbf{e}}^\alpha)^\dagger = (-1)^{\frac{|\alpha|(|\alpha|+1)}{2}}\underline{\mathbf{e}}^\alpha.$$

In particular, if $x$ is a vector, we obtain $x^\dagger = x^{\text{‘}} = -x$ and $x^* = x$.

Using the above operations, we can define the subalgebra $\mathbb{R}^E_{p,q}$ (respectively, $\mathbb{R}^O_{p,q}$) as the set of all Clifford numbers $\mathbf{a}$ satisfying $\mathbf{a}^{‘} = \mathbf{a}$ (respectively, $\mathbf{a}^{‘} = -\mathbf{a}$). In the Clifford language $\mathbb{R}^E_{p,q}$ is usually known as the even (respectively) Clifford subalgebra. In particular we can split $\mathbb{R}_{p,q}$ as a direct sum between $\mathbb{R}^E_{p,q}$ and $\mathbb{R}^O_{p,q}$, i.e. $\mathbb{R}_{p,q} = \mathbb{R}^E_{p,q} \oplus \mathbb{R}^O_{p,q}$.

The $\dagger-$conjugation leads to the Clifford inner product and its associated norm on $\mathbb{R}_{p,q}$ given by

$$(\mathbf{a}, \mathbf{b}) = [\mathbf{a}^\dagger \mathbf{b}]_0, \quad |\mathbf{a}|^2 = (\mathbf{a}, \mathbf{a}). \tag{3.6}$$

Notice that when $\mathbf{a}$ and $\mathbf{b}$ belong to $\mathbb{R}^1_{p,q}$, their inner product and associated norm reduces to the classical inner product and norms on the ambient space $\mathbb{R}^n$, respectively.

### 3.1.2 Complexified Clifford Algebras and Spinor Spaces

Now we fix our attention on the real Clifford algebra of signature $(n, n)$, $\mathbb{R}_{n,n}$, in particular in its realization as the algebra of endomorphisms $\mathrm{End}(\mathbb{R}_{0,n})$.

Let us observe that, from (3.5) for any Clifford element $\mathbf{a} \in \mathbb{R}_{0,n}$, the *inner* and *wedge* products, $\mathbf{e}_j \bullet \mathbf{a}$ and $\mathbf{e}_j \wedge \mathbf{a}$, respectively, correspond to

$$\mathbf{e}_j \bullet \mathbf{a} = \frac{1}{2}\left(\mathbf{e}_j \mathbf{a} - \mathbf{a}^* \mathbf{e}_j\right), \quad \mathbf{e}_j \wedge \mathbf{a} = \tfrac{1}{2}\left(\mathbf{e}_j \mathbf{a} + \mathbf{a}^* \mathbf{e}_j\right), \tag{3.7}$$

where $*$ is the main involution defined in the above section. This suggests the introduction of the basic endomorphisms acting on $\mathbb{R}_{0,n}$ :

$$\xi_j : \mathbf{a} \mapsto \mathbf{e}_j \mathbf{a}, \quad \xi_{j+n} : F(\underline{x}) \mapsto \mathbf{a}^* \mathbf{e}_j.$$

It is clear that $\xi_j$ and $\xi_{j+n}$ endow the generators of the algebra $\mathbb{R}_{n,n}$ since $\xi_j(\xi_j \mathbf{a}) = \mathbf{e}_j^2 \mathbf{a} = -\mathbf{a}$ and $\xi_{j+n}(\xi_{j+n}\mathbf{a}) = (\mathbf{a}^* \mathbf{e}_j)^* \mathbf{e}_j = \mathbf{a}\mathbf{e}_j^* \mathbf{e}_j = \mathbf{a}$. On the other hand, straightforward computations reveal that

$$\xi_j(\xi_k \mathbf{a}) + \xi_k(\xi_j \mathbf{a}) = 0, \quad \text{for} \quad j, k = 1, \ldots, 2n \quad \text{with } j \neq k. \tag{3.8}$$

Hence the isomorphism between $\mathbb{R}_{n,n}$ and $\mathrm{End}(\mathbb{R}_{0,n})$ is thus obtained. Moreover the operator actions $\mathbf{e}_j \bullet (\cdot)$ and $\mathbf{e}_j \wedge (\cdot)$ given by

$$\mathbf{e}_j \wedge (\cdot) = \frac{1}{2}\left(\xi_j - \xi_{j+n}\right), \quad \mathbf{e}_j \bullet (\cdot) = \tfrac{1}{2}\left(\xi_j + \xi_{j+n}\right),$$

respectively, form a new basis for $\mathrm{End}(\mathbb{R}_{0,n})$, usually know as the Witt basis for $\mathbb{R}_{n,n}$ and the relations follow straightforward when acting on $\mathbb{R}_{0,n}$:

$$\begin{aligned} &\text{Grassmann identities:} && \{\mathbf{e}_j \wedge (\cdot), \mathbf{e}_k \wedge (\cdot)\} = 0 = \{\mathbf{e}_j \bullet (\cdot), \mathbf{e}_k \bullet (\cdot)\}, \\ &\text{duality identities:} && \{\mathbf{e}_j \bullet (\cdot), \mathbf{e}_k \wedge (\cdot)\} = \delta_{jk}\mathbf{id}. \end{aligned}$$

Hence the complexified Clifford algebra $\mathbb{C}_{2n} = \mathbb{C} \otimes \mathbb{R}_{0,2n}$ may be obtained as a complexification of $\mathbb{R}_{n,n}$ leading to the isomorphisms

$$\mathbb{C}_{2n} \cong \mathbb{C} \otimes \mathbb{R}_{n,n} \cong \mathbb{C} \otimes End(\mathbb{R}_{0,n}) \cong \mathbb{C} \otimes End(\mathbb{C}_n).$$

In concrete terms, from the basic endomorphisms $\xi_j, \xi_{n+j}$ the original basis $\mathbf{e}_1, \ldots, \mathbf{e}_{2n}$ of $\mathbb{C}_{2n}$ is reformulated in terms of the identifications $\mathbf{e}_j \leftrightarrow \xi_j$ and $\mathbf{e}_{j+n} \leftrightarrow -i\xi_{j+n}$. The corresponding Witt basis for $\mathbb{C}_{2n}$ then reads

$$\mathfrak{f}_j = \frac{1}{2}(\mathbf{e}_j - i\mathbf{e}_{n+j}), \quad \mathfrak{f}_j^\dagger = -\tfrac{1}{2}(\mathbf{e}_j + i\mathbf{e}_{n+j}),$$

and satisfies the following anti-commuting identities

$$\begin{array}{ll} \text{Grassmann identities:} & \{\mathfrak{f}_j, \mathfrak{f}_k\} = 0 = \{\mathfrak{f}_j^\dagger, \mathfrak{f}_k^\dagger\}, \\ \text{duality identities:} & \{\mathfrak{f}_j, \mathfrak{f}_k^\dagger\} = \delta_{jk}. \end{array} \tag{3.9}$$

From the relations established above, we introduce the complexified Grassmann algebras as

$$\begin{array}{ll} \mathbb{C}\Lambda_n = \mathrm{Alg}_{\mathbb{C}}\left\{\mathfrak{f}_j \ : \ j = 1, \ldots, n\right\}, & \mathbb{C}\Lambda_n^\dagger = \mathrm{Alg}_{\mathbb{C}}\left\{\mathfrak{f}_j^\dagger \ : \ j = 1, \ldots, n\right\}, \\ \mathbb{C}\Lambda_n^r = \mathbb{C}_{2n}^r \cap \mathbb{C}\Lambda_n, & \mathbb{C}(\Lambda_n^r)^\dagger = \mathbb{C}_{2n}^r \cap (\mathbb{C}\Lambda_n)^\dagger. \end{array}$$

The vectors in these Grassmann algebras may be characterized by means of the primitive idempotent $I$ introduced in the following manner:

For each $j = 1, \ldots, n$, we set by $I_j$ the products

$$I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2}\left(1 + i\mathbf{e}_{n+j}\mathbf{e}_j\right) = \frac{1}{2}\left(1 - i\mathbf{e}_j\mathbf{e}_{n+j}\right).$$

From the relations (3.9) we observe that $I_j$ are mutually commuting idempotents, i.e. $[I_j, I_k] = 0$ and $I_j^2 = I_j$. Moreover, we also have $I_j I_j^\dagger = 0 = I_j^\dagger I_j$. Now we define $I$ as

$$I = \prod_{j=1}^{n} I_j = \prod_{j=1}^{n} \mathfrak{f}_j \mathfrak{f}_j^\dagger$$

for which we obviously have $I^2 = I$ and $II^\dagger = I^\dagger I = 0$. The conversion relations are thus given by

$$\begin{array}{l} \mathbf{e}_j I = i\mathbf{e}_{j+n} I = -\mathfrak{f}_j^\dagger I, \quad \mathfrak{f}_j I = 0, \quad j = 1, 2, \ldots, n, \\ I\mathbf{e}_j = -i\mathbf{e}_{j+n} I = I\mathfrak{f}_j, \quad I\mathfrak{f}_j^\dagger = 0, \quad j = 1, 2, \ldots, n. \end{array}$$

Moreover a vector $\mathbf{a} \in \mathbb{C}_{2n}^1$ belongs to $\mathbb{C}\Lambda_n^r$ (respectively, $\mathbb{C}(\Lambda_n^r)^\dagger$) if and only if $\mathbf{a}I = 0$ (respectively, $I\mathbf{a} = 0$).

Using the idempotent element $I$ we introduce the complexified spinor spaces $\mathbb{C}S_n$ and $\mathbb{C}S_n$ as $\mathbb{C}S_n = \mathbb{C}_n I$ and $\mathbb{C}S_n^r = \mathbb{C}_n^r I$, respectively. Then it readily follows the identifications

$$\mathbb{C}S_n \cong \mathbb{C}\Lambda_n^\dagger \cong \mathbb{C}_{2n} I, \quad \mathbb{C}S_n^r \cong \mathbb{C}(\Lambda_n^r)^\dagger \neq \mathbb{C}_{2n}^r I.$$

From the above isomorphisms, there exists a unique element $\hat{\mathbf{a}} \in \mathbb{C}_n$ such that $\mathbf{a}I = \hat{\mathbf{a}}I$. Since the dimension of $\mathbb{C}_n$ does not exceed $n$, we obtain that $I[\hat{\mathbf{a}}]_r I = 0$ for $r > 0$. This implies

$$I\mathbf{a}I = I\hat{\mathbf{a}}I = I[\hat{\mathbf{a}}]_0 I = [\hat{\mathbf{a}}]_0 I.$$

### 3.1.3 Group Structures

Now we will explore the Clifford algebras $\mathbb{R}_{p,q}$ and $\mathbb{C}_{0,2n}$ from the group theoretical point of view.

We start to consider the *pseudo-orthogonal group* $O(p,q)$ as the set of all linear mappings $T : \mathbb{R}^1_{p,q} \mapsto \mathbb{R}^1_{p,q}$ for which $\mathcal{B}(Tx, Ty) = \mathcal{B}(x, y)$ holds for all $x, y \in \mathbb{R}^1_{p,q}$. Alternatively $O(p,q)$ may be realized as the set of all invertible matrices $M$ satisfying $M^T AM = A$ where $A = (a_{jk})_{1\leq j,k\leq n}$ is the matrix obtained from $\mathcal{B}$ relative to the basis of $\mathbb{R}^1_{p,q}$.

We introduce $SO(p,q)$ as a subgroup of $O(p,q)$ consisting of those matrices $M$ whose determinant is equal to 1. This subgroup is well known as the *special orthogonal group* or the *rotation group*. From the relations (3.1), we can see that Clifford algebras induce two-fold covering groups for $O(p,q)$ and $SO(p,q)$. In particular, the following two groups are of special interest:

- The *Pin group* $\mathrm{Pin}(p,q)$: Group generated by finite products of the form $s = \prod_{j=1}^k w^j$ with $w_k^2 = \pm 1$.
- The *Spin group* $\mathrm{Spin}(p,q)$: The subgroup obtained from the intersection of $\mathrm{Pin}(p,q)$ with the even Clifford subalgebra $\mathbb{R}^E_{p,q}$.

From the above description it follows that $\mathrm{Pin}(p,q)$ and $\mathrm{Spin}(p,q)$ can be described as

$$Pin(p,q) = \left\{ s = \prod_{j=1}^{k} w^j \ : \ ss^\dagger = s^\dagger s = \pm 1 \right\},$$
$$Spin(p,q) = \left\{ s = \prod_{j=1}^{2k} w^j \ : \ ss^\dagger = s^\dagger s = \pm 1 \right\}.$$

Next we turn our attention to the complexified Clifford algebras $\mathbb{C}_{2n}$. We define the complexification of the spin group $Spin(0,2n)$, i.e. $\mathrm{Spin}(2n;\mathbb{C}) = \mathbb{C} \otimes Spin(0,2n)$. We would like to stress that $\mathrm{Spin}(2n;\mathbb{C})$ is too large, since it contains all the spin groups $\mathrm{Spin}(p,q)$ with $p+q=2n$. This suggests to get a refinement of $\mathrm{Spin}(2n;\mathbb{C})$. More precisely, we will introduce the group $\tilde{U}(n)$ as being

$$\tilde{U}(n)=\{s\in \mathrm{Spin}(2n;\mathbb{C})\ :\ s^\dagger I=\exp(i\theta)I,\ \text{for some } \theta\geq 0\}.$$

This clearly forms a representation for the unitary group of matrices $U(n)=\{M\in\mathbb{R}^{n\times n}\ :\ MM^T=\mathbf{id}\}$.

In particular, the elements $s\in\tilde{U}(n)$ for which $sI=I$ form a subgroup $\tilde{SU}(n)$ of $\tilde{U}(n)$, usually called the *special unitary subgroup*. The two main transformations associated to the traditional representations of $\tilde{U}(n)$, $H(s),L(s)\in \mathrm{End}(\tilde{U}(n))$ given by

$$H(s):\mathbf{a}\mapsto s\mathbf{a}s^\dagger,\quad L(s):\mathbf{a}\mapsto s\mathbf{a},\quad \text{with}\ \ \mathbf{a}\in\mathbb{C}_{0,2n}.$$

When $s$ is restricted to the space $\mathrm{Spin}(p,q)$ with $p+q=2n$, the maps $H(s)$ and $L(s)$ are coincide with the so-called spin-1 and spin-1/2 representations, respectively.

It is well know that $H(s)$ preserves the multi-vector structure of $\mathbb{C}\Lambda_n$ and $\mathbb{C}\Lambda_n^\dagger$, i.e. $H(s):\mathbb{C}\Lambda_n^r\to\mathbb{C}\Lambda_n^r$ and $H(s):\mathbb{C}(\Lambda_n^r)^\dagger\to\mathbb{C}(\Lambda_n^r)^\dagger$ while $L(s)$ leaves the complexified spinor subspaces of $\mathbb{C}S_n$ invariant, i.e. $L(s):\mathbb{C}S_n^r\to\mathbb{C}S_n^r$ (see e.g. [12, 8]).

## 3.2 Fock Space Representation of the Algebra of Endomorphisms

We will look therefore for an Clifford Exterior Calculus that is a transcription of the classical Clifford Exterior Calculus. To this end we will start to describe the algebra of endomorphisms acting on $\Lambda^*\mathbb{R}[\underline{x}]$ as a realization of Fock spaces. Indeed $\mathbb{R}[\underline{x}]$ may be viewed as realization of the Bose algebra while the Grassmann algebra may be viewed as a realization of the Fermi algebra. Afterwards, the resulting exterior differential calculus will be obtained as an extension of Umbral Calculus to the language of differential forms.

Before we proceed, we first we recall some basic facts on Quantum Field Theory:

A Fock space is the free algebra generated from the vacuum vector $\Phi$ by the $2n+1$ elements $\mathbf{a}_1,\ldots,\mathbf{a}_n,\mathbf{a}_1^\dagger,\ldots,\mathbf{a}_n^\dagger,\mathbf{id}$ satisfying $\mathbf{a}_j\Phi=0,\ j=1,\ldots,n$ and the canonical commutation (respectively, anti-commutation) relations between the operators $\mathbf{a}_j,\mathbf{a}_j^\dagger$:

$$[\mathbf{a}_j,\mathbf{a}_k]=0=[\mathbf{a}_j^\dagger,\mathbf{a}_k^\dagger],\quad [\mathbf{a}_j^\dagger,\mathbf{a}_k]=\delta_{jk}\mathbf{id} \tag{3.10}$$

for the space $\mathcal{F}^+$ (bosons), and

$$\{\mathbf{a}_j,\mathbf{a}_k^\dagger\}=\delta_{jk}\mathbf{id},\quad \{\mathbf{a}_j,\mathbf{a}_k\}=0=\{\mathbf{a}_j^\dagger,\mathbf{a}_k^\dagger\} \tag{3.11}$$

for the space $\mathcal{F}^-$ (fermions).

We now equip $\mathcal{F}^\pm$ with an euclidian inner product $\langle\cdot|\cdot\rangle$ such that $\langle\Phi|\Phi\rangle = 1$ and the operators $\mathbf{a}_j^\dagger$ are adjoint to $\mathbf{a}_j$, i.e. $\langle\mathbf{a}_j^\dagger x|y\rangle = \langle x|\mathbf{a}_j y\rangle$.

The later linear spaces $(\mathcal{F}^+, \langle\cdot|\cdot\rangle)$ and $(\mathcal{F}^-, \langle\cdot|\cdot\rangle)$ becomes the Bose algebra and the Fermi algebra, respectively, where the elements $\mathbf{a}_j^\dagger$ and $\mathbf{a}_j$ (creation and annihilation operators, respectively) form a basis for the $n-$particle space. For a sake of simplicity, we will identify these linear spaces with $\mathcal{F}^\pm$.

The explicit expression from the basic vectors in the Fock spaces $\eta_\alpha \in \mathcal{F}^\pm$ can be derived from the standard lemma of elementary Quantum Field Theory (c.f. [58]) as the multi-index product of creation operators acting on the vacuum vector $\Phi$, i.e.

$$\eta_\alpha = \prod_{j=1}^n (\mathbf{a}_j^\dagger)^{\alpha_j}\Phi$$

and moreover, inner products between basic vector fields has the form $\langle\eta_\alpha|\eta_\beta\rangle = \alpha!\delta_{\alpha,\beta}$.

From the canonical relations (3.10) and (3.11) for the Fock spaces $\mathcal{F}^\pm$, we end up with the following constraints for the multi-indices $\alpha$:

- $\alpha_j = 0, 1, \ldots$ for the case of bosons $\mathcal{F}^+$,
- $\alpha_j = 0, 1$ for the case of fermions $\mathcal{F}^-$.

The aim of this subsection shall be concerned to the description of the algebra of endomorphisms $\mathrm{End}(\Lambda^*\mathbb{R}[\underline{x}])$ as the tensor product between the Bose and Fermi algebra $\mathcal{F}^+ \otimes \mathcal{F}^-$. In particular, we will construct and study a (possibly) noncommutative exterior differential calculus *viz* representations of commutation relations (3.10) by raising and lowering operators, $x_j'$ and $O_{x_j}$, respectively and *viz* the anti-commutation relations (3.11) in terms of left representations of contraction and inflation operators $\mathfrak{f}_j$ and $\mathfrak{f}_j^\dagger$ on certain polynomial forms $\omega_\alpha^\beta(\underline{x}) = V_\alpha(\underline{x})(\mathbf{d}_\Psi\underline{x})^\beta$.

Among the infinity of possible representations of (3.10) and (3.11) the simplest one are the multiplication $x_j$ and the derivation operator $\partial_{x_j}$ (bosons), exterior product $\mathbf{d}x_j$ and interior product $\mathbf{i}_{\partial_{x_j}}$ (fermions).

Evidently $x_j$ and $\partial_{x_j}$ acting on $\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta$ gives

$$\begin{aligned} x_j(\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta) &= \underline{x}^{\alpha+\mathbf{v}_j}(\mathbf{d}\underline{x})^\beta, \\ \partial_{x_j}(\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta) &= \alpha_j\underline{x}^{\alpha-\mathbf{v}_j}(\mathbf{d}\underline{x})^\beta, \end{aligned} \tag{3.12}$$

while the action of $\mathbf{d}x_j$ and $\mathbf{i}_{\partial_{x_j}}$ on $\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta$ ( see Definition 2.1.2, Proposition 2.1.10 and

Lemma 2.1.12 ) gives

$$\begin{array}{rcl} \mathbf{d}x_j(\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta) & = & \sum_{k=1}^{|\beta|}(-1)^{k-1}(1-\delta_{j,j_k})\underline{x}^\alpha(\mathbf{d}\underline{x})^{\beta+\mathbf{v}_{j_k}}, \\ \mathbf{i}_{\partial_{x_j}}(\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta) & = & \sum_{k=1}^{|\beta|}(-1)^{k-1}\delta_{j,j_k}\ \underline{x}^\alpha(\mathbf{d}\underline{x})^{\beta-\mathbf{v}_{j_k}}. \end{array} \tag{3.13}$$

We now construct the extension of Umbral Calculus to differential forms through representations of the commutating and anti-commuting relations, (3.12) and (3.13), respectively. In order to do that, we introduce three type of operators actions on the basic polynomial sequence of forms $\omega_\alpha^\beta(\underline{x}) = V_\alpha(\underline{x})(\mathbf{d}_\Psi\underline{x})^\beta$.

**Definition 3.2.1 (Sheffer operator)** *Let $\{V_\alpha\}_\alpha$ be a basic polynomial sequence for the multivariate delta operator $O_{\underline{x}}$ and $\mathbf{d}\underline{x}$, $\mathbf{d}_\Psi\underline{x}$ two vectors of coordinate differentials dual to $\partial_{\underline{x}}$ and $O_{\underline{x}}$, respectively. The operator $\Psi_{\underline{x}}$ defined as the linear extension of*

$$\Psi_{\underline{x}} : \underline{x}^\alpha(\mathbf{d}\underline{x})^\beta \mapsto V_\alpha(\underline{x})(\mathbf{d}_\Psi\underline{x})^\beta$$

*is called the Sheffer operator for $(\mathbf{d}_\Psi, \Lambda^*\mathbb{R}[\underline{x}])$.*

**Definition 3.2.2 (Sheffer shifts)** *Let $\{V_\alpha\}_\alpha$ be a basic polynomial sequence for the multivariate delta operator $O_{\underline{x}}$ and $\mathbf{d}_\Psi\underline{x}$ be a vector of coordinate differentials dual to $O_{\underline{x}}$.*

*For $\omega_\alpha^\beta(\underline{x}) = V_\alpha(\underline{x})(\mathbf{d}_\Psi\underline{x})^\beta$, the operators $x_j'$ and $\mathfrak{f}_j^\dagger$ defined as the linear extension of*

$$\begin{array}{c} x_j' : \omega_\alpha^\beta(\underline{x}) \mapsto \omega_{\alpha+\mathbf{v}_j}^\beta(\underline{x}), \\ \mathfrak{f}_j^\dagger : \omega_\alpha^\beta(\underline{x}) \mapsto \sum_{k=1}^{|\beta|}(1-\delta_{j,j_k})(-1)^{k-1}\omega_\alpha^{\beta+\mathbf{v}_{j_k}}(\underline{x}), \end{array} \tag{3.14}$$

*are called the Sheffer shifts for $(\mathbf{d}_\Psi, \Lambda^*\mathbb{R}[\underline{x}])$.*

**Definition 3.2.3 (Delta operators)** *Let $\{V_\alpha\}_\alpha$ be a basic polynomial sequence for the multivariate delta operator $O_{\underline{x}}$ and $\mathbf{d}_\Psi\underline{x} = (\mathbf{d}_\Psi x_1, \mathbf{d}_\Psi x_2, \ldots, \mathbf{d}_\Psi x_n)$ be a vector of coordinate differentials dual to $O_{\underline{x}}$.*

*For $\omega_\alpha^\beta(\underline{x}) = V_\alpha(\underline{x})(\mathbf{d}_\Psi\underline{x})^\beta$, the operators $O_{x_j}$ and $\mathfrak{f}_j$ defined as the linear extension of*

$$\begin{array}{c} O_{x_j} : \omega_\alpha^\beta(\underline{x}) \mapsto \alpha_j\omega_{\alpha-\mathbf{v}_j}^\beta(\underline{x}), \\ \mathfrak{f}_j : \omega_\alpha^\beta(\underline{x}) \mapsto \sum_{k=1}^{r}(-1)^{k-1}\delta_{j,j_k}\ \omega_\alpha^{\beta-\mathbf{v}_{j_k}}(\underline{x}), \end{array} \tag{3.15}$$

*are called the delta operators for $(\mathbf{d}_\Psi, \Lambda^*\mathbb{R}[\underline{x}])$.*

From the Definition 3.2.1, $\Psi_{\underline{x}}$ is obviously invertible and when restricted to the space of zero forms, it coincides with the Umbral operator which maps the multivariate monomials $\underline{x}^\alpha$ onto the multivariate basic polynomial sequence $V_\alpha(\underline{x})$ (c.f. [60]), so it is clear that each basic polynomial sequence of differential forms $\omega_\alpha^\beta(\underline{x})$ uniquely determine a Sheffer operator and vice-versa.

On the other hand, the operators introduced in Definitions 3.2.2 and 3.2.3 can be immediately recognized as raising and lowering operators, respectively, acting on basic polynomial sequence of differential forms $\omega_\alpha^\beta(\underline{x}) = V_\alpha(\underline{x})(\mathbf{d}_\Psi \underline{x})^\beta$.

The above construction allow us to derive some intertwining properties that will be important on the sequel. The first lemma follows as a consequence of the setting introduced in Chapter 1 while the second one follows from the framework developed in Section 2.1, Chapter 2.

**Lemma 3.2.4** *The Sheffer operator* $\Psi_{\underline{x}} : (\mathbf{d}, \Lambda^*\mathbb{R}[\underline{x}]) \to (\mathbf{d}_\Psi, \Lambda^*\mathbb{R}[\underline{x}])$ *intertwines* $\partial_{x_j}$ *and* $O_{x_j}$ *and (respectively,* $x_j$ *and* $x_j'$*) i.e.*

$$O_{x_j}\Psi_{\underline{x}} = \Psi_{\underline{x}}\partial_{x_j} \quad \textit{and} \quad x_j'\Psi_{\underline{x}} = \Psi_{\underline{x}} x_j. \tag{3.16}$$

*Moreover, the raising and lowering operators,* $x_j'$ *and* $O_{x_j}$*, respectively, satisfy the commutation relations (3.10).*

**Proof:** From the definition of $x_j'$ and $O_{x_j}$, we obtain from the definition of $\Psi_{\underline{x}}$ the relations

$$\Psi_{\underline{x}}(x_j\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta) = x_j'(\Psi_{\underline{x}}\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta) \quad \text{and} \quad \Psi_{\underline{x}}(\partial_{x_j}\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta) = O_{x_j}(\Psi_{\underline{x}}\underline{x}^\alpha(\mathbf{d}\underline{x})^\beta).$$

By linearity arguments the above extends to arbitrary forms $\omega \in \Lambda^*\mathbb{R}[\underline{x}]$. This proves the relations (3.16).

The bosonic character of the operators $O_{x_j}$ and $x_j'$ then follows as a consequence of the above intertwining relations and from the fact that $\partial_{x_j}$ and $x_j$ satisfy the commutation relations (3.10) when acting on $\Lambda^*\mathbb{R}[\underline{x}]$.

■

**Lemma 3.2.5** *The Sheffer operator* $\Psi_{\underline{x}} : (\mathbf{d}, \Lambda^*\mathbb{R}[\underline{x}]) \to (\mathbf{d}_\Psi, \Lambda^*\mathbb{R}[\underline{x}])$ *intertwines* $\mathbf{d}x_j$ *(respectively,* $\mathbf{i}_{\partial_{x_j}}$*) and* $\mathbf{d}_\Psi x_j(O_{x_j}')^{-1}$ *(respectively* $\mathbf{i}_{O_{x_j}}$*), i.e.*

$$\mathbf{d}_\Psi x_j(O_{x_j}')^{-1}\ \Psi_{\underline{x}} = \Psi_{\underline{x}}\ \mathbf{d}x_j \quad \textit{and} \quad \mathbf{i}_{O_{x_j}}\ \Psi_{\underline{x}} = \Psi_{\underline{x}}\ \mathbf{i}_{\partial_{x_j}}. \tag{3.17}$$

*Moreover, the operators* $\mathfrak{f}_j^\dagger = \mathbf{d}_\Psi x_j(O_{x_j}')^{-1}$ *and* $\mathfrak{f}_j = \mathbf{i}_{O_{x_j}}$ *satisfy the anti-commutation relations (3.11).*

**Proof:** For the polynomial differential forms $\omega = \underline{x}^\alpha(\mathbf{d}\underline{x})^\alpha$ the proof for the intertwining relations (3.17) follows from the definitions of $\Psi_{\underline{x}}$, $\mathbf{i_u}$, $\varepsilon_\theta$ and from the property (2.11). By linearity argument the above extends to arbitrary polynomial differential forms $\omega \in \Lambda^*\mathbb{R}[\underline{x}]$.

Finally, the proof of the fermionic character of the operators $\mathfrak{f}_j^\dagger = \mathbf{d}_\Psi x_j(O'_{x_j})^{-1}$ and $\mathfrak{f}_j = \mathbf{i}_{O_{x_j}}$ naturally follows from (3.17) and from the fermionic character of the operators $\mathbf{d}x_j$ and $\mathbf{i}_{\partial_{x_j}}$.

■

At this stage it is now interesting to make the bridge between differential forms and Clifford algebras:

**Remark 3.2.6** *As it was explained in [36, 10], in continuum (i.e. for $O_{x_j} = \partial_{x_j}$) the languages of differential forms and multi-vector functions may be identified in the natural way. Indeed there the bijective correspondence between the Clifford algebra of signature $(n,n)$ $\mathbb{R}_{n,n}$ and the algebra of endomorphisms $End(\Lambda^*\mathbb{R}[\underline{x}])$ is given by the bijective correspondences $\xi_j \leftrightarrow \Upsilon^-_{\mathbf{d}x_j}$ and $\xi_{j+n} \leftrightarrow \Upsilon^+_{\mathbf{d}x_j}$, where $\Upsilon^\pm_{\mathbf{d}x_j} \in End(\Lambda^*\mathbb{R}[\underline{x}])$ are defined viz (2.29).*

*We must note, however from Lemma 3.2.5 that there is also another natural possibility still, that is to consider the basic endomorphisms $\Psi_{\underline{x}}\Upsilon^\pm_{\mathbf{d}x_j}\Psi_{\underline{x}}^{-1} = \mathbf{d}_\Psi x_j(O'_{x_j})^{-1} \pm \mathbf{i}_{O_{x_j}}$.*

The above remark allow us to conclude that the isomorphism $\mathbb{R}_{n,n} \cong \mathrm{End}(\Lambda^*\mathbb{R}[\underline{x}])$ is invariant under the action of the Sheffer map. In particular this allows us to sift the group structure of the spin group $Spin(n,n)$ from continuous to discrete structures (c.f. [34]). The formal analogy between discrete differential forms with chains and cochains obtained in Section 2.2 should makes it possible to reinterpret the Ising model in relationship with the existence of Dirac spinors near to critically (c.f. [56], Section 4).

Now we will point out the correspondence between the formulae given in (3.5) and the complexified Witt basis (i.e. the fermionic setting).

