# Non-existence of perfect 2-error correcting Lee codes of word length 7 over $\mathbb{Z}$

*Catarina Cruz and Ana Breda* *

**Abstract**

The Golomb-Welch conjecture states that there is no perfect $r$-error correcting Lee code of word length $n$ over $\mathbb{Z}$ for $n \geq 3$ and $r \geq 2$. This problem has received great attention due to its importance in applications in several areas beyond mathematics and computer sciences. Here, we give a contribution for the proof of the Golomb-Welch conjecture which reinforces it, proving the non-existence of perfect 2-error correcting Lee codes of word length 7 over $\mathbb{Z}$.

Perfect Lee codes, Golomb-Welch conjecture, tilings, Lee metric.

## 1 Introduction

Tiling problems are common in coding theory, in fact, certain tilings can be seen as error correcting codes, see [7] and [10]. Here, we are interested in dealing with tilings of spaces by Lee spheres. The study of these tilings was introduced by Golomb and Welch ([6] and [7]) which related them with error correcting codes, considering the center of a Lee sphere as a codeword and the other elements of the sphere as words which are decoded by the central codeword. When a Lee sphere of radius $r$ tiles the $n$-dimensional space, the set of all centers of the Lee spheres, that is, the set of all codewords, produces a perfect $r$-error correcting Lee code of word length $n$. There exists an extensive literature on codes in the Lee metric due to their several applications, see, for instance, [1] and [2].

Golomb and Welch have conjectured that there is no perfect $r$-error correcting Lee code of word length $n$ over $\mathbb{Z}$ for $n \geq 3$ and $r \geq 2$. Many partial results on this subject have been achieved. In [4] we present a proof of the Golomb-Welch conjecture for the case $n = 7$ and $r = 2$, one of the cases of the conjecture that has resisted for a long time. Later, Kim [9] has proved the non-existence of perfect 2-error correcting Lee codes for a certain values of $n$, including $n = 7$. Our idea to prove the case $n = 7$ and $r = 2$ of the Golomb-Welch conjecture differs to the one presented in [9]. While Kim [9] has used an algebraic process, our method is faithful to the geometric idea of the problem. In our strategy we were faced with a huge amount of hypotheses to try to cover certain words

*C. Cruz and A. Breda are with Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal. cmcruz@ua.pt

by codewords, being the hard work of the proof. The proof is very extensive, and so, we present here only its generic idea as well as some key proofs of the achieved results, showing the line of reasoning. It should be pointed out that all proofs can be conferred in [4].

## 2 Definitions and previous results

Let $(\mathcal{S}, \mu)$ be a metric space, where $\mathcal{S}$ is a nonempty set and $\mu$ a metric on $\mathcal{S}$. Any subset $\mathcal{M}$ of $\mathcal{S}$ satisfying $|\mathcal{M}| \geq 2$ is a **code**. The elements of $\mathcal{S}$ are called **words** and, in particular, the elements of a code $\mathcal{M}$ are called **codewords**. A sphere centered at $W \in \mathcal{S}$ with radius $r$, will be denoted by $S(W, r)$. If $W \in \mathcal{M}$ and $V \in S(W, r)$, with $V \neq W$, then we say that **the codeword $W$ covers the word $V$**.

**Definition 1.** *A code $\mathcal{M}$ is a **perfect $r$-error correcting code** if:*

*i)* $S(W, r) \cap S(V, r) = \varnothing$ *for any two distinct codewords $W$ and $V$ in $\mathcal{M}$;*

*ii)* $\bigcup_{W \in \mathcal{M}} S(W, r) = \mathcal{S}$.

Here, we deal with metric spaces $(\mathbb{Z}^n, \mu_L)$, where $\mathbb{Z}^n$ is the $n$-fold Cartesian product of the set of the integer numbers, with $n$ a positive integer number, and $\mu_L$ is the **Lee metric**. If $\mathcal{M} \subset \mathbb{Z}^n$ is a perfect $r$-error correcting code of $(\mathbb{Z}^n, \mu_L)$, then $\mathcal{M}$ is called a **perfect $r$-error correcting Lee code of word length $n$ over $\mathbb{Z}$**, shortly a **PL($n$,$r$) code**.

Having in mind the Golomb-Welch conjecture, we intend to prove the non-existence of PL$(7, 2)$ codes. Our strategy is based on the assumption of their existence. Let us assume the existence of a PL$(7, 2)$ code $\mathcal{M} \subset \mathbb{Z}^7$, and suppose, without loss of generality, that $O \in \mathcal{M}$, with $O = (0, \ldots, 0)$. Thus, all words $W \in \mathbb{Z}^7$ such that $\mu_L(W, O) \leq 2$ are covered by the codeword $O$. Taking into account Definition 1, for each word $W \in \mathbb{Z}^7$ satisfying $\mu_L(W, O) = 3$ there exists a unique codeword $V \in \mathcal{M}$ such that $\mu_L(W, V) \leq 2$, where $V$ is such that $\mu_L(V, O) = 5$. We focus our attention on these codewords, being our idea mostly based in cardinality restrictions on sets of these codewords. This strategy is a natural adaptation of the one given by Horak [8] and follows the same notation.

The words $W \in \mathbb{Z}^7$ satisfying $\mu_L(W, O) = 3$ are of types $[\pm 3]$, $[\pm 2, \pm 1]$ and $[\pm 1^3]$. Note that, for instance, $V = (v_1, ..., v_7)$ is a word of type $[\pm 2, \pm 1]$, if $|v_i| = 2$ and $|v_j| = 1$ for some $i, j \in \{1, ..., 7\}$, and $|v_k| = 0$ for all $k \in \{1, ..., 7\} \backslash \{i, j\}$. We denote by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{G}$ the sets of codewords $W$ satisfying $\mu_L(W, O) = 5$ of types $[\pm 5]$, $[\pm 4, \pm 1]$, $[\pm 3, \pm 2]$, $[\pm 3, \pm 1^2]$, $[\pm 2^2, \pm 1]$, $[\pm 2, \pm 1^3]$ and $[\pm 1^5]$, respectively. Note that: $[\pm 3]$ must be covered by codewords of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D}$; $[\pm 2, \pm 1]$ must be covered by codewords of $\mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F}$; $[\pm 1^3]$ must be covered by codewords of $\mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$. The conditions for the existence of PL(7,2) codes derive essentially from the analysis of the cardinality of subsets of $\mathcal{A} \cup \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{E} \cup \mathcal{F} \cup \mathcal{G}$, in particular, of their **index subsets**.

Let $\mathcal{I} = \{+1, +2, ..., +7, -1, -2, ..., -7\}$ be the set of **signed coordinates**. By index subsets of $\mathcal{H} \subset \mathbb{Z}^7$ we consider, for $i, j \in \mathcal{I}$, with $|i| \neq |j|$, and $k$ a positive integer number, the sets: $\mathcal{H}_i = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0\}$; $\mathcal{H}_{ij} = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0 \ \wedge \ jw_{|j|} > 0\}$; $\mathcal{H}_i^{(k)} = \{W \in \mathcal{H} \ : \ iw_{|i|} > 0 \wedge |w_{|i|}| = k\}$.

Next, we present some necessary conditions for the existence of PL(7, 2) codes proved in [5].