**Remark 3.2.7** *Note that the operators $\mathbf{d}_\Psi x_j(O'_{x_j})^{-1}$ and $\mathbf{i}_{O_{x_j}}$, $j = 1, \ldots, n$ are nothing else than identifications of the so-called Witt basis of the algebra of endomorphisms $End(\Lambda^*\mathbb{R}[\underline{x}])$.*

*Indeed, we can write the operators $\mathbf{d}_\Psi x_j(O'_{x_j})^{-1}$ and $\mathbf{i}_{O_{x_j}}$ in terms of the Clifford-like basis $\Upsilon^\pm_{\mathbf{d}_\Psi x_j}$ as*

$$\begin{aligned}
\mathbf{d}_\Psi x_j(O'_{x_j})^{-1}\omega &= \frac{1}{2}\left(\Upsilon^+_{\mathbf{d}_\Psi x_j}(\omega) + \Upsilon^-_{\mathbf{d}_\Psi x_j}((O'_{x_j})^{-1}\omega)\right), \\
\mathbf{i}_{O_{x_j}}\omega &= \frac{1}{2}\left(\Upsilon^+_{\mathbf{d}_\Psi x_j}(O'_{x_j}\omega) + \Upsilon^-_{\mathbf{d}_\Psi x_j}(\omega)\right).
\end{aligned}$$

*This leads to the bijective correspondences $\mathbf{e}_j \bullet (\cdot) \leftrightarrow \mathbf{i}_{O_{x_j}}$ and $\mathbf{e}_j \wedge (\cdot) \leftrightarrow \mathbf{d}_\Psi x_j(O'_{x_j})^{-1}$.*

*Since the real Clifford algebra $\mathbb{R}_{n,n} \cong End(\Lambda^*\mathbb{R}[\underline{x}])$ is contained in the complex Clifford algebra $\mathbb{C}_{2n} = \mathbb{C} \bigotimes \mathbb{R}_{0,2n}$ as a special subalgebra (see Section 3.1), the isomorphism between the $\mathbb{R}_{n,n}$ and $\mathbb{C}_{2n} = \mathbb{C} \bigotimes \mathbb{R}_{0,2n}$ is thus obtained through the identifications $\mathbf{e}_j \leftrightarrow \xi_j$ and $\mathbf{e}_{j+n} \leftrightarrow -i\xi_{j+n}$. Moreover, we can identify $\mathbb{C}_{2n}$ with the set of endomorphisms $\left\{\mathbf{d}_\Psi x_j(O'_{x_j})^{-1}, \mathbf{i}_{O_{x_j}} \ : \ j = 1, \dots, n\right\} \subset End(\Lambda^*\mathcal{A})$ viz $\mathfrak{f}_j \leftrightarrow \mathbf{i}_{O_{x_j}}$ and $\mathfrak{f}_j^\dagger \leftrightarrow \mathbf{d}_\Psi x_j(O'_{x_j})^{-1}$.*

The above identities in particular allow us to recast the discrete counterpart of exterior derivative $\mathbf{d}$ and Hodge coderivative $\mathbf{d}^*$ in terms of left and right actions of Dirac operators acting on oriented and symmetric lattices [68, 49]. On the other hand, from the isomorphisms $\mathbb{C}_{2n} \cong \mathbb{R}_{n,n} \cong End(\mathbb{R}_{0,n})$, the groups $\tilde{U}(n)$ and $\tilde{SU}(n)$ are the right induced representations for the algebra of endomorphisms $\mathrm{End}(\Lambda^*\mathbb{R}[\underline{x}])$.

These facts have several important consequences, some of which we will discuss in the next subsection. We have now at hand the building blocks to construct basic operators in Discrete Clifford Analysis.

## 3.3 Basic Clifford Operators on Lattices

### 3.3.1 Umbral Dirac Operators and Lattice Structure

In order to make the transcription from differential forms to multi-vector calculus, we give the definition of $\mathbb{R}^r_{0,n}$-valued and $\mathbb{R}_{0,n}$-valued functions.

**Definition 3.3.1** *A function with values in $\mathbb{R}^r_{0,n}$ is expressed as*

$$F^{[r]}(\underline{x}) = \sum_{|\alpha|=r} F^\alpha(\underline{x})\underline{\mathbf{e}}^\alpha, \tag{3.18}$$

*where $F^\alpha(\underline{x})$ are real-valued functions.*

*The set of functions with values in $\mathbb{R}_{0,n}$ is expressed as a linear combination of $\mathbb{R}^r_{0,n}-$valued functions, i.e.*

$$F(\underline{x}) = \sum_{r=0}^{n} F^{[r]}(\underline{x}). \tag{3.19}$$

Let us restrict ourselves to the space of Clifford-valued polynomials $\mathcal{P} = \mathbb{R}[\underline{x}] \otimes \mathbb{R}_{0,n}$. We denote by $\mathcal{P}^{[r]} = \mathbb{R}[\underline{x}] \otimes \mathbb{R}^r_{0,n}$ the space of $\mathbb{R}^r_{0,n}-$valued polynomials. Using the same notation introduced on Section 3.1, we denote by $[\cdot]_r : \mathcal{P} \to \mathcal{P}^{[r]}$ the projection operator from $\mathcal{P}$ onto $\mathcal{P}^{[r]}$. Each $\mathbb{R}^r_{0,n}-$valued polynomial $F^{[r]}(\underline{x})$ is given by $F^{[r]}(\underline{x}) = [F(\underline{x})]_r$ so the following decomposition holds

$$\mathcal{P} = \sum_{r=0}^{n} \bigoplus \mathcal{P}^{[r]}.$$

We introduce the left and right delta operators $Q_{x_j}|$ and $Q_{x_j}$, respectively, as the operators which satisfy

$$Q_{x_j}(F_\alpha(\underline{x}))\mathbf{d}_\Psi x_j = \mathbf{d}_\Psi x_j Q_{x_j}|(F_\alpha(\underline{x})). \tag{3.20}$$

By combining the Clifford basis $\mathbf{e}_j$ with $Q_{x_j}$ and $Q_{x_j}|$, we introduce the left and right multiplications $D'(\cdot), (\cdot)D' \in \mathrm{End}(\mathcal{P})$ *viz*

$$\begin{array}{rcl} D'F(\underline{x}) & = & \sum_{j=1}^{n}\sum_{|\alpha|=0}^{n} Q_{x_j}|F_\alpha(\underline{x})\mathbf{e}_j\underline{\mathbf{e}}^\alpha, \\ (F(\underline{x}))D' & = & \sum_{j=1}^{n}\sum_{|\alpha|=0}^{n} Q_{x_j}F_\alpha(\underline{x})\underline{\mathbf{e}}^\alpha\mathbf{e}_j. \end{array} \tag{3.21}$$

According to (2.11) (see Section 2.1), the operators $Q_{x_j}, Q_{x_j}| \in \mathrm{End}(\mathbb{R}[\underline{x}])$ are interrelated with the Pincherle derivative $O'_{x_j}$, namely

$$Q_{x_j}| = (O'_{x_j})^{-1}Q_{x_j}, \quad Q_{x_j} = O'_{x_j}Q_{x_j}|. \tag{3.22}$$

These actions are the umbral counterpart of left and right multiplications of a Clifford-valued function $F(\underline{x})$ with the classical Dirac operator $D = \sum_{j=1}^{n}\mathbf{e}_j\partial_{x_j}$. Furthermore, we can also introduce the concepts of left and right Umbral monogenicity:

**Definition 3.3.2 (Umbral Monogenic Function)** *A Clifford-valued function $F(\underline{x})$ defined viz (3.19) is Umbral left monogenic (respectively, Umbral right monogenic) if and only if $D'F(\underline{x}) = 0$ (respectively, $F(\underline{x})D' = 0$).*

From the geometric point of view, this constraint can be interpreted in connection with rhombic lattices (c.f. [28]), critical maps (c.f. [56, 47]) or quad-graphs (c.f [6, 31]).

We would like to point out that the actions (3.21) can be written in terms basic multipliers acting on the algebra $\mathrm{End}(\mathcal{P})$. Indeed by considering $\xi_j, \xi_{j+n} \in \mathrm{End}(\mathbb{R}_{0,n})$ introduced in Section 3.1, the coordinate expressions for the operators $D' : F(\underline{x}) \mapsto D'F(\underline{x})$ and $D'| : F(\underline{x}) \mapsto F(\underline{x})D'$ are given by

$$D' = \sum_{j=1}^{n}\xi_j Q_{x_j}| \quad \text{and} \quad D'| = \sum_{j=1}^{n}\xi_{j+n}Q_{x_j}. \tag{3.23}$$

Since $(O'_{x_j})^{-1}$ commutes with $Q_{x_j}$ (c.f. [60], pp. 18), one can easily verify that the operators $Q_{x_j}$ and $Q_{x_j}|$ commute when acting on $\mathcal{P}$, i.e. $[Q_{x_j}, Q_{x_k}|] = 0$. Moreover the operators $D'$ and $D'|$ satisfy the following anti-commuting relations when acting on $\mathcal{P}$:

$$\{D', D'\} = -2\sum_{j=1}^{n}(Q_{x_j}|)^2, \quad \{D'|, D'|\} = 2\sum_{j=1}^{n}Q_{x_j}^2, \quad \{D', D'|\} = 0. \tag{3.24}$$

From the identities (3.24), it is clear that the square of the operators $(D')^2 = \frac{1}{2}\{D', D'\}$ and $(D'|)^2 = \frac{1}{2}\{D'|, D'|\}$ are scalar operators and hence it can be defined as the umbral counterparts of the Laplacian, i.e. $\Delta' = -(D')^2$ and $\Delta'| = (D'|)^2$. This indeed what we expected from a "true" Dirac operator.

On the other hand, the anti-commutator between $D'$ and $D'|$ means that the operators $D'$ and $D'|$ are orthogonal. Moreover the subalgebra of $\mathrm{End}(\mathcal{P})$ generated by the endomorphisms $D', D'| \in \mathrm{End}(\mathcal{P})$ induces a radial algebra representation (see formula (3.3), Section 3.1).

**Remark 3.3.3** *We would like to stress however that from the coordinate expressions (3.23) the notions of left and right monogenicity only coincide if the left and right delta operators coincide (i.e. $Q_{x_j}$ is symmetric). Hence the transcription of Clifford Analysis breaks in general when we sift from the classical setting to its umbral counterparts.*

There are still at least two possibilities to consider alternative definitions for the operators $D'$ and $D'|$. One of them can be obtained by the replacements $\mathbf{e}_j \rightarrow \Upsilon^-_{\mathbf{d}x_j}$ and $\mathbf{e}_{j+n} \rightarrow \Upsilon^+_{\mathbf{d}x_j}$, where $\Upsilon^\pm_{\mathbf{d}x_j}$ are the Clifford-like basis defined *viz* formulae (2.28) (see Subsection 2.1.2, Chapter 2). Taking into account this replacement, we formalize the construction of Dirac operators on the oriented lattice proposed by Vaz [68] and Kanamori and Kawamoto [49] in terms of delta operators acting on the lattice.

Another possibility can be obtained by considering a symmetric extension for the operators $D'$ and $D'|$. In order to proceed, let us restrict the products (3.21) to the space $\mathcal{P}^{[r]}$. By applying the definition we can split $D'F^{[r]}(\underline{x})$ and $F^{[r]}(\underline{x})D'$ into

$$\begin{aligned} D'F^{[r]}(\underline{x}) &= \sum_{j=1}^n \sum_{|\alpha|=r} Q_{x_j}|F^\alpha(\underline{x})\mathbf{e}_j\underline{\mathbf{e}}^\alpha = \sum_{j=1}^n \mathbf{e}_j \bullet Q_{x_j}|(F^{[r]}(\underline{x})) + \sum_{j=1}^n \mathbf{e}_j \wedge Q_{x_j}|(F^{[r]}(\underline{x})), \\ F^{[r]}(\underline{x})D' &= \sum_{j=1}^n \sum_{|\alpha|=r} Q_{x_j}F^\alpha(\underline{x})\underline{\mathbf{e}}^\alpha\mathbf{e}_j = \sum_{j=1}^n Q_{x_j}|(F^{[r]}(\underline{x})) \bullet \mathbf{e}_j + \sum_{j=1}^n Q_{x_j}(F^{[r]}(\underline{x})) \wedge \mathbf{e}_j. \end{aligned}$$

In particular, the projections $[D'F^{[r]}(\underline{x})]_{r-1}$ and $[F^{[r]}(\underline{x})D']_{r+1}$ thus give rise to two differential operators $D' \bullet (\cdot) \ : \mathcal{P}^{[r]} \rightarrow \mathcal{P}^{[r-1]}$ and $(\cdot) \ \wedge D' : \mathcal{P}^{[r]} \rightarrow \mathcal{P}^{[r+1]}$ defined *viz*

$$\begin{aligned} D' \bullet (\cdot) : F^{[r]}(\underline{x}) &\mapsto [D'F^{[r]}(\underline{x})]_{r-1} = \sum_{|\alpha|=r} \sum_{j=1}^n \mathbf{e}_j \bullet Q_{x_j}|(F^{[r]}(\underline{x})) \\ (\cdot) \wedge D' : F^{[r]}(\underline{x}) &\mapsto [F^{[r]}(\underline{x})D']_{r+1} = \sum_{|\alpha|=r} \sum_{j=1}^n Q_{x_j}(F^{[r]}(\underline{x})) \wedge \mathbf{e}_j \end{aligned}$$

From Remark 3.2.7 it readily follows that these operators may be identified as

$$D' \bullet (\cdot) \leftrightarrow \partial'| = \textstyle\sum_{j=1}^n \mathfrak{f}_j Q_{x_j}|, \quad (\cdot) \wedge D' \leftrightarrow \partial' = \sum_{j=1}^n \mathfrak{f}_j^\dagger Q_{x_j}. \tag{3.25}$$

Furthermore, by combining the operators $\partial'$ and $\partial'|$ we introduce two new operators $\partial'_\pm$ *viz* $\partial'_+ = \partial'| - \partial'$ and $\partial'_- = -i(\partial'| + \partial')$.

In terms of the Clifford generators of $\mathbb{C}_{2n}$, the operators $\partial'_\pm$ then correspond to

$$\begin{array}{rclcl} \partial'_+ &=& \sum_{j=1}^n \mathfrak{f}_j Q_{x_j}| - \mathfrak{f}_j^\dagger Q_{x_j} &=& \sum_{j=1}^n \mathbf{e}_j Q^+_{x_j} + \mathbf{e}_{j+n} Q^-_{x_j} \\ \partial'_- &=& -i\sum_{j=1}^n \mathfrak{f}_j Q_{x_j}| + \mathfrak{f}_j^\dagger Q_{x_j} &=& \sum_{j=1}^n \mathbf{e}_j Q^-_{x_j} - \mathbf{e}_{j+n} Q^+_{x_j} \end{array} \tag{3.26}$$

where $Q^+_{x_j} = \frac{1}{2}\left(Q_{x_j}| + Q_{x_j}\right)$ and $Q^-_{x_j} = \frac{1}{2i}\left(Q_{x_j}| - Q_{x_j}\right)$ are the symmetric and the skew-symmetric versions of the operator $Q_{x_j}$, respectively.

Since $Q_{x_j}| = Q^+_{x_j} + iQ^-_{x_j}$ and $Q_{x_j} = Q^+_{x_j} - iQ^-_{x_j}$, the left and right delta operators can be identified as the umbral counterparts of the classical Cauchy-Riemann operators and their conjugates.

All the above identifications clearly suggests the Hermitian Clifford setting as the natural multi-vector setting to build up the symmetric extension of the operators $D'$ and $D'|$ (see e.g. [12, 8]). So the following extension of the concept of Umbral monogenicity (see definition 3.3.2) becomes rather natural

**Definition 3.3.4 (Umbral Hermitian monogenic)** *A polynomial function $F(\underline{x}) \in \mathcal{P}$ taking values in $\mathbb{C}_{2n}$ is Umbral Hermitian monogenic if and only if $F(\underline{x})$ is a solution of the coupled systems of equations*

$$\left\{\begin{array}{l} \partial'(F(\underline{x})) = 0 \\ \partial'|(F(\underline{x})) = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \partial'_+(F(\underline{x})) = 0 \\ \partial'_-(F(\underline{x})) = 0 \end{array}\right.$$

The space of Umbral Hermitian monogenic functions is given by the subspace $\ker\partial' \cap \ker\partial'| = \ker\partial'_+ \cap \ker\partial'_-$. In particular in the case where $Q_{x_j}$ is symmetric, $\ker\partial'_+$ and $\ker\partial'_-$ are isomorphic to the space of Umbral left and right monogenic functions, respectively. Moreover from the correspondence (3.25), $\ker\partial' = \ker\partial'|$.

We now take a close look at the case when the polynomial functions $F(\underline{x})$ take values in $\mathbb{C}_n$. From the isomorphism $\mathbb{C}S_n = \mathbb{C}_n I \cong \mathbb{C}_{2n} I$, the set of polynomial functions $F(\underline{x})I$ with values in the spinor space may be solutions of the coupled system of equations

$$\left\{\begin{array}{l} \partial'(F(\underline{x})I) = 0 \\ \partial'|(F(\underline{x})I) = 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{l} \partial'_+(F(\underline{x})I) = 0 \\ \partial'_-(F(\underline{x})I) = 0 \end{array}\right.$$

Using the conversion relations obtained in Section 3.1, we can obtain an alternative characterization for function $F(\underline{x})$.

**Proposition 3.3.5** *A polynomial function with values on $\mathbb{C}_n$ is Umbral Hermitian monogenic if and only if*

$$\begin{cases} (D')^+(F(\underline{x})) - i(F(\underline{x})^*)(D')^- = 0 \\ (F(\underline{x}))(D')^+ - i(D')^-(F(\underline{x})^*) = 0 \end{cases}$$

*where $(D')^\pm := \sum_{j=1}^n \mathbf{e}_j Q^\pm_{x_j}$ are the discrete orthogonal Dirac operators.*

**Proof:** By take into account the conversion relation $i\mathbf{e}_{j+n}F(\underline{x})I = F(\underline{x})^*\mathbf{e}_j I$ for $j = 1, \ldots, n$ we have

$$\partial'_+(F(\underline{x})I) = \sum_{j=1}^n \mathbf{e}_j Q^+_{x_j}(F(\underline{x}))I + \mathbf{e}_{j+n}Q^-_{x_j}(F(\underline{x}))I = \sum_{j=1}^n \mathbf{e}_j Q^+_{x_j}(F(\underline{x}))I - iQ^-_{x_j}(F(\underline{x})^*)\mathbf{e}_j I.$$

This corresponds to $\partial'_+(F(\underline{x})I) = ((D')^+F(\underline{x}) - iF(\underline{x})^*(D')^-)\, I$.

On the other hand, using again the converse relation we obtain after straightforward manipulations the identity $\partial'_-(F(\underline{x})I) = (F(\underline{x})^*(D')^+ - i(D')^-F(\underline{x}))\, I$.

Hence the equations $\partial'_\pm(F(\underline{x})I) = 0$ are equivalent to

$$(D')^+(F(\underline{x})) - i(F(\underline{x})^*)(D')^- = 0 = (F(\underline{x}))(D')^+ - i(D')^-(F(\underline{x})^*).$$

■

The later proposition can be viewed as an extension of the concept of discrete holomorphy proposed by Bobenko, Mercat and Suris in [6] on the elementary squares of $\mathbb{Z}^n$. According to the proposed setting, for a given quasicristallic rhombic embedding $\mathcal{D}$ with set of labels $\{\pm\alpha_1, \ldots, \pm\alpha_n\}$, a function $f : \mathbb{Z}^n \mapsto \mathbb{C}$ is called discrete holomorphic, if it satisfies, on each elementary square of $\mathbb{Z}^n$, the equations

$$\frac{f(m + \mathbf{v}_j + \mathbf{e}_k) - f(m)}{f(m + \mathbf{v}_j) - f(m + \mathbf{v}_k)} = \frac{\alpha_j + \alpha_k}{\alpha_j - \alpha_k}$$

holds for all $j$ and $k$.

Faustino, Kähler and Sommen's ideas in [31] give a geometric interpretation of the above proposition. According to this end we will consider $n$-rhombohedric embeddings $\mathcal{D}$ with set of labels $\{\pm\alpha_1, \ldots, \pm\alpha_n\}$. Extending the labeling $\alpha : \vec{E}(\mathcal{D}) \mapsto \mathbb{C}$ to all edges of $\mathbb{Z}^n$, a new basis is created (locally for each rhombohedron) via

$$\begin{aligned} \mathbf{e}_1 &= \alpha_1 + \ldots + \alpha_{n-2} + \alpha_{n-1} + \alpha_n \\ \mathbf{e}_2 &= \alpha_1 + \ldots + \alpha_{n-2} + \alpha_{n-1} - \alpha_n \\ \mathbf{e}_3 &= \alpha_1 + \ldots + \alpha_{n-2} - \alpha_{n-1} - \alpha_n \\ \mathbf{e}_4 &= \alpha_1 + \ldots - \alpha_{n-2} - \alpha_{n-1} - \alpha_n \\ &\vdots \end{aligned}$$

and, afterwards, we decompose it into $\mathbf{e}_j = \mathfrak{f}_j + \mathfrak{f}_j^\dagger$. Therefore, joining the center points of the rhombohedrons we create two graphs, $\mathcal{G}$ ("white" vertices) and $\mathcal{G}^*$ ("black" vertices), which leads to the definition of a discrete monogenic function on the graph $\mathcal{D}$, constructed from $\mathcal{G}$ and $\mathcal{G}^*$ ( $V(\mathcal{D}) = V(\mathcal{G}) \cup V(\mathcal{G}^*)$ ) by means of Definition 3.3.4.

Hence Proposition 3.3.5 follows by construction and in particular for a discrete Hermitian monogenic function which is additionally complex-valued, is also a discrete holomorphic function in the sense of [6].

The next corollary follows by combining the conversion relations obtained in Section 3.1 with the last proposition:

**Corollary 3.3.6** *A polynomial function $F(\underline{x})$ with values in $\mathbb{C}_n$ is Umbral Hermitian monogenic if its projections $[F(\underline{x})]_r$, $r = 0, 1, \ldots, n$ are Umbral monogenic in the spinor space $\mathbb{C}S_n^r = \mathbb{C}_n^r I$, i.e.*

$$\partial_+([F(\underline{x})]_r I) = 0, \quad \text{for} \;\; r = 0, 1, \ldots, n.$$

There are some formulae holding from direct application of the coordinate expressions (3.25) and (3.26) that follows, namely

1. Isotropy condition: $\{\partial', \partial'\} = 0 = \{\partial'|, \partial'|\}$.

2. Orthogonality condition: $\{\partial'_+, \partial'_-\} = 0$.

3. Second order operator splitting:

   - Using $\partial'$ and $\partial'|$: $\{\partial', \partial'|\} = \sum_{j=1}^n Q_{x_j}|Q_{x_j}$.
   - Using $\partial'_\pm$: $\{\partial'_+, \partial'_+\} = -2\sum_{j=1}^n Q_{x_j}|Q_{x_j} = \{\partial'_-, \partial'_-\}$.

From the above conditions, it is easy to see that the subalgebra generated by the operators $\partial'_\pm \in \mathrm{End}(\mathcal{P})$ coincide with the subalgebra generated by $\partial', \partial'| \in \mathrm{End}(\mathcal{P})$ and moreover induces a radial algebra (see formula (3.3), Section 3.1).

Among many examples of this construction we will explore the case where $Q_{x_j}$ are finite difference operators. We will start to consider the case of Dirac operators acting on the oriented lattice.

**Example 3.3.7 (Dirac operators on the oriented lattice)** *We consider the difference Dirac operators $D_h^\pm$ introduced by N. Faustino and U. Kähler in [30]. In the terminology of that paper, $D_h^\pm = \sum_{j=1}^n \mathbf{e}_j \partial_h^{\pm j}$ are the forward/backward versions of the Dirac operator,* *where $\partial_h^{\pm j}$ where $\partial_h^{\pm j}$ are the forward (respectively, backward) differences defined in Example 1.2.9 (see Section 1.2).*

*Notice that the square of these operators does not split the star Laplacian operator*

$$\Delta_h = \sum_{j=1}^{n} \frac{\tau_{h\mathbf{v}_j} + \tau_{-h\mathbf{v}_j} - 2\mathbf{id}}{h^2} = \sum_{j=1}^{n} \partial_h^{-j}\partial_h^{+j}.$$

*However if we consider a central version of the Dirac operator acting on the lattice with mesh-size $h$, i.e. $D_h = \frac{1}{2}\left(D_h^+ + D_h^-\right)$, we obtain the splitting of the star Laplacian $\Delta_h$ as $D_{h/2}^2 = -\Delta_h$.*

*Next we introduce new basis elements $b_1, b_2, \ldots, b_n$ via the relation $b_j = \mathbf{e}_j\tau_{h/2\mathbf{v}_j}$. Hence the operator $D_{h/2}$ can be rewritten in terms of $b_j$ as $D_{h/2} = \sum_{j=1}^n b_j\partial_h^{-j}$.*

*Since $b_jb_k + b_kb_j = -2\delta_{jk}\tau_{h\mathbf{v}_k}$, these generators can be identified with the basic endomorphisms $\Upsilon^-_{\mathbf{d}x_j}$ (see (2.28), Subsection 2.1.2). From the isomorphism between $b_j$ and $\Upsilon^-_{\mathbf{d}x_j}$, the product $b_jF(\underline{x})$ is no longer commutative, however, from formulae (2.11) (see Subsection 2.1.1) it induces shifts on the real-valued functions $F^\alpha(\underline{x})$, i.e.*

$$b_jF^\alpha(\underline{x}) = F^\alpha(\underline{x} + h\mathbf{v}_j)b_j. \tag{3.27}$$

*This suggest the introduction of basic multipliers $b_j, b_{j+n} \in End(\mathcal{P})$,*

$$b_j : F(\underline{x}) \mapsto \mathbf{e}_j\tau_{h/2\mathbf{v}_j}F(\underline{x}), \quad b_{j+n} : F(\underline{x}) \mapsto \tau_{-h/2\mathbf{v}_j}(F(\underline{x}))^*\mathbf{e}_j.$$

*In terms of $b_j$ and $b_{j+n}$, the identity (3.27) can be extended to Clifford-valued functions as $b_jF(\underline{x}) = b_{j+n}F(\underline{x} + h\mathbf{v}_j)$.*

In the above example we build up a Dirac operator acting on the oriented lattice. This operator coincide with the Dirac operators introduced in [68, 49]. Moreover, for the left and right products $D_{h/2}F(\underline{x}) := \sum_{j=1}^n b_j\partial_h^{-j}F(\underline{x})$ and $F(\underline{x})D_{h/2} := \sum_{j=1}^n b_{j+n}\partial_h^{+j}F(\underline{x})$, respectively, the next lemma readily follows:

**Lemma 3.3.8** *When acting on polynomial $r-$vectors $\mathcal{P}^{[r]}$ we have*

$$\left[D_{h/2}F^{[r]}(\underline{x})\right]_{r-1} = (-1)^{r+1}\left[F^{[r]}(\underline{x})D_{h/2}\right]_{r-1}, \quad \left[D_{h/2}F^{[r]}(\underline{x})\right]_{r+1} = (-1)^{r}\left[F^{[r]}(\underline{x})D_{h/2}\right]_{r+1}.$$

*Hereby $[\cdot]_r : \mathbb{R}_{0,n} \to \mathbb{R}_{0,n}^r$ stands for the projection operator from $\mathbb{R}_{0,n}$ onto $\mathbb{R}_{0,n}^r$ (see formula (3.2) Section 3.1).*

**Proof:** By applying the definition we can split $D_{h/2}F^{[r]}(\underline{x})$ into

$$\begin{aligned} D_{h/2}F^{[r]}(\underline{x}) &= \sum_{j=1}^{n}\sum_{|\alpha|=r} \partial_h^{-j}F^\alpha(\underline{x} + h/2\mathbf{v}_j)\mathbf{e}_j\underline{\mathbf{e}}^\alpha \\ &= \left[D_{h/2}F^{[r]}(\underline{x} + h/2\mathbf{v}_j)\right]_{r-1} + \left[D_{h/2}F^{[r]}(\underline{x} + h/2\mathbf{v}_j)\right]_{r+1}, \end{aligned}$$

where $\left[D_{h/2}F^{[r]}(\underline{x})\right]_{r-1}$ and $\left[D_{h/2}F^{[r]}(\underline{x})\right]_{r+1}$ are given by

$$\left[D_{h/2}F^{[r]}(\underline{x})\right]_{r-1} = \sum_{j=1}^{n}\sum_{|\alpha|=r} \frac{\partial_{h/2}^{-j}+\partial_{h/2}^{+j}}{2}F^{\alpha}(\underline{x})\mathbf{e}_j \bullet \underline{\mathbf{e}}^{\alpha},$$
$$\left[D_{h/2}F^{[r]}(\underline{x})\right]_{r+1} = \sum_{j=1}^{n}\sum_{|\alpha|=r} \frac{\partial_{h/2}^{-j}+\partial_{h/2}^{+j}}{2}F^{\alpha}(\underline{x})\mathbf{e}_j \wedge \underline{\mathbf{e}}^{\alpha}.$$

Take into account formulae (3.5), the terms $\left(\frac{\partial_{h/2}^{-j}+\partial_{h/2}^{+j}}{2}\right)F^{\alpha}(\underline{x})\mathbf{e}_j\bullet\underline{\mathbf{e}}^{\alpha}$ and $\left(\frac{\partial_{h/2}^{-j}+\partial_{h/2}^{+j}}{2}\right)F^{\alpha}(\underline{x})\mathbf{e}_j\wedge \underline{\mathbf{e}}^{\alpha}$ are equal to $(-1)^{r+1}\partial_h^{+j}F^{\alpha}(\underline{x}-h/2\mathbf{v}_j)\underline{\mathbf{e}}^{\alpha}\bullet\mathbf{e}_j$ and $(-1)^{r}\partial_h^{+j}F^{\alpha}(\underline{x}-h/2\mathbf{v}_j)\underline{\mathbf{e}}^{\alpha}\wedge\mathbf{e}_j$, respectively. Hence, we get

$$\left[D_{h/2}F^{[r]}(\underline{x})\right]_{r-1} = (-1)^{r+1}\left[F^{[r]}(\underline{x})D_{h/2}\right]_{r-1} \quad \text{and} \quad \left[D_{h/2}F^{[r]}(\underline{x})\right]_{r+1} = (-1)^{r}\left[F^{[r]}(\underline{x})D_{h/2}\right]_{r}$$

This proves (3.21).

■

Moreover, direct application of the above lemma leads to the following corollary:

**Corollary 3.3.9** *On the oriented lattice with meshsize $h>0$, the Clifford-valued polynomial $F(\underline{x})$ is discrete left monogenic if and only if it is discrete right monogenic.*

Next we will show how to construct Dirac operators on symmetric lattices.

**Example 3.3.10 (Dirac operators on the symmetric lattice)** *Our starting point is to consider again the difference Dirac operators $D_h^{\pm}$ introduced in [30].*

*Notice that $D_h^{+}| = \sum_{j=1}^{n}\mathbf{e}_{j+n}\partial_h^{-j}$, where the forward/backward differences $\partial_h^{\pm j}$ corresponds to the left/right delta operators acting on the lattice [23], i.e. $\mathbf{d}_{\Psi}x_j\ \partial_h^{+j}F(\underline{x}) = \partial_h^{-j}F(\underline{x})\mathbf{d}_{\Psi}x_j$. Thus the operators $\partial'|$ and $\partial'$ given by the identifications $\partial'| \leftrightarrow \left(D_h^{+}\right)^{\dagger}\bullet(\cdot)$ and $\partial' \leftrightarrow (\cdot)\wedge D_h^{+}$ correspond to $\partial'| = \sum_{j=1}^{n}\mathfrak{f}_j\partial_h^{-j}$ and $\partial' = \sum_{j=1}^{n}\mathfrak{f}_j^{\dagger}\partial_h^{+j}$ and hence the operators $\partial'_{\pm}$ are given by*

$$\begin{array}{rcl} \partial'_{+} &=& \sum_{j=1}^{n}\mathbf{e}_j\frac{1}{2}\left(\partial_h^{+j}+\partial_h^{-j}\right)+\mathbf{e}_{j+n}\frac{1}{2i}(\partial_h^{-j}-\partial_h^{+j}),\\ \partial'_{-} &=& \sum_{j=1}^{n}\mathbf{e}_j\frac{1}{2i}\left(\partial_h^{+j}-\partial_h^{-j}\right)+\mathbf{e}_{j+n}\frac{1}{2}(\partial_h^{-j}+\partial_h^{+j}). \end{array} \tag{3.28}$$

*Physically speaking, the operator $\partial'_{+}$ is obtained from $D_h=\frac{1}{2}\left(D_h^{+}+D_h^{-}\right)$ by adding the extra terms of the type $\mathbf{e}_{j+n}\frac{1}{2i}\left(\partial_h^{+j}-\partial_h^{-j}\right)$. It should be noted however that the difference $\partial'_{+}-D_h$ is a second order operator which converges to zero, which can be understood as an extra fermion term which acquires a mass of the order of the cut off (c.f. [72]).*

*Furthermore according to Proposition 3.3.5, the polynomial functions $F(\underline{x})$ with values on $\mathbb{C}_n$ are Umbral Hermitian monogenic on the lattice with mesh-width $h>0$ if and only if they are solutions of the coupled system*

$$\begin{cases} \frac{1}{2}\left(D_h^- + D_h^+\right)(F(\underline{x})) = \frac{1}{2}(F(\underline{x})^*)\left(D_h^- - D_h^+\right) \\ \frac{1}{2}(F(\underline{x}))\left(D_h^- + D_h^+\right) = \frac{1}{2}\left(D_h^- - D_h^+\right)(F(\underline{x})^*) \end{cases}$$

*while from Corollary 3.3.6, the Umbral Hermitian monogenic on the lattice with mesh-width $h>0$, $F(\underline{x})$ can be recovered viz projections on the space of $r-$vectors satisfying*

$$[F(\underline{x})]_r I \in \ker \partial'_+, \quad r = 0, 1, \ldots, n.$$

*Since the difference $D_h^- - D_h^+$ tends to zero when $h$ tends to zero, we can see the Umbral Hermitian monogenic polynomial functions on the lattice with mesh-width $h>0$ can be viewed as deformed versions of Umbral monogenic polynomial functions on the lattice with mesh-width $h>0$. Such deformations can be splitted into Umbral monogenic functions on the lattice $h>0$ with values on the spinor spaces $\mathbb{C}S_n^r$, $r=0,1,\ldots,n$.*

With the above construction, we also formalize the approach proposed introduced by Kanamori, Kanamoto in [49] and Forgy, Schreiber in [34] for the symmetric lattice in terms left and right delta operators as well as establish a contact with the Wilson approach [72]. Indeed the operator $\partial'_-$ on the lattice with mesh-width $h$ is closely related the Dirac operators on the symmetric lattice introduced in [49] (see formula (4.10), page 21) and in [34] (see formula (5.34), page 71).

**Remark 3.3.11** *It should be pointed that here, we establish in particular a correspondence between the symmetry of the lattice and the Hermitian setting (see [12, 8] and the references given there), the departure point will be the Hermitian operators $\partial|'$ and $\partial'$ instead of the orthogonal operator $\partial'_+$. This is indeed the main difference between our philosophy and the philosophy proposed by Kanamori, Kanamoto in [49] and Forgy, Schreiber in [34].*

There also exist an alternative approach to define discrete Dirac operators obtained by Faustino, Kähler, and Sommen in [31], where the Witt basis $\mathfrak{f}_j$ and $\mathfrak{f}_j^\dagger$ are replaced by $\mathbf{e}_j^-$ and $\mathbf{e}_j^+$, respectively, such that

1. $\mathbf{e}_j^\pm \mathbf{e}_k^\pm + \mathbf{e}_k^\pm \mathbf{e}_j^\pm = -2\mathfrak{g}_{jk}^\pm, \quad \mathbf{e}_j^+ \mathbf{e}_k^- + \mathbf{e}_k^- \mathbf{e}_j^+ = -2M_{jk}, \quad$ and $\quad (\mathbf{e}_j^\pm)^2 = 0$
2. $\mathbf{e}_j^+$ and $\mathbf{e}_j^-$ has the same direction, that is, $\mathfrak{g}_{jk}^+ = \mathfrak{g}_{jk}^-$ and $M_{jk} = M_{kj}$
3. $\mathfrak{g}_{jk}^+ + \mathfrak{g}_{jk}^- + M_{jk} + M_{kj} = \delta_{jk}$

hold for two symmetric matrices $\mathfrak{g}_{jk}^{+}, \mathfrak{g}_{jk}^{-}$ and one general matrix $M_{jk}$. These relations completely determine the metric of the underlying $2n-$dimensional space as well as $\mathbf{e}_j^- + \mathbf{e}_j^+$ the Clifford generators of $\mathbb{R}_{0,n}$.

As it was pointed in [31], the square of the operators $D_h^{-+} = \sum_{j=1}^n \mathbf{e}_j^- \partial_h^{-j} + \mathbf{e}_j^+ \partial_h^{+j}$ and $D_h^{+-} = \sum_{j=1}^n \mathbf{e}_j^+ \partial_h^{-j} + \mathbf{e}_j^- \partial_h^{+j}$ (i.e. the discrete Laplacians) in general involve terms which do not respect the neighborhood of $\underline{x}$ in $h\mathbb{Z}^n$.

### 3.3.2 Umbral Vector Operators and Lattice Structure

Having introduced the umbral versions of Dirac operators as well as established the interplay between Dirac operators and lattice structure, we arrive now at the problem of introducing the umbral counterpart of the coordinate variables $x = \sum_{j=1}^n \mathbf{e}_j x_j$ as well as establish the interplay between multiplication vector operators belonging to the Clifford algebra and the lattice structure.

We define the products $x' F(\underline{x})$ and $F(\underline{x}) x'$ as

$$x' F(\underline{x}) = \sum_{j=1}^n \mathbf{e}_j x_j' | F(\underline{x}), \quad F(\underline{x}) x' = \sum_{j=1}^n x_j' F(\underline{x}) \mathbf{e}_j. \tag{3.29}$$

In the above setting, the operators $x_j'|$ and $x_j'$ are left and right Sheffer shifts (see Definition 3.2.2).

According to Subsection 1.2, Chapter 1 these operators are completely determined viz the commutation relations (see Remarks 1.2.7 and 1.2.8)

$$\begin{array}{cc} \left[Q_{x_j}, Q_{x_k}\right] = 0, & \left[Q_{x_j}|, Q_{x_k}|\right] = 0, \\ {[x_j', x_k'] = 0,} & {[x_j'|, x_k'|] = 0,} \\ \left[Q_{x_j}, x_k'\right] = \delta_{jk}\mathbf{id}, & \left[Q_{x_j}|, x_k'|\right] = \delta_{jk}\mathbf{id}. \end{array} \tag{3.30}$$

i.e. the operators $Q_{x_j}, x_j' \in \mathrm{End}(\mathcal{P})$ (respectively, $Q_{x_j}|, x_j'| \in \mathrm{End}(\mathcal{P})$) satisfy the following Bose commutation relations (see relations (3.10) and Lemma 3.2.4).

Indeed by considering again the basic multipliers $\xi_j, \xi_{j+n} \in \mathrm{End}(\mathbb{R}_{0,n})$ defined in Section 3.1, the endomorphisms $x', x'| \in \mathrm{End}(\mathcal{P})$ defined viz the mappings $x' : F(\underline{x}) \mapsto x' F(\underline{x})$ and $x'|F(\underline{x}) \mapsto F(\underline{x}) x'$ are given by the coordinate expressions

$$x' = \sum_{j=1}^n \xi_j x_j'|, \quad x'| = \sum_{j=1}^n \xi_{j+n} x_j'. \tag{3.31}$$

We would like to point out the left and right Sheffer shifts only coincide when the operator $Q_{x_j}$ is symmetric. From the above construction, it is clear that the square of the operators

$x'$ and $x'|$, $(x')^2=\frac{1}{2}\{x',x'\}$ and $(x'|)^2=\frac{1}{2}\{x'|,x'|\}$, respectively, are scalar operators since

$$(x')^2=-\sum_{j=1}^n (x_j'|)^2, \quad (x'|)^2=\sum_{j=1}^n (x_j')^2.$$

This allows us to define the umbral counterparts of the norm operator $|\underline{x}|^2=-x^2$ as $|\underline{x}'|^2=-(x')^2\mathbf{1}$ and $|\underline{x}'|\ |^2=(x'|)^2\,\mathbf{1}$, respectively.

**Remark 3.3.12** *We would like to stress that for the non-symmetric case, (i.e. $Q_{x_j}\neq Q_{x_j}|$), the operators $x_j'$ and $x_j'|$ no longer commute when acting on End($\mathcal{P}$). As a further consequence, the commutator $\{x',x'|\}$ corresponds to*

$$\{x',x'|\}=\sum_{j,k=1}^n \mathbf{e}_j\mathbf{e}_{k+n}[x_j',x_k'|]=\sum_{j,k=1}^n \mathbf{e}_{k+n}\mathbf{e}_j[x_k'|,x_j'].$$

**Remark 3.3.13** *It should be also pointed out that the Pincherle derivative no longer commute with the operators $x_j'$ and $x_j'|$ (see e.g. Example 1.2.9) and hence the basic endomorphisms $\Upsilon^{\pm}_{\mathbf{d}x_j}$ defined in 2.28 (see Subsection 2.1.2, Chapter 2).*

As a further consequence of the above remarks, the operators $x'$ and $x'|$ are no longer orthogonal and, moreover, the subalgebra generated by endomorphisms $x',x'|\in \mathrm{End}(\mathcal{P})$ does not induce a radial algebra representation (see formula (3.3), Section 3.1) but a deformation of it.

With the above remarks we conclude that the basic structure of Clifford Analysis in terms of induced representations for the spin group $Spin(0,n)$ can not be sifted from the classical case to the case where the delta operators are not symmetric. This is the particular case of Dirac operators defined on oriented lattices [5, 68, 49] (see also Example 3.3.7).

Next we will take the symmetric extension of the operators $x',x'|\in \mathrm{End}(\mathcal{P})$. By restricting the products (3.29) to the space $\mathcal{P}^{[r]}$, the projections $[F^{[r]}(\underline{x})x']_{r-1}$ and $[x'F^{[r]}(\underline{x})]_{r+1}$ thus give rise to two differential operators $(\cdot)\bullet x'\ :\mathcal{P}^{[r]}\to\mathcal{P}^{[r-1]}$ and $x'\ \wedge(\cdot):\mathcal{P}^{[r]}\to\mathcal{P}^{[r+1]}$ defined *viz*

$$(\cdot)\bullet x': F^{[r]}(\underline{x})\mapsto [D'F^{[r]}(\underline{x})]_{r-1}=\sum_{j=1}^n x_j'(F^{[r]}(\underline{x}))\bullet\mathbf{e}_j$$

$$x'\wedge(\cdot): F^{[r]}(\underline{x})\mapsto [F^{[r]}(\underline{x})D']_{r+1}=\sum_{j=1}^n \mathbf{e}_j\wedge x_j'|(F^{[r]}(\underline{x}))$$

Hence, the following "formal" vector variable identifications naturally follow:

$$(\cdot)\bullet x'\leftrightarrow z'=\sum_{j=1}^n \mathfrak{f}_j x_j',\quad x'\wedge(\cdot)\leftrightarrow z'|=\sum_{j=1}^n \mathfrak{f}_j^\dagger x_j'|. \tag{3.32}$$

Next we define the Clifford vector multiplication operators $x'_\pm$ associated to $\partial'_\pm$ as the operators which takes the form $x'_+ = z' - z'|$ and $x'_- = -i(z' + z'|)$, respectively. In terms of generators of the complexified Clifford algebra $\mathbb{C}_{2n}$, we obtain the following coordinate expressions

$$\begin{array}{lclcl} x'_+ & = & z' - z'| & = & \sum_{j=1}^n \mathbf{e}_j \, \frac{x'_j + x'_j|}{2} + \mathbf{e}_{j+n} \, \frac{x'_j - x'_j|}{2i}, \\ x'_- & = & -i(z + z^\dagger) & = & \sum_{j=1}^n \mathbf{e}_j \, \frac{x'_j - x'_j|}{2i} - \mathbf{e}_{j+n} \frac{x'_j + x'_j|}{2} \ . \end{array}$$

Formally the operators $x'_j, \ x_j'| \in \mathrm{End}(\mathcal{P})$ can be interpreted as the complex variables $z_j = x_j + iy_j$ and their conjugates $\overline{z_j} = x_j - iy_j$, respectively while the vector operators $z' \in \mathrm{End}(\mathcal{P})$ and $z'| \in \mathrm{End}(\mathcal{P})$ can be interpreted as the Hermitian vector variable $z = \sum_{j=1}^n \mathfrak{f}_j^\dagger z_j$ and its Hermitian conjugate $z^\dagger = \sum_{j=1}^n \mathfrak{f}_j^\dagger \overline{z_j}$, respectively.

There are some formulae holding for the vector variables $x'_\pm$, $z'$, and $z'|$ that will be of interest, namely

1. $(z')^2 = 0 = (z'|)^2$;

2. $\{x'_+, x'_-\} = -i\{z' - z'|, z' + z'|\} = 0$;

3. $\{z', z'|\} = \sum_{j=1}^n x'_j x'_j| + \sum_{j,k=1}^n [x'_j, x'_k|]\mathfrak{f}_j \mathfrak{f}_k^\dagger$;

4. $\{x'_\pm, x'_\pm\} = -2\{z', z'|\}$.

Notice that equations 1. and 2. correspond to the isotropy and orthogonality conditions, respectively, while the equations 3. and 4. immediately show that $(x'_+)^2$ and $(x'_-)^2$ are not scalar operators. Hence the subalgebra generated by the operators $x'_\pm \in \mathrm{End}(\mathcal{P})$ coincides with the subalgebra generated by $z', z'| \in \mathrm{End}(\mathcal{P})$ but, however, this algebra does not induce a radial algebra representation (see formula (3.3), Section 3.1).

This means that contrary to [12], the induced representation behind the symmetric extension of the operators $x'$ and $x'|$ is not the unitary group $\widetilde{U}(n)$. Indeed due to the discreteness of the formalism and, as a consequence, the inevitable non-commutativity between $x'_j$ and $x'_j|$ (see also Remark 3.3.12) there is no chance to get a radial algebra representation like in [12]. In terms of group structures, the above phenomena can be interpreted as quantum deformations of $\widetilde{U}(n)$ [69] that we do not explore in this thesis, but that is likely to change in a future research, namely its connections with noncommutative geometry [16].

Below we will explore the above construction in the context of the Quantum Harmonic Oscillator (see Chapter 5):

**Example 3.3.14** *The continuum Hamiltonian $\mathcal{H} = \frac{1}{2}\left(-\Delta + |\underline{x}|^2\right)$ can we rewritten in Clifford language as*

$$\mathcal{H} = \frac{1}{2}(D^2 - x^2),$$

*where the quantity $\frac{1}{2}|\underline{x}|^2 = -\frac{1}{2}x^2$ should be interpreted as a spherical symmetric potential operator (see Chapter 5).*

*Next we turn our attention for the discrete counterpart of the $\mathcal{H}$, where the Laplace operator is replaced by the star Laplacian $\Delta_h$ and the Dirac operator $D$ is replaced by its central version $D_h$ (see Example 3.3.7). From the star Laplacian splitting $D_{h/2}^2 = -\Delta_h$, the operators $x'$ and $x'|$ are build in terms of Pincherle derivative of the central difference operator $Q_{x_j} = \frac{1}{2}\left(\partial_{h/2}^{+j} + \partial_{h/2}^{-j}\right) = Q_{x_j}|$ (see Example 1.2.10, Section 1.2) and, moreover, $-\frac{1}{2}(x')^2$ corresponds to spherical potential operator on the lattice with mesh-width $h$.*

Next we consider discrete counterpart of the Hamiltonian operator $\mathcal{H}'$ written in terms of the operators acting on the symmetric lattice.

**Example 3.3.15** *We shall now consider discrete counterpart of the operator $\mathcal{H}$, where $D'$ is replaced by one of the operators $\partial'_\pm$ obtained in (3.28) and the operator $x'$ is replaced by one of the above operators*

$$\begin{aligned} x'_+ &= \sum_{j=1}^n \mathbf{e}_j \ \frac{x_j(\tau_{h\mathbf{v}_j} + \tau_{-h\mathbf{v}_j})}{2} + \mathbf{e}_{j+n} \ \frac{x_j(\tau_{h\mathbf{v}_j} - \tau_{-h\mathbf{v}_j})}{2i}, \\ x'_- &= \sum_{j=1}^n \mathbf{e}_j \ \frac{x_j(\tau_{h\mathbf{v}_j} - \tau_{-h\mathbf{v}_j})}{2i} - \mathbf{e}_{j+n} \frac{x_j(\tau_{h\mathbf{v}_j} + \tau_{-h\mathbf{v}_j})}{2} \ . \end{aligned}$$

*In the limit case, the operators $x'_+$ and $x'_-$ approximate the left and right multiplication actions $x(\cdot)$ and $(\cdot)x$ on End($\mathcal{P}$), respectively and hence $-\frac{1}{2}(x'_\pm)^2$ approximates the spherical potential $-\frac{1}{2}(x'_\pm)^2$.*

In Examples 3.3.14 and 3.3.15, the operators of the form $-\frac{1}{2}(x')^2$ describe a spherical potencial of the lattice while the equation $-(x')^2\mathbf{1} = r$ describes a lattice sphere on the $n-$dimensional ambient space.

We would like to point out the spherical potentials on the lattice described in Example 3.3.14 possess the symmetric property like in *continuum* only if we take the special choice (1.10) (see Example 1.2.10, Section 1.2). However the same does not hold under the conditions of Example 1.2.9. On the other hand, it would be stressed that the symmetric extension of the operators $x'$ and $x'|$ never lead to spherical potentials on the lattice but to deformations of it. In particular, the lattice sphere described in terms of the operators $x'_\pm$ derived in Example 3.3.15 correspond to $(x'_\pm)^2\mathbf{1} = \sum_{j=1}^n x_j(x_j+h) + \sum_{j=1}^n 2hx_j\mathfrak{f}_j\mathfrak{f}_j^\dagger$.

**Remark 3.3.16** *In Example 3.3.14 we can also see that contrary to the operators* $\Delta_h$ *and* $(x')^2$*, the discrete operators* $D_{h/2}$ *and* $x'$ *are not 'local operators' since the shift operators* $\tau_{\pm\frac{h}{2}\mathbf{v}_j}$ *when acting on some lattice functions, they not only concern with the nearest neighbor points but also all the points contained in the direction* $h\mathbf{v}_j$*. So we can call the combinations* $D_{h/2}$ *and* $x'$ *'quasi-local' operators.*

*In Example 3.3.15, all the operators are 'local' operators but, however, the lattice sphere described by the equation* $r = -(x'_\pm)^2\mathbf{1}$ *only take values on* $\mathbb{C}_{0,2n}I$*.*

*The symmetric and quasi-local property is the price for getting the above mentioned description.*

## 3.4 Discrete Clifford Analysis

Introducing the umbral counterparts of the Dirac operator and the coordinate vector variable, $D'$ and $x'$, respectively, one may now study in general the operators belonging to the algebra of Clifford differential operators $\text{Alg}\{x', D', \mathbf{e}_j \ : \ j = 1, \ldots, n\}$ which are in particular $\text{Spin}(0,n)-$invariant.

By take into account the representations $L(s)$ and $H(s)$ defined in Section 3.1, we extend its action to $\mathbb{R}_{0,n}$ functions as being

$$L(s)F(x) = sF(s^\dagger x s), \quad H(s)F(x) = sF(s^\dagger x s)s^\dagger.$$

In the classical case it was proved by Sommen and Van Acker in [64] that the algebra of polynomial differential operators $P(x,D) \in \text{Alg}\{x, D, \mathbf{e}_j \ : \ j = 1, \ldots, n\}$ satisfying $H(s)P(x,D) = P(x,D)$ (i.e. $\text{Spin}(0,n)-$invariant operators) is generated by the left and right operators $F \mapsto xF$, $F \mapsto Fx$, $F \mapsto DF$, $F \mapsto FD$, the pseudoscalar $F \mapsto \mathbf{e}_1\mathbf{e}_2 \ldots \mathbf{e}_n F$ and the projection operators $F \mapsto [F]_r$.

By using the Sheffer operator $\Psi_{\underline{x}}$ introduced in Definition 3.2.1, the above result can be sifted from the classical case to the Umbral setting only in the case where the delta operators $Q_{x_j}$ are symmetric (see Lemma 3.2.4 and Remarks 3.3.12 and 3.3.13) and under the special choice for $x'_j$ (1.10) ( Example 1.2.10, Section 1.2). Indeed the intertwining operator $\Psi_{\underline{x}}$ maps the spin group $\text{Spin}(0,n)$ onto the group

$$\Psi_{\underline{x}}\text{Spin}(0,n) = \left\{ s = \prod_{j=1}^{2k} w^j \ : \ (s')(s')^\dagger\mathbf{1} = 1, \ \Psi_{\underline{x}} s = s'\Psi_{\underline{x}} \right\}.$$

In particular, the symmetries between $\text{Spin}(0,n)$ and $\Psi_{\underline{x}}\text{Spin}(0,n)$ are preserved. This in particular leads to $(-s')(-s')^\dagger = (s')(s')^\dagger$ and hence

$$\begin{aligned} H(-s')P(x',D') &= -s'P\left((-s')^\dagger x'(-s'), (-s')^\dagger x'(-s')\right)(-s')^\dagger \\ &= s'P\left((s')^\dagger x's', (s')^\dagger x'(s')\right)(s')^\dagger \\ &= H(s')P(x',D') \end{aligned}$$

fulfils like in *continuum*. Furthermore, the natural transcription of [64] to the Umbral setting is the following proposition

**Proposition 3.4.1 (c.f. [64])** *When the delta operators $Q_{x_j}$ are symmetric and under the special choice for $x'_j$ (1.10) ( Example 1.2.10, Section 1.2), any $\mathbb{R}_{0,n}-$valued polynomial operator $P(x',D')$ belonging to the algebra*

$$Alg\{x', D', \mathbf{e}_j \ : \ j=1,\ldots,n\}$$

*such that $H(s')P(x',D') = P(x',D')$ is generated by the left and right operators*

$$F \mapsto x'F, \quad F \mapsto Fx', \quad F \mapsto D'F, \quad F \mapsto FD',$$

*the pseudoscalar $F \mapsto \mathbf{e}_1\mathbf{e}_2\ldots\mathbf{e}_nF$ and the projection operators $F \mapsto [F]_r$.*

The above proposition suggests the following definition for (orthogonal) Discrete Clifford Analysis.

**Definition 3.4.2 (Orthogonal Discrete Clifford Analysis)** *For a finite difference operator $Q_{x_j}$ of umbral type satisfying $\mathbf{d}_\Psi x_j Q_{x_j}F(\underline{x}) = Q_{x_j}F(\underline{x})\mathbf{d}_\Psi x_j$, we define Orthogonal Discrete Clifford analysis as the study of operators belonging to the algebra of operators*

$$Alg\{x', D', \mathbf{e}_j \ : \ j=1,\ldots,n\} = Alg\{x'_j, Q_{x_j}, \mathbf{e}_j \ : \ j=1,\ldots,n\}$$

*where the Sheffer shifts $x'_j$ are given by $x'_j = \frac{1}{2}\left(x_j(Q'_{x_j})^{-1} + (Q'_{x_j})^{-1}x_j\right)$.*

Using the *wedge* and *dot* products, $\wedge$ and $\bullet$, respectively, we can introduce formally the umbral counterparts of Euler and Gamma operators, $E'$ and $\Gamma'$, respectively, as follows:

$$\begin{array}{rll} E' = & -x' \bullet D' = & \sum_{j=1}^n x'_j Q_{x_j}, \\ \Gamma' = & -x' \wedge D' = & -\sum_{j<k} \mathbf{e}_j\mathbf{e}_k(x'_jQ_{x_k} - x'_kQ_{x_j}). \end{array} \tag{3.33}$$

The $E'$ and $\Gamma'$ are nothing else than the umbral counterparts of the radial and angular derivatives [18]. Indeed $E'$ and $\Gamma'$ have radial and angular character (see Lemma 4.2.4, Section 4.2). The next lemma give some formulae holding from combinations between $x'$, $D'$ and $E'$ that will be of interest in the discrete Clifford setting.

**Lemma 3.4.3** *When acting on the space of Clifford-valued polynomials $\mathcal{P}$, the operators $x', D', E'$ satisfy the following (anti-)commuting relations*

$$\{x', D'\} = -2E' - n\mathbf{id}, \quad [E', D'] = -D', \quad [E', x'] = x'. \tag{3.34}$$

**Proof:** Recall that the operators $x'_1, \ldots, x'_n, Q_{x_1}, Q_{x_2}, \ldots, Q_{x_n}$ satisfy the Boson commutation relations (3.10) (see Lemma 3.2.4, Section 3.2) and $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ are the Clifford generators satisfying the anti-commutation relations $\{\mathbf{e}_j, \mathbf{e}_k\} = -2\delta_{jk}$. Hence, straightforward computations leads to

$$\begin{aligned} x'D' + D'x' &= \sum_{j,k=1}^{n} \mathbf{e}_j\mathbf{e}_k x'_j Q_{x_k} + \mathbf{e}_k\mathbf{e}_j Q_{x_k} x'_j \\ &= \sum_{j,k=1}^{n} (\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j) x'_j Q_{x_k} + \mathbf{e}_k\mathbf{e}_j \delta_{jk}\mathbf{id} \\ &= \sum_{j,k=1}^{n} -2\delta_{jk} x'_j Q_{x_k} + \mathbf{e}_j^2 \delta_{jk}\mathbf{id} \\ &= -2E' - n\mathbf{id}. \end{aligned}$$

$$\begin{aligned} E'D' - D'E' &= \sum_{j,k=1}^{n} \mathbf{e}_k (x'_j Q_{x_j} Q_{x_k} - Q_{x_k} x'_j Q_{x_j}) \\ &= \sum_{j,k=1}^{n} \mathbf{e}_k (x'_j Q_{x_j} Q_{x_k} - \delta_{jk} Q_{x_j} - x'_j Q_{x_k} Q_{x_j}) \\ &= -\sum_{j,k=1}^{n} \mathbf{e}_k \delta_{jk} Q_{x_j} \\ &= -D'. \end{aligned}$$

$$\begin{aligned} E'x' - x'E' &= \sum_{j,k=1}^{n} \mathbf{e}_k (x'_j Q_{x_j} x'_k - x'_k x'_j Q_{x_j}) \\ &= \sum_{j,k=1}^{n} \mathbf{e}_k (x'_j \delta_{jk} + x'_j x'_k Q_{x_j} - x'_k x'_j Q_{x_j}) \\ &= \sum_{j,k=1}^{n} \mathbf{e}_k x'_j \delta_{jk} \\ &= x'. \end{aligned}$$

■

The following identities also involving the operators $\Gamma'$ and $\Delta'$ will be also useful in the sequel

**Lemma 3.4.4** *When acting on the space of Clifford-valued polynomials $\mathcal{P}$, we have*

$$[\Delta', x'] = 2D', \quad x'D' = -E' - \Gamma', \quad [E', \Gamma'] = 0, \quad \Gamma' = n\mathbf{id} + E' + D'x'.$$

**Proof:** From the relations Lemma 3.4.3 it immediately follows that $[\Delta', x'] = 2[D', E'] = 2D'$.

In order to prove $x'D' = -E' - \Gamma'$, we start to split $x'D'$ into

$$x'D' = -\sum_j x'_j Q_{x_k} + \sum_{j<k} \mathbf{e}_j\mathbf{e}_k x'_j Q_{x_k} + \mathbf{e}_k\mathbf{e}_j x'_k Q_{x_j}.$$

The first term of the sum coincides with $-E'$ while from the relation $\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j$ the second term coincides with $-\Gamma'$. Hence the relation $\Gamma' = n\mathbf{id} + E' + D'x'$ then follows from the identities $x'D' = -E' - \Gamma'$ and $\{x', D'\} = -2E' - n\mathbf{id}$ (see Lemma 3.4.3) while $[E', \Gamma'] = 0$ follows from the fact that $[E', D'x'] = 0$.

■

From the above relations, when we apply the Umbral Euler and Gamma operators, $E'$ and $\Gamma'$, respectively to the polynomial $\Phi = \mathbf{1}$, we obtain $\Gamma'(\mathbf{1}) = 0 = E'(\mathbf{1})$ and hence from the above lemma, we can recast the dimension of the ambient space $\mathbb{R}^n$ as $n = -(D'x')\mathbf{1}$.

In physical terms, the polynomial $\Phi = \mathbf{1}$ (the vacuum vector) is the corresponding ground level eigenstate while the dimension of the ambient space is nothing else than twice of the ground level energy associated to the Harmonic Oscillator containing $n$ degrees of freedom (see Chapter 5).

## 3.5 Clifford Differential Forms and Discrete Integration

### 3.5.1 Basic Operators and Relations

In Section 3.2, we have shown that the Fock space setting emboided in equations (3.14) and (3.15) provides a natural description for the algebra of endomorphisms. This setting plays an important role in a developing a basis-free exterior differential calculus instead of an exterior differential calculus based on a fixed set of variables and coordinate differentials (see [22, 24] and the references therein). The most important properties underlying the operators $x'_j$, $Q_{x_j}$, $\mathfrak{f}_j^\dagger$ and $\mathfrak{f}_j$ and hence, the Exterior Differential Calculus as a whole, can be derived from intertwining relations.

Using a scheme analogous to the *continuum*, we are now in conditions to determine the umbral counterparts to proceed to the transcription of Clifford Exterior Differential Calculus.

We start to define $\mathbf{d}_\Psi \in \mathrm{End}(\Lambda^*\mathbb{R}[\underline{x}])$ by imposing the shift-invariance relation

$$[\mathbf{d}_\Psi, \exp(\underline{y} \cdot \partial_{\underline{x}})] = 0, \quad \text{for all} \quad \underline{y} \in \mathbb{R}^n. \tag{3.35}$$

where on the left hand side of (3.35) the exponentiation $\exp(\underline{y} \cdot \partial_{\underline{x}})$ acts as a multiplication operator. The relation (3.35) is justified by Corollary 2.1.4 and assures that the gradient of tangent vector fields $O_{\underline{x}}$ uniquely define a basic polynomial sequence (see Theorem 1.1.13, Chapter 1).

The Exterior Differential Calculus described by $(\mathbf{d}_\Psi, \Lambda^*\mathbb{R}[\underline{x}])$ is now extended to $(\mathbf{d}_\Psi, \Lambda^*\mathcal{P})$ by imposing the additional constraint

$$[\mathbf{d}_\Psi x_j, \mathbf{e}_k] = 0 \tag{3.36}$$

Next, we introduce formally $\mathbf{d}_\Psi x_j' \in \mathrm{End}(\Lambda^*\mathcal{P})$ as the mapping $\mathbf{d}_\Psi x_j' : x_j' \mapsto \mathbf{d}_\Psi x_j (O'_{x_j})^{-1}$. As a further consequence, the operators given in Definitions 3.2.1, 3.2.2 and 3.2.3 are well defined and, moreover, from Lemmata 3.2.4 and 3.2.5, $\mathbf{d}_\Psi \in \mathrm{End}(\Lambda^*\mathcal{P})$ is uniquely defined as

$$\mathbf{d}_\Psi = \Psi_{\underline{x}} \mathbf{d} \Psi_{\underline{x}}^{-1} = \sum_{j=1}^{n} \mathbf{d}_\Psi x_j' Q_{x_j} \tag{3.37}$$

Analogously, the umbral version of the Euler contractor $\mathbf{i}_{\mathbf{d}_\Psi}$ is uniquely determined as

$$\mathbf{i}_{\mathbf{d}_\Psi} = \Psi_{\underline{x}} \mathbf{i}_{\mathbf{d}} \Psi_{\underline{x}}^{-1} \quad = \quad \sum_{j=1}^{n} \mathbf{d}_\Psi x_j' \mathbf{i}_{O_{x_j}}. \tag{3.38}$$

The Sheffer operator $\Psi_{\underline{x}}$ allows also to transpose the fundamental formula of Lemma 4.2, [10] about the Lie derivative associated to the Euler operator to the Umbral setting:

**Lemma 3.5.1** *When acting on the space $\Lambda^*\mathcal{P}$, one has*

$$\mathcal{L}_{E'} = \{\mathbf{d}_\Psi, \mathbf{i}_{E'}\} = E' + \mathbf{i}_{\mathbf{d}_\Psi}.$$

**Proof:** By applying the definition, we obtain $\mathcal{L}_{E'} = \sum_{j,k=1}^{n} \mathbf{d}_\Psi x_j' Q_{x_j} x_k' \mathbf{i}_{O_{x_k}}$.