**Lemma 1.** *For each $i \in \mathcal{I}$, $|\mathcal{A}_i \cup \mathcal{B}_i^{(4)} \cup \mathcal{C}_i^{(3)} \cup \mathcal{D}_i^{(3)}| = 1$.*

**Lemma 2.** *For each $i, j \in \mathcal{I}$, with $|i| \neq |j|$,*

$$|\mathcal{B}_i^{(4)} \cap \mathcal{B}_j^{(1)}| + |\mathcal{C}_i \cap \mathcal{C}_j| + |\mathcal{D}_i^{(3)} \cap \mathcal{D}_j^{(1)}| + |\mathcal{E}_i^{(2)} \cap \mathcal{E}_j| + |\mathcal{F}_i^{(2)} \cap \mathcal{F}_j^{(1)}| = 1.$$

**Lemma 3.** *For each $i, j, k \in \mathcal{I}$, $|\mathcal{D}_{ijk} \cup \mathcal{E}_{ijk} \cup \mathcal{F}_{ijk} \cup \mathcal{G}_{ijk}| = 1$, with $|i|$, $|j|$ and $|k|$ pairwise distinct.*

Horak [8] has deduced the following results, one of them involving the parameters $a = |\mathcal{A}|$, $b = |\mathcal{B}|$, $c = |\mathcal{C}|$, $d = |\mathcal{D}|$, $e = |\mathcal{E}|$, $f = |\mathcal{F}|$ and $g = |\mathcal{G}|$.

**Proposition 1.** *The parameters $a$, $b$, $c$, $d$, $e$, $f$ and $g$ satisfy the system of equations*

$$\begin{cases} a + b + c + d = 14 \\ b + 2c + 2d + 4e + 3f = 168 \\ d + e + 4f + 10g = 280. \end{cases}$$

**Lemma 4.** *For each $i \in \mathcal{I}$, $|\mathcal{D}_i \cup \mathcal{E}_i| + 3|\mathcal{F}_i| + 6|\mathcal{G}_i| = 60$. Consequently, $|\mathcal{D}_i \cup \mathcal{E}_i| \equiv 0 \,(mod\, 3)$.*

**Lemma 5.** *For each $i, j \in \mathcal{I}$, $|i| \neq |j|$, $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 3|\mathcal{G}_{ij}| = 10$.*

Next results impose conditions in subsets of $\mathcal{F}$. The first one is derived from Lemmas 3 and 5. The proof of the second one can be conferred in [4].

**Lemma 6.** *For any $i, j \in \mathcal{I}$, with $|i| \neq |j|$, $|\mathcal{F}_{ij}| \leq 5$. Furthermore, if $|\mathcal{F}_{ij}| = 5$, then $|\mathcal{F}_{ijk}| = 1$ for all $k \in \mathcal{I} \backslash \{i, -i, j, -j\}$.*

**Lemma 7.** *For each $i \in \mathcal{I}$, $|\mathcal{F}_i^{(2)}| \leq 4$. If $|\mathcal{F}_i^{(2)}| = 4$, then $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_j| = 1$ for all $j \in \mathcal{I} \backslash \{i, -i\}$.*

Our aim is to prove that any nonnegative integer solutions of the system of equations presented in Proposition 1 contradicts the definition of PL(7,2) code. In this sense, particular attention will be given to the sets $\mathcal{G}$ and $\mathcal{F}$ in which the codewords have more nonzero coordinates. In [3] and [5] we have proved:

**Theorem 1.** *For each $i \in \mathcal{I}$, $3 \leq |\mathcal{G}_i| \leq 7$.*

Next, are presented conditions that must be satisfied when $|\mathcal{G}_i|$ assumes one of the possible values for some $i \in \mathcal{I}$. The proofs can be checked in [4].

**Lemma 8.** *If $|\mathcal{G}_i| = 3$, $i \in \mathcal{I}$, then $|\mathcal{A}_i| = 1$, $|\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{E}_i| = 0$, $|\mathcal{D}_i| = 3$ and $|\mathcal{F}_i| = 13$. More precisely, $|\mathcal{D}_i^{(3)}| = 0$, $|\mathcal{D}_i^{(1)}| = 3$, $|\mathcal{F}_i^{(2)}| = 4$ and $|\mathcal{F}_i^{(1)}| = 9$.*

**Lemma 9.** *If $|\mathcal{G}_i| = 4$, $i \in \mathcal{I}$, then one and only one of the following conditions must occurs: i) $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ and $|\mathcal{F}_i| = 11$; ii) $|\mathcal{D}_i| = 6$, $|\mathcal{E}_i| = 0$ and $|\mathcal{F}_i| = 10$. Besides, if ii) is satisfied, then $|\mathcal{A}_i| = 1$, $|\mathcal{B}_i \cup \mathcal{C}_i| = 0$, $|\mathcal{D}_i^{(1)}| = 6$, $|\mathcal{F}_i^{(2)}| = 4$ and $|\mathcal{F}_i^{(1)}| = 6$.*

**Lemma 10.** *If $|\mathcal{G}_i| = 5$, $i \in \mathcal{I}$, then one and only one of the following conditions must occurs: i) $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$ and $|\mathcal{F}_i| = 10$; ii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ and $|\mathcal{F}_i| = 9$; iii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$, $|\mathcal{F}_i| = 8$ and $|\mathcal{D}_i| \geq 3$; iv) $|\mathcal{D}_i| = 9$, $|\mathcal{E}_i| = 0$ and $|\mathcal{F}_i| = 7$. Besides, if iv) is satisfied, then $|\mathcal{A}_i| = 1$, $|\mathcal{B}_i \cup \mathcal{C}_i| = 0$, $|\mathcal{D}_i^{(1)}| = 9$, $|\mathcal{F}_i^{(2)}| = 4$ and $|\mathcal{F}_i^{(1)}| = 3$.*

**Lemma 11.** *If $|\mathcal{G}_i| = 6$, $i \in \mathcal{I}$, then one and only one of the following conditions must occurs: i) $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$ and $|\mathcal{F}_i| = 8$; ii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ and $|\mathcal{F}_i| = 7$; iii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$ and $|\mathcal{F}_i| = 6$; iv) $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$, $|\mathcal{F}_i| = 5$ and $|\mathcal{D}_i| \geq 6$; v) $|\mathcal{D}_i| = 12$, $|\mathcal{E}_i| = 0$ and $|\mathcal{F}_i| = 4$. Besides, if v) is satisfied, then $|\mathcal{A}_i| = 1$, $|\mathcal{B}_i \cup \mathcal{C}_i| = 0$, $|\mathcal{D}_i^{(1)}| = 12$ and $|\mathcal{F}_i^{(2)}| = 4$.*

**Lemma 12.** *If $|\mathcal{G}_i| = 7$, $i \in \mathcal{I}$, then one and only one of the following conditions must occurs: i) $|\mathcal{D}_i \cup \mathcal{E}_i| = 3$ and $|\mathcal{F}_i| = 5$; ii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 6$ and $|\mathcal{F}_i| = 4$; iii) $|\mathcal{D}_i \cup \mathcal{E}_i| = 9$, $|\mathcal{F}_i| = 3$ and $|\mathcal{D}_i| \geq 3$; iv) $|\mathcal{D}_i \cup \mathcal{E}_i| = 12$, $|\mathcal{F}_i| = 2$ and $|\mathcal{D}_i| \geq 9$.*

**Lemma 13.** *Let $\mathcal{G}_i$ for $i \in \mathcal{I}$. For all $j \in \mathcal{I}\backslash\{i, -i\}$, $|\mathcal{G}_{ij}| \leq 3$. If $|\mathcal{G}_{ij}| = 3$ for some $j \in \mathcal{I}\backslash\{i, -i\}$, then $|\mathcal{F}_i^{(2)}| \leq 3$. Besides: i) $|\mathcal{G}_i| \neq 3$; ii) if $|\mathcal{G}_i| = 4$, then $|\mathcal{F}_i| = 11$; iii) if $|\mathcal{G}_i| = 5$, then $8 \leq |\mathcal{F}_i| \leq 10$; iv) if $|\mathcal{G}_i| = 6$, then $5 \leq |\mathcal{F}_i| \leq 8$.*

Theorem 1 restricts the variation of $g = |\mathcal{G}|$. Since $g = \frac{1}{8}\sum_{i\in\mathcal{I}} |\mathcal{G}_i|$, with $|\mathcal{I}| = 14$, and by Theorem 1, $3 \leq |\mathcal{G}_i| \leq 7$, for all $i \in \mathcal{I}$, then $9 \leq g \leq 19$. Our strategy to prove the non-existence of PL(7, 2) codes relies on restricting more and more the variation of $|\mathcal{G}_i|$, for any $i \in \mathcal{I}$.