Adding and subtracting the quantities $\sum_{j,k=1}^{n} \mathbf{d}_\Psi x_j' x_k' Q_{x_j} \mathbf{i}_{O_{x_k}}$, we obtain after straightforward algebraic manipulations, the identity

$$\mathcal{L}_{E'} = \sum_{j,k=1}^{n} \mathbf{d}_\Psi x_j' [Q_{x_j}, x_k'] \mathbf{i}_{O_{x_k}} + \sum_{j,k=1}^{n} \{\mathbf{d}_\Psi x_j', \mathbf{i}_{O_{x_k}}\} x_k' Q_{x_j}.$$

Finally, direct application of Lemmata 3.2.4 and 3.2.5 completes the proof of Proposition 3.5.1.

■

Let us turn our attention to the concept Lie derivative introduced in Subsection 2.1.2. In the language of Clifford analysis, Umbral Clifford vector-fields are objects of the form

$$\mathbf{u} = \sum_{j=1}^{n} u_j(\underline{x})\mathbf{e}_j.$$

For these, we introduce the Lie derivative in the same manner as in 2.1.15 by taking Clifford vector-fields instead of classical vector-fields. Particular examples of Umbral Clifford vector-fields are the Umbral Dirac operator $D' = \sum_{j=1}^{n} \mathbf{e}_j Q_{x_j}$ and the Umbral coordinate variable $x' = \sum_{j=1}^{n} \mathbf{e}_j x'_j$.

We define the umbral version of the Dirac contractor $\mathbf{i}_{D'}$ as

$$\mathbf{i}_{D'} \quad = \quad \sum_{j=1}^{n} \mathbf{e}_j \mathbf{i}_{O_{x_j}}. \tag{3.39}$$

Using the $\mathbf{i}_{D'}$, we define the Umbral Dirac operator acting on forms as the Lie derivative in the direction $D'$ as

$$\mathcal{L}_{D'} = \{\mathbf{d}_\Psi, \mathbf{i}_{D'}\}.$$

In the case when $Q_{x_j}$ is symmetric, from the fermionic character of $\mathbf{d}_\Psi x'_j$ and $\mathbf{i}_{O_{x_j}}$ (see Lemma 3.2.5), the action of $\mathcal{L}_{D'}$ in the space of Exterior Algebra of Clifford-valued forms $\Lambda^*\mathcal{P}$ corresponds to

$$\mathcal{L}_{D'}\omega^r = \sum_\alpha (D'F^\alpha(\underline{x}))(\mathbf{d}_\Psi \underline{x})^\alpha.$$

Hereby $\omega^r = \sum_{|\alpha|=r} F^\alpha(\underline{x})(\mathbf{d}\underline{x})^\alpha \in \Lambda^*\mathcal{P}$, i.e. the component functions $F^\alpha(\underline{x})$ are Clifford-valued.

This means that the restriction of $\mathcal{L}_{D'}$ to the space $\mathcal{P}$ corresponds to the Dirac operator $D' = \mathbf{i}_{D'}\mathbf{d}_\Psi$. From constraint (3.36), the action of $\mathbf{d}_\Psi$ on $x' = \sum_{j=1}^{n} \mathbf{e}_j x'_j$ corresponds to the coordinate expression

$$\mathbf{d}_\Psi x' = \sum_{j=1}^{n} \mathbf{d}_\Psi(\mathbf{e}_j x'_j) = \sum_{j=1}^{n} \mathbf{e}_j \mathbf{d}_\Psi x'_j.$$

There are some formulae involving the operators $x', \mathcal{L}_{D'}, \mathbf{d}_\Psi x', \mathbf{i}_{D'}, \mathbf{i}_{\mathbf{d}_\Psi}$ that will be interesting in the context of discrete Clifford Differential forms. This corresponds to the following Lemma:

**Lemma 3.5.2** *For the operators* $\mathbf{d}_\Psi x', \mathbf{i}_{D'}, \mathbf{i}_{\mathbf{d}_\Psi}, \mathcal{L}_{D'}$ *we have the following commuting relations*

$$[\mathbf{d}_\Psi x', \mathbf{i}_{D'}] = -2\mathbf{i}_{\mathbf{d}_\Psi} - n\mathbf{id}, \quad [\mathbf{i}_{\mathbf{d}_\Psi}, \mathbf{i}_{D'}] = -\mathbf{i}_{D'}, \quad [\mathbf{i}_{\mathbf{d}_\Psi}, \mathbf{d}_\Psi x'] = \mathbf{d}_\Psi x',$$
$$[\mathcal{L}_{D'}, \mathbf{d}_\Psi] = 0, \qquad [\mathcal{L}_{D'}, \mathbf{i}_{D'}] = 0, \qquad [\mathcal{L}_{D'}, \mathbf{i}_{E'}] = \mathcal{L}_{D'}.$$

The proof of Lemma 3.5.2 is analogue to the proof of Lemma 3.4.3, by using the Bosonic and the Fermionic commutation relations, (3.10) and (3.11), respectively (see Lemmata 3.2.4 and 3.2.5, Section 3.2). In the same order of ideas, we get alternative descriptions for the operators $\mathbf{d}_\Psi$, $\mathbf{i}_{E'}$ and $\mathbf{i}_{\mathbf{d}_\Psi}$ in terms of anti-commuting relations involving the operators $\mathcal{L}_{D'}, \mathbf{d}_\Psi x', \mathbf{i}_{D'}, x' \in \mathrm{End}(\Lambda^*\mathcal{P})$.

**Lemma 3.5.3** *When acting on the space* $\Lambda^*\mathcal{P}$ *we have*

$$\mathbf{d}_\Psi = -\frac{1}{2}\{\mathbf{d}_\Psi x', \mathcal{L}_{D'}\}, \quad \mathbf{i}_{E'} = -\tfrac{1}{2}\{x', \mathbf{i}_{D'}\}, \quad \mathbf{i}_{\mathbf{d}_\Psi} = -\frac{1}{2}\{\mathbf{d}_\Psi x', \mathbf{i}_{D'}\}.$$

The above setting clearly suggest the extensions of the Discrete Orthogonal setting introduced in Section 3.4 to the space $\Lambda^*\mathcal{P}$.

### 3.5.2 Discrete Clifford-Stokes Formula

Having obtained the basic features in the language of Clifford differential forms, we have now the key ingredients to derive a discrete version of the Stokes formula for Clifford-valued functions as follows:

First we start to define the volume and surface element forms as

$$\mathrm{Vol}(\underline{x}, \mathbf{d}_\Psi \underline{x}) = (\mathbf{d}_\Psi \underline{x})^{\underline{1}},$$
$$\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x}) = \mathbf{i}_{D'}(V(\underline{x}, \mathbf{d}_\Psi \underline{x})) = \sum_{j=1}^{n} (-1)^{j-1} \mathbf{e}_j (\mathbf{d}_\Psi \underline{x})^{\underline{1}-\mathbf{v}_j},$$

respectively. The lemma below will be important on the sequel.

**Lemma 3.5.4** *For two functions* $F = \sum_\alpha F_\alpha \underline{\mathbf{e}}^\alpha$ *and* $G = \sum_\alpha G_\alpha \underline{\mathbf{e}}^\alpha$ *we have*

$$\mathbf{d}_\Psi (G(\underline{x})\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})((O'_{\underline{x}})^{-1}F)(\underline{x})) = (G(\underline{x})D')F(\underline{x}) + G(\underline{x})D'F(\underline{x}))V(\underline{x}, \mathbf{d}_\Psi \underline{x}),$$

*where* $(O'_{\underline{x}})^{-1} := (O'_{x_n})^{-1} \ldots (O'_{x_2})^{-1}(O'_{x_1})^{-1}$.

**Proof:** By applying the Leibniz rule (2.2), we obtain

$$\mathbf{d}_\Psi (G(\underline{x})\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})((O'_{\underline{x}})^{-1}F)(\underline{x})) =$$
$$= \mathbf{d}_\Psi G(\underline{x})\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})((O'_{\underline{x}})^{-1}F)(\underline{x})) + (-1)^n G(\underline{x})\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})\mathbf{d}_\Psi ((O'_{\underline{x}})^{-1}F)(\underline{x})$$

From the fact that $\mathbf{d}_\Psi x_j \sigma(\underline{x}, \mathbf{d}_\Psi \underline{x}) = \mathbf{e}_j \mathrm{Vol}(\underline{x}, \mathbf{d}_\Psi \underline{x})$ and $\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})\mathbf{d}_\Psi x_j = (-1)^n \mathbf{e}_j V(\underline{x}, \mathbf{d}_\Psi \underline{x})$ and from recursive application of relation (2.11) (see also Subsection 2.1.1), we immediately get

$$\mathbf{d}_\Psi G(\underline{x})\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})((O'_{\underline{x}})^{-1}F)(\underline{x}) = (G(\underline{x})D')F(\underline{x})\mathrm{Vol}(\underline{x}, \mathbf{d}_\Psi \underline{x}),$$
$$G(\underline{x})\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})\mathbf{d}_\Psi((O'_{\underline{x}})^{-1}F)(\underline{x}) = (-1)^n G(\underline{x})D'F(\underline{x})\mathrm{Vol}(\underline{x}, \mathbf{d}_\Psi \underline{x}).$$

Adding up the above terms we conclude the proof of our assertion.

■

Before giving the discrete Clifford-Stokes formula, we still need to make the correspondence with the discrete differential forms approach described in Section 2.2.

We start to consider the *continuum* volume form $\mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x})$, and assign the $n-$simplex $\sigma^n = [\mathbf{p}^0, \mathbf{p}^1, \ldots, \mathbf{p}^n]$. This induces the discrete volume form $\mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x})_\sharp = (\mathbf{d}\underline{x})^{\mathbf{1}}_\sharp$.

Recall that from Subsection 2.2.3, the interconnecting structure of discrete vector-fields is given in terms of cochains. Hence the discrete surface element $\sigma(\underline{x}, \mathbf{d}\underline{x})_\sharp$ is uniquely determined in terms of the boundary of $\mathrm{supp}(\sigma^n)$, i.e.

$$\sigma(\underline{x}, \mathbf{d}\underline{x})_\sharp := \sum_{j=1}^n (-1)^{j-1}\mathbf{e}_j (\mathbf{d}\underline{x})^{\mathbf{1}-\mathbf{v}_j}_\sharp.$$

This clearly shows that

$$\mathrm{Vol}(\underline{x}, \mathbf{d}_\Psi \underline{x})|_{\mathrm{supp}(\sigma^n)} = \mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x})_\sharp, \quad \text{and} \quad \sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})|_{\mathrm{supp}(\sigma^n)} = \sigma(\underline{x}, \mathbf{d}\underline{x})_\sharp.$$

Moreover, since $\omega^r \mapsto \mathbf{d}_\Psi \omega^r$ is completely determined by $\sigma^r \mapsto \partial\sigma^r$, from the discrete Stokes theorem (Theorem 2.2.23, Section 2.2) the discrete versions of Stokes theorem and Cauchy integral formula in the Clifford setting naturally follows:

**Theorem 3.5.5 (Discrete Clifford-Stokes theorem)** *For two functions $F(\underline{x})$ and $G(\underline{x})$ acting on supp($\sigma^n$), we have*

$$\int_{\partial\ supp(\sigma^n)} G_\sharp(\underline{x})\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})((O'_{\underline{x}})^{-1}F_\sharp)(\underline{x}) =$$
$$= \int_{supp(\sigma^n)} (G_\sharp(\underline{x})D')F_\sharp(\underline{x}) + G_\sharp(\underline{x})D'F_\sharp(\underline{x}))V(\underline{x}, \mathbf{d}_\Psi \underline{x}).$$

**Corollary 3.5.6 (Discrete Cauchy integral formula)** *If $G_\sharp(\underline{x})$ is discrete left monogenic in supp($\sigma^n$) and $F_\sharp(\underline{x})$ is discrete right monogenic in supp($\sigma^n$), then*

$$\int_{\partial\ supp(\sigma^n)} G_\sharp(\underline{x})\sigma(\underline{x}, \mathbf{d}_\Psi \underline{x})((O'_{\underline{x}})^{-1}F_\sharp(\underline{x})) = 0.$$

One can view this theorem as follows: Since at the level of chains $\mathbf{d}_\Psi$ is the adjoint of the boundary operator $\partial$, as a particular case of the discrete Clifford Stokes theorem (Theorem 3.5.5), we see that $\mathbf{d}_\Psi(G(\underline{x})\sigma(\underline{x},\mathbf{d}_\Psi\underline{x})(O'_{\underline{x}})^{-1}F(\underline{x})) = 0$ whenever $\mathrm{supp}(\sigma^n)$ is a closed discrete manifold, i.e. $\partial\sigma^n = \emptyset$. Thus the discrete Cauchy integral formula (Corollary 3.5.6) asserts that $\mathrm{supp}(\sigma^n)$ is closed. The converse is also true and the answer is given as a special case of discrete Poincaré lemma (see Theorem 4.4.4, Subsection 4.4).

**Remark 3.5.7** *Contrary to discrete integral formulae obtained in [45, 40, 38], the resulting formulae obtained in Theorem 2.2.23 and Corollary 2.2.24 are purely coordinate free and encode the connectivity of the discrete space.*

Next we turn our attention for the umbral version of Leray form

$$L(\underline{x},\mathbf{d}_\Psi\underline{x}) = \mathbf{i}_{E'}(\mathrm{Vol}(\underline{x},\mathbf{d}_\Psi\underline{x})) = \sum_{j=1}^{n}(-1)^{j-1}x_j(\mathbf{d}_\Psi\underline{x})^{\underline{1}-\mathbf{v}_j}. \tag{3.40}$$

From the coordinate expression of $\mathbf{d}_\Psi$, we can easily derive the following two formulae which relates Umbral Leray form with the Umbral volume form.

**Proposition 3.5.8** *We have*

$$Vol(\underline{x},\mathbf{d}_\Psi\underline{x}) = \frac{1}{r^2}\{r,\mathbf{d}_\Psi r\}L(\underline{x},\mathbf{d}_\Psi\underline{x}).$$

**Proof:** Notice that from the relations $[Q_{x_j},(x'_k)^2] = 2\delta_{jk}x'_k$ and $[\mathbf{d}_\Psi x'_j, x'_j] = 0$ we immediately get $[\mathbf{d}_\Psi, -(x')^2] = 2\sum_{j=1}^n \mathbf{d}_\Psi x'_j\ x'_j$. Hence for $r^2 = -(x')^2\mathbf{1}$,

$$\mathbf{d}_\Psi(r^2) = 2\sum_{j=1}^{n}\mathbf{d}_\Psi x_j(O'_{x_j})^{-1}x'_j\mathbf{1} = 2\sum_{j=1}^{n}x_j\mathbf{d}_\Psi x_j,$$

i.e. $\{r,\mathbf{d}_\Psi r\} = 2\sum_{j=1}^n x_j\mathbf{d}_\Psi x_j$. By applying the Leibniz rule to the left-hand side of the above equality, one gets

$$\begin{aligned}
\{r,\mathbf{d}_\Psi r\}\ L(\underline{x},\mathbf{d}_\Psi\underline{x}) &= \left(\sum_{j=1}^{n}x_j\mathbf{d}_\Psi x_j\right)\left(\sum_{j=1}^{n}(-1)^{j-1}x_j(\mathbf{d}_\Psi\underline{x})^{\underline{1}-\mathbf{v}_j}\right)\\
&= \sum_{j=1}^{n}x_j^2(-1)^{j-1}\mathbf{d}_\Psi x_j(\mathbf{d}_\Psi\underline{x})^{\underline{1}-\mathbf{v}_j} + \sum_{j\neq k}x_jx_k(-1)^{k-1}\mathbf{d}_\Psi x_j(\mathbf{d}_\Psi\underline{x})^{\underline{1}-\mathbf{v}_k}\\
&= r^2\ \mathrm{Vol}(\underline{x},\mathbf{d}_\Psi\underline{x}).
\end{aligned}$$

■

We will come back to this approach in Section 4.2 when we introduce an inner product on the discrete sphere.

# Chapter 4

# Discrete Monogenic Function Theory

"*We cannot direct the wind but we can adjust the sails.*"
Bertha Calloway

## 4.1 Umbral Fischer Decomposition

For the algebra of Clifford-valued polynomials $\mathcal{P} = \mathbb{R}[\underline{x}] \otimes \mathbb{R}_{0,n}$ a natural inner product on $\mathcal{P}$ is the so-called *Fischer inner product*

$$(P(\underline{x}), Q(\underline{x})) = ([P^\dagger(\partial_{\underline{x}})Q(\underline{x})]_0)_{\underline{x}=\underline{0}}. \tag{4.1}$$

where on the right-hand side of (4.1), $P^\dagger(\partial_{\underline{x}}) \in \mathrm{End}(\mathcal{P})$ is an the operator obtained from the Clifford-valued polynomial $P(\underline{x})$ by computing Hermitian conjugation $P^\dagger(\underline{x})$ followed by the replacement $\underline{x} \to \partial_{\underline{x}}$ and finally evaluating the scalar part at the point $\underline{x} = \underline{0}$.

The spaces of homogeneous Clifford-valued polynomials of degree $k$

$$\mathcal{P}_k = \{P_k(\underline{x}) \in \mathcal{P} \ : \ P_k(t\underline{x}) = t^k P_k(\underline{x})\} \tag{4.2}$$

are orthogonal with respect to (4.1) (see [18], page 206) and the reproducing kernel related to this inner product on $\mathcal{P}_k$ is $\frac{1}{k!}(\underline{y} \cdot \underline{x})^k$ since for each $P(\underline{x}) \in \mathcal{P}_k$

$$\left(\frac{1}{k!}(\underline{y} \cdot \underline{x})^k, P(\underline{x})\right) = P(\underline{y}).$$

Afterwards, using (4.1) the basic idea of a Fischer decomposition (see e.g. [18], pages 204-207) is to decompose the space $\mathcal{P}_k$ into homogeneous monogenic polynomials of lower degrees.

In the classic case such a decomposition is based on the fact that the multivariate basis $\underline{x}^\alpha$ is homogeneous and satisfies the *lowering* property $\partial_{x_j}\underline{x}^\alpha = \alpha_j \underline{x}^{\alpha - \mathbf{v}_j}$.

According to Definition 1.1.11 given in Chapter 1 such transcription can be obtained by replacing the standard basis $\{\underline{x}^\alpha \ : \ |\alpha| = k\}$ by a basic polynomial sequence $\{V_\alpha(\underline{x}) \ : \ |\alpha| = k\}$.

Let us emphasize in the end the major advantage of formulating the Fischer decomposition in the umbral setting: While in the classical case we are restricted to the gradient operator $\partial_{\underline{x}}$, there are some degrees of freedom in the choice of the multivariate delta operator $Q_{\underline{x}}$. Indeed, we can further postulate them by requiring the existence of the associated basic polynomial sequence $\{V_\alpha(\underline{x}) \ : \ |\alpha| \in \mathbb{N}_0\}$. Moreover, each operator $Q_{\underline{x}}$ turns out to be connected to a unique sequence of basic polynomials (see Theorem 1.1.13, Chapter 1).

### 4.1.1 Inner Products for Clifford-valued polynomials

With the goal of constructing a Fischer inner product we will start to introduce a generalization of the notion of homogeneity to the Umbral setting:

**Definition 4.1.1 (Umbral homogeneous polynomial)** *A Clifford-valued polynomial $P_k \in \mathcal{P}$ is Umbral homogeneous of degree $k$ if it is an eigenfunction of the Umbral Euler operator $E'$, i.e.*

$$E' P_k(\underline{x}) = k P_k(\underline{x}).$$

*Moreover we denote by $\mathcal{P}_k[\Psi]$ the space of all Umbral homogeneous polynomials of degree $k$.*

Notice that the Clifford homogeneous polynomials of degree $k$ are eigenfunctions of the classical Euler operator $E$. On the other hand, from a direct application of Lemma 3.2.4, the Sheffer operator $\Psi_{\underline{x}}$ intertwines the Umbral Euler operator $E'$ and the classical Euler operator $E$. Hence, the space of all Umbral homogeneous polynomials of degree $k$, $\mathcal{P}_k[\Psi]$, may be characterized as

$$\mathcal{P}_k[\Psi] = \Psi_{\underline{x}} \mathcal{P}_k := \{\Psi_{\underline{x}} P_k(\underline{x}) \ : \ P_k(\underline{x}) \in \mathcal{P}_k\}. \tag{4.3}$$

The above characterization together with the *Rodrigues* formula (see Corollary 1.2.6, Section 1.2) allows us to conclude that the basic polynomial sequence $\{V_\alpha(\underline{x}) = (\underline{x}')^\alpha \mathbf{1} \ : \ |\alpha| = k\}$ forms a basis for $\mathcal{P}_k[\Psi]$ and moreover the following graded decomposition

$$\mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{P}_k[\Psi]$$

holds for $\mathcal{P}$.

Having introduced the notion of Umbral homogeneity, it now raises the question how to define the umbral counterpart for the Fischer inner product (4.1). Motivated from the Quantum Field setting, we observe that the bilinear form (4.1) is an inner product since it consists in the composition of two inner products: The Bose algebra inner product emboided in the space of multivariate polynomials $\mathbb{R}[\underline{x}]$ by $[P(\partial_{\underline{x}})Q(\underline{x})]_{\underline{x}=\underline{0}}$ and the Clifford inner product (3.6).

On the other hand, notice that each polynomial $Q(\underline{x}) \in \oplus_{k=0}^{\infty}\mathcal{P}_k = \mathcal{P}$ can be viewed as a series representation in terms of position operators acting on the vacuum vector $\mathbf{1}$, i.e. $Q(\underline{x}) = Q(\underline{x})\mathbf{1}$. Hence the natural transcription of the right hand side of (4.1) can be obtained using the replacement $([P^\dagger(\partial_{\underline{x}})Q(\underline{x})]_0)_{\underline{x}=\underline{0}} \to ([P^\dagger(Q_{\underline{x}})Q(\underline{x}')\mathbf{1}]_0)_{\underline{x}=\underline{0}}$.

With other words, we can construct a Fischer inner product between Clifford-valued polynomials by applying the *lowering* polynomial operator $P^\dagger(Q_{\underline{x}})$ to the polynomial $Q(\underline{x}')\mathbf{1}$ and evaluate the scalar part of the product $P(Q_{\underline{x}})^\dagger Q(\underline{x}')\mathbf{1}$ at the point $\underline{x} = \underline{0}$. This in particular leads to the following lemma:

**Lemma 4.1.2** *For $P, Q \in \bigoplus_{k=0}^{\infty}\mathcal{P}_k$ we have*

$$(\Psi_{\underline{x}}P, Q) = ([P^\dagger(Q_{\underline{x}})Q(\underline{x}')\mathbf{1}]_0)_{\underline{x}=\underline{0}}.$$

**Proof:** In fact, by letting act the Sheffer operator $\Psi_{\underline{x}}$ on $Q(\underline{x}) \in \mathcal{P}_k$, one obtains $Q(\underline{x}')\mathbf{1} = \Psi_{\underline{x}}Q(\underline{x})$. Hence for $P, Q \in \mathcal{P}_k$, direct application of (4.1) combined with Lemma 3.2.4 (see Subsection 3.2), leads to

$$(\Psi_{\underline{x}}P, Q) = ([(\Psi_{\underline{x}}P)^\dagger(\partial_{\underline{x}})\Psi_{\underline{x}}Q(\underline{x})]_0)_{\underline{x}=\underline{0}} = ([P^\dagger(Q_{\underline{x}})\Psi_{\underline{x}}Q(\underline{x})]_0)_{\underline{x}=\underline{0}} = ([P^\dagger(Q_{\underline{x}})Q(\underline{x}')\mathbf{1}]_0)_{\underline{x}=\underline{0}}.$$

By linearity the above extends to arbitrary polynomials $P(\underline{x}), Q(\underline{x}) \in \mathcal{P} = \bigoplus_{k=0}^{\infty}\mathcal{P}_k$.

■

Using the following lemma, we are now in conditions to define the umbral counterpart of (4.1).

**Definition 4.1.3 (Umbral Fischer inner product)** *For two polynomials $P, Q \in \mathcal{P} = \bigoplus_{k=0}^{\infty}\mathcal{P}_k[\Psi]$, we define the Umbral Fischer inner product as the bilinear form*

$$(P(\underline{x}), Q(\underline{x}))_\Psi = (\Psi_{\underline{x}}^{-1}P(\underline{x}), Q(\underline{x})) = ([([\Psi_{\underline{x}}^{-1}P]^\dagger(Q_{\underline{x}})Q(\underline{x}))]_0)_{\underline{x}=\underline{0}} \tag{4.4}$$

The Umbral Fischer inner product can be viewed as the combination of the Isomorphism theorem of Umbral Calculus (Theorem 1.1.3, Chapter 1) with the invertibility condition of shift-invariant operators (Corollary 1.1.4, Chapter 1). Here we define the Umbral Fischer inner product by means of the classical Fischer inner product. Of course, one could also define $(\cdot,\cdot)_\Psi$ directly in terms of spaces of homogeneous polynomials by setting $(P(\underline{x}),Q(\underline{x}))_\Psi = ([P^\dagger(Q_{\underline{x}})Q(\underline{x}')\mathbf{1}]_0)_{\underline{x}=\underline{0}}$.

The next results establish some important properties for the Umbral Fischer inner product $(\cdot,\cdot)_\Psi$.

**Proposition 4.1.4** *The spaces of umbral homogeneous polynomials of degree $k$, $\mathcal{P}_k[\Psi]$, are orthogonal with respect to the Umbral Fischer inner product (4.4), i.e*

$$\mathcal{P}_k[\Psi] \ \perp_{(\cdot,\cdot)_\Psi} \ \mathcal{P}_l[\Psi], \quad \textit{for} \ \ k\neq l. \tag{4.5}$$

*Moreover the spaces $\mathcal{P}_k$ and $\mathcal{P}_l[\Psi]$ are bi-orthogonal with respect to the classical Fischer inner product (4.1).*

**Proof:** First we recall that every polynomial $P_k(\underline{x})\in\mathcal{P}_k[\Psi]$, $Q_l(\underline{x})\in\mathcal{P}_l[\Psi]$ is of the form

$$P_k(\underline{x}) = \sum_{|\alpha|=k} V_\alpha(\underline{x})\mathbf{a}_\alpha, \quad Q_r(\underline{x}) = \textstyle\sum_{|\beta|=l} V_\beta(\underline{x})\mathbf{b}_\beta.$$

Hence by a direct computation we obtain $(P_k(\underline{x}),Q_l(\underline{x}))_\Psi = \sum_{|\alpha|=k}\sum_{|\beta|=l}[\mathbf{a}_\alpha^\dagger\mathbf{b}_\beta]_0[Q_{\underline{x}}^\alpha V_\beta(\underline{x})]_{\underline{x}=\underline{0}}$.

By recursive application of Definition 1.1.11 (see Section 1.1), the term $[Q_{\underline{x}}^\alpha V_\beta(\underline{x})]_{\underline{x}=\underline{0}}$ is equal to $\beta!\delta_{\alpha,\beta}$ for $|\beta|\leq|\alpha|$ while for $|\alpha|<|\beta|$ it is equal to $O_{\underline{x}}^\beta V_\alpha(\underline{x})|_{\underline{x}=\underline{0}}=0$.

Thus we have

$$(P_k(\underline{x}),Q_l(\underline{x}))_\Psi = \delta_{k,l}\sum_{|\alpha|=k}[\mathbf{a}_\alpha^\dagger\mathbf{b}_\alpha]_0.$$

This proves the orthogonality with respect to the Umbral Fischer inner product (4.4). The proof of the bi-orthogonality between the spaces $\mathcal{P}_k$ and $\mathcal{P}_l[\Psi]$ follows from the right-hand side of (4.4).

■

As a direct consequence of Proposition 4.1.4, we obtain the following corollary:

**Corollary 4.1.5** *The bilinear form $(\cdot,\cdot)_\Psi$ is an inner product on $\mathcal{P}$.*

**Proof:** From the proof of Proposition 4.1.4, the bilinear form $(\cdot,\cdot)_\Psi$ acting on the polynomials $P_k(\underline{x})\in\mathcal{P}_k[\Psi]$, $Q_l(\underline{x})\in\mathcal{P}_l[\Psi]$ is given by $(P_k(\underline{x}),Q_l(\underline{x}))_\Psi=\delta_{k,l}\sum_{|\alpha|=k}[\mathbf{a}_\alpha^\dagger\mathbf{b}_\alpha]_0$.

The $\dagger$-conjugation ensures that the term $[\mathbf{a}_\alpha^\dagger\mathbf{b}_\alpha]_0$ is an inner product in the Clifford algebra of signature $(0,n)$ and so the axioms for the inner product are satisfied for the polynomials $P_k(\underline{x})\in\mathcal{P}_k[\Psi]$, $Q_l(\underline{x})\in\mathcal{P}_l[\Psi]$.

By a linearity argument the above extends to arbitrary polynomials $P(\underline{x}), Q(\underline{x}) \in \mathcal{P} = \bigoplus_{k=0}^{\infty} \mathcal{P}_k[\Psi]$.

$\blacksquare$

We will see afterwards that according to Theorem 4.2.16, Proposition 4.1.4 is equivalent to say that $\{V_\alpha(\underline{x}) \,:\, |\alpha| = k\}$ is indeed an orthogonal set with respect to the inner product on the discrete sphere $\langle \cdot, \cdot \rangle_{\Psi S^{n-1}}$ (formula (4.16)).

The next lemma will play an important role in the sequel:

**Lemma 4.1.6 (Fischer duality)** *The operators $x'$ and $-D'$ are dual with respect to the umbral Fischer inner product $(\cdot, \cdot)_\Psi$, i.e.*

$$(x' P(\underline{x}), Q(\underline{x}))_\Psi = -(P(\underline{x}), D'Q)_\Psi. \tag{4.6}$$

**Proof:** First notice that from the Lemma 3.2.4, for two polynomials $P, Q \in \mathcal{P}$ the Fischer inner product (4.4) leads to

$$(P(\underline{x}), Q(\underline{x}))_\Psi = (\Psi_{\underline{x}}^{-1} P(\underline{x}), Q(\underline{x})) = (P(\underline{x}), \Psi_{\underline{x}}^{-1} Q(\underline{x})), \tag{4.7}$$

where $(\cdot, \cdot)$ stands for the standard Fischer inner product given by formula (4.1).

By taking the replacement $P(\underline{x}) \to x' P(\underline{x})$ on both sides of (4.7), one obtains

$$(x' P(\underline{x}), Q(\underline{x}))_\Psi = (x P(\underline{x}), \Psi_{\underline{x}}^{-1} Q(\underline{x})).$$

From the classical Fischer inner product (c.f. Corollary 1.10.3. [18]), the term $(x P(\underline{x}), \Psi_{\underline{x}}^{-1} Q(\underline{x}))$ is equal to $-(P(\underline{x}), D\Psi_{\underline{x}}^{-1} Q(\underline{x}))$. Furthermore, by applying again formula (4.7), we end up with

$$(x' P(\underline{x}), Q(\underline{x}))_\Psi = -(P(\underline{x}), D\Psi_{\underline{x}}^{-1} Q(\underline{x})) = -(\Psi_{\underline{x}}^{-1} P(\underline{x}), D'Q(\underline{x})) = -(P(\underline{x}), D'Q(\underline{x}))_\Psi.$$

$\blacksquare$

We are now in conditions to formulate the umbral counterpart of Fischer decomposition:

**Theorem 4.1.7 (Umbral Fischer decomposition)** *We have*

$$\mathcal{P}_k[\Psi] = \mathcal{M}_k[\Psi] \oplus_{(\cdot,\cdot)_\Psi} x' \mathcal{P}_{k-1}[\Psi].$$

*where $\mathcal{M}_k[\Psi] = \mathcal{P}_k[\Psi] \cap \ker D'$.*

*Moreover, the subspaces $\mathcal{M}_k[\Psi]$ and $x'\mathcal{P}_{k-1}[\Psi]$ are orthogonal with respect to the Umbral Fischer inner product $(\cdot, \cdot)_\Psi$.*

**Proof:** Because of $\mathcal{P}_k[\Psi] = x'\mathcal{P}_{k-1}[\Psi] \oplus_{(\cdot,\cdot)_\Psi} (x'\mathcal{P}_{k-1}[\Psi])^\perp$ it is enough to prove that $(x'\mathcal{P}_k[\Psi])^\perp = \mathcal{M}_k[\Psi]$. For this we choose $P_{k-1} \in \mathcal{P}_{k-1}[\Psi]$ arbitrarily and assume that for some $P_k \in \mathcal{P}_k[\Psi]$ we have $(x'P_{k-1}(\underline{x}), P_k(\underline{x}))_\Psi = 0$. Due to Lemma 4.1.6 we have

$$(P_{k-1}(\underline{x}), D'P_k(\underline{x}))_\Psi = 0, \quad \text{for all } P_{k-1}(\underline{x}).$$

As $D'P_k(\underline{x}) \in \mathcal{P}_{k-1}[\Psi]$, by the intertwining property $D'\Psi_{\underline{x}} = \Psi_{\underline{x}}D$, $D'P_k(\underline{x}) = 0$ i.e. $P_k(\underline{x}) \in \mathcal{M}_k[\Psi]$. This means that $(x'\mathcal{P}_{k-1}[\Psi])^\perp \subset \mathcal{M}_k[\Psi]$. Now, let $P_k(\underline{x}) \in \mathcal{M}_k[\Psi]$.

Then we have for each $P_{k-1}(\underline{x}) \in \mathcal{P}_{k-1}[\Psi]$ the relation

$$(x'P_{k-1}(\underline{x}), P_k(\underline{x}))_\Psi = -(P_{k-1}(\underline{x}), D'P_k(\underline{x}))_\Psi = 0$$

and, therefore, $(x'\mathcal{P}_{k-1}[\Psi])^\perp = \mathcal{M}_k[\Psi]$.

■

From this theorem we obtain the following decomposition for the Umbral homogeneous polynomials $\mathcal{P}_k[\Psi]$ in terms of Umbral monogenic polynomials of lower degrees.

**Theorem 4.1.8 (Umbral monogenic decomposition)** *For the space $\mathcal{P}_k[\Psi]$ we get that*

$$\mathcal{P}_k[\Psi] = \sum_{s=0}^{k} \oplus_{(\cdot,\cdot)_\Psi} (x')^s \mathcal{M}_{k-s}[\Psi].$$

*Moreover, this decomposition is unique.*

In a similar way we can decompose the space of Umbral homogeneous polynomials of degree $k$, $\mathcal{P}_k[\Psi]$, in terms of Umbral monogenic polynomials of lower degrees. This corresponds to the following theorem

**Theorem 4.1.9** *$\mathcal{P}_k[\Psi]$ admits a unique decomposition in terms of Umbral harmonic polynomials. $\mathcal{H}_{k-2s}[\Psi] = \mathcal{P}_{k-2s}[\Psi] \cap \ker \Delta'$ as*

$$\mathcal{P}_k[\Psi] = \sum_{s=0}^{\lfloor k/2 \rfloor} \oplus_{(\cdot,\cdot)_\Psi} (x')^{2s} \mathcal{H}_{k-2s}[\Psi].$$

**Proof:** From the splitting $\Delta' = -(D')^2$ and Lemma 4.1.6, we have $((x')^2P, Q)_\Psi = -(P, \Delta'Q)_\Psi$.

Following the same order of ideas as in the proof of Lemma 4.1.7, $\mathcal{P}_k[\Psi]$ may be uniquely decomposed as $\mathcal{P}_k[\Psi] = \mathcal{H}_k[\Psi] \oplus_{(\cdot,\cdot)_\Psi} (x')^2\mathcal{P}_{k-2}[\Psi]$.

Recursive application of the above decomposition then leads to

$$\mathcal{P}_k[\Psi] = \sum_{s=0}^{\lfloor k/2 \rfloor} \oplus_{(\cdot,\cdot)_\Psi} (x')^{2s} \mathcal{H}_{k-2s}[\Psi].$$

■

**Remark 4.1.10** *Let $Q_{x_j}$ to be the usual continuous derivative $\partial_{x_j}$. Since $(\partial_{x_j})' = \mathbf{id}$, $D'$ is exactly the classical Dirac operator and hence our main theorem then recover the Fischer decomposition as in [18].*

With the above remark we show that the Fischer decomposition obtained in [18] can be viewed as a particular case of Theorem 4.1.8.

The next remark will show that the Umbral Fischer inner product (4.4) is valid for polynomial functions taking values on the spinor space $\mathbb{R}_{0,n}I$.

**Remark 4.1.11** *In order to define a Umbral Fischer inner product for the spinor space*

$$\mathbb{R}_{0,n}I \subset \mathbb{C}_{0,n}I \cong \mathbb{C}\Lambda_n^\dagger,$$

*we start to consider two spinor-valued polynomials $P(\underline{x})I$ and $Q(\underline{x})I$ with $P(\underline{x}), Q(\underline{x}) \in \mathcal{P}$.*

*Recall that $I$ is a primitive idempotent satisfying $I^\dagger = I$ (see Section 3.1). Hence the Hermitian conjugation of $P(\underline{x})I$ corresponds to $(P(\underline{x})I)^\dagger = IP^\dagger(\underline{x})$.*

*By taking the replacements $P(\underline{x}) \to P(\underline{x})I$ and $Q(\underline{x}) \to Q(\underline{x})I$ on the Umbral Fischer inner product (4.4) we obtain*

$$(P(\underline{x})\ I, Q(\underline{x})\ I)_\Psi = ([(I(\Psi_{\underline{x}}^{-1}P)^\dagger(Q_{\underline{x}})Q(\underline{x})I)]_0)_{\underline{x}=\underline{0}} = ([((\Psi_{\underline{x}}^{-1}P)^\dagger(Q_{\underline{x}})Q(\underline{x})I)]_0)_{\underline{x}=\underline{0}}. \quad (4.8)$$

*Because for every $\mathbf{a} \in \mathbb{C}\Lambda_n^\dagger$, $I\mathbf{a}I = [\mathbf{a}]_0 I$ (see Section 3.1), the right-hand side of (4.8) is non-negative and hence $(\cdot,\cdot)_\Psi$ is also an inner product for the spinor space $\mathbb{R}_{0,n}I$.*

From the above remark, we can conclude that a result similar to Theorem 4.1.8 follows then automatically in the spinor space $\mathbb{R}_{0,n}I$.

We will end this subsection by presenting some examples of the application of the above scheme on lattices. First we will show how to recover the Fischer decomposition obtained by N. Faustino and U. Kähler in [30].

**Example 4.1.12 (Discrete counterpart of Fischer Decomposition)** *First we consider the difference Dirac operators $D_h^\pm$ introduced in [30]. In the terminology of that paper, $D_h^\pm = \sum_{j=1}^n \mathbf{e}_j \partial_h^{\pm j}$ are the forward/backward versions of the Dirac operator, where $\partial_h^{\pm j}$ are the forward/backward differences introduced in Example 1.2.9.*

*Additionally, we put $x^\pm = \Psi_{\underline{x}}^\pm x (\Psi_{\underline{x}}^\pm)^{-1} = \sum_{j=1}^n \mathbf{e}_j\ x_j \tau_{\mp h \mathbf{v}_j}$ and denote by $\mathcal{P}_k^\mp$ the set*

$$\mathcal{P}_k^\mp = \Psi_{\underline{x}}^\pm \mathcal{P}_k = \left\{ f \in \mathcal{P} : E_h^\mp f = kf \right\},$$

*where* $E_h^{\mp} = \sum_{j=1}^{n} x_j \partial_h^{\mp j}$ *stands the difference Euler operators.*

*According to [30], these spaces correspond to the spaces of Clifford-valued polynomials generated by the basic polynomial sequences* $(\underline{x})_{\mp}^{(\alpha)}$ *of degree* $|\alpha| = k$.

*By applying Theorem 3.2.4, Section 3.2, the Sheffer maps* $\Psi_{\underline{x}}^{\pm} : \underline{x}^{\alpha} \mapsto (\underline{x})_{\mp}^{(\alpha)}$ *intertwine the operators* $D$ *and* $D_h^{\pm}$ *(respectively,* $x$ *and* $x^{\pm}$*). So* $-D_h^{\pm}$ *is the dual of* $x^{\pm}$ *with respect to the umbral Fischer inner product (4.4) (see also Theorem 4.1.6) and hence the Fischer decomposition for the Difference Dirac Operators* $D_h^{\pm}$ *then correspond to*

$$\mathcal{P}_k^{\mp} = \bigoplus_{j=0}^{k} (x^{\pm})^j \mathcal{M}_{k-j}^{\mp}, \quad \text{where } \mathcal{M}_{k-j}^{\mp} := \mathcal{P}_{k-j}^{\mp} \cap \ker D_h^{\pm}. \tag{4.9}$$

Next we will formulate the Fischer decomposition obtained in terms of simplicial complexes.

**Example 4.1.13 (Fischer decomposition on simplicial complexes)** *According to the framework obtained in Subsection 2.2.3, the generalization of the Difference Dirac operators* $D_h^{\pm}$ *to 0-chains is defined locally viz*

$$D_{\sigma^0} F(\underline{x}) = \sum_{j=1}^{n} \mathbf{e}_j \frac{O'_{x_j} F_{\sharp}(\underline{x}) - F_{\sharp}(\underline{x})}{O'_{x_j}(x_j)_{\sharp} - (x_j)_{\sharp}},$$

*where* $O'_{x_j}$ *is the left action of the Cayley graph on* $supp(\sigma^0)$ *(see formula (2.41)).*

*We have observed in Remark 2.2.28 (Subsection 2.2.3) how to construct the right action* $(O'_{x_j})^{-1}$ *against reversing the orientation of* $1-$*paths of the form* $[\mathbf{p}, \mathbf{q}]$. *Roughly speaking, for the discrete path* $\underline{x}(\cdot) : supp(\sigma^0) \mapsto \mathbb{R}^n$ *given by the ordered set of points* $\underline{x}(\mathbf{p}^0), \underline{x}(\mathbf{p}^1), \ldots, \underline{x}(\mathbf{p}^m)$, *the operators* $x_j(O'_{x_j})^{-1}$ *are build up by reversing the orientation of* $\underline{x}(\cdot) : supp(\sigma^0) \mapsto \mathbb{R}^n$, *i.e.* $(x_j(O'_{x_j})^{-1}F)(\underline{x}(\mathbf{p}^k)) = x_j(\mathbf{p}^k)F(\underline{x}(\mathbf{p}^{k-1}))$.

*Since* $D_{\sigma^0}$ *can be described in terms discrete integration, the Fischer inner product property obtained in Proposition 4.1.6 makes* $x'$ *appear as a discrete integration operator obtained by reversing the orientation of* $\underline{x}(\cdot) : supp(\sigma^0) \mapsto \mathbb{R}^n$. *Thus the extension of* $(\underline{x})_{\mp}^{\alpha}$ *can be easily grasped locally first, by extracting the first* $\alpha_j$ *coordinates of the reversed path* $\underline{x}(\cdot) : supp(\sigma^0) \mapsto \mathbb{R}^n$ *viz projection on the* $x_j-$*axis, say* $x_j(\mathbf{p}^0), x_j(\mathbf{p}^0 + (-\mathbf{v}_j)), \ldots, x_j(\mathbf{p}^0 + (\alpha_j - 1)(-\mathbf{v}_j))$ *and afterwards by concatenating all the selected points in each direction on the reversed order, i.e.*

$$(\underline{x}')^{\alpha}\mathbf{1} = \prod_{j=1}^{n} x_j(\mathbf{p}^0)x_j(\mathbf{p}^0 + (-\mathbf{v}_j)) \ldots x_j(\mathbf{p}^0 + (\alpha_j - 1)(-\mathbf{v}_j))$$

*and hence the spaces of discrete polynomials* $\mathcal{P}_k[\Psi]$ *are generated locally viz interpolation by taking all the possible combinations of paths of length* $|\alpha| = k$.

*Moreover, from the discrete Cauchy integral formula (Corollary 3.5.6, Subsection 3.5.2) the $n-$dimensional discrete manifolds*

$$\bigcup\left\{ supp[\mathbf{q},\mathbf{q}+(-\mathbf{v}_1),\ldots\mathbf{q}+(-\mathbf{v}_n)]\ :\ \mathbf{q}=\mathbf{p}+\sum_{j=1}^{n}\alpha_j(+\mathbf{v}_j),\ \textit{and}\ |\alpha|=k\right\}$$

*are closed.*

*This makes $\mathcal{M}_k[\Psi]$ appear as the space of piecewise linear Clifford-valued polynomials supported on closed $n-$simplexes.*

We have thus shown in the above example, as a generalization of Example 4.1.12, how to construct the space of discrete monogenic polynomials against reversed oriented paths and discrete closed manifolds. Physically speaking, this construction can also be formalized as a path integral acting on a phase space (c.f. [51]) establishing a bridge between enumerative combinatorics (c.f. [62]) and quantum statistical mechanics (c.f. [16], Chapter 3).

However, we have to stress that the Umbral Fischer Decomposition for Umbral harmonic polynomials (Theorem 4.2.2) obtained in above examples does not allow to obtain the Fischer decomposition in terms of discrete harmonic polynomials (see also Example 3.3.7). As it was pointed out in [6, 47, 31], discrete monogenic functions are supported on primal graphs (i.e. $1-$simplex) while discrete harmonic functions are supported on dual graphs (i.e. the geometric dual[1] of the $1-$simplex).

We will turn our attention back to this problem in Section 4.2 when we obtain the space of discrete spherical harmonics as a refinement of the space of discrete spherical monogenics (see afterwards Example 4.2.3).

### 4.1.2 Umbral Monogenic Projectors

As it was shown in the later section, the Umbral Fischer inner product (4.4) provides a way to split the space of umbral homogeneous polynomials in umbral homogeneous monogenic polynomials of lower degrees.

According to Proposition 4.1.4, the Umbral monogenic components of the Umbral Fischer decomposition (Theorem 4.1.8) are mutually orthogonal with respect to $(\cdot,\cdot)_\Psi$. However finding the monogenic components *viz* the orthogonalization procedure remains a hard problem.

[1] For further details related with the construction of the geometric dual, we refer the reader to take a look e.g. to [44], Chapter 2.

In the subsequent results, we will derive explicit expressions for the umbral monogenic components by means of the (anti-)commutation relations obtained in Section 3.4.

The following lemma gives the key information to derive some explicit formula to compute the action of the umbral Dirac operator $D'$ when acting on the functions of the form $(x')^s F(\underline{x})$.

**Lemma 4.1.14** *For any Clifford-valued function $F(\underline{x})$, the following relations hold*

$$D'((x')^s F(\underline{x})) = -2(x')^{s-1} T'_s F(\underline{x}) + (-1)^s (x')^s D' F(\underline{x}), \tag{4.10}$$

*where*

$$T'_s = \begin{cases} k\ \mathbf{id}, & \text{if } s = 2k \\ E' + (\frac{n}{2} + k)\mathbf{id}, & \text{if } s = 2k+1 \end{cases}.$$

**Proof:** We use induction to prove (4.10). First notice for $s = 0$ formula (4.10) is automatically satisfied and for $s = 1$, the relation

$$D(x' F(\underline{x})) = -2E' F(\underline{x}) - nF(\underline{x}) = -2T'_1 F(\underline{x})$$

follows from the first anti-commutation relation given by Lemma 3.4.3.

Next we assume that (4.10) holds for any $s \in \mathbb{N}$, i.e.

$$\begin{aligned} D'((x')^{2k} F(\underline{x})) &= -2(x')^{2k-1} F(\underline{x}) + (x')^{2k} D' F(\underline{x}), \quad \text{for } s = 2k, \\ D'((x')^{2k+1} F(\underline{x})) &= -2(x')^{2k} T'_{2k+1} F(\underline{x}) - (x')^{2k+1} D' F(\underline{x}), \quad \text{for } s = 2k+1. \end{aligned}$$

Hence, the induction assumption together with the first and third relations given by Lemma 3.4.3 leads to

$$\begin{aligned} D'((x')^{2k+2} F(\underline{x})) &= -2T'_1((x')^{2k+1} F(\underline{x})) - x' D'((x')^{2k+1} F(\underline{x})) \\ &= -2(x')^{2k+1}(T'_{2(2k+1)+1} F(\underline{x}) - T'_{2k+1} F(\underline{x})) + (x')^{2k+2} D' F(\underline{x}) \\ &= -2(k+1)(x')^{2k+1} F(\underline{x}) + (x')^{2k+2} D' F(\underline{x}), \end{aligned}$$

$$\begin{aligned} D'((x')^{2k+3} F(\underline{x})) &= -2T'_1(x')^{2k+2} F(\underline{x}) - x' D'((x')^{2k+2} F(\underline{x})) \\ &= -2(x')^{2k+2}(T'_{2(2k+2)+1} F(\underline{x}) - (k+1) F(\underline{x})) - (x')^{2k+3} D' F(\underline{x}) \\ &= -2(x')^{2k+2} T'_{2(k+1)+1} F(\underline{x}) - (x')^{2k+3} D' F(\underline{x}). \end{aligned}$$

This proves (4.10).

■

**Lemma 4.1.15** *Let $(t)_s = t(t+1)\dots(t+s-1)$ denote the Pochhammer symbol. Then the map given by*

$$\pi_{\mathcal{M}_k[\Psi]}F(\underline{x}) = \sum_{j=0}^{k} c_{j,k}(x')^j(D')^jF(\underline{x})$$

*is the projection from $\mathcal{P}_k[\Psi]$ to $\mathcal{M}_k[\Psi]$.*

*Moreover, the constants are given by*

$$c_{j,k} = \begin{cases} 1 & , \textit{if} \;\; j=0 \\ \frac{(-1)^s}{4^s s!(\frac{n}{2}+k-s)_s}, & \textit{if} \;\; j=2s \\ \frac{(-1)^s}{4^s s!(\frac{n}{2}+k-s-1)_{s+1}}, & \textit{if} \;\; j=2s+1 \end{cases} \qquad (4.11)$$

**Proof:** Let us consider the *ansatz* $R(\underline{x}) = \sum_{j=0}^{k} a_j(x')^k(D')^kP(\underline{x})$. Recall that from Theorem 4.1.7, we can split $R(\underline{x})$ as

$$R(\underline{x}) = P(\underline{x}) + Q(\underline{x}),$$

where $Q(\underline{x}) = \sum_{j=1}^{k} a_j(x')^k(D')^kP(\underline{x}) \in x'\mathcal{P}_{k-1}[\Psi]$.

By applying Lemma 4.10 we get

$$\begin{aligned} D'R(\underline{x}) &= \sum_{j=0}^{k} a_jD'[(x')^j(D')^jP(\underline{x})] \\ &= \sum_{j=0}^{k}(-2a_j(x')^{j-1}T'_j(D')^jP(\underline{x}) + (-1)^ja_j(x')^j(D')^{j+1}P(\underline{x}) \\ &= \sum_{j=0}^{k}(-2a_jd_{j,k}(x')^{j-1}(D')^jP(\underline{x}) + (-1)^ja_j(x')^j(D')^{j+1}P(\underline{x})) \end{aligned}$$

where the coefficients $d_{j,k}$ are eigenvalues of the operator $T'_j(D')^j$ relative to the polynomial $P(\underline{x}) \in \mathcal{P}_k[\Psi]$, i.e $T'_j(D')^jP(\underline{x}) = d_{j,k}P(\underline{x})$.

Indeed, for $j = 2s$, $d_{2s,k} = s$ follows from definition of $T'_j$ ; for $j = 2s+1$ the coefficients $d_{2s+1,k} = k + \frac{n}{2} - s - 1$ result from the relation $T'_{2s+1}(D')^{2s+1} = (D')^{2s+1}T'_{-2(s+1)+1}$ (see Lemma 3.4.3, Section 3.4 ).

Hence $R(\underline{x})$ is Umbral monogenic if and only if the equation

$$(-1)^{j-1}a_{j-1} + a_jd_{j,k-j} = 0$$

holds for all $j = 1, \dots, k$.

By induction on $j$, we get

$$a_j = \frac{(-1)^{\lfloor j/2 \rfloor}}{\prod_{s=1}^{j} -d_{s,k-s}}$$

and hence by putting $c_{j,k} = a_j$ we get the coefficients (4.11).

■

**Theorem 4.1.16** *The individual components in the decomposition in Lemma 4.1.8 are given by*

$$M_{k-s}(\underline{x}) = c'_{k,s} \sum_{j=0}^{k-s} c_{j,k-s}(x')^j (D')^{j+s} P_k(\underline{x}), \quad s = 0, \ldots, k, \tag{4.12}$$

*where the constants $c'_{k,s}$ are uniquely determined by*

$$c'_{k,s} = \begin{cases} 1, & \text{if } s = 0 \\ \frac{1}{4^s s! (\frac{n}{2}+k-2j)_s}, & \text{if } s = 2j \\ \frac{1}{2 \quad 4^s s! (\frac{n}{2}+k-2j-1)_{s+1}}, & \text{if } s = 2j+1 \end{cases}$$

**Proof:** From the Umbral Fischer decomposition (Theorem 4.1.7) we know that every $P_k(\underline{x}) \in \mathcal{P}_k[\Psi]$ can be written as $P_k(\underline{x}) = \sum_{s=0}^{k-1} (x')^s M_{k-s}(\underline{x})$, where each $M_{k-s}(\underline{x})$ are in $\mathcal{M}_{k-s}[\Psi]$.

In order to compute $M_{k-s}(\underline{x})$ explicitly, we apply $(D')^k$ on both sides of the relation, i.e.

$$(D')^k P_k(\underline{x}) \quad = \quad \sum_{s=0}^{k-1} (D')^k ((x')^s M_{k-s}(\underline{x})). \tag{4.13}$$

By recursive application of Lemma 4.10, the summands on the right-hand side belong, in turn, to the spaces $(x')^{k-s-j}\mathcal{M}_j[\Psi]$ for each $j = 0, 1, \ldots, k-s$.

Hence $(D')^k[(x')^k P_k(\underline{x})]$ is equal to $\pi_{\mathcal{M}_k[\Psi]}((D')^k \mathcal{P}_k)$, so we can now use the expression for the monogenic projection proved in Lemma 4.1.15. Thus, in order to get the statement, it is enough to prove that $(D')^s[(x')^s M_{k-s}(\underline{x})] = \frac{1}{c'_{k,s}} M_s(\underline{x})$.

Using Lemma 4.10 we get by recursion the constants $c'_{k,s} = \frac{1}{\prod_{j=1}^{s} d_{j,k-j}}$, where $d_{j,k-j}$ are determined in the proof of Lemma 4.1.15. This finishes the proof.

■

**Remark 4.1.17** *The same scheme obtained in Lemma 4.1.15 and Theorem 4.1.16 can be applied to obtain the extension to the so-called discrete polymonogenic functions of degree $k$ (i.e. solutions of the equation $(D')^k F(\underline{x}) = 0$).*

*Indeed, it is interesting to note that in [54, 59], the application of iterated differential operators play a central role in these approaches and the introduction of the Fischer inner product is not required a priori. According to [46], this type of decomposition obtained can be described by using the Howe dual pair technique (see also Theorems 5.4.2 and 5.4.3, Chapter 5).*

## 4.2 Discrete Spherical Monogenics

### 4.2.1 Basic Definitions and Properties

According to Examples 1.2.9 and 1.2.10 (see Section 1.2) and , it is now natural to emboid the Umbral calculus formalism to obtain the discrete analogues of spherical monogenics and, moreover, the discrete spherical harmonics as a refinement of discrete spherical monogenics.

In order to proceed, let us start to introduce the following definitions:

**Definition 4.2.1 (Discrete Spherical Harmonics/ Discrete Spherical Monogenics)** *Let $P_k(\underline{x}) \in \mathcal{P}$ be a Clifford-valued polynomial.*

- *$P_k(\underline{x})$ is a discrete spherical harmonic polynomial of degree $k$ if the equation*

$$\Delta' P_k(\underline{x}) = 0$$

*holds on the discrete sphere $\Psi_{\underline{x}} S^{n-1} = \{x \in \mathbb{R}^1_{0,n} \ : \ -(x')^2 \mathbf{1} = 1\}$.*

- *$P_k(\underline{x})$ is a discrete spherical monogenic polynomial of degree $k$ if the equation*

$$D' P_k(\underline{x}) = 0$$

*holds on the discrete sphere $\Psi_{\underline{x}} S^{n-1} = \{x \in \mathbb{R}^1_{0,n} \ : \ -(x')^2 \mathbf{1} = 1\}$.*

In order to describe the discrete counterparts of spherical harmonics and spherical monogenics, our starting point is again the Fischer decomposition (Theorems 4.1.7 and 4.1.8) obtained in Subsection 4.1.1.

First we will show the interplay between the spaces of Umbral harmonic polynomials with the spaces of Umbral monogenic polynomials. This corresponds to the following proposition:

**Proposition 4.2.2** *We have*

$$\mathcal{H}_k[\Psi] = \mathcal{M}_k[\Psi] \oplus_{(\cdot,\cdot)_\Psi} x' \mathcal{M}_{k-1}[\Psi].$$

**Proof:** Let $H_k \in \mathcal{P}_k[\Psi] \cap \ker \Delta'$. From the Umbral Fischer decomposition (Theorem 4.1.7) we get $H_k = M_k + x' H_{k-1}$, where $x' H_k \in \mathcal{P}_k[\Psi] \cap \ker \Delta'$.

Since from (3.4.3), Subsection 3.4, the commuting relation $[\Delta', x'] = 2[D', E'] = 2D'$ holds in $\mathcal{P}$, we get

$$0 = \Delta' H_k = -D'(D' M_k) + \Delta'(x' H_{k-1}) = 2D' H_{k-1}.$$

Thus $H_{k-1} \in \mathcal{M}_{k-1}[\Psi]$, as desired.

■

From the above result and Proposition 4.2.2 the Umbral Fischer decomposition (Theorem 4.1.8) appears as a refinement of Proposition 4.1.9. This nicely shows, as in *continuum* [18], that the theory of discrete spherical monogenics appear as a refinement of the theory of discrete spherical harmonics.

In the next example we will explore the above result in terms of lattice structure.

**Example 4.2.3** *The coordinate-free formulation of the Difference Dirac operator* $D_h = \frac{1}{2}\left(D_h^+ + D_h^-\right)$ *(see also Examples 3.3.7 and 3.3.14) in terms of 0-chains corresponds to*

$$D_{\sigma^0}F(\underline{x}) = \sum_{j=1}^{n} \mathbf{e}_j Q_{x_j} F_\sharp(\underline{x}),$$

*where* $Q_{x_j}F_\sharp(\underline{x}) = \frac{O'_{x_j}F_\sharp(\underline{x}) - (O'_{x_j})^{-1}F_\sharp(\underline{x})}{O'_{x_j}(x_j)_\sharp - (O'_{x_j})^{-1}(x_j)_\sharp}$ *denotes the central difference operator (see also Example 4.1.13).*

*According to Example 1.2.10, the operator* $(Q'_{x_j})^{-1}$ *is computed by means of alternating series expansion in terms of left-actions on a periodic Cayley graph (i.e. a cyclic discrete group). This makes the spaces of discrete monogenics* $\mathcal{M}_k[\Psi]$ *appear as spaces of piecewise periodic functions and moreover, from Theorem 4.2.2, the space of discrete harmonics* $\mathcal{H}_k[\Psi]$ *is also a space with periodic functions.*

*We would like to stress that the elements underlying the spaces* $\mathcal{M}_k[\Psi]$ *and* $\mathcal{H}_k[\Psi]$ *are supported on Voronoi cells instead of simplicial complexes (c.f. [44]). Indeed for* $\sigma^1 = [\mathbf{p}, \mathbf{q}]$*, the Pincherle derivative* $Q'_{x_j}$ *acting on the discrete path* $\underline{x}(\cdot) : supp(\sigma^0) \mapsto \mathbb{R}^n$ *gives* $\frac{1}{2}(\underline{x}(\mathbf{p}) + \underline{x}(\mathbf{q}))$ *and so the average value along the edge* $[\mathbf{p}, \mathbf{q}]$ *is taken.*

Next we turn our attention for the actions of the operators $E'$ and $\Gamma'$ defined in Subsection 3.4. In *continuum*, when passing the coordinate vector variable $x = \sum_{j=1}^n x_j \mathbf{e}_j$ to polar coordinates i.e. $x = r\theta$ with $r = |\underline{x}|$ and $\theta = \frac{1}{r}x$, the *continuum* Euler operator $E$ is equal to $E = r\partial_r$, where $\partial_r$ is a directional derivative. This allow us to split the Dirac operator $D = \sum_{j=1}^n \mathbf{e}_j \partial_{x_j}$ as

$$D = \theta\left(\partial_r + \frac{1}{r}\Gamma\right).$$

Taking the square of both sides, the Laplace operator corresponds to the splitting

$$\Delta = -D^2 = \partial_r^2 + \frac{n-1}{r} + \frac{1}{r^2}\Delta_{LB},$$

where $\Delta_{LB} = ((n-2)\mathbf{id} - \Gamma)\Gamma$ is the classical Laplace-Beltrami operator.

From the above splitting, it is usual to call to the operator $\Gamma$ the *spherical Dirac operator*, so $\Gamma'$ corresponds to the discrete *spherical Dirac operator*. This suggests to introduce the discrete counterpart of the Laplace-Beltrami operator $\Delta_{LB}$ as being

$$\Delta'_{LB} = ((n-2)\mathbf{id} - \Gamma')\Gamma'. \tag{4.14}$$

Next we will show the radial and the angular character of the operators $E'$ and $\Gamma'$, respectively, and moreover the angular character of $\Delta'_{LB}$.

**Lemma 4.2.4** *One has*

$$[E', (x')^2] = 2(x')^2, \quad [\Gamma', (x')^2] = 0, \quad [\Delta'_{LB}, (x')^2] = 0.$$

**Proof:** By means of the relation $[E', x'] = x'$ (see Lemma 3.4.3, Subsection 3.4), it immediately follows that $[E', (x')^2] = 2(x')^2$.

In order to prove the relation $[\Gamma', (x')^2] = 0$, is it enough to show that

$$[x'D', (x')^2] = -[E', (x')^2].$$

In fact, by combining the identities $\{x', D'\} = -2E' - n\mathbf{id}$ and $[E', x'] = x'$ obtained in Lemma 3.4.3, we get

$$\begin{aligned}
(x'D')((x')^2) &= x'(-2E' - n\mathbf{id} - x'D')x' \\
&= -2x'E'x' - n(x')^2 - (x')^2(-2E' - n\mathbf{id} - x'D') \\
&= -2x'[E', x'] + (x')^2(x'D') \\
&= -2(x')^2 + (x')^2(x'D').
\end{aligned}$$

Thus $[x'D', (x')^2] = -[E', (x')^2]$, as desired.

Finally, from the relation (4.14), the statement $[\Delta'_{LB}, (x')^2] = 0$ is then immediate.