## 3 Proof of $|\mathcal{G}_i| \neq 3$ for any $i \in \mathcal{I}$

In this section we present the general idea of the proof of $|\mathcal{G}_i| \neq 3$ for any $i \in \mathcal{I}$. Under the assumption $|\mathcal{G}_i| = 3$, for some $i \in \mathcal{I}$, we derive conditions that necessarily must be satisfied by the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ and which will lead to contradictions in the definition of PL(7, 2) code.

Let us suppose $|\mathcal{G}_i| = 3$ for some $i \in \mathcal{I}$. Under this condition, by Lemma 8, we have $|\mathcal{F}_i| = 13$ and, in particular, $|\mathcal{F}_i^{(2)}| = 4$. The following results impose conditions in the index distribution of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$.

**Proposition 2.** *If $|\mathcal{G}_i| = 3$, for some $i \in \mathcal{I}$, then there are $\alpha, \beta, \gamma$ in $\mathcal{I}\backslash\{i, -i\}$, with $\alpha$, $\beta$ and $\gamma$ pairwise distinct, such that, $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$. Furthermore, $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$.*

*Proof.* Let $i \in \mathcal{I}$ be such that $|\mathcal{G}_i| = 3$. The three codewords $W_1, W_2, W_3$ of $\mathcal{G}_i$ satisfy $W_1 \in \mathcal{G}_{iw_1w_2w_3w_4}$, $W_2 \in \mathcal{G}_{iw_5w_6w_7w_8}$ and $W_3 \in \mathcal{G}_{iw_9w_{10}w_{11}w_{12}}$, with $w_1, \ldots, w_{12} \in \mathcal{I}\backslash\{i, -i\}$ and not necessarily pairwise distinct.

As $|\mathcal{F}_i| = \frac{1}{3}\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\omega}|$ and $|\mathcal{F}_i| = 13$ one has,

$$\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\omega}| = 39. \tag{3.1}$$

Since $|\mathcal{I}\backslash\{i,-i\}| = 12$ and, by Lemma 5, $|\mathcal{F}_{i\omega}| \leq 5$ for all $\omega \in \mathcal{I}\backslash\{i,-i\}$, the equation (3.1) implies the existence of, at least, two elements $\alpha, \beta \in \mathcal{I}\backslash\{i,-i\}$, with $\alpha \neq \beta$, such that, $|\mathcal{F}_{i\alpha}|, |\mathcal{F}_{i\beta}| \geq 4$.

Let us show, now, that there are, at most, three elements $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i,-i\}$, distinct between them, such that, $|\mathcal{F}_{i\alpha}|, |\mathcal{F}_{i\beta}|, |\mathcal{F}_{i\gamma}| \geq 4$. Suppose, by contradiction, that there exist $\alpha, \beta, \gamma, \delta \in \mathcal{I}\backslash\{i,-i\}$, distinct between them, such that, $|\mathcal{F}_{i\alpha}|, |\mathcal{F}_{i\beta}|, |\mathcal{F}_{i\gamma}|, |\mathcal{F}_{i\delta}| \geq 4$. By Lemma 5, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = |\mathcal{G}_{i\delta}| = 0$ and having in account the index distribution of $W_1, W_2, W_3 \in \mathcal{G}_i$, we may conclude that $w_1, \ldots, w_{12} \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma,\delta\}$. As $|\mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma,\delta\}| = 8$, there are $\omega, \theta \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma,\delta\}$ such that $|\mathcal{G}_{i\omega\theta}| \geq 2$, contradicting Lemma 3. Thus, there are, at most, three distinct elements $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i,-i\}$ satisfying $|\mathcal{F}_{i\alpha}|, |\mathcal{F}_{i\beta}|, |\mathcal{F}_{i\gamma}| \geq 4$.

Next, we prove that there is no $\omega \in \mathcal{I}\backslash\{i,-i\}$ satisfying $|\mathcal{F}_{i\omega}| = 4$. By contradiction, assume that $\alpha \in \mathcal{I}\backslash\{i,-i\}$ is such that $|\mathcal{F}_{i\alpha}| = 4$.

In view of (3.1) and in spite of the conditions established until now, one and only one of the following conditions is verified:

*i*) there is $\beta \in \mathcal{I}\backslash\{i,-i,\alpha\}$ such that $|\mathcal{F}_{i\beta}| = 5$ and $|\mathcal{F}_{i\omega}| = 3$ for any $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta\}$;

*ii*) there are $\beta, \gamma \in \mathcal{I}\backslash\{i,-i,\alpha\}$, with $\beta \neq \gamma$, such that $|\mathcal{F}_{i\beta}|, |\mathcal{F}_{i\gamma}| \geq 4$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$.

As $|\mathcal{G}_i| = 3$ and $|\mathcal{G}_i| = \frac{1}{4}\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}|$, then $\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}| = 12$.

Let us analyze the hypothesis *i*). By Lemma 5, $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 0$ and $|\mathcal{G}_{i\omega}| \leq 1$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta\}$. As $|\mathcal{I}\backslash\{i,-i,\alpha,\beta\}| = 10$, it follows that $\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}| \leq 10$, which is a contradiction.

Now assume that the conditions stated in *ii*) are fulfilled. In these conditions, $|\mathcal{F}_{i\alpha}| + |\mathcal{F}_{i\beta}| + |\mathcal{F}_{i\gamma}| \leq 14$, then having in consideration (3.1) we get $\sum_{\omega\in\mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}} |\mathcal{F}_{i\omega}| \geq 25$. Since $|\mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}| = 9$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$, then $|\mathcal{F}_{i\omega}| \geq 1$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$, furthermore, there are, at most, two distinct elements $\theta, \theta' \in \mathcal{I}\backslash\{i,-i,\alpha,\beta,\gamma\}$ so that $1 \leq |\mathcal{F}_{i\theta}|, |\mathcal{F}_{i\theta'}| \leq 2$. Thus, by Lemma 5, $\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}| \leq 11$, contradicting our assumption.

Accordingly:

- there are exactly two distinct elements $\alpha, \beta \in \mathcal{I}\backslash\{i,-i\}$ so that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = 5$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i,-i,\alpha,\beta\}$;

- there are exactly three distinct elements $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i\}$ such that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$.

Let us assume first that there are only two distinct elements $\alpha, \beta \in \mathcal{I}\backslash\{i, -i\}$ such that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = 5$. By (3.1), there exists a unique element $\theta \in \mathcal{I}\backslash\{i, -i, \alpha, \beta\}$ such that $|\mathcal{F}_{i\theta}| = 2$ and $|\mathcal{F}_{i\omega}| = 3$ for all $\omega \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \theta\}$. Consequently, by Lemma 5, we conclude that $\sum_{\omega\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\omega}| \leq 11$, which is a contradiction.