■

We must note that the above characterization is completely analogous in *continuum* upon application of the Sheffer operator $\Psi_{\underline{x}}$ (see Definition 3.2.1, Section 3.2). Using the above lemma we can get the following eigenvalue properties for the discrete *spherical Dirac operator* $\Gamma'$:

**Theorem 4.2.5** *For each $M_k(\underline{x}) \in \mathcal{M}_k[\Psi]$, we have $\Gamma'((x')^s M_k) = g_{k,s}(x')^s M_k$, where*

$$g_{k,s} = \begin{cases} -k\,, & \text{if } s \text{ even} \\ k+n-1, & \text{if } s \text{ odd} \end{cases}.$$

**Proof:** If $s$ is even, i.e. $s = 2t$ with $t = 0, 1, \ldots$ by combining the previous lemma with the relation $x'D' = -E' - \Gamma'$ we find by recursion

$$\Gamma'((x')^{2t}M_k) = (x')^{2t}(\Gamma' M_k) = -(x')^{2t}(E' M_k) = -k(x')^{2t}M_k.$$

If $s = 2t+1$ with $t = 0, 1, \ldots$, recall that from $\Gamma' = -x'D' - E'$, and $\{x', D'\} = -2E' - n\mathbf{id}$ it follows that

$$\Gamma'(x'M_k) = -x'D'(x'M_k) - E'(x'M_k) = x'(2E' + n\mathbf{id})M_k - E'(x'M_k) = (k + n - 1)M_k.$$

Hence, by applying again the previous lemma, we end up with

$$\Gamma'((x')^{2t+1}M_k) = (x')^{2t}(\Gamma'(x'M_k)) = (k + n - 1)(x')^{2t+1}M_k.$$

■

In order to study the spectra of the discrete Laplace-Beltrami operator, the following characterization will be useful in the sequel:

**Lemma 4.2.6** *When acting on the space of Clifford-valued polynomials $\mathcal{P}$, the $\Delta'_{LB}$ is given by*

$$\Delta'_{LB} = -(x')^2\Delta' - E'((n-2)\mathbf{id} + E').$$

**Proof:** By taking into account the relation $\Gamma' = -x'D' - E'$ (see Lemma 4.2.6, Subsection 3.4), the square of the operator $\Gamma'$ can be splited into

$$(\Gamma')^2 = \frac{1}{2}\{\Gamma', \Gamma'\} = (E')^2 + (x'D')^2 + \{E', x'D'\}.$$

By applying the basic identities obtained in Lemma 3.4.3 we obtain

$$\begin{aligned}
(x'D')^2 &= x'(-2E' - n\mathbf{id} - x'D')D' \\
&= -2x'E'D' - nx'D' - (x')^2(D')^2 \\
&= (x')^2\Delta' - (n - 2 + 2E')x'D',
\end{aligned}$$

$$\begin{aligned}
\{E', x'D'\} &= E'(x'D') + x'(D' + E'D') \\
&= E'(x'D') + x'D' + (-x' + E'x')D' \\
&= 2E'(x'D').
\end{aligned}$$

Thus $(\Gamma')^2 = (x')^2\Delta' - (n-2)x'D' + (E')^2$ and hence the operator $\Delta'_{LB}$ corresponds to

$$\begin{aligned}
\Delta'_{LB} &= -(n-2)(x'D' + E') - (x')^2\Delta' + (n-2)x'D' - (E')^2 \\
&= -(x')^2\Delta' - E'((n-2)\mathbf{id} + E').
\end{aligned}$$

■

Naturally from the above lemma, the following characterization for the spectrum of the discrete Laplace-Beltrami operator follows:

**Theorem 4.2.7** *For $k \geq 2s \geq 0$ the subspaces of $\mathcal{P}_k[\Psi]$, $(x')^{2s}\mathcal{H}_{k-2s}[\Psi]$, are eigenspaces of the discrete Laplace-Beltrami operator $\Delta'_{LB}$ corresponding to the eigenvalues $-k(n-2+k)$.*

*Moreover $\mathcal{P}_k[\Psi]$ is also an eigenspace for $\Delta'_{LB}$.*

**Proof:** Recall that a discrete harmonic polynomial $H_{k-2s}(\underline{x}) \in \mathcal{H}_{k-2s}[\Psi]$ satisfies the equation $E'((x')^{2s}H_{k-2s}(\underline{x})) = kH_{k-2s}(\underline{x})$ and hence the equation

$$\Delta'_{LB}((x')^{2s}H_{k-2s}(\underline{x})) = -k(n-2+k)(x')^{2s}H_{k-2s}(\underline{x}),$$

naturally follows from the above corollary.

In order to prove that $\mathcal{P}_k[\Psi]$ is also an eigenspace for $\Delta'_{LB}$, we will use the decomposition in terms of discrete spherical harmonics. Indeed, in the view of Theorem 4.1.9, each element $P_k(\underline{x}) \in \mathcal{P}_k[\Psi]$ may be written as

$$P_k(\underline{x}) = \sum_{s=0}^{\lfloor \frac{k}{2} \rfloor} (x')^{2s}H_{k-2s}(\underline{x}), \tag{4.15}$$

with $H_{k-2s}(\underline{x}) \in \mathcal{H}_{k-2s}[\Psi]$.

Furthermore by applying the discrete Laplace-Beltrami operator on both sides of (4.15), we obtain $\Delta'_{LB}(P_k(\underline{x})) = -k(n-2+k)P_k(\underline{x})$, as desired.

■

We are now in conditions to determine explicitly the summands of Theorem 4.1.8 in terms of discrete spherical harmonics. This corresponds to the following theorem:

**Theorem 4.2.8** *For $k \geq 2s$ all elements of $(x')^{2s}\mathcal{M}_{k-2s}[\Psi]$ and $(x')^{2s+1}\mathcal{M}_{k-2s-1}[\Psi]$ are uniquely determined by*

$$\begin{aligned}
(x')^{2s+1}M_{k-2s-1}(\underline{x}) &= \frac{1}{2k-2+n}(k(x')^{2s} + (x')^{2s}\Gamma')(H_{k-2s}(\underline{x})), \\
(x')^{2s}M_{k-2s}(\underline{x}) &= \frac{1}{2k-2+n}((k+n-2)(x')^{2s} - (x')^{2s}\Gamma')(H_{k-2s}(\underline{x})).
\end{aligned}$$

**Proof:** First we will determine the explicit formula for $s = 0$. Using the same order of ideas of the above proof , $H_k(\underline{x}) = M_k(\underline{x}) + x'M_{k-1}(\underline{x})$ with $M_k(\underline{x}) \in \mathcal{M}_k[\Psi]$ and $M_{k-1}(\underline{x}) \in \mathcal{M}_{k-1}[\Psi]$.

By acting $x'D' = -E' - \Gamma'$ on both sides of the above identity we get

$$(k\mathbf{id} + \Gamma')(H_k(\underline{x})) = -x'D'(x'M_{k-1}(\underline{x})) = (2k-2+n)x'M_{k-1}(\underline{x}).$$

This leads to

$$\begin{array}{rcl} x'M_{k-1}(\underline{x}) & = & \frac{1}{2k-2+n}(k\mathbf{id} + \Gamma')(H_k(\underline{x})), \\ M_k(\underline{x}) & = & \frac{1}{2k-2+n}((k+n-2)\mathbf{id} - \Gamma')(H_k(\underline{x})). \end{array}$$

Hence by taking the replacement $H_k(\underline{x}) \to (x')^{2s}H_{k-2s}(\underline{x})$ and using the fact that $[\Gamma', (x')^2] = 0$ (see Lemma 4.2.4), we get

$$\begin{array}{rcl} (x')^{2s+1}M_{k-2s-1}(\underline{x}) & = & \dfrac{1}{2k-2+n}(k(x')^{2s} + (x')^{2s}\Gamma')(H_{k-2s}(\underline{x})), \\ (x')^{2s}M_{k-2s}(\underline{x}) & = & \dfrac{1}{2k-2+n}((k+n-2)(x')^{2s} - (x')^{2s}\Gamma')(H_{k-2s}(\underline{x})). \end{array}$$

■

**Remark 4.2.9** *When restricted to the sphere $S^{n-1}$ ($\Psi_{\underline{x}} = \mathbf{id}$), the spaces $(x')^{2s}\mathcal{M}_{k-2s}[\Psi]$ and $(x')^{2s+1}\mathcal{M}_{k-2s-1}[\Psi]$ are usually known as the spaces of inner and outer spherical monogenics, respectively (see e.g. [18]).*

### 4.2.2 Integral Representation Formulae

It becomes now natural to take a look for integral formulae representations in terms of discrete spherical harmonics and discrete spherical monogenics.

For this purpose, we introduce for two polynomials $F(\underline{x}), G(\underline{x}) \in \mathcal{P}$ on the discrete sphere $\Psi_{\underline{x}}S^{n-1}$, the functional $\langle \cdot, \cdot \rangle_{\Psi S^{n-1}}$ as being

$$\langle F(\underline{x}), G(\underline{x}) \rangle_{\Psi S^{n-1}} = \int_{S^{n-1}} [(\Psi_{\underline{x}}^{-1}F)^{\dagger}(\underline{x})(\Psi_{\underline{x}}^{-1}G)(\underline{x})]_0 L(\underline{x}, \mathbf{d}\underline{x}), \tag{4.16}$$

where $L(\underline{x}, \mathbf{d}\underline{x})$ represents the continuous version of the Leray form defined *viz* formula (3.40) (see Subsection 3.5.2).

Using the continuous version of the Clifford-Stokes theorem (see also Subsection 3.5.2) we derive the following relation:

**Theorem 4.2.10** *For $M_k(\underline{x}) \in \mathcal{M}_k[\Psi]$ and $M_l(\underline{x}) \in \mathcal{M}_l[\Psi]$, we obtain*

$$\langle M_k(\underline{x}), M_l(\underline{x}) \rangle_{\Psi S^{n-1}} = 0, \qquad \textit{for all } \ k, l = 1, \ldots, n;$$

$$\langle x'M_k(\underline{x}), M_l(\underline{x}) \rangle_{\Psi S^{n-1}} = 0 = \langle x'M_k(\underline{x}), x'M_l(\underline{x}) \rangle_{\Psi S^{n-1}} \quad \textit{for all } \ k, l = 1, \ldots, n \ \textit{ with } k \neq l.$$

**Proof:** Recall that since the operators $\mathbf{i}_{\partial_{x_j}}$ and $x_j$ commute, we get $[\mathbf{i}_D, x] = 0$. Hence, from Lemma 3.5.3 (i.e. $\Psi_{\underline{x}} = \mathbf{id}$) we obtain $x\mathbf{i}_D = \mathbf{i}_D x = \frac{1}{2}\{x, \mathbf{i}_D\} = -\mathbf{i}_E$. In particular, when $\underline{x} \in S^{n-1}$, i.e. $x^2 = -1$, it follows straightforward the relations

$$-x\sigma(\underline{x}, \mathbf{d}\underline{x}) = -\sigma(\underline{x}, \mathbf{d}\underline{x})x = L(\underline{x}, \mathbf{d}\underline{x}), \quad -xL(\underline{x}, \mathbf{d}\underline{x}) = -L(\underline{x}, \mathbf{d}\underline{x})x = \sigma(\underline{x}, \mathbf{d}\underline{x}). \tag{4.17}$$

On the other hand, for two functions $F(\underline{x}), G(\underline{x}) \in \mathcal{P}$, Clifford-Stokes theorem in *continuum* states that

$$\int_{S^{n-1}} G(\underline{x})^\dagger \sigma(\underline{x}, \mathbf{d}\underline{x}) F(\underline{x}) = \int_{\mathbb{B}^n} \Big( (G(\underline{x})^\dagger D) F(\underline{x}) + G(\underline{x})^\dagger D F(\underline{x}) \Big) \mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x}),$$

where $\mathbb{B}^n = \left\{ x = \sum_{j=1}^n x_j \mathbf{e}_j \ : \ -x^2 \leq 1 \right\}$ denotes the unit ball.

Taking the replacements $F(\underline{x}) \to \Psi_{\underline{x}}^{-1} F(\underline{x})$ and $G(\underline{x}) \to \Psi_{\underline{x}}^{-1} G(\underline{x})$, it follows from Lemma 3.2.4 (see Section 3.2) that

$$\begin{aligned} &\int_{S^{n-1}} (\Psi_{\underline{x}}^{-1} G)^\dagger(\underline{x}) \sigma(\underline{x}, \mathbf{d}\underline{x}) (\Psi_{\underline{x}}^{-1} F)(\underline{x}) = \\ &= \int_{\mathbb{B}^n} \Big( \Psi_{\underline{x}}^{-1} (G^\dagger(\underline{x}) D')(\Psi_{\underline{x}}^{-1} F)(\underline{x}) + (\Psi_{\underline{x}}^{-1} G)^\dagger \Psi_{\underline{x}}^{-1} D' F(\underline{x}) \Big) \mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x}), \end{aligned} \tag{4.18}$$

and, hence, we immediately get $\langle x' M_k(\underline{x}), M_l(\underline{x}) \rangle_{\Psi S^{n-1}} = 0$.

Next we consider the integral

$$\int_{\partial \mathbb{B}^n(r)} (\Psi_{\underline{x}}^{-1} M_k)^\dagger(\underline{x}) (\Psi_{\underline{x}}^{-1} M_l)(\underline{x}) L(\underline{x}, \mathbf{d}\underline{x}) = \int_{\partial \mathbb{B}^n(r)} (\Psi_{\underline{x}}^{-1} x' M_k)^\dagger(\underline{x}) \sigma(\underline{x}, \mathbf{d}\underline{x}) (\Psi_{\underline{x}}^{-1} M_l)(\underline{x}),$$

where $\mathbb{B}^n(r)$ denotes the closed ball with radius $r > 0$.

Notice that the above integral is independent of the radius $r$. Hence direct application of Cauchy's theorem to the compact manifold $\Omega = \mathbb{B}^n(r) \setminus \mathbb{B}^n$ with boundary $\partial\Omega = \partial\mathbb{B}^n(r) \cup S^{n-1}$ yields

$$\int_{\partial\Omega} (\Psi_{\underline{x}}^{-1} x' G)^\dagger(\underline{x}) \sigma(\underline{x}, \mathbf{d}\underline{x}) (\Psi_{\underline{x}}^{-1} F)(\underline{x}) = 0,$$

or equivalently,

$$\int_{\partial \mathbb{B}^n(r)} (\Psi_{\underline{x}}^{-1} x' G)^\dagger(\underline{x}) \sigma(\underline{x}, \mathbf{d}\underline{x}) (\Psi_{\underline{x}}^{-1} F)(\underline{x}) = \int_{S^{n-1}} (\Psi_{\underline{x}}^{-1} x' G)^\dagger(\underline{x}) \sigma(\underline{x}, \mathbf{d}\underline{x}) (\Psi_{\underline{x}}^{-1} F)(\underline{x}). \tag{4.19}$$

By changing to spherical coordinates $\underline{x} = r\underline{\theta}$ with $\underline{\theta} \in S^{n-1}$ the left-hand side of (4.19) then corresponds to

$$\int_{\partial \mathbb{B}^n(r)} (\Psi_{\underline{x}}^{-1} x' M_k)^\dagger(\underline{x}) \sigma(\underline{x}, \mathbf{d}\underline{x}) (\Psi_{\underline{x}}^{-1} M_l)(\underline{x}) = r^{l-k} \int_{S^{n-1}} (\Psi_{\underline{x}}^{-1} \theta' M_k)^\dagger(\underline{\theta}) \sigma(\underline{\theta}, \mathbf{d}\underline{\theta}) (\Psi_{\underline{\theta}}^{-1} M_l)(\underline{\theta})$$

As by assumption $k \neq l$, by inserting the above equation into (4.19) we obtain

$$\int_{S^{n-1}} (\Psi_{\underline{\theta}}^{-1}\theta' M_k)^\dagger(\underline{\theta})\sigma(\underline{\theta}, \mathbf{d}\underline{\theta})(\Psi_{\underline{\theta}}^{-1} M_l)(\underline{\theta}) = 0$$

and hence $\langle M_k(\underline{x}), M_l(\underline{x})\rangle_{\Psi S^{n-1}} = 0$.

Finally, by taking into account the replacement $M_l(\underline{x}) \to x' M_l(\underline{x})$ in equation (4.19) we prove in a similar way that $\langle x' M_k(\underline{x}), x' M_l(\underline{x})\rangle_{\Psi S^{n-1}} = 0$.

■

By combining the Umbral Fischer decomposition (Theorem 4.1.8), with the above Theorem, the next corollary follows naturally

**Corollary 4.2.11** *For $M_k(\underline{x}) \in \mathcal{M}_k[\Psi]$ and $P_l(\underline{x}) \in \mathcal{P}_l[\Psi]$, with $l < k$ we obtain*

$$\langle M_k(\underline{x}), P_l(\underline{x})\rangle_{\Psi S^{n-1}} = 0$$

Next, for two polynomial functions $F(\underline{x}), G(\underline{x})$, we define $\langle F(\underline{x}), G(\underline{x})\rangle_\Psi$ by setting

$$\langle F(\underline{x}), G(\underline{x})\rangle_\Psi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left[(\Psi_{\underline{x}}^{-1} F)^\dagger(\underline{x})(\Psi_{\underline{x}}^{-1} G)(\underline{x})\right]_0 \exp\left(-\frac{|\underline{x}|^2}{2}\right) \mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x}), \quad (4.20)$$

where $\mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x})$ denotes the *continuous* volume form.

The right-hand side of (4.20) is nothing else than an inner product on the weighted $L_2-$space $L_2\left(\mathbb{R}^n, \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{|\underline{x}|^2}{2}\right)\right)$ which is nothing else than the inner product in the real Bargmann space (c.f. [33, 59]). In particular the next lemma readily follows:

**Lemma 4.2.12** *$x_j' - Q_{x_j}$ is the adjoint of the operator $Q_{x_j}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\Psi$.*

**Proof:** Let $F(\underline{x}), G(\underline{x}) \in \mathcal{P}$. From Lemma 3.2.4, Section 3.2

$$\langle Q_{x_j} F(\underline{x}), G(\underline{x})\rangle_\Psi = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} [\partial_{x_j}(\Psi_{\underline{x}}^{-1} F)^\dagger(\underline{x})(\Psi_{\underline{x}}^{-1} G)(\underline{x})]_0 \exp\left(-\frac{|\underline{x}|^2}{2}\right) \mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x}).$$

Integrating by parts, the right-hand side of the above identity corresponds to

$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} [\partial_{x_j}(\Psi_{\underline{x}}^{-1} F)^\dagger(\underline{x})(\Psi_{\underline{x}}^{-1} G)(\underline{x})]_0 \exp\left(-\frac{|\underline{x}|^2}{2}\right) \mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x}) = I_1 + I_2,$$

where the constants $I_1$ and $I_2$ are given by

$$I_1 = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} [(\Psi_{\underline{x}}^{-1} F)^\dagger(\underline{x})(\Psi_{\underline{x}}^{-1} G)(\underline{x})]_0 \exp\left(-\frac{|\underline{x}|^2}{2}\right) \mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x}),$$

$$I_2 = -\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \left[(\Psi_{\underline{x}}^{-1} F)^\dagger(\underline{x})\partial_{x_j}\left(\exp\left(-\frac{|\underline{x}|^2}{2}\right)(\Psi_{\underline{x}}^{-1} G)(\underline{x})\right)\right]_0 \mathrm{Vol}(\underline{x}, \mathbf{d}\underline{x}).$$

Using standard arguments, the term $I_1$ vanishes while direct application of the Leibniz rule, leads to

$$\partial_{x_j}\left(\exp\left(-\frac{|\underline{x}|^2}{2}\right)(\Psi_{\underline{x}}^{-1}G)(\underline{x})\right)=\left(\partial_{x_j}-x_j\right)(\Psi_{\underline{x}}^{-1}G)(\underline{x})\exp\left(-\frac{|\underline{x}|^2}{2}\right).$$

Finally, direct application of Lemma 3.2.4 leads to $I_2=\langle F(\underline{x}),(x_j'-Q_{x_j})G(\underline{x})\rangle_\Psi$, as desired.

■

The subsequent propositions will be important on the sequel, namely in the interplay between between Umbral homogeneous polynomials (see relation (4.3)) and the classical Clifford-Stokes formula (see afterwards Remark 4.2.17).

We start to prove the following proposition:

**Proposition 4.2.13** *When acting on Clifford-valued polynomials $\mathcal{P}$, we obtain*

$$\langle\Delta' F(\underline{x}),G(\underline{x})\rangle_\Psi-\langle F(\underline{x}),\Delta' G(\underline{x})\rangle_\Psi=\langle E' F(\underline{x}),G(\underline{x})\rangle_\Psi-\langle F(\underline{x}),E' G(\underline{x})\rangle_\Psi.$$

**Proof:** From the coordinate expressions of $\Delta'$ and $E'$ it is enough to show that

$$\langle Q_{x_j}^2 F(\underline{x}),G(\underline{x})\rangle_\Psi-\langle F(\underline{x}),Q_{x_j}^2 G(\underline{x})\rangle_\Psi=\langle x_j' Q_{x_j}F(\underline{x}),G(\underline{x})\rangle_\Psi-\langle F(\underline{x}),x_j' Q_{x_j}G(\underline{x})\rangle_\Psi$$

holds for each $j=1,2,\ldots,n$.

Recall that from direct application of Lemma 4.2.12 we have

$$\langle x_j' F(\underline{x}),G(\underline{x})\rangle_\Psi=\langle Q_{x_j}F(\underline{x}),G(\underline{x})\rangle_\Psi+\langle F(\underline{x}),Q_{x_j}G(\underline{x})\rangle_\Psi.$$

Hence the replacements $F(\underline{x})\to Q_{x_j}F(\underline{x})$ and $G(\underline{x})\to Q_{x_j}G(\underline{x})$ on the inner products $\langle x_j' F(\underline{x}),G(\underline{x})\rangle_\Psi$ and $\langle F(\underline{x}),x_j' G(\underline{x})\rangle_\Psi$, respectively, leads to

$$\langle x_j' Q_{x_j}F(\underline{x}),G(\underline{x})\rangle_\Psi-\langle F(\underline{x}),x_j' Q_{x_j}G(\underline{x})\rangle_\Psi \quad = \quad \langle Q_{x_j}^2 F(\underline{x}),G(\underline{x})\rangle_\Psi-\langle F(\underline{x}),Q_{x_j}^2 G(\underline{x})\rangle_\Psi.$$

■

From the coordinate expressions of $\Gamma'$ and $\Delta'_{LB}$ the next proposition follows straightforward using the same order of ideas as in the proof of Proposition 4.2.13.

**Proposition 4.2.14** *When acting on Clifford-valued polynomials $\mathcal{P}$, we obtain*

$$\langle\Gamma' F(\underline{x}),G(\underline{x})\rangle_\Psi=\langle F(\underline{x}),\Gamma' G(\underline{x})\rangle_\Psi.$$

*Moreover* $\langle\Delta'_{LB}F(\underline{x}),G(\underline{x})\rangle_\Psi=\langle F(\underline{x}),\Delta'_{LB}G(\underline{x})\rangle_\Psi$.

At this level it's now interesting to make the bridge between our approach and the approach recently proposed by H. Render in [59]:

**Remark 4.2.15** *From Proposition 4.2.13 it readily follows that Umbral harmonic polynomials of different degrees (see also (4.3), Subsection 4.1.1) are obviously orthogonal with respect to the space inner product* $\langle \cdot, \cdot \rangle_\Psi$.

*Moreover, with respect to this inner product, we can establish a correspondence between Umbral homogeneous polynomials and the Umbral version of polyharmonic functions of degree* $k$*, i.e. the null solutions of the equation* $(\Delta')^k P_l(\underline{x}) = 0$.

*Indeed, following the same order of ideas of proof of Theorem 2 obtained in [59], it follows upon of the application of the Sheffer operator (see Definition 3.2.1, Section 3.2) that for* $l \geq 2(k-1)$, $(\Delta')^k P_l(\underline{x}) = 0$ *if and only if* $\langle P_l(\underline{x}), Q_m(\underline{x}) \rangle_\Psi = 0$ *for all* $Q_m(\underline{x}) \in \mathcal{P}_m[\Psi]$ *with* $2(k-1) + m < l$.

With the former approach, we have established a first contact between the spaces of umbral homogeneous polynomials as elements of the real Bargmann space. The passage from the real Bargmann space to the classical Bargmann space can be obtained by taking the umbral counterpart of holomorphic extension (c.f. [33]) or, alternatively, an umbral version of the Cauchy-Kovaleskaya product (c.f. [15]). Indeed a description of the Cauchy-Kovaleskaya product in terms of Hypercomplex Bernoulli polynomials was recently obtained by Malonek and Tomaz in [55], but we shall not explore it here.

Having obtained the above characterization umbral homogeneous polynomials in terms of Bargmann spaces, the next step in the development of the theory of discrete spherical monogenics is the passage from the inner product on real Bargmann spaces to the inner product on the discrete sphere $\Psi_{\underline{x}} S^{n-1}$.

Let us now take a closer look at the inner product $\langle \cdot, \cdot \rangle_\Psi$ written in terms of polar coordinates:

By taking the change of variable $x = r\theta$ with $r = |x|$ and $\theta = \frac{x}{r} \in S^{n-1}$, take into account that for each $F(\underline{x}) \in \mathcal{P}_k[\Psi]$, $G(\underline{x}) \in \mathcal{P}_l[\Psi]$, $\Psi_{\underline{x}}^{-1} F$ and $\Psi_{\underline{x}}^{-1} G$ are Clifford-valued homogeneous polynomials of degree $k$ and $l$, respectively, we get

$$(\Psi_{\underline{x}}^{-1} F)(r\theta) = r^k (\Psi_{\underline{x}}^{-1} F)(\theta) \quad \text{and} \quad (\Psi_{\underline{x}}^{-1} G)(r\theta) = r^l (\Psi_{\underline{x}}^{-1} G)(\theta)$$

On the other hand, from Proposition 3.5.8 (see Subsection 3.5.2) and due to the homogeneity of $L(\underline{x}, \mathbf{d}\underline{x})$, the continuous Leray form on the unit sphere $S^{n-1}$, $L(\underline{\theta}, \mathbf{d}\underline{\theta})$, is given by

$$L(\underline{\theta}, \mathbf{d}\underline{\theta}) = \frac{1}{r^n} L(\underline{x}, \mathbf{d}_\Psi \underline{x}) = \frac{2}{r^{n-1}} \mathbf{d}r \ \mathrm{Vol}(\underline{x}, \mathbf{d}_\Psi \underline{x}).$$

This leads to

$$\begin{aligned}\langle F(\underline{x}), G(\underline{x})\rangle_\Psi &= \frac{2}{(2\pi)^{n/2}}\int_0^\infty r^{k+l+n-1}\exp\left(-\frac{r^2}{2}\right)\mathbf{d}r\int_{S^{n-1}}(\Psi_{\underline{x}}^{-1}F)^\dagger(\underline{\theta})(\Psi_{\underline{x}}^{-1}G)(\underline{\theta})L(\underline{\theta},\mathbf{d}\underline{\theta})\\ &= 2^{\frac{k+l}{2}}\frac{\Gamma\left(\frac{k+l+n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\langle F(\theta), G(\theta)\rangle_{\Psi S^{n-1}}. \qquad (4.21)\end{aligned}$$

where $\Gamma(\cdot)$ denotes the classical Gamma function.

We are now in conditions to derive the following integral representation for the Umbral Fischer inner product $(\cdot,\cdot)_\Psi$. This corresponds to the following theorem:

**Theorem 4.2.16** *If $P_k(\underline{x}) \in \mathcal{P}_k[\Psi]$ and $M_k(\underline{x}) \in \mathcal{M}_k[\Psi]$*

$$(P_k(\underline{x}), M_k(\underline{x}))_\Psi = \langle M_k(\underline{x}), P_k(\underline{x})\rangle_\Psi = 2^k\frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\langle M_k(\underline{x}), P_k(\underline{x})\rangle_{\Psi S^{n-1}}.$$

**Proof:** By setting $\mathbf{d}\mu(\underline{x}) = \frac{1}{(2\pi)^{n/2}}\exp\left(-\frac{|\underline{x}|^2}{2}\right)\mathrm{Vol}(\underline{x},\mathbf{d}\underline{x})$, we immediately get $\int_{\mathbb{R}^n}\mathbf{d}\mu(\underline{x}) = 1$.

Since $(P_k(\underline{x}), M_k(\underline{x}))_\Psi$ is a constant, by using Lemma 4.2.12 we can rewrite the umbral Fischer inner product as the integral

$$\begin{aligned}(P_k(\underline{x}), M_k(\underline{x}))_\Psi &= \frac{1}{\int_{\mathbb{R}^n}\mathbf{d}\mu(\underline{x})}\int_{\mathbb{R}^n}[([\Psi_{\underline{x}}^{-1}P_k]^\dagger(Q_{\underline{x}})M_k(\underline{x}))]_0\mathbf{d}\mu(\underline{x})\\ &= \int_{\mathbb{R}^n}([(M_k^\dagger(\underline{x})[\Psi_{\underline{x}}^{-1}P_k](\underline{x}'-Q_{\underline{x}})\mathbf{1})]_0)\mathbf{d}\mu(\underline{x}).\end{aligned}$$

By recursion, we can split up $(\Psi_{\underline{x}}P_k)^{-1}(\underline{x}'-Q_{\underline{x}})\mathbf{1}$ as

$$(\Psi_{\underline{x}}P_k)^{-1}(\underline{x}'-Q_{\underline{x}})\mathbf{1} = (\Psi_{\underline{x}}^{-1}P_k)(\underline{x}')\mathbf{1} + (\Psi_{\underline{x}}^{-1}Q)(\underline{x}')\mathbf{1} = P_k(\underline{x}) + Q(\underline{x}),$$

where $Q(\underline{x})$ is a polynomial of degree less than $k$.

Since $M_k(\underline{x})$ is a discrete monogenic polynomial, we get from Corollary 4.2.11 that $\langle M_k(\underline{\theta}), Q(\underline{\theta})\rangle_{\Psi S^{n-1}} = 0$ and hence $\langle M_k(\underline{x}), Q(\underline{x})\rangle_\Psi = 0$. This leads to

$$(P_k(\underline{x}), M_k(\underline{x}))_\Psi = \langle M_k(\underline{x}), P_k(\underline{x})\rangle_\Psi = 2^k\frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\langle M_k(\underline{\theta}), P_k(\underline{\theta})\rangle_{\Psi S^{n-1}}.$$

■

**Remark 4.2.17** *Since the discrete Euler operator $E'$ has radial character and the discrete Gamma operator $\Gamma'$ has angular character (see Lemma 4.2.4) and $\langle\cdot,\cdot\rangle_\Psi$ is independent of $r^2 = -x^2$, from the above relation and upon the action of the Sheffer operator, we conclude that Propositions 4.2.13 and 4.2.14 are equivalent to the second Green's formula and to the*

*Cauchy's integral formula on the discrete sphere* $\Psi_{\underline{x}}S^{n-1}$*, where* $E'$ *plays the role of the normal derivative in the discrete sphere* $\Psi_{\underline{x}}S^{n-1}$ *while* $\Gamma'$ *plays the role of the spherical Dirac operator on* $\Psi_{\underline{x}}S^{n-1}$*. Indeed, for each* $F, G \in \mathcal{P}$ *Propositions 4.2.13 and 4.2.14 are equivalent to*

$$\langle \Delta' F(\underline{x}), G(\underline{x})\rangle_{\Psi\mathbb{B}^n} - \langle F(\underline{x}), \Delta' G(\underline{x})\rangle_{\Psi\mathbb{B}^n} = \langle E' F(\underline{x}), G(\underline{x})\rangle_{\Psi S^{n-1}} - \langle F(\underline{x}), E' G(\underline{x})\rangle_{\Psi S^{n-1}},$$
$$\langle \Gamma' F(\underline{x}), G(\underline{x})\rangle_{\Psi S^{n-1}} - \langle F(\underline{x}), \Gamma' G(\underline{x})\rangle_{\Psi S^{n-1}} = 0,$$

*where* $\langle \cdot, \cdot \rangle_{\Psi\mathbb{B}^n}$ *denotes the inner product with respect to the discrete unit ball*

$$\Psi_{\underline{x}}\mathbb{B} := \left\{ x \ : \ -\Psi_{\underline{x}}(x^2) \leq 1 \right\}.$$

With the above remark we conclude that the theory of discrete spherical harmonics and discrete spherical monogenics are equivalent to the theory of discrete harmonic and discrete monogenics polynomials in $\mathbb{R}^n$.

In the spirit of operational calculus, we transfer these results from the Fock space model to the spaces $L_2\left(\mathbb{R}^n, \frac{1}{(2\pi)^{n/2}} \exp\left(-\frac{|x|^2}{2}\right)\right)$ and $L_2(S^{n-1})$. The scope of the above results can be easily extended to the superspace setting (see [17] and references given there). Contrary to the standard sphere, the supersphere is not a compact manifold neither the probability measure on the superspace is radial. However bear in mind that translation of these relations to the superspace take into account the probabilistic jargon in the sense that square integrable functions on the supersphere are nothing else than random variables with finite mean.