Summarizing, if $|\mathcal{G}_i| = 3$, there are exactly three distinct elements $\alpha, \beta, \gamma$ in $\mathcal{I}\backslash\{i, -i\}$, such that, $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$ and $|\mathcal{F}_{i\omega}| \leq 3$ for all $\omega \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$. □

**Proposition 3.** *Let $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i\}$ such that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$. Then, $|\alpha|$, $|\beta|$ and $|\gamma|$ are pairwise distinct and there exist $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$ whose index distributions satisfy:*

| | | | | |
|---|---|---|---|---|
| $U_1$ | $i$ | $\alpha$ | $\beta$ | $x_1$ |
| $U_2$ | $i$ | $\alpha$ | $\gamma$ | $x_2$ |
| $U_3$ | $i$ | $\beta$ | $\gamma$ | $x_3$ |
| $U_4$ | $i$ | $y_1$ | $y_2$ | $y_3$ |

*where $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$.*

*Proof.* Let $\alpha, \beta, \gamma \in \mathcal{I}\backslash\{i, -i\}$ so that $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$. Let us assume, by contradiction, that $|\alpha|$, $|\beta|$ and $|\gamma|$ are not pairwise distinct. Without loss of generality we may assume that $\alpha = -\beta$. Thus, $\mathcal{F}_{i\alpha} \cap \mathcal{F}_{i\beta} = \varnothing$ and, consequently, $|\mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta}| = 10$. As $|\mathcal{F}_i| = 13$, then $|\mathcal{F}_{i\alpha\gamma}| = |\mathcal{F}_{i\beta\gamma}| = 1$ and so $\mathcal{F}_i = \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}$. From Lemma 8, $|\mathcal{F}_i^{(2)}| = 4$. That is, $|\mathcal{F}_i^{(2)} \cap (\mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma})| = 4$. Consequently, there exists $\omega \in \{\alpha, \beta, \gamma\}$ such that $|\mathcal{F}_i^{(2)} \cap \mathcal{F}_{i\omega}| \geq 2$, contradicting Lemma 7. Therefore, $|\alpha|$, $|\beta|$ and $|\gamma|$ are pairwise distinct.

We have just seen that if $\mathcal{F}_i = \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}$, then Lemma 7 is contradicted. Thus, $\mathcal{F}_i \supset \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}$ which implies $|\mathcal{F}_{i\alpha\beta\gamma}| = 0$. As $|\mathcal{F}_{i\omega}| = 5$ for all $\omega \in \{\alpha, \beta, \gamma\}$, by Lemma 6 we get $|\mathcal{F}_{i\omega u}| = 1$ for all $u \in \mathcal{I}\backslash\{i, -i, \omega, -\omega\}$. As a consequence, $|\mathcal{F}_{i\alpha\beta}| = |\mathcal{F}_{i\alpha\gamma}| = |\mathcal{F}_{i\beta\gamma}| = 1$. That is, there are $U_1, U_2, U_3 \in \mathcal{F}_i$ satisfying:

| | | | | |
|---|---|---|---|---|
| $U_1$ | $i$ | $\alpha$ | $\beta$ | $x_1$ |
| $U_2$ | $i$ | $\alpha$ | $\gamma$ | $x_2$ |
| $U_3$ | $i$ | $\beta$ | $\gamma$ | $x_3$ |

Tab. 1: Partial index distribution of $U_1, U_2, U_3 \in \mathcal{F}_i$.

where $x_1, x_2, x_3 \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$. As $|\mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}| = 12$ and $|\mathcal{F}_i| = 13$, there exists $U_4 \notin \mathcal{F}_{i\alpha} \cup \mathcal{F}_{i\beta} \cup \mathcal{F}_{i\gamma}$, that is, $U_4 \in \mathcal{F}_{iy_1y_2y_3}$ where $y_1, y_2, y_3 \in \mathcal{I}\backslash\{i, -i, \alpha, \beta, \gamma\}$. □

Next results are proved in [4] and characterize partially the codewords of the sets $\mathcal{F}_i^{(2)}$ and $\mathcal{G}_i$, respectively.

**Corollary 1.** *In the considered conditions* $\mathcal{F}_i^{(2)} = \{U_4, U', U'', U'''\}$, *where* $U' \in \mathcal{F}_{i\alpha} \backslash (\mathcal{F}_\beta \cup \mathcal{F}_\gamma)$, $U'' \in \mathcal{F}_{i\beta} \backslash (\mathcal{F}_\alpha \cup \mathcal{F}_\gamma)$ *and* $U''' \in \mathcal{F}_{i\gamma} \backslash (\mathcal{F}_\alpha \cup \mathcal{F}_\beta)$.

**Proposition 4.** *If* $|\mathcal{F}_{i\alpha}| = |\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 5$, *then* $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = |\mathcal{G}_{i\gamma}| = 0$. *Furthermore, there are* $\delta, \varepsilon, \theta \in \mathcal{I} \backslash \{i, -i, \alpha, \beta, \gamma\}$ *so that* $|\mathcal{G}_{i\delta}| = |\mathcal{G}_{i\varepsilon}| = |\mathcal{G}_{i\theta}| = 2$ *and* $|\mathcal{G}_{i\omega}| = 1$ *for all* $\omega \in \mathcal{I} \backslash \{i, -i, \alpha, \beta, \gamma, \delta, \varepsilon, \theta\}$. *The index distributions of the three codewords* $W_1, W_2, W_3 \in \mathcal{G}_i$ *satisfy:*

| | | | | | |
|---|---|---|---|---|---|
| $W_1$ | $i$ | $\delta$ | $\varepsilon$ | $w_1$ | $w_2$ |
| $W_2$ | $i$ | $\delta$ | $\theta$ | $w_3$ | $w_4$ |
| $W_3$ | $i$ | $\varepsilon$ | $\theta$ | $w_5$ | $w_6$ |

*where* $\delta, \varepsilon, \theta, w_1, \ldots, w_6 \in \mathcal{I} \backslash \{i, -i, \alpha, \beta, \gamma\}$ *are pairwise distinct.*

Let us consider

$$\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\}.$$

Since the index distribution of $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$ is the one illustrated in Proposition 3, we may assume, without loss of generality, that $\alpha = j$, $\beta = k$ and $U_1 \in \mathcal{F}_{ijkl}$, that is:

| | | | | |
|---|---|---|---|---|
| $U_1$ | $i$ | $j$ | $k$ | $l$ |
| $U_2$ | $i$ | $j$ | $\gamma$ | $x_1$ |
| $U_3$ | $i$ | $k$ | $\gamma$ | $x_2$ |
| $U_4$ | $i$ | $y_1$ | $y_2$ | $y_3$ |

Tab. 2: Partial index distribution of $U_1, \ldots, U_4 \in \mathcal{F}_i$.

where $x_1, x_2 \in \mathcal{I} \backslash \{i, -i, j, k, \gamma, l\}$ and $y_1, y_2, y_3 \in \mathcal{I} \backslash \{i, -i, j, k, \gamma\}$.

In what follows, the index distribution of the codewords of $\mathcal{G}_i$ and $U_1, U_2, U_3$, $U_4$ codewords of $\mathcal{F}_i$ are the ones given in Proposition 4 and Table 2, respectively. Next results, proved in [4], allow us to analyze how the codewords of $\mathcal{G}_i$ and $\mathcal{F}_i$ fit together.

**Proposition 5.** *If* $l \neq \delta, \varepsilon, \theta$, *then, without loss of generality,* $W_1 \in \mathcal{G}_{i\delta\varepsilon l}$, *and either* $\theta = -l$ *or* $\theta = -j$ *or* $\theta = -k$.

**Proposition 6.** $U_4 \in \mathcal{F}_{iy_1y_2y_3}$ *for* $y_1, y_2, y_3 \in \{-j, -k, -\gamma, l, x_1, x_2\}$, *where* $x_1$ *and* $x_2$ *are such that* $U_2 \in \mathcal{F}_{ij\gamma x_1}$ *and* $U_3 \in \mathcal{F}_{ik\gamma x_2}$ *(see Table 2).*

**Proposition 7.** *If* $|\mathcal{G}_{i\omega}| = 2$ *for some* $\omega \in \mathcal{I} \backslash \{i, -i, j, -j, k, -k, \gamma, -\gamma, l\}$, *then either* $\omega = x_1$ *or* $\omega = x_2$.

**Proposition 8.** *The indices* $\delta, \varepsilon, \theta \in \{-j, -k, -\gamma, l, x_1, x_2\} \backslash \{y_1, y_2, y_3\}$ *furthermore* $|\{-j, -k, -\gamma, l, x_1, x_2\}| = 6$.

We have characterized the partial index distributions of the codewords of $\mathcal{G}_i$ and the codewords $U_1, U_2, U_3, U_4 \in \mathcal{F}_i$. From this characterization we can get the complete index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ which apparently do not contradict the definition of PL(7,2) code. There exist only two possible index distributions, however, considering other elements $\omega \in \mathcal{I}\backslash\{i\}$, analyzing the complete index distribution of all codewords of $\mathcal{G}_\omega \cup \mathcal{F}_\omega$, we conclude that is not possible to describe the set $\mathcal{G} \cup \mathcal{F}$ without superposition between codewords. This analysis is extensive and can be checked in [4].

## 4 Proof of $|\mathcal{G}_i| \neq 4$ for any $i \in \mathcal{I}$

Here, we restrict even more the range of variation of $|\mathcal{G}_i|$ for any $i \in \mathcal{I}$. Such as in the previous section, we assume, without loss of generality, that there exists an $i \in \mathcal{I}$ such that $|\mathcal{G}_i| = 4$. We focus our attention on the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ and deduce some necessary conditions which must be satisfied by them.

Let $i \in \mathcal{I}$ be so that $|\mathcal{G}_i| = 4$, by Lemma 9, $10 \leq |\mathcal{F}_i| \leq 11$. The first result characterize the partial index distribution of the four codewords of $\mathcal{G}_i$.

**Proposition 9.** *If $|\mathcal{G}_i| = 4$, for $i \in \mathcal{I}$, then $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{i, -i\}$.*

*Proof.* By Lemma 5 we know that $|\mathcal{G}_{i\alpha}| \leq 3$ for all $\alpha \in \mathcal{I}\backslash\{i, -i\}$. Suppose, by contradiction, that $j \in \mathcal{I}\backslash\{i, -i\}$ is such that $|\mathcal{G}_{ij}| = 3$.

As $|\mathcal{G}_i| = 4$, from Lemma 9 it follows that $|\mathcal{F}_i| = 10$ or $|\mathcal{F}_i| = 11$. Next, we analyze, separately, these two hypotheses: $|\mathcal{F}_i| = 10$ and $|\mathcal{F}_i| = 11$.

Suppose first that $|\mathcal{F}_i| = 10$. Then, by Lemma 9, $|\mathcal{B}_i \cup \mathcal{C}_i \cup \mathcal{E}_i| = 0$. Considering Lemma 5 and taking into account that, by hypothesis, $|\mathcal{G}_{ij}| = 3$, then $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| = 1$. Since $|\mathcal{E}_i| = 0$, it follows that $|\mathcal{D}_{ij}| = 1$ and $|\mathcal{F}_{ij}| = 0$.

Let us consider two words $V_1 = (v_{11}, v_{12}, ..., v_{17})$ and $V_2 = (v_{21}, v_{22}, ..., v_{27})$ such that $|v_{1i}| = 2$, $|v_{1j}| = 1$, $|v_{2i}| = 1$ and $|v_{2j}| = 2$. These words must be covered by codewords of $\mathcal{B}_{ij} \cup \mathcal{C}_{ij} \cup \mathcal{D}_{ij} \cup \mathcal{E}_{ij} \cup \mathcal{F}_{ij}$. As $|\mathcal{B}_{ij}| = |\mathcal{C}_{ij}| = |\mathcal{E}_{ij}| = |\mathcal{F}_{ij}| = 0$, then $V_1$ and $V_2$ must be covered by the unique codeword in $\mathcal{D}_{ij}$, which is not possible since the codewords of $\mathcal{D}$ are of type $[\pm 3, \pm 1^2]$.

Now assume that $|\mathcal{F}_i| = 11$. Since we are under the assumption $|\mathcal{G}_{ij}| = 3$, let us consider $W_1, W_2, W_3 \in \mathcal{G}_{ij}$ such that $W_1 \in \mathcal{G}_{ijw_1w_2w_3}$, $W_2 \in \mathcal{G}_{ijw_4w_5w_6}$ and $W_3 \in \mathcal{G}_{ijw_7w_8w_9}$, with $w_1, \ldots, w_9 \in \mathcal{I}\backslash\{i, -i, j, -j\}$. We note that, by Lemma 3, $w_1, \ldots, w_9$ must be pairwise distinct. As $|\mathcal{G}_i| = 4$, let $W_4 \in \mathcal{G}_i\backslash\mathcal{G}_j$ so that $W_4 \in \mathcal{G}_{iw_{10}w_{11}w_{12}w_{13}}$, where $w_{10}, w_{11}, w_{12}, w_{13} \in \mathcal{I}\backslash\{i, -i, j\}$. In Table 3, the codewords $W_1, \ldots, W_4 \in \mathcal{G}_i$ are schematically represented.

Since $w_1, \ldots, w_9 \subset \mathcal{I}\backslash\{i, -i, j, -j\}$ with $w_1, \ldots, w_9$ pairwise distinct, taking into account that $|\mathcal{I}| = 14$, let $\{\beta\} = \mathcal{I}\backslash\{i, -i, j, -j, w_1, \ldots, w_9\}$. Note that, $\mathcal{I}\backslash\{i, -i\} = \{j\} \cup \{-j\} \cup \{\beta\} \cup \{w_1, \ldots, w_9\}$. Considering $W_4 \in \mathcal{G}_{iw_{10}w_{11}w_{12}w_{13}}$, $w_{10}, \ldots, w_{13} \in \mathcal{I}\backslash\{i, -i, j\}$, we conclude that $|\{w_{10},...,w_{13}\} \cap \{w_1,...,w_9\}| \geq 2$. On the other hand, $|\{w_{10},...,w_{13}\} \cap \{w_1,...,w_9\}| \leq 3$, otherwise Lemma 3 is contradicted. Consider the cases: *i)* $|\{w_{10}, \ldots, w_{13}\} \cap \{w_1, \ldots, w_9\}| = 2$; *ii)* $|\{w_{10}, \ldots, w_{13}\} \cap \{w_1, \ldots, w_9\}| = 3$.

Suppose that $|\{w_{10},...,w_{13}\} \cap \{w_1,...,w_9\}| = 2$, in these conditions $W_4 \in \mathcal{G}_{i,-j,\beta,w_{10},w_{11}}$, with $w_{10}, w_{11} \in \{w_1, \ldots, w_9\}$. Accordingly, we have

| | | | | | |
|---|---|---|---|---|---|
| $W_1$ | $i$ | $j$ | $w_1$ | $w_2$ | $w_3$ |
| $W_2$ | $i$ | $j$ | $w_4$ | $w_5$ | $w_6$ |
| $W_3$ | $i$ | $j$ | $w_7$ | $w_8$ | $w_9$ |
| $W_4$ | $i$ | $w_{10}$ | $w_{11}$ | $w_{12}$ | $w_{13}$ |

Tab. 3: Partial index distribution of the codewords of $\mathcal{G}_i$.

$|\mathcal{G}_{ij}| = 3$, $|\mathcal{G}_{iw_{10}}| = |\mathcal{G}_{iw_{11}}| = 2$ and $|\mathcal{G}_{iw}| = 1$ for all $w \in \mathcal{I}\backslash\{i, -i, j, w_{10}, w_{11}\}$. Consequently, from Lemma 5 it follows that $|\mathcal{F}_{ij}| = 0$, $|\mathcal{F}_{iw_{10}}|, |\mathcal{F}_{iw_{11}}| \leq 2$ and $|\mathcal{F}_{iw}| \leq 3$ for all $w \in \mathcal{I}\backslash\{i, -i, j, w_{10}, w_{11}\}$. As $|\mathcal{F}_i| = \frac{1}{3}\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}|$ and we are assuming $|\mathcal{F}_i| = 11$, then $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| = 33$. However, taking into account what was been said before, $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| \leq 31$, which is a contradiction.