**Remark 4.2.18** *While in the above example the discrete spherical harmonics and discrete spherical monogenics are in fact only defined on the discrete unit sphere, the extension for discrete counterpart of simply connected star-like (or star-shaped) domains* $\Omega$ *proposed by Malonek and Ren in [54] being now rather obvious.*

*In fact, by combining Theorem 1.1.17 with the Sheffer operator (see Definition 3.2.1, Section 1.1), we build up the operator* $I'_s = \Psi_{\underline{x}} I_s \Psi_{\underline{x}}^{-1}$ *(the inverse of* $s\mathbf{id} + E'$*) as a formal power series in terms of* $\underline{x}'$ *and* $Q_{\underline{x}}$*, respectively. However, bear in mind that* $I'_s$ *is defined locally as a series expansion.*

## 4.3 Generating Functions and Reproducing Kernels

The problem of representing any function $F(\underline{x})$ in terms of reproducing kernels can be stated as follows:

In the sequel, we assume that $F$ is square integrable on the discrete sphere $\Psi_{\underline{x}}S^{n-1}$, i.e. $\langle F(\underline{x}), F(\underline{x})\rangle_{\Psi S^{n-1}} < \infty$ (see formula 4.16). According to Proposition 4.1.4 and Theorem 4.2.16, we can consider the Fourier orthogonal series expansion is terms of Umbral homogeneous polynomials $P_k(\underline{x}) \in \mathcal{P}_k[\Psi]$, i.e.

$$F(\underline{x}) = \sum_{k=0}^{\infty} c_k[F]P_k(\underline{x}), \quad \text{with} \quad c_k[F] = \langle F(\underline{x}), P_k(\underline{x})\rangle_{\Psi S^{n-1}}. \tag{4.22}$$

Furthermore, under suitable density arguments, the series expansion (4.22) can be recasted by projecting $F(\underline{x})$ onto the the spaces of umbral homogeneous polynomials $\mathcal{P}_k[\Psi]$ i.e.

$$F(\underline{x}) = \sum_{k=0}^{\infty} \pi_{\mathcal{P}_k[\Psi]}F(\underline{x}), \quad \text{with} \quad \pi_{\mathcal{P}_k[\Psi]}F(\underline{x}) = c_k[F]P_k(\underline{x}).$$

In particular, by putting $C_k(\underline{t},\underline{x}) = P_k^{\dagger}(\underline{t})P_k(\underline{x})$, the projector $\pi_{\mathcal{P}_k[\Psi]}$ is nothing else than $\pi_{\mathcal{P}_k[\Psi]}F(\underline{x}) = \langle F(\underline{x}), C_k(\underline{t},\underline{x})\rangle_{\Psi S^{n-1}}$. The function $\underline{x} \mapsto C_k(\underline{t},\underline{x})$ can be viewed as the discrete counterpart of the so-called *zonal spherical monogenic* of degree $k$ with pole at $\underline{t}$ (c.f. [18]).

Thus the right hand side of the series expansion (4.22) admits the following integral representation on the discrete sphere

$$F(\underline{x}) = \sum_{k=0}^{\infty} c_k[F]P_k(\underline{x}) = \langle F(\underline{x}), C(\underline{t},\underline{x})\rangle_{\Psi S^{n-1}},$$

where the kernel $C(\underline{t},\underline{x})$ is defined *viz* $C(\underline{t},\underline{x}) = \sum_{k=0}^{\infty} C_k(\underline{t},\underline{x})$ is known as the reproducing kernel. We will see for reasons to be given below that $C(\underline{t},\underline{x})$ corresponds to the series expansion of the discrete Cauchy kernel.

Now we turn our attention for the exponential generating function obtained in Corollary 1.1.15. According to its definition, $V(\underline{t},\underline{x})$ may be characterized as

$$V(\underline{t},\underline{x}) = \sum_{k=0}^{\infty} V_k(\underline{t},\underline{x}),$$

where the $k$-term $V(\underline{t},\underline{x})$ is given by $V_k(\underline{t},\underline{x}) = \frac{1}{k!}(\underline{t}\cdot O^{-1}(\underline{x}))^k$.

Monogenic extension of $V(\underline{t},\underline{x})$ using the Cauchy-Kovaleskaya extension gives rise to an exponential generating function which is umbral monogenic, whose $k-$terms are given e.g. in terms of hypercomplex Bernoulli or Euler polynomials (see [55] and references given there).

The next lemma shows that the $k-$term of the exponential generating function $V(\underline{t},\underline{x})$ is a reproducing kernel for the space of polynomials.

**Proposition 4.3.1** *$V_k(\underline{x},\underline{t}) = \frac{1}{k!}(\underline{x}\cdot O^{-1}(\underline{t}))^k$ is the reproducing kernel of the umbral Fischer inner product (4.4), i.e.*

$$(V_k(\underline{t},\underline{x}), P(\underline{x}))_\Psi = P(\underline{t}), \quad \textit{for all} \quad P(\underline{t}) \in \mathcal{P}.$$

**Proof:** If $P(\underline{x}) \in \mathcal{P}_k$, then $(\frac{1}{k!}(\underline{t}\cdot\underline{x})^k, P(\underline{x})) = P(\underline{t})$. Applying the Sheffer operator $\Psi_{\underline{t}}$ one gets

$$(V_k(\underline{t},\underline{x}), P(\underline{x})) = \left[\,\left[V_k(\underline{t},\partial_{\underline{x}})P^\dagger(\partial_{\underline{x}})Q(\underline{x})\right]_0\right]_{\underline{x}=\underline{0}} = \Psi_{\underline{t}}P(\underline{t}).$$

By taking into account that the right hand side of the above equation is independent of the variable $\underline{t}$, and taking into account Lemma 4.1.2 and formula (4.4), the desired result holds in $\mathcal{P}[\Psi]$ by applying $\Psi_{\underline{x}}$ on both sides.

■

In fact what we used here is the fact that the generating function $\exp(\underline{t}\cdot\underline{x})$ from Umbral Calculus also represents the reproducing kernel for the classical Fischer inner product (4.1).

The subsequent results provide an alternative description for the discrete Cauchy kernel obtained in [41, 39]. Using the umbral Fischer decomposition, we will start to show that the $k-$term of $C(\underline{t},\underline{x})$ can be decomposed in terms of the reproducing kernel for the umbral Fischer inner product.

**Theorem 4.3.2** *For $k = 0, 1, \ldots$*

$$C_k(\underline{t},\underline{x}) \quad = \quad 2^k\frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\sum_{j=0}^{k} c_{j,k}(x')^j t^j V_{k-j}(\underline{t},\underline{x}),$$

*where the constants are given by formulae (4.11).*

**Proof:** From Proposition 4.3.1, the kernel $C_k(\underline{t},\underline{x})$ of the Cauchy integral formula is uniquely determined by the reproducing property for the space $\mathcal{M}_k[\Psi]$, i.e. $F(\underline{t}) = (V_k(\underline{t},\underline{x}), F(\underline{x}))_\Psi$.

From the umbral version of Fischer decomposition (4.1.8), we can decompose the reproducing kernel $V_k(\underline{t},\underline{x})$ as

$$V_k(\underline{t},\underline{x}) = \sum_{j=0}^{n}(x')^j M_{k-j}(\underline{t},\underline{x}), \quad \text{for all } \underline{t}\in\mathbb{R}^n \quad \text{with} \quad M_{k-j}(\underline{t},\underline{x}) = \pi_{\mathcal{M}_{k-j}[\Psi]}V_k(\underline{t},\underline{x}),$$

and, hence, by applying Theorem 4.2.16, linearity arguments lead to

$$F(\underline{t}) = (M_k(\underline{t},\underline{x}), F(\underline{x}))_\Psi = 2^k\frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\langle M_k(\underline{x}), F(\underline{x})\rangle_{\Psi S^{n-1}}.$$

Thus the reproducing property assures that $C_k(\underline{t},\underline{x}) = 2^k\frac{\Gamma(k+\frac{n}{2})}{\Gamma(\frac{n}{2})}\pi_{\mathcal{M}_k[\Psi]}V_k(\underline{t},\underline{x})$.

Using Theorem 4.1.15, from the property $(D')^j V_k(\underline{t},\underline{x}) = t^j V_{k-j}(\underline{t},\underline{x})$, with $t = \sum_{l=1}^n t_l \mathbf{e}_l$, we get

$$C_k(\underline{t},\underline{x}) \quad = \quad 2^k\frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\sum_{j=0}^{k} c_{j,k}(x')^j t^j V_{k-j}(\underline{t},\underline{x}),$$

where the constants are given by formulae (4.11).

■

**Theorem 4.3.3** *For each $|\underline{t}|<1$, $C(\underline{t},\underline{x})$ corresponds to*

$$C(\underline{t},\underline{x}) = \Psi_{\underline{x}}\left(\frac{\overline{x-t}}{|x-t|^{n-1}}\right),$$

*where $\frac{\overline{x-t}}{|x-t|^{n-1}}$ stands for the continuous Cauchy kernel.*

**Proof:** Like in [18], by straightforward computations we can obtain $C_k(\underline{t},\underline{x})$ in terms of Gegenbauer polynomials. In fact, since $\Psi_{\underline{x}}$ is linear and $\underline{x}\mapsto(\underline{x}\cdot\underline{t})^k$ is an homogeneous polynomial of degree $k$, for $|\underline{x}|=1$ we write

$$C_k(\underline{t},\underline{x}) = \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}-k+1\right)}|\underline{t}|^k\Psi_{\underline{x}}\left((k+n-2)C_k^{\frac{n}{2}-1}\left(\frac{\underline{x}\cdot\underline{t}}{|\underline{t}|}\right)+(n-2)\frac{x\wedge t}{|x\wedge t|}C_k^{\frac{n}{2}}\left(\frac{\underline{x}\cdot\underline{t}}{|\underline{t}|}\right)\right)$$

with $|\underline{t}|\leq|\underline{x}|=1$.

Furthermore, using similar arguments to [18], Subsection 1.6.1, the discrete Cauchy kernel is given by

$$C(\underline{t},\underline{x}) = \sum_{k=0}^{\infty}C_k(\underline{t},\underline{x}) = \Psi_{\underline{x}}\left(\frac{\overline{x-t}}{|x-t|^{n-1}}\right).$$

■

**Remark 4.3.4** *Theorems 4.3.2 and 4.3.3 provides a new way to compute discrete fundamental solutions by giving series expansions valid for all points on the grid and not only on the axis and diagonals like in [45]. Moreover by approximating the kernels of the involved discrete integral operators via the above obtained series expansions we can develop discrete multipole methods as a numerical acceleration of the discrete function theoretical methods for solving the Navier-Stokes equations and the non-linear Schrödinger equation. (see e.g. [29, 14]).*

Since the discrete Cauchy kernel can be obtained using the $k-$terms $V_k(\underline{t},\underline{x})$ of the exponential generating function, it becomes interesting at this stage to describe $V(\underline{t},\underline{x})$ in terms of the Fock space formalism presented in Section 3.2:

Formally, a generating function is an infinite linear combination of monomials, where the variables $\underline{x}$ and $\underline{t}$ are used to denote formal power series whose coefficients are polynomials. The Euclidean inner product between $\underline{t}$ and the the multivariate raising operator $x'$ can be denoted formally as $\underline{t}\cdot\underline{x}' = \sum_{j=1}^{n} t_j x_j'$.

Due to Lemma 3.2.4, the Sheffer operator $\Psi_{\underline{x}}$ intertwines the propagators $\exp(\underline{t}\cdot\underline{x})$ and $\exp(\underline{t}\cdot\underline{x}')$, i.e. $\Psi_{\underline{x}}\exp(\underline{t}\cdot\underline{x}) = \exp(\underline{t}\cdot\underline{x}')\Psi_{\underline{x}}$. So $\exp(\underline{t}\cdot\underline{x}')$ can be used to compute the exponential generating function.

In the language of representation theory, the propagator $\exp(\underline{t}\cdot\underline{x}')$ is nothing else than a representation for the Heisenberg group (c.f. [33]). So, the former approach, in the border points of combinatorics and quantum mechanics, gives us a constructive framework to develop discrete Clifford analysis on the lattice in connection with Fourier analysis on the lattice (c.f. [45, 41, 39]).

Indeed the bridge between the two approaches follows from the fact that the discrete Cauchy kernel $C(\underline{t},\underline{x})$ as well as the group characters $\underline{\xi}\mapsto exp(i\underline{\xi}\cdot\underline{t})$ possess token structure (c.f. [51]). This suggests to take $\xi\mapsto V(\underline{t},i\underline{\xi})$ as the discrete Fourier transform on the phase space (c.f. [50]).

We would also like to point out that the connection between the discrete Fourier transform with the discrete harmonic oscillator is already explored in [24] for the real line while the connection between the discrete Wigner transform and Kravchuk polynomials of discrete variable is already established in [53].

We will come to this question in Section 5.10, when we compute the eigenstates for the discrete harmonic oscillator.

## 4.4 Discrete Poincaré Lemma

In this section we will construct an alternative proof for the discrete Poincaré lemma (see [19]) based on the notion of umbral homogeneity. This approach was inspired in [10] and motivated from the correspondence between positive quasi-homogeneity and the analytic analogue of contractibility in the classical setting explored in [27].

Let us turn again our attention to the results obtained in Section 3.5. Because $\Psi_{\underline{x}}$ intertwines $E'$ (respectively, $\mathbf{i}_{\mathbf{d}_\Psi}$) and $E=\sum_{j=1}^{n} x_j\partial_{x_j}$ (respectively, $\mathbf{i}_{\mathbf{d}} = \sum_{j=1}^{n}\mathbf{d}x_j\mathbf{i}_{\partial_{x_j}}$), the polynomial differential forms $\omega_\alpha^\beta(\underline{x}) = V_\alpha(\underline{x})(\mathbf{d}_\Psi\underline{x})^\beta$ satisfy the eigenvalue properties

$$E'\omega_\alpha^\beta(\underline{x}) = |\alpha|\omega_\alpha^\beta(\underline{x}) \quad \text{and} \qquad \mathbf{i}_{\mathbf{d}_\Psi}\omega_\alpha^\beta(\underline{x}) = |\beta|\omega_\alpha^\beta(\underline{x}).$$

In other words: $E'$ measures the degree of the polynomial while $\mathbf{i}_{\mathbf{d}_\Psi}$ measures the order of a differential form. The last operator is sometimes called *fermionic* Euler operator.

So it still makes sense to speak of $r-$form which are umbral homogeneous of degree $k$. Indeed $\Lambda^r\mathcal{P}_k[\Psi] = \{\omega \in \Lambda^*\mathcal{P} \ : \ E'\omega = k\omega, \ \mathbf{i}_{\mathbf{d}_\Psi}\omega = r\omega\}$ is an eigenspace for $E'$ and $\mathbf{i}_{\mathbf{d}_\Psi}$. Furthermore we obtain the following decomposition

$$\Lambda^*\mathcal{P} = \bigoplus_{r=0}^{n}\bigoplus_{k=0}^{\infty}\Lambda^r\mathcal{P}_k[\Psi].$$

Moreover, from Lemma 3.5.1, the Lie derivative $\mathcal{L}_{E'}$ measures the total degree of homogeneity of the Clifford-valued polynomial differential forms $\omega_\alpha^\beta(\underline{x})$ since

$$\mathcal{L}_{E'}(\omega_\alpha^\beta)(\underline{x}) = (|\alpha| + |\beta|)\omega_\alpha^\beta(\underline{x}).$$

We are now in conditions to prove the discrete Poincaré lemma. This relies on the fact that from construction $\mathbf{d}_\Psi, \mathbf{i}_{\mathbf{d}_\Psi}$ are mappings between the following spaces

$$\mathbf{d}_\Psi : \Lambda^r\mathcal{P}_k[\Psi] \to \Lambda^{r+1}\mathcal{P}_{k-1}[\Psi], \quad \mathbf{i}_{\mathbf{d}_\Psi} : \Lambda^r\mathcal{P}_k[\Psi] \to \Lambda^{r-1}\mathcal{P}_{k+1}[\Psi]$$

We start to prove the discrete counterpart of the homotopy mapping property:

**Lemma 4.4.1 (Homotopy mapping property)** *There exists $\chi_\Psi : \Lambda^*\mathcal{P} \to \Lambda^*\mathcal{P}$ such that*

$$\{\mathbf{d}_\Psi, \chi_\Psi\} = \mathbf{id}.$$

**Proof:** Take $\omega_k^r \in \Lambda^r\mathcal{P}_k$, from Lemma (3.5.1) we have $\mathcal{L}_{E_\Psi}\omega_k^r = (k+r)\omega_k^r$. This shows that $\Lambda^r\mathcal{P}_k$ is the eigenspace for $\mathcal{L}_{E_\Psi}$, which for $r \geq 1$, has only positive eigenvalues.

Hence for the ansatz $\omega = \sum_{r=1}^n\sum_{k=0}^\infty \omega_k^r \in \Lambda^*\mathcal{P}$, the inverse for $\mathcal{L}_{E_\Psi} : \Lambda^*\mathcal{P} \mapsto \Lambda^*\mathcal{P}$ is given by the series expansion

$$\mathcal{L}_{E_\Psi}^{-1}\omega = \sum_{r=1}^{n}\sum_{k=0}^{\infty}\frac{1}{k+r}\omega_k^r.$$

From construction, the operator $\mathcal{L}_{E_\Psi}^{-1}$ commutes with the operators $\mathbf{d}_\Psi$ and $\mathbf{i}_{E_\Psi}$ (see relations (2.25) and (2.26) from Subsection 2.1.2). Finally, by taking $\chi_\Psi = \mathcal{L}_{E_\Psi}^{-1}\mathbf{i}_{E_\Psi}$, we thus have

$$\omega = \mathcal{L}_{E_\Psi}^{-1}\mathcal{L}_{E_\Psi}\omega = \mathcal{L}_{E_\Psi}^{-1}(\mathbf{d}_\Psi \ (\mathbf{i}_{E_\Psi}\omega) + \mathbf{i}_{E_\Psi}(\mathbf{d}_\Psi\omega)) = \mathbf{d}_\Psi(\chi_\Psi\omega) + \chi_\Psi(\mathbf{d}_\Psi\omega).$$

■

**Proposition 4.4.2 (discrete Poincaré lemma)** *Given a closed $r-$form, i.e. $\mathbf{d}_\Psi\omega^r = 0$, there exists an $(r-1)-$form such that $\omega^r = \mathbf{d}_\Psi\omega^{r-1}$.*

**Proof:** Applying Lemma 3.5.1 at the level of polynomial differential forms we have for each $\omega_k^r \in \Lambda^r\mathcal{P}_k$ the relation

$$\mathcal{L}_{E_\Psi}\omega_k^r = (k+r)\omega_k^r. \tag{4.23}$$

Thus $\Lambda^r\mathcal{P}_k$ is the eigenspace for $\mathcal{L}_{E'}$, which for $r \geq 1$, has only positive eigenvalues.

On the other hand, restricting $\omega_k^r$ to the space $\ker \mathbf{d}_\Psi$, the left hand side of (4.23) remains $\mathcal{L}_{E'}\omega_k^r = \mathbf{d}_\Psi(\mathbf{i}_{E'}\omega_k^r)$. Therefore, $\omega_{k+1}^{r-1} = \frac{1}{k+r}\mathbf{i}_{E'}\omega_k^r$ satisfies $\omega_k^r = \mathbf{d}_\Psi\omega_{k+1}^{r-1}$.

Extension of the above relation to $r-$forms $\omega^r = \sum_{k=0}^{\infty}\omega_k^r$ follows by taking $\omega^{r-1} = \mathbf{i}_{E'}\omega^r$.

■

**Remark 4.4.3** *The above lemma proves the discrete Poincaré lemma for closed Clifford-valued polynomial differential forms of degree $r \geq 1$.*

Now we study the interplay between the properties of polynomial differential forms and the interconnection structure of discrete differential forms. According to Lemma 2.2.16 ( see Chapter 2) there exists an homotopy mapping (i.e. the cocone operator) $\chi_{\mathbf{q}} : C^r(\mathcal{K},\mathbb{R}) \to C^{r-1}(\mathcal{K},\mathbb{R})$ at the level of cochains satisfying $\chi_{\mathbf{q}}\delta + \delta\chi_{\mathbf{q}} = \mathbf{id}$.

So, in order to construct $H_{\mathbf{q}} : \Lambda^r\mathcal{A}(\mathcal{M}) \to \Lambda^r\mathcal{A}(\mathcal{M})$ at the level of discrete differential forms, from Proposition 2.2.20 (see Section 2.2.2) it remains to show the intertwining relations

$$H_{\mathbf{q}}W = W\chi_{\mathbf{q}} \quad \text{and} \quad \chi_{\mathbf{q}}R = H_{\mathbf{q}}R.$$

By taking into account Lemma 4.4.1, the restriction of $\chi_\Psi = \mathcal{L}_{E'}^{-1}\mathbf{i}_{E'}$ to the $r-$chain $\sigma^r$ coincides with $H_{\mathbf{q}}$. Hence linear interpolation then makes $\chi_\Psi$ appear as a dual of the cone operator $\kappa_{\mathbf{q}}$ (see Definition 2.2.13, Subsection 2.2.1).

The next theorem shows the correspondence between discrete closed differential forms and discrete monogenic forms.

**Theorem 4.4.4** *Let $\omega^r \in \Lambda^r\mathcal{P}$ be a Clifford-valued differential $r-$form.*

1. *If $\omega^r$ is closed and discrete monogenic, then there exists a differential $r-$form $\mu^r \in \Lambda^r\mathcal{P}$ such that $\mathbf{i}_{D'}\omega^r = \mathcal{L}_{D'}\mu^r$.*

2. *If $\omega^r$ and $\mathbf{i}_{D'}(\omega^r)$ are closed differential forms, then $\omega^r$ is discrete monogenic and $\omega^r = \mathcal{L}_{D'}\mu^r$ for some $\mu^r \in \Lambda^r\mathcal{P}$ .*

**Proof:** For the proof of 1. notice that from the definition of $\mathcal{L}_{D'}$ it follows that $\mathbf{i}_{D'}\omega^r$ is closed. Hence, direct application of the discrete Poincaré lemma (Theorem 4.4.2) asserts that $\mathbf{i}_{D'}\omega^r$ is equal to $\mathbf{d}_\Psi(\mathcal{L}_{E'}^{-1}\mathbf{i}_{E'}(\mathbf{i}_{D'}\omega^r))$.

From the relations $\mathbf{d}_\Psi = -\frac{1}{2}\{\mathbf{d}_\Psi x', \mathcal{L}_{D'}\}$ and $[\mathcal{L}_{D'}, \mathbf{i}_{E'}] = \mathcal{L}_{D'}$ (see Lemmata 3.5.2 and 3.5.3 from Section 3.5, respectively) we get

$$\mathbf{i}_{D'}\omega^r = -\frac{1}{2}\mathcal{L}_{D'}(\mathbf{d}_\Psi x'(\mathcal{L}_{E'}^{-1}\mathbf{i}_{E'}(\mathbf{i}_{D'}\omega^r)).$$

The proof of statement 1. then follows by taking $\mu^r = -\frac{1}{2}\mathbf{d}_\Psi x'\mathcal{L}_{E'}^{-1}\mathbf{i}_{E'}(\mathbf{i}_{D'}\omega^r)$.

For the proof of 2. definition of $\mathcal{L}_{D'}$ asserts that $\omega^r$ is a discrete monogenic form. Moreover, the proof of assertion 2. then follows by the same order of ideas of the above proof.

■

**Remark 4.4.5** *The homotopy mapping $\chi_\Psi = \mathcal{L}_{E'}^{-1}\mathbf{i}_{E'}$ used in the proof of the above theorems is defined in a natural way by using the language of Clifford differential forms introduced in Section 3.5.*

In summary, we have given an alternative framework to prove the discrete Poincaré lemma. More specifically, the introduction of a shift-invariant exterior calculus allows one to construct the homotopy mapping that can be defined in a very natural way by combining the Clifford setting with the shift-invariant exterior calculus developed in Section 3.2.

The usefulness of this construction stems from the fact the mimetic transcription of the homotopy mapping can be viewed as the pairing between Clifford-valued polynomial differential forms and star-shaped complexes and from the action of the Sheffer operator $\Psi_{\underline{x}}$. This give us in a constructive way a contact with a De Rham cohomology on a singular set (c.f. [27, 19]).

However, while the classical proof relies heavily on the use of integration by parts which in turn relies the Leibniz rule on vector calculus, our approach combines the languages of Quantum Field Theory (c.f. [58]) with Algebraic Topology (c.f. [57]) and, hence, the lack of the Leibniz rule is not an obstacle for our approach. Indeed the interplay between both languages in terms of spectral triples was recently explored in [16] using the framework of (metric) noncommutative geometry (see Chapter 10, pp. 178-196).

# Chapter 5

# The Discrete Harmonic Oscillator

"*To know the road ahead, ask those coming back.*"
attributed to Confucius

In this section we will establish the bridge between Discrete Clifford Analysis and the physical model of the discrete harmonic oscillator. The approach that we will consider is motivated by the observation that classes of Wigner Quantum systems are described by means of the Lie superalgebra $osp(1|2n)$ (c.f. [35]) and from the fact that Clifford Analysis in its minimal form corresponds to a realization of the Lie superalgebra osp(1|2) (c.f. [18]).

## 5.1 The Quantum Harmonic Oscillator

The Hamiltonian of the system consists of a chain of $n$ independent one-dimensional harmonic oscillators, each having the same mass $m$ and frequency $f$, i.e.

$$\mathcal{H}=\sum_{j=1}^{n}-\frac{1}{2m}(\mathbf{a}_j)^2+\frac{mf^2}{2}(\mathbf{a}_j^\dagger)^2. \tag{5.1}$$

where $\mathbf{a}_j^\dagger$ and $\mathbf{a}_j$ are position and momentum operators, respectively, corresponding to the canonical generators of the Bose algebra (see (3.10), Subsection 3.2). The terms $-\frac{1}{2m}(\mathbf{a}_j)^2$ represents the kinetic energy of the one-dimensional Harmonic Oscillator while the term $\frac{mf^2}{2}(\mathbf{a}_j^\dagger)^2$ represents the potential energy.

Such system based upon the compatibility of the Hamiltonian with the infinite-dimensional Fock space is a particular case of a Wigner quantum system [71]. The energy levels $\epsilon$ and and the corresponding energy eigenstates $F(\underline{x})$ are described by the solution of the time-independent Schrödinger equation,

$$\mathcal{H}F(\underline{x}) = \epsilon F(\underline{x}). \tag{5.2}$$

In order to solve the above equation, we build up the so called creation and annihilation operators of the Bose algebra (see (3.10), Section 3.2). These are defined by

$$\mathbf{a}_j^{\pm} = \sqrt{\frac{mf}{2}}\left(\mathbf{a}_j^{\dagger} \mp \frac{1}{mf}\mathbf{a}_j\right).$$

Indeed, the operators $\mathbf{a}_j^{\pm}$ satisfy the commutation relations

$$[\mathbf{a}_j^{\pm}, \mathbf{a}_k^{\pm}] = 0, \quad [\mathbf{a}_j^{-}, \mathbf{a}_k^{+}] = \delta_{jk}\mathbf{id}.$$

So we can split the Hamiltonian $\mathcal{H}$ as

$$\mathcal{H} = E^{+-} + \frac{n}{2}\mathbf{id}, \tag{5.3}$$

where $E^{+-} := \sum_{j=1}^n \mathbf{a}_j^{+}\mathbf{a}_j^{-}$ denotes the Hamiltonian of a field of free (non-interacting) bosons.

Furthermore assuming that $\mathbf{a}_j^{+}$ and $\mathbf{a}_j^{-}$ are *raising* and *lowering* operators, respectively, the Hamiltonian operator $E^{+-}$ plays the same role as the discrete Euler operator (see (3.33), Subsection 3.4). Thus, the problem of finding the energy levels and the energy eigenstates *viz* equation (5.2) is analogous to the problem of determining basic polynomial sequences as described in Chapter 1.

## 5.2 Superalgebra Representation

We first review the classical Harmonic Oscillator and take a look a the discrete harmonic oscillator afterwards. For a sake of simplicity we consider the case $m = f = 1$. Put

$$\mathcal{H} = \frac{1}{2}(-\Delta + |\underline{x}|^2)$$

and

$$\mathbf{a}_j^{\pm} = \frac{1}{\sqrt{2}}(x_j \mp \partial_{x_j}). \tag{5.4}$$

For the standard Dirac operator $D = \sum_{j=1}^n \mathbf{e}_j\partial_{x_j}$ and coordinate variable $x = \sum_{j=1}^n \mathbf{e}_j x_j$, we set

$$D_{\pm} = \frac{1}{\sqrt{2}}(x \mp D) = \sum_{j=1}^n \mathbf{e}_j\mathbf{a}_j^{\pm}.$$

Hence, the operators $D_\pm$ and $\mathcal{H}$ are related by the anti-commuting relations

$$\{D_\pm, D_\pm\} = 2D_\pm^2 \quad , \{D_+, D_-\} = -2\mathcal{H}. \tag{5.5}$$

As in Lemma 3.4.3, the proof of (5.5) follows straightforward from the Bose algebra representation of (5.3).

The next lemma describes the operators $D_\pm$ and $\mathcal{H}$ by means of representations of a Lie superalgebra:

**Lemma 5.2.1** *The operators $D_\pm$ and $\mathcal{H}$ generate a finite-dimensional Lie superalgebra in End($\mathcal{P}$). The remaining commutation relations are*

$$\left[D_+, -D_+^2\right] = 0, \quad \left[D_+, -D_-^2\right] = -2D_-, \quad [\mathcal{H}, D_+] = D_+ \tag{5.6}$$

$$\left[D_-, -D_+^2\right] = 2D_+, \qquad \left[D_-, -D_-^2\right] = 0, \qquad [\mathcal{H}, D_-] = -D_- \tag{5.7}$$

$$\left[D_+^2, D_-^2\right] = 4\mathcal{H}, \quad \left[\mathcal{H}, -D_+^2\right] = -2D_+^2, \quad \left[\mathcal{H}, D_-^2\right] = -2D_-^2 \tag{5.8}$$

**Proof:** Notice that the relations $\left[D_-, -D_-^2\right] = 0 = \left[D_+, -D_+^2\right]$ are then fulfilled since $D_\pm^2$ commutes with all elements of End($\mathcal{P}$) (first relation of (5.5)).