Now consider that $|\{w_{10}, \ldots, w_{13}\} \cap \{w_1, \ldots, w_9\}| = 3$. Thus, we have $W_4 \in \mathcal{G}_{ixw_{10}w_{11}w_{12}}$, with $x \in \{-j, \beta\}$ and $w_{10}, w_{11}, w_{12} \in \{w_1, \ldots, w_9\}$. In these conditions, $|\mathcal{G}_{ij}| = 3$, $|\mathcal{G}_{iw_{10}}| = |\mathcal{G}_{iw_{11}}| = |\mathcal{G}_{iw_{12}}| = 2$, $|\mathcal{G}_{iy}| = 0$ for $\{y\} = \{-j, \beta\}\backslash\{x\}$ and $|\mathcal{G}_{iw}| = 1$ for all $w \in \mathcal{I}\backslash\{i, -i, j, y, w_{10}, w_{11}, w_{12}\}$. Consequently, by Lemma 5, we get $|\mathcal{F}_{ij}| = 0$, $|\mathcal{F}_{iw_{10}}|, |\mathcal{F}_{iw_{11}}|, |\mathcal{F}_{iw_{12}}| \leq 2$, $|\mathcal{F}_{iy}| \leq 5$ and $|\mathcal{F}_{iw}| \leq 3$ for all $w \in \mathcal{I}\backslash\{i, -i, j, y, w_{10}, w_{11}, w_{12}\}$. Accordingly, $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| \leq 32$, obtaining again a contradiction. □

We have just proved that for any $\alpha \in \mathcal{I}\backslash\{i, -i\}$ we get $|\mathcal{G}_{i\alpha}| \leq 2$. Let us consider the subset $\mathcal{J} \subset \mathcal{I}\backslash\{i, -i\}$ so that: $\mathcal{J} = \{\alpha \in \mathcal{I}\backslash\{i, -i\} : |\mathcal{G}_{i\alpha}| = 2\}$. The following result, proved in [4], restricts the variation of $|\mathcal{J}|$.

**Proposition 10.** *The cardinality of $\mathcal{J}$ satisfies* $4 \leq |\mathcal{J}| \leq 6$.

Next, we establish conditions which must be verified by the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ when $|\mathcal{J}|$ assumes each one of the possible values.

**Proposition 11.** *If $|\mathcal{J}| = 4$, then $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ and* $|\mathcal{F}_i| = 10$.

*Proof.* By assumption $|\mathcal{G}_i| = 4$, consequently $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{G}_{i\alpha}| = 16$. That is, $\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})} |\mathcal{G}_{i\alpha}| + \sum_{\alpha\in\mathcal{J}} |\mathcal{G}_{i\alpha}| = 16$. As $|\mathcal{G}_{i\alpha}| = 2$ for all $\alpha \in \mathcal{J}$ and, by assumption, $|\mathcal{J}| = 4$, it follows that $\sum_{\alpha\in\mathcal{I}\backslash(\{i,-i\}\cup\mathcal{J})} |\mathcal{G}_{i\alpha}| = 8$. Taking into account Proposition 9, $|\mathcal{G}_{i\alpha}| \leq 1$ for all $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$. Since that $|\mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})| = 8$, we must impose $|\mathcal{G}_{i\alpha}| = 1$ for all $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$.

From Lemma 9 we know that $|\mathcal{F}_i| = 10$ or $|\mathcal{F}_i| = 11$. Let us suppose that $|\mathcal{F}_i| = 11$. In these conditions, taking into account that $|\mathcal{F}_i| = \frac{1}{3}\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}|$, we have $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| = 33$. Having in mind what was proved before, from Lemma 5, we get $|\mathcal{F}_{i\alpha}| \leq 2$ for all $\alpha \in \mathcal{J}$ and $|\mathcal{F}_{i\alpha}| \leq 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$. That is, $\sum_{\alpha\in\mathcal{I}\backslash\{i,-i\}} |\mathcal{F}_{i\alpha}| \leq 32$, which is an absurdity. Therefore, $|\mathcal{F}_i| = 10$. □

Next, following a similar reasoning, we derive equivalent results whose proofs can be checked in [4].

**Proposition 12.** *If $|\mathcal{J}| = 4$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon\}$, then $|\mathcal{F}_{i\alpha}| \geq 1$ for all $\alpha \in \mathcal{I}\backslash\{i, -i\}$ and there exist, at least, two elements $\alpha \in \mathcal{J}$ such that $|\mathcal{F}_{i\alpha}| = 2$. Furthermore:*

*i) if $\beta, \gamma \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 2$, then $|\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for all $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$;*

*ii) if $\beta, \gamma, \delta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ which satisfy $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$, then $|\mathcal{F}_{i\varepsilon}| = 1$ and there are seven elements $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ such that $|\mathcal{F}_{i\alpha}| = 3$;*

*iii) if $|\mathcal{F}_{i\alpha}| = 2$ for all $\alpha \in \mathcal{J}$, then there are, at least, six elements $\alpha \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ satisfying $|\mathcal{F}_{i\alpha}| = 3$.*

**Proposition 13.** *If $|\mathcal{J}| = 5$, there exists $x \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ so that $|\mathcal{G}_{ix}| = 0$. Furthermore, $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$.*

Since by Lemma 9 we have $|\mathcal{F}_i| = 11$ or $|\mathcal{F}_i| = 10$, the following two propositions give us conditions for the index distribution of the codewords of $\mathcal{F}_i$ when $|\mathcal{J}| = 5$ and $|\mathcal{F}_i|$ assumes each one of these values.

**Proposition 14.** *Let $|\mathcal{J}| = 5$ and $x \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ be such that $|\mathcal{G}_{ix}| = 0$. If $|\mathcal{F}_i| = 11$, then: $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$; $|\mathcal{F}_{ix}| = 5$; $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$.*

**Proposition 15.** *Let $|\mathcal{J}| = 5$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta\}$, and $x \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ such that $|\mathcal{G}_{ix}| = 0$. If $|\mathcal{F}_i| = 10$, then there are, at least, two elements $\alpha \in \mathcal{J}$ such that $|\mathcal{F}_{i\alpha}| = 2$. Furthermore:*
*i) if $\beta, \gamma \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = 2$, then $|\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 1$, $|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*

*ii) if $\beta, \gamma, \delta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ which satisfy $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$, then one of the following conditions must occurs: $|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$; $|\mathcal{F}_{ix}| = 4$, $|\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*

*iii) if $\beta, \gamma, \delta, \varepsilon \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 2$, then one of the following conditions must occurs: $|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, four elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$; $|\mathcal{F}_{ix}| = 4$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$; $|\mathcal{F}_{ix}| = 3$, $|\mathcal{F}_{i\theta}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$;*

*iv) if $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$, then one of the following conditions must occurs: $|\mathcal{F}_{ix}| = 5$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, three elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$; $|\mathcal{F}_{ix}| = 4$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, four elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$; $|\mathcal{F}_{ix}| = 3$ and $|\mathcal{F}_{i\alpha}| = 3$ for, at least, five elements $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$; $|\mathcal{F}_{ix}| = 2$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x\} \cup \mathcal{J})$.*

**Proposition 16.** *If $|\mathcal{J}| = 6$, then there exist $x, y \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ such that $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$. Furthermore, $|\mathcal{G}_{i\alpha}| = 1$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$.*

**Proposition 17.** *Let $|\mathcal{J}| = 6$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta, \mu\}$, and $x, y \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ such that $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$. If $|\mathcal{F}_i| = 11$, then there are, at least, five elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Furthermore, if there exist exactly five elements in these conditions, then: $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 2$; $|\mathcal{F}_{i\mu}| = 1$; $|\mathcal{F}_{ix}| = |\mathcal{F}_{iy}| = 5$; $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$.*

**Proposition 18.** *Let $|\mathcal{J}| = 6$, with $\mathcal{J} = \{\beta, \gamma, \delta, \varepsilon, \theta, \mu\}$, and $x, y \in \mathcal{I}\backslash(\{i, -i\} \cup \mathcal{J})$ be such that $|\mathcal{G}_{ix}| = |\mathcal{G}_{iy}| = 0$. If $|\mathcal{F}_i| = 10$, then there are, at least, three elements $\alpha \in \mathcal{J}$ satisfying $|\mathcal{F}_{i\alpha}| = 2$. Furthermore:*

*i) if $\beta, \gamma, \delta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = 2$, then $|\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = |\mathcal{F}_{i\mu}| = 1$, $|\mathcal{F}_{ix}| = 5$, $|\mathcal{F}_{iy}| = 4$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$;*

*ii) if $\beta, \gamma, \delta, \varepsilon \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = 2$, then $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| \geq 8$; if $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| = 8$, then $|\mathcal{F}_{i\theta}| = |\mathcal{F}_{i\mu}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$;*

*iii) if $\beta, \gamma, \delta, \varepsilon, \theta \in \mathcal{J}$ are the unique elements in $\mathcal{J}$ satisfying $|\mathcal{F}_{i\beta}| = |\mathcal{F}_{i\gamma}| = |\mathcal{F}_{i\delta}| = |\mathcal{F}_{i\varepsilon}| = |\mathcal{F}_{i\theta}| = 2$, then $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| \geq 7$; if $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| = 7$, then $|\mathcal{F}_{i\mu}| = 1$ and $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$;*

*iv) if $|\mathcal{F}_{i\alpha}| = 2$ for any $\alpha \in \mathcal{J}$, then $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| \geq 6$; if $|\mathcal{F}_{ix}| + |\mathcal{F}_{iy}| = 6$, then $|\mathcal{F}_{i\alpha}| = 3$ for any $\alpha \in \mathcal{I}\backslash(\{i, -i, x, y\} \cup \mathcal{J})$.*

From the presented results is possible to describe the index distribution of all codewords of $\mathcal{G}_i \cup \mathcal{F}_i$, $i \in \mathcal{I}$. The strategy applied to verify that any index distribution of such codewords contradicts the definition of PL$(7, 2)$ code is the same referred in the last part of the Section 3 and can be conferred in [4].

## 5 Proof of $|\mathcal{G}_i| \neq 5$ for any $i \in \mathcal{I}$

Here, we analyze the hypothesis $|\mathcal{G}_i| = 5$ for some $i \in \mathcal{I}$. Let us assume $|\mathcal{G}_i| = 5$ for $i \in \mathcal{I}$. Since, from Lemma 5, $|\mathcal{G}_{i\alpha}| \leq 3$ for any $\alpha \in \mathcal{I}\backslash\{i, -i\}$, we distinguish the cases:

1) $|\mathcal{G}_{i\alpha}| = 3$ for some $\alpha \in \mathcal{I}\backslash\{i, -i\}$;

2) $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I}\backslash\{i, -i\}$.

For each one of these cases, we derive, initially, some conditions which will be useful for the characterization of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$. Here, we only present the analysis of the hypothesis 1), being the hypothesis 2) analyzed in [4].

Let us consider $|\mathcal{G}_i| = 5$ and $|\mathcal{G}_{ij}| = 3$ for some $j \in \mathcal{I}\backslash\{i, -i\}$. Then, by Lemma 10, $7 \leq |\mathcal{F}_i| \leq 10$. The following proposition restricts even more the variation of $|\mathcal{F}_i|$.

**Proposition 19.** *If* $|\mathcal{G}_i| = 5$ *and* $|\mathcal{G}_{ij}| = 3$*, then* $|\mathcal{F}_{ij}| = 0$ *and* $8 \leq |\mathcal{F}_i| \leq 9$.

*Proof.* Since $|\mathcal{G}_{ij}| = 3$, from Lemma 5 it follows that $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| + 2|\mathcal{F}_{ij}| + 9 = 10$, implying $|\mathcal{F}_{ij}| = 0$ and $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$.

As $|\mathcal{G}_i| = 5$ and $|\mathcal{G}_{ij}| = 3$, by Lemma 13 we get $8 \leq |\mathcal{F}_i| \leq 10$. Supposing $|\mathcal{F}_i| = 10$, by Lemma 4 we must impose $|\mathcal{D}_i \cup \mathcal{E}_i| = 0$, which contradicts $|\mathcal{D}_{ij} \cup \mathcal{E}_{ij}| = 1$. Therefore, $8 \leq |\mathcal{F}_i| \leq 9$. □

It is possible, up to an equivalent index distributions, to characterize all codewords of $\mathcal{G}_{ij}$, as we will see in the next proposition proved in [4].

**Proposition 20.** *The index distribution of the codewords* $W_1, W_2, W_3 \in \mathcal{G}_{ij}$ *satisfies:*

| | | | | | |
|---|---|---|---|---|---|
| $W_1$ | $i$ | $j$ | $k$ | $l$ | $m$ |
| $W_2$ | $i$ | $j$ | $-k$ | $-l$ | $n$ |
| $W_3$ | $i$ | $j$ | $-m$ | $-n$ | $o$ |

The index distribution of the codewords of $\mathcal{G}_{ij}$ presented in Proposition 20 induces a partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i, -i, j\}$:

$$\mathcal{P}_1 = \{k, l, m\};\ \mathcal{P}_2 = \{-k, -l, n\};\ \mathcal{P}_3 = \{-m, -n, o\};\ \mathcal{P}_4 = \{-o\};\ \mathcal{P}_5 = \{-j\}. \tag{5.1}$$

Let us consider the subsets of, respectively, $\mathcal{G}_i$ and $\mathcal{F}_i$:

$$\mathcal{H} = \{W \in \mathcal{G}_{iw_1w_2w_3w_4} : w_1 \in \mathcal{P}_1 \ \wedge\ w_2 \in \mathcal{P}_2 \ \wedge\ w_3 \in \mathcal{P}_3 \ \wedge\ w_4 \in \{-o, -j\}\}$$

and

$$\mathcal{J} = \{U \in \mathcal{F}_{iu_1u_2u_3} : u_1 \in \mathcal{P}_1 \ \wedge\ u_2 \in \mathcal{P}_2 \ \wedge\ u_3 \in \mathcal{P}_3\}.$$

Taking into account the partition of $\mathcal{I}\backslash\{i, -i, j\}$, see (5.1), we get $\mathcal{G}_i\backslash\mathcal{G}_j = \mathcal{H} \cup \mathcal{G}_{i,-o,-j}$ and $\mathcal{F}_i = \mathcal{J} \cup \mathcal{F}_{i,-o} \cup \mathcal{F}_{i,-j}$.

Next results, proved in [4], impose conditions on the index distribution of the codewords of $(\mathcal{G}_i\backslash\mathcal{G}_j) \cup \mathcal{F}_i$ by the establishment of relations between the cardinality of the sets $\mathcal{H}$, $\mathcal{J}$, $\mathcal{F}_{i,-o}$ and $\mathcal{F}_{i,-j}$. The following proposition will be useful to obtain the refereed relations.

**Proposition 21.** *The set* $\mathcal{D}_{i,j,-o} \cup \mathcal{E}_{i,j,-o}$ *satisfies* $|\mathcal{D}_{i,j,-o} \cup \mathcal{E}_{i,j,-o}| = 1$.