The proof of $[\mathcal{H}, D_+] = D_+$ and $[\mathcal{H}, D_-] = -D_-$ follows from the relations (5.3) as in Lemma 3.4.3. Straightforward application of the above relations naturally lead to

$$\left[\mathcal{H}, D_+^2\right] = (D_+ + D_+\mathcal{H})D_+ - D_+(-D_+ + \mathcal{H}D_+) = 2D_+^2,$$
$$\left[\mathcal{H}, D_-^2\right] = (-D_- + D_-\mathcal{H})D_- - D_-(D_- + \mathcal{H}D_-) = -2D_-^2.$$

Furthermore the relations $[\mathcal{H}, D_+] = D_+$, $[\mathcal{H}, D_-] = -D_-$ together with the anti-commuting relation (5.5) lead to

$$\begin{aligned} D_-D_+^2 - D_+^2D_- &= (-2\mathcal{H} - D_+D_-)D_+ - D_+(-2\mathcal{H} - D_-D_+) \\ &= -2[\mathcal{H}, D_+] \\ &= -2D_+ \end{aligned}$$

$$\begin{aligned} D_+D_-^2 - D_-^2D_+ &= (-2\mathcal{H} - D_-D_+)D_- - D_-(-2\mathcal{H} - D_+D_-) \\ &= -2[\mathcal{H}, D_-] \\ &= 2D_- \end{aligned}$$

Finally, the combination of one of the above relations with the anti-commuting relation leads to

$$D_-^2D_+^2 = D_-(-2D_+ + D_+^2D_-) = -2\{D_+, D_-\} + D_+^2D_-^2 = 4\mathcal{H} + D_+^2D_-^2.$$

$\blacksquare$

From the set of relations obtained in Lemma 5.2.1, there is such a kind of Clifford Analysis framework involving the operators $D_\pm$ since

$$\mathrm{span}\left\{\frac{1}{2}D_-^2, \frac{1}{2}D_+^2, \mathcal{H}\right\} \oplus \mathrm{span}\left\{D_-, D_+\right\}$$

equipped with the standard graded commutator $[\cdot,\cdot]$ is a isomorphic to a Lie superalgebra of type osp(1|2) (see e.g. [35]). Indeed the normalization

$$\mathbf{p}^- = \tfrac{1}{2}D_-^2, \quad \mathbf{p}^+ = -\tfrac{1}{2}D_+^2, \quad \mathbf{q} = \tfrac{1}{2}\mathcal{H} \quad \text{and} \quad \mathbf{r}^\pm = \tfrac{1}{2\sqrt{2}}iD_\pm$$

leads to the standard commutation relations for osp(1|2) (see e.g. [35]).

$$\begin{aligned}
\left[\mathbf{q}, \mathbf{p}^\pm\right] &= \pm\mathbf{p}^\pm, & [\mathbf{p}^+, \mathbf{p}^-] &= 2\mathbf{q},\\
\left[\mathbf{q}, \mathbf{r}^\pm\right] &= \pm\frac{1}{2}\mathbf{p}^\pm, & [\mathbf{r}^+, \mathbf{r}^-] &= \tfrac{1}{2}\mathbf{q},\\
\left[\mathbf{p}^\pm, \mathbf{r}^\mp\right] &= -\mathbf{r}^\pm, & [\mathbf{r}^\pm, \mathbf{r}^\pm] &= \pm\tfrac{1}{2}\mathbf{p}^\pm.
\end{aligned}$$

In particular, $\mathbf{p}^\pm = \frac{1}{2}D_\pm^2$, and $\mathbf{q} = \mathcal{H}$ are the canonical generators of the Lie algebra $sl_2(\mathbb{R})$. Indeed, they satisfy the graded commutation relations

$$\left[\mathbf{p}^-, \mathbf{p}^+\right] = \mathbf{q}, \quad [\mathbf{q}, \mathbf{p}^+] = \mathbf{p}^+, \quad \left[\mathbf{q}, \mathbf{p}^-\right] = -\mathbf{p}^-.$$

and hence we have a representation of harmonic analysis for the continuous Harmonic Oscillator (see e.g. [46]).

Now we turn our attention to the discrete Harmonic Oscillator written in terms of the language of Discrete Clifford Analysis. This corresponds to the operator

$$\mathcal{H}' = \frac{1}{2}\left(\ (D')^2 - (x')^2\ \right),$$

where $D'$ and $x'$ are discrete Dirac operator and the discrete coordinate variable introduced in Section 3.4. Next we put

$$D'_\pm := \frac{1}{\sqrt{2}}(x' \mp D'). \tag{5.9}$$

The representations of the Lie superalgebra osp(1|2) and the Lie algebra $\mathrm{sl}_2(\mathbb{R})$ also fulfil in the discrete Clifford setting. Indeed, if we consider the Sheffer operator $\Psi_{\underline{x}}$ defined in Definition 3.2.1, the discrete operators

$$\frac{1}{\sqrt{2}}(x'_j \mp Q_{x_j}) = \Psi_{\underline{x}}\mathbf{a}_j^\pm\Psi_{\underline{x}}^{-1}$$

preserve the commutation relations (5.3). The relations (5.5)-(5.8) can be lifted for the operators $D'_\pm$, $(D'_\pm)^2$ and $\mathcal{H}'$. This correspond to the following Proposition:

**Proposition 5.2.2** *The operators $D'_\pm$ and $\mathcal{H}'$ generate a finite-dimensional Lie superalgebra in $End(\mathcal{P})$ isomorphic to $osp(1|2)$ The remaining commutation relations are*

$$\begin{array}{ccc} \left[D'_+, (D'_+)^2\right] = 0, & \left[D'_+, (D'_-)^2\right] = 2D'_-, & [\mathcal{H}', D'_+] = D'_+ \\ \left[D'_-, (D'_+)^2\right] = 2D'_+, & \left[D'_-, (D'_-)^2\right] = 0, & [\mathcal{H}', D'_-] = -D'_- \\ \left[(D'_+)^2, (D'_-)^2\right] = 4\mathcal{H}', & \left[\mathcal{H}', (D'_+)^2\right] = 2(D'_+)^2, & \left[\mathcal{H}', (D'_-)^2\right] = -2(D'_-)^2. \end{array}$$

The proof of the above proposition relies on the relations $D'_\pm = \Psi_{\underline{x}} D_\pm \Psi_{\underline{x}}^{-1}$ and $\mathcal{H}' = \Psi_{\underline{x}} \mathcal{H} \Psi_{\underline{x}}^{-1}$.

## 5.3 Spectrum of the Discrete Hamiltonian

In order to analyze the spectrum of the discrete Hamiltonian operator $\mathcal{H}'$, we further consider the auxiliar operator

$$\mathcal{J}' = -\frac{\Delta'}{2} + E' + \frac{n}{2}\mathbf{id}$$

acting on the space $\mathcal{P}$. The subsequent theorem gives a complete description of the spectral properties of $\mathcal{H}'$ and $\mathcal{J}'$ and sifts the well know facts from the classical theory to the Umbral setting, and moreover to the discrete setting as a whole (see e.g. [33]).

We start with the following lemma:

**Lemma 5.3.1** *The operators $\mathcal{J}', \mathcal{H}' \in End(\mathcal{P})$ are related by*

$$\mathcal{H}' = \exp\left(-\frac{(x')^2}{2}\right) \mathcal{J}' \exp\left(\frac{(x')^2}{2}\right).$$

**Proof:** Following the same order of ideas as in Lemma 5.2.1, it follows from the relations

$$\{x', D'\} = -2E' - n\mathbf{id}, \quad [\Delta', x'] = 2D', \quad [E', x'] = x' \quad \text{and} \quad [E', D'] = -D'$$

(see Lemmata 3.4.3 and 4.2.6, Section 3.4) that $-\frac{\Delta'}{2}$, $\frac{(x')^2}{2}$ and $E' + \frac{n}{2}\mathbf{id}$ are the canonical generators of $sl_2(\mathbb{R})$, i.e.

$$\left[-\frac{\Delta'}{2}, \frac{(x')^2}{2}\right] = E' + \frac{n}{2}\mathbf{id}, \quad \left[E' + \tfrac{n}{2}\mathbf{id}, -\tfrac{\Delta'}{2},\right] = \tfrac{\Delta'}{2}, \quad \left[E' + \frac{n}{2}\mathbf{id}, \frac{(x')^2}{2}\right] = \frac{(x')^2}{2}.$$

From the above relations, it follows *viz* induction over $k$ that

$$\left[-\frac{\Delta'}{2}, \left(\frac{(x')^2}{2}\right)^k\right] = -k\left(\frac{(x')^2}{2}\right)^{k-1}\left(E' + \frac{n+k-1}{2}\mathbf{id}\right)$$

and therefore $\left[-\frac{\Delta'}{2}, \exp\left(-\frac{(x')^2}{2}\right)\right] = \exp\left(-\frac{(x')^2}{2}\right)\left(E' + \frac{n}{2}\mathbf{id}\right) + \frac{(x')^2}{2}\exp\left(-\frac{(x')^2}{2}\right)$.

Multiplying both sides by $\exp\left(\frac{(x')^2}{2}\right)$ on the right and after straightforward simplifications we immediately get $\mathcal{H}' = \exp\left(\frac{(x')^2}{2}\right)\mathcal{J}'\exp\left(-\frac{(x')^2}{2}\right)$.

■

**Theorem 5.3.2** *For $k$ integer, we define*

$$\begin{aligned} \mathcal{P}_k^{\Delta'}[\Psi] &= \left\{\exp\left(-\frac{\Delta'}{2}\right)P_k(\underline{x}) \ : \ P_k(\underline{x}) \in \mathcal{P}_k[\Psi]\right\}, \\ \mathcal{P}_k^{+-}[\Psi] &= \left\{\exp\left(\frac{(x')^2}{2}\right)P_k(\underline{x}) \ : \ P_k(\underline{x}) \in \mathcal{P}_k^{\Delta'}[\Psi]\right\}. \end{aligned}$$

*Then the polynomial space $\mathcal{P}$ admits the decompositions*

$$\mathcal{P} = \sum_{k=0}^{\infty} \bigoplus \mathcal{P}_k^{\Delta'}[\Psi] = \sum_{k=0}^{\infty} \bigoplus \mathcal{P}_k^{+-}[\Psi].$$

*Hereby $\mathcal{P}_k^{\Delta'}[\Psi]$ and $\mathcal{P}_k^{+-}[\Psi]$ are eigenspaces for $\mathcal{J}'$ and $\mathcal{H}'$, respectively, corresponding to the eigenvalue $k + \frac{n}{2}$.*

**Proof:** Recall that $-\frac{\Delta'}{2}$, $\frac{(x')^2}{2}$ and $E' + \frac{n}{2}\mathbf{id}$ are the canonical generators of the Lie algebra $sl_2(\mathbb{R})$ (see proof of Lemma 5.3.1). Induction over $k$ yield the commuting relation

$$\left[E' + \frac{n}{2}\mathbf{id}, \left(\frac{\Delta'}{2}\right)^k\right] = -k\left(\frac{\Delta'}{2}\right)^k,$$

and, hence, $\left[E' + \frac{n}{2}\mathbf{id}, \exp\left(-\frac{\Delta'}{2}\right)\right] = \frac{\Delta'}{2}\exp\left(-\frac{\Delta'}{2}\right)$.

Next for $P_k(\underline{x}) \in \mathcal{P}_k[\Psi]$ we put $R_k(\underline{x}) = \exp(\frac{\Delta'}{2})P_k(\underline{x})$. Then it follows that

$$\left(k + \frac{n}{2}\right)P_k(\underline{x}) = \left(E' + \frac{n}{2}\mathbf{id}\right)\left(\exp\left(-\frac{\Delta'}{2}\right)R_k(\underline{x})\right) = \exp\left(-\frac{\Delta'}{2}\right)E'R_k(\underline{x}) + \frac{\Delta'}{2}P_k(\underline{x}).$$

With the above arguments, the equation $\mathcal{J}'R_k(\underline{x}) = \epsilon R_k(\underline{x})$ is equivalent to

$$\left(E' + \frac{n}{2}\mathbf{id}\right)P_k(\underline{x}) = \epsilon P_k(\underline{x}).$$

This leads to $\epsilon = k + \frac{n}{2}$ and, hence, it immediately follows that

$$\mathcal{P} = \sum_{k=0}^{\infty} \bigoplus \mathcal{P}_k^{\Delta'}[\Psi].$$

The statements for $\mathcal{H}'$ are then immediate by the previous lemma.

■

**Remark 5.3.3** *Part of the above theorem implies that the operator $\mathcal{J}'$ has a unique polynomial eigenfunction of the form $R_k(\underline{x}) = P_k(\underline{x}) + S(\underline{x})$, where the degree of $S(\underline{x})$ is strictly less than $k$ and it is given by $R_k(\underline{x}) = \exp\left(-\frac{\Delta'}{2}\right) P_k(\underline{x})$.*

*On the other hand, the other part shows that spectrum of $\mathcal{H}'$ corresponds to the set of values $\epsilon_k = k + \frac{n}{2}$ with $k \in \mathbb{N}_0$ giving $\frac{n}{2}$ as the minimum eigenvalue for $\mathcal{H}'$. Physically speaking, this is nothing else than the Planck's radiation law, with the correction $\frac{n}{2}$ giving $\frac{n}{2}$ for the energy of the ground state $\Psi(\underline{x}) = V(-\underline{x}', \underline{x})\mathbf{1}$. (see also end of Section 3.4).*

Using the above theorem allows us that we can recast the integral representation for the Umbral Fischer inner product $(\cdot,\cdot)_\Psi$ (Theorem 4.2.16, Subsection 4.2.2) in terms of elements of the eigenspace $\mathcal{P}_k^{\Delta'}[\Psi]$. This corresponds to the following theorem:

**Theorem 5.3.4** *For two polynomials $P, Q \in \mathcal{P}$, we have*

$$\begin{aligned}(P(\underline{x}), Q(\underline{x}))_\Psi &= \left\langle \exp\left(-\frac{\Delta'}{2}\right) P(\underline{x}), \exp\left(-\frac{\Delta'}{2}\right) Q(\underline{x}) \right\rangle_\Psi \\ &= 2^k \frac{\Gamma\left(k+\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \left\langle \exp\left(-\frac{\Delta'}{2}\right) P(\underline{x}), \exp\left(-\frac{\Delta'}{2}\right) Q(\underline{x}) \right\rangle_{\Psi S^{n-1}}.\end{aligned}$$

**Proof:** By Theorem 4.1.9, it is enough to show the identity for $P, Q \in \mathcal{P}$ of the form

$$P(\underline{x}) = (x')^{2s} P_{k-2s}(\underline{x}), \quad Q(\underline{x}) = (x')^{2s} Q_{k-2s}(\underline{x}) \quad \text{with} \quad P_{k-2s}(\underline{x}), Q_{k-2s}(\underline{x}) \in \mathcal{H}_{k-2s}[\Psi].$$

By recursive application of Proposition 4.1.6 (see Subsection 4.1.1), the operators $(x')^{2j}$ and $(-\Delta')^j$ are dual with respect to the Umbral Fischer inner product $(\cdot,\cdot)_\Psi$. Hence direct application of Theorem 4.1.16, Subsection 4.1.2 leads to

$$(P(\underline{x}), Q(\underline{x}))_\Psi = 4^s\ s! \left(k - 2s + \frac{n}{2}\right)_s (P_{k-2s}(\underline{x}), Q_{k-2s}(\underline{x}))_\Psi.$$

Finally by applying Theorem 4.2.16, Subsection 4.2.2, the statement $(P(\underline{x}), Q(\underline{x}))_\Psi$ is then immediate.

■

With the above corollary we establish a contact with the MacDonald formula. Contrary to the classical approach, the spectrum of the operators $\mathcal{J}'$ and $\mathcal{H}'$ is determined upon the action of the reproducing kernel for the Umbral Fischer inner product (see Proposition 4.3.1, Section 4.3) and *a priori* no knowledge about special functions is required.

We would like to remark that extension of the above result can be easily extended to the superspace [17]. In particular, this allows us to define formally integration on the superspace as the recipe equivalent to the Berezin integral.

## 5.4 Discrete Clifford-Hermite Functions

We will finish this chapter by determining the discrete counterparts of the Clifford-Hermite functions in a combinatorial way by using the Fock space representation of the Bose algebra. From Lemma 3.2.4 (Section 3.2) and equation (5.3) amounts the study the ladder structure of the *raising* and *lowering* operators $\frac{1}{\sqrt{2}}(x_j' \mp Q_{x_j})$, respectively.

Using the operators defined in (5.4) and Lemma 3.2.4, we see that $\frac{1}{\sqrt{2}}(x_j' + Q_{x_j})$ annihilates the function $\Phi(\underline{x}) = \exp\left(\frac{(x')^2}{2}\right)\mathbf{1} = V(-\underline{x}', \underline{x})\mathbf{1}$, where $V(\cdot,\cdot)$ denotes the generating function for $Q_{\underline{x}}$ (see Corollary 1.1.15, Section 1.1).

Using the Quantum Field Lemma (see Section 3.2), the corresponding basis of polynomials is given by

$$W_\alpha(\underline{x}) = \prod_{j=1}^{n} \left(\frac{1}{\sqrt{2}}(x_j' - Q_{x_j})\right)^{\alpha_j} \Phi(\underline{x}) \tag{5.10}$$

For these multivariate polynomials the following raising and lowering properties then follow

$$\frac{1}{\sqrt{2}}(x_j' - Q_{x_j})W_\alpha(\underline{x}) = W_{\alpha+\mathbf{v}_j}(\underline{x}), \quad \frac{1}{\sqrt{2}}(x_j' + Q_{x_j})W_\alpha(\underline{x}) = \alpha_j W_{\alpha-\mathbf{v}_j}(\underline{x}). \tag{5.11}$$

These relations are nothing else than the discrete counterpart of *raising* and *lowering* properties for the Hermite functions, respectively (c.f. [33], pages 51-55).

Incidentally using Theorem 4.2.12 (see Subsection 4.2.2), the dual of $Q_{x_j}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\Psi$ nothing else than the *raising* operator $x_j' - Q_{x_j}$. So, from the intertwining property given in Lemma 3.2.4 (see Section 3.2) the computation of the discrete Hermite functions (5.10) and, moreover, the span of $\mathcal{P}_k^{+-}[\Psi]$ as a whole involves the exponentiation operator $\exp\left(\frac{(x')^2}{2}\right)$. This corresponds to the following theorem:

**Theorem 5.4.1** *Every $R_k(\underline{x}) = \exp\left(-\frac{\Delta'}{2}\right) P_k(\underline{x}) \in \mathcal{P}_k^{\Delta'}[\Psi]$ corresponds to*

$$R_k(\underline{x}) = \exp\left(\frac{(x')^2}{2}\right) (\Psi_{\underline{x}}^{-1} P_k)(\underline{x}' - Q_{\underline{x}})\Phi(\underline{x}).$$

*Moreover, every $R_k(\underline{x}) = \exp\left(\frac{(x')^2}{2}\right) \exp\left(-\frac{\Delta'}{2}\right) P_k(\underline{x}) \in \mathcal{P}_k^{+-}[\Psi]$ is of the form*

$$R_k(\underline{x}) = \left(\sqrt{2}\right)^k (\Psi_{\underline{x}}^{-1} P_k) \left(\frac{1}{\sqrt{2}}(\underline{x}' - Q_{\underline{x}})\right) \Phi(\underline{x}).$$

**Proof:** Recall that $x_j' - Q_{x_j}$ is the dual of $Q_{x_j}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\Psi$. (see Theorem 4.2.12, Subsection 4.2.2).

On the other hand, using the Bosonic character of the operators $Q_{x_j}$ and $x'_j$ (see Lemma 3.2.4, Subsection 3.2) we can recast the operator $x'_j - Q_{x_j}$ as

$$x'_j - Q_{x_j} = -\exp\left(\frac{(x')^2}{2}\right) Q_{x_j} \exp\left(-\frac{(x')^2}{2}\right).$$

Thus our assertion is therefore equivalent to

$$\exp\left(-\frac{\Delta'}{2}\right) P_k(\underline{x}) = (\Psi_{\underline{x}}^{-1} P_k)(\underline{x}' - Q_{\underline{x}})\mathbf{1}. \tag{5.12}$$

This identity is now easily checked by induction with respect to the degree $k$. The case $k = 0$ is clear. Moreover, if (5.12) holds, the relation $\left[x'_j, \exp\left(-\frac{\Delta'}{2}\right)\right] = Q_{x_j} \exp\left(-\frac{\Delta'}{2}\right)$ leads to

$$(x'_j - Q_{x_j})(\Psi_{\underline{x}}^{-1} P_k)(\underline{x}' - Q_{\underline{x}})\mathbf{1} = (x'_j - Q_{x_j}) \exp\left(-\frac{\Delta'}{2}\right) P_k(\underline{x}) = \exp\left(-\frac{\Delta'}{2}\right)(x'_j P_k(\underline{x})).$$

Then it follows that $R_k(\underline{x}) = (\Psi_{\underline{x}}^{-1} P_k)(\underline{x}' - Q_{\underline{x}})\Phi(\underline{x})$.

To describe the elements $R_k(\underline{x}) \in \mathcal{P}_k^{+-}[\Psi]$, we write

$$R_k(\underline{x}) = \exp\left(\frac{(x')^2}{2}\right)(\Psi_{\underline{x}}^{-1} P_k)(-Q_{\underline{x}}) \exp\left(-\frac{(x')^2}{2}\right)\Phi(\underline{x}).$$

From the relation $x'_j - Q_{x_j} = -\exp\left(\frac{(x')^2}{2}\right) Q_{x_j} \exp\left(-\frac{(x')^2}{2}\right)$ we have

$$\begin{aligned} \exp\left(\frac{(x')^2}{2}\right)(\Psi_{\underline{x}}^{-1} P_k)(-Q_{\underline{x}}) \exp\left(-\frac{(x')^2}{2}\right)\Phi(\underline{x}) &= (\Psi_{\underline{x}}^{-1} P_k)(\underline{x}' - Q_{\underline{x}})\Phi(\underline{x}) \\ &= \left(\sqrt{2}\right)^k (\Psi_{\underline{x}}^{-1} P_k)\left(\frac{1}{\sqrt{2}}(\underline{x}' - Q_{\underline{x}})\right)\Phi(\underline{x}). \end{aligned}$$

This yields the desired assertion.

■

Now we have the key ingredients to formulate the Fischer-type decomposition for the discrete Harmonic Oscillator.

We denote by $\mathcal{M}_k^-[\Psi] = \mathcal{P}_k^{+-}[\Psi] \cap \ker D'_-$. Starting from Lemma 5.2.1 and using the same methods as in [11], we get the following result

**Theorem 5.4.2** *The space of Clifford-valued polynomials $\mathcal{P}$ decompose as $Pin(n) \times osp(1|2)-$module to the direct sum*

$$\oplus_{k=0}^{\infty} \oplus_{j=0}^{\infty} (D'_+)^j \mathcal{M}_k^-[\Psi]$$

*This decomposition has a form of an infinite triangle*

$$\begin{array}{cccccccc}
\mathcal{P}_0^{+-}[\Psi] & & \mathcal{P}_1^{+-}[\Psi] & & \mathcal{P}_2^{+-}[\Psi] & & \mathcal{P}_3^{+-}[\Psi] & \dots \\
 & & & & & & & \\
\mathcal{M}_0^-[\Psi] & \xleftarrow{D'_-} & D'_+\mathcal{M}_0^-[\Psi] & \xleftarrow{D'_-} & (D'_+)^2\mathcal{M}_0^-[\Psi] & \xleftarrow{D'_-} & (D'_+)^3\mathcal{M}_0^-[\Psi] & \dots \\
 & & \oplus & & \oplus & & \oplus & \\
 & & \mathcal{M}_1^-[\Psi] & \xleftarrow{D'_-} & D'_+\mathcal{M}_1^-[\Psi] & \xleftarrow{D'_-} & (D'_+)^2\mathcal{M}_1^-[\Psi] & \dots \\
 & & & & \oplus & & \oplus & \\
 & & & & \mathcal{M}_2^-[\Psi] & \xleftarrow{D'_-} & D'_+\mathcal{M}_2^-[\Psi] & \dots \\
 & & & & & & \oplus & \\
 & & & & & & \mathcal{M}_3^-[\Psi] & \dots
\end{array} \tag{5.13}$$

In the above diagram, all the summands in the same row are isomorphic to $Pin(n)-$modules and each row is an irreducible module for the Howe dual pair $Pin(n) \times \mathrm{osp}(1|2)$. The operator $D'_-$ behaves as the discrete Dirac operator $D'$, shifts all the spaces in the same row to the left, the multiplication by $D'_+$ behaves as the discrete coordinate variable $x'$ shifts them to both these actions are isomorphisms of $Pin(n)-$modules. Similarly as for discrete harmonic polynomials, any spinor-valued polynomial can be uniquely written as a sum of null solutions of $D'_-$ and a products of powers of $D'_+$.

Consequently, we get the following Fischer-type decomposition involving the operators $D'_\pm$

**Theorem 5.4.3 (Fischer-type decomposition for the Umbral Harmonic Oscillator)** *We have*

$$\mathcal{P}_k^{+-}[\Psi] = \bigoplus_{j=0}^{k} (D'_+)^j \mathcal{M}_{k-j}^-[\Psi].$$

**Remark 5.4.4** *In the last theorem, the Fischer decomposition results as a direct consequence of the Howe dual pair technique (c.f. [46, 11]).*

Using the same kind of arguments as in Subsection 4.1.2, it is also possible to explicitly determine the Fischer-type decomposition for the subspaces $\mathcal{P}_k^{+-}[\Psi]$. Motivated by Lemma 4.1.15, this amounts to constructing projection operators $\pi_{\mathcal{M}_k^-[\Psi]}$ from $\mathcal{P}_k^{+-}[\Psi]$ to $\mathcal{M}_k^-[\Psi]$ satisfying the property

$$\pi_{\mathcal{M}_k^-[\Psi]}((D'_-)^k M_{l-k}^-(\underline{x})) = \delta_{jk} M_{l-k}^-(\underline{x}), \quad \text{for all } M_{l-k}^-(\underline{x}) \in \mathcal{M}_{k-l}^-[\Psi]. \tag{5.14}$$

From the relations (5.5) and Lemma 5.2.1, after straightforward computations the operator $\pi_{\mathcal{M}_k^-[\Psi]}$ is explicitly given by

$$\pi_{\mathcal{M}_k^-[\Psi]}F(\underline{x}) = \sum_{j=1}^{k} c_{j,k}(D'_+)^j(D'_-)^j F(\underline{x}), \tag{5.15}$$

and, moreover, the individual components are given by

$$M_{k-s}^-(\underline{x}) = c'_{k,s}\sum_{j=0}^{k-s} c_{j,k-s}(D'_+)^j(D'_-)^{j+s}P_k^{+-}(\underline{x}), \quad \text{with } P_k^{+-} \in \mathcal{P}_k^{+-}[\Psi]. \tag{5.16}$$

The proof of formulae (5.15) and (5.16) follow the same line as the proofs of Lemma 4.1.15 and Theorem 4.1.16, respectively.

The above construction allow us to to determine formally the so-called discrete Clifford-Hermite functions. In fact, the operators $D'_\pm$ are linear combinations of raising and lowering operators of the form $\frac{1}{\sqrt{2}}\left(x'_j \mp Q_{x_j}\right)$. So the Clifford extension of the recursive relations (5.11) becomes then

$$D'_+W_k(\underline{x}) = W_{k+1}(\underline{x}), \quad D'_-W_k(\underline{x}) = kW_{k-1}(\underline{x}). \tag{5.17}$$

By means of the linear extension of Lemma 4.2.12 (Subsection 4.2.2), $D'_+$ appears as the dual of the discrete Dirac operator $D'$ with respect to the inner product $(\cdot,\cdot)_\Psi$.

Moreover, combining *Rodrigues* formula

$$(D'_+)^k = (-1)^k \exp\left(\frac{(x')^2}{2}\right)(D')^k \exp\left(-\frac{(x')^2}{2}\right)$$

(see Lemma 3.2.4, Subsection 3.2) with Theorems 4.2.10 and 4.2.16, the spaces $\mathcal{P}_l^{+-}[\Psi]$ are mutually orthogonal with respect to $(\cdot,\cdot)_\Psi$ (c.f. Lemma 4.1.4, Subsection 4.1.1) while for $l < k$, the spaces $\mathcal{M}_k^-[\Psi]$ and $\mathcal{P}_l^{+-}[\Psi]$ are orthogonal with respect to $(\cdot,\cdot)_\Psi$ (c.f. Corollary 4.2.11, Subsection 4.2.2). So Theorem 5.4.3, and moreover, relations (5.15) and (5.16) are the building blocks for the discrete Clifford-Hermite functions as solutions of the discrete counterpart of stationary Schrödinger equation 5.2 (see Example 3.3.14, Subsection 3.3.2).

We will end this chapter with important remarks concerns the Gauged version of Harmonic Oscillator written in terms of the operators described in the Example 3.3.14, Subsection 3.3.2

**Remark 5.4.5 (The gauged version of the Discrete Harmonic Oscillator)** *The Gauged version of the Hamiltonian operator described in Example 3.3.14 (see also Example 1.2.10,* *Section 1.2) is local operator is defined on the uniform lattice with mesh-width $h$ and corresponds to a splitting in terms of 'quasi-local' operators. So, we can split the eigenvectors of $\mathcal{H}'$ in terms of null solutions for the operator $(D_{h/2})_-$.*

*We would like to emphasize that the corresponding eigenfunctions of $\mathcal{H}'$ are closely related with the Kravchuk polynomials of discrete variable (see e.g. [48, 53]).*

In summary, we shown that the discrete version of Clifford-Hermite polynomials are eigenstates for the discrete Harmonic oscillator.

Such kind of eigenstates can be constructed by means of the inverse of the Bargmann transform (c.f. [15]). In fact, when restricted to the space of polynomials, the Bargmann transform coincides with the differential operator $\exp\left(\frac{\Delta'}{2}\right)$ of infinite degree (c.f. [33]), and hence, the Wick operator $\exp\left(-\frac{\Delta'}{2}\right)$ (i.e. the inverse of the Bargmann transform) have the mapping property

$$\exp\left(-\frac{\Delta'}{2}\right) : \mathcal{P}_k[\Psi] \to \mathcal{P}_k^{\Delta'}[\Psi].$$

One way to substantiate these claims is to consider the differential-difference heat equation

$$\partial_s F(s,\underline{x}) = \frac{1}{2}\Delta' F(s,\underline{x}) \tag{5.18}$$

with initial value $F(0,\underline{x}) = P(\underline{x}) \in \mathcal{P}[\Psi]$.

Since the solution of equation (5.18) has the formal representation $F(s,\underline{x}) = \exp\left(s\frac{\Delta'}{2}\right) P(\underline{x})$, the Bargmann transform is build up as the solution of the above differencial-difference heat equation for time-step $s = 1$.

This sifts for the discrete setting the constructions proposed by Sommen in [65] and Cnops and Kisil in [15]. In the view of Remark 5.3.3, the parametric solutions $s \mapsto F(s,\underline{x})$ are the discrete monogenic extension to the upper half plane $(0,\infty) \times \mathbb{R}^n$. For example, the hypercomplex Bernoulli polynomials constructed by Malonek and Tomaz in [55] appear as natural solutions of the above differential-difference heat equation.

# Conclusion and Outlook

> "*It is long - said Knight - but it is very, very beautiful.*
> *All the people who hear me sing ... or stands with tears in the eyes ...*
> *- Or what? - Alice asked, as the Knight had made a sudden break.*
> *- Or is not..*"
> Lewis Carol, On the Other Side of the Mirror

This dissertation presents the basics of of a higher dimensional discrete function theory from different border points, given a core of promising applications.

Some preliminary ideas on this direction were presented along the dissertation, namely on Chapter 2.1, where the description of Clifford algebras as a subalgebra of the algebra of endomorphisms only takes into account combinatorial arguments in the sense that it described as a set of shift-invariant automorphisms (i.e. the Pincherle derivative) acting on the algebra of polynomials. It was already established the correspondence between Finite Difference methods and Finite Element methods by means of barycentric coordinates. As a result, basic polynomial sequences are piecewise linear polynomials which assigns the connectivity of the discrete space.

Connections with the theory of special functions were presented in Chapters 4 and 5 by means of Fischer decomposition. In particular, the reproducing kernel property combined with the integral representation of the Fischer inner product allow us to obtain the Cauchy kernel as an asymptotic expansion while the study of the spectrum of the discrete harmonic oscillator gives us a contact with the MacDonald formula.

The flexibility of the approach proposed, as a general case of [31], rests mostly on the possibility to emboid monogenic functions on graphs and dual graphs. Geometrically these structures can be interpreted by joining a simplicial complex with its dual [56, 44] that we did not explore in detail in this dissertation, but that is likely to change in future researches.

The major contribution of this thesis to the development of discrete function theory rests mostly in its description as a special case of Wigner quantum systems [71]. This was tactically explored along Chapter 5 by means of Lie superalgebras of the type osp(1|2) establishing a parallel with continuous Clifford analysis setting presented in [18].

In conclusion, the powerf of this approach lies at its constructive isomorphism showing Umbral calculus and differential forms as the toolkits unifying continuous and discrete Clifford analysis as a whole.

We do not claim to have established a complete theory for discrete Clifford analysis but rather a basic framework on which future research could be based. Hereby we gave the theory of Gürlebeck and Hommel [45, 38, 39], in itself constructed by complicated lengthy calculations and ad-hoc constructions without giving any clear idea on its overall mathematical structure, a clear foundation.

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