**Proposition 22.** *The sets* $\mathcal{H}$ *and* $\mathcal{J}$ *satisfy* $|\mathcal{H} \cup \mathcal{J}| \leq 6$*. Furthermore,* $1 \leq |\mathcal{H}| \leq 2$ *and* $|\mathcal{J}| \leq 5$.

**Proposition 23.** *If* $|\mathcal{H}| = 1$*, then* $3 \leq |\mathcal{J}| \leq 5$ *and* $|\mathcal{F}_{i,-o,-j}| = 0$*. In particular, considering* $W_4 \in \mathcal{H}$ *one has:*

*i) if* $W_4 \in \mathcal{G}_{i,-o}$*, then* $4 \leq |\mathcal{J}| \leq 5$*. Moreover, if* $|\mathcal{J}| = 4$*, then* $|\mathcal{F}_{i,-o}| = 1$*,* $|\mathcal{F}_{i,-j}| = 3$ *and* $|\mathcal{F}_i| = 8$*;*

*ii)* *if* $W_4 \in \mathcal{G}_{i,-j}$, *then* $3 \leq |\mathcal{J}| \leq 5$. *Moreover, if* $|\mathcal{J}| = 3$, *then* $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 2$ *and* $|\mathcal{F}_i| = 8$.

**Proposition 24.** *If* $|\mathcal{H}| = 2$, *then* $|\mathcal{G}_{i,-o,-j}| = 0$ *and* $3 \leq |\mathcal{J}| \leq 4$. *In particular, if* $|\mathcal{J}| = 3$, *then* $|\mathcal{F}_i| = 8$ *and considering* $W_4, W_5 \in \mathcal{H}$:

*i)* *if* $W_4, W_5 \in \mathcal{G}_{i,-o}$, *then either* $|\mathcal{F}_{i,-j}| = 5$, *or,* $|\mathcal{F}_{i,-j}| = 4$, $|\mathcal{F}_{i,-o}| = 1$ *and* $|\mathcal{F}_{i,-o,-j}| = 0$;

*ii)* *if* $W_4, W_5 \in \mathcal{G}_{i,-j}$, *then either* $|\mathcal{F}_{i,-o}| = 4$, $|\mathcal{F}_{i,-j}| = 2$ *and* $|\mathcal{F}_{i,-o,-j}| = 1$, *or,* $|\mathcal{F}_{i,-o}| = 4$, $|\mathcal{F}_{i,-j}| = 1$ *and* $|\mathcal{F}_{i,-o,-j}| = 0$;

*iii)* *if* $W_4 \in \mathcal{G}_{i,-o}$ *and* $W_5 \in \mathcal{G}_{i,-j}$, *then either* $|\mathcal{F}_{i,-o}| = 3$, $|\mathcal{F}_{i,-j}| = 3$ *and* $|\mathcal{F}_{i,-o,-j}| = 1$, *or,* $|\mathcal{F}_{i,-o}| = 2$, $|\mathcal{F}_{i,-j}| = 3$ *and* $|\mathcal{F}_{i,-o,-j}| = 0$.

The index characterization of the codewords of $\mathcal{G}_i \cup \mathcal{F}_i$ is mostly based in Propositions 23 and 24. However, as in the previous cases, all possible index characterizations lead to contradictions, as can be conferred in [4]. The proof of $|\mathcal{G}_i| \neq 5$ for any $i \in \mathcal{I}$ is completed with the analysis of the condition $|\mathcal{G}_{i\alpha}| \leq 2$ for any $\alpha \in \mathcal{I} \backslash \{i, -i\}$. The study of this condition can be checked in [4], where its impossibility is proved.

## 6 Conclusion of the proof of the non-existence of PL$(7, 2)$ codes

In the previous sections we have proved that if there exists a PL$(7, 2)$ code $\mathcal{M}$, then $\mathcal{G} \subset \mathcal{M}$ is such that $6 \leq |\mathcal{G}_i| \leq 7$ for any $i \in \mathcal{I}$. Here, we show that this assumption leads us to contradictions, proving thus the non-existence of PL$(7, 2)$ codes. Under this condition we derive the following result.

**Proposition 25.** *There exists* $\alpha \in \mathcal{I}$ *such that* $|\mathcal{G}_\alpha| = 7$. *Furthermore, if* $|\mathcal{G}_\alpha| = 7$, *for some* $\alpha \in \mathcal{I}$, *then there exist, at least, four elements* $\beta \in \mathcal{I} \backslash \{\alpha, -\alpha\}$ *satisfying* $|\mathcal{G}_{\alpha\beta}| = 3$.

*Proof.* We have proved that $6 \leq |\mathcal{G}_\alpha| \leq 7$ for any $\alpha \in \mathcal{I}$. We recall that

$$g = |\mathcal{G}| = \frac{1}{5} \sum_{\alpha \in \mathcal{I}} |\mathcal{G}_\alpha|. \tag{6.1}$$

Let us suppose, by contradiction, that $|\mathcal{G}_\alpha| = 6$ for any $\alpha \in \mathcal{I}$. As $|\mathcal{I}| = 14$, by (6.1) we conclude that $g = \frac{84}{5}$, which it is not possible since $g$ must be an integer number. Therefore, there exists $\alpha \in \mathcal{I}$ such that $|\mathcal{G}_\alpha| = 7$.

Let $\alpha \in \mathcal{I}$ be such that $|\mathcal{G}_\alpha| = 7$. We note that $|\mathcal{G}_\alpha| = \frac{1}{4} \sum_{\beta \in \mathcal{I} \backslash \{\alpha, -\alpha\}} |\mathcal{G}_{\alpha\beta}|$, that is,

$$\sum_{\beta \in \mathcal{I} \backslash \{\alpha, -\alpha\}} |\mathcal{G}_{\alpha\beta}| = 28. \tag{6.2}$$

From Lemma 5 it follows that $|\mathcal{G}_{\alpha\beta}| \leq 3$ for any $\beta \in \mathcal{I}\backslash\{\alpha, -\alpha\}$. If we suppose, by contradiction, that, at most, there are three elements $\beta \in \mathcal{I}\backslash\{\alpha, -\alpha\}$ satisfying $|\mathcal{G}_{\alpha\beta}| = 3$, then from (6.2) it follows that $\sum_{\beta\in\mathcal{I}\backslash\{\alpha,-\alpha\}} |\mathcal{G}_{\alpha\beta}| \leq 27$, facing up a contradiction. Accordingly, there are, at least, four elements $\beta \in \mathcal{I}\backslash\{\alpha, -\alpha\}$ such that $|\mathcal{G}_{\alpha\beta}| = 3$. □

Consider $\mathcal{I} = \{i, -i, j, -j, k, -k, l, -l, m, -m, n, -n, o, -o\}$. Taking into account the previous proposition, let us assume $|\mathcal{G}_i| = 7$ and $|\mathcal{G}_{ij}| = 3$. From Proposition 20 it follows that the codewords $W_1, W_2, W_3 \in \mathcal{G}_{ij}$ satisfy the following index distribution:

| | | | | | |
|---|---|---|---|---|---|
| $W_1$ | $i$ | $j$ | $k$ | $l$ | $m$ |
| $W_2$ | $i$ | $j$ | $-k$ | $-l$ | $n$ |
| $W_3$ | $i$ | $j$ | $-m$ | $-n$ | $o$ |

Tab. 4: Index distribution of the codewords of $\mathcal{G}_{ij}$.

The index distribution of the codewords of $\mathcal{G}_{ij}$ induces the following partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i, -i, j\}$:

$$\mathcal{P}_1 = \{k, l, m\};\ \ \mathcal{P}_2 = \{-k, -l, n\};\ \ \mathcal{P}_3 = \{-m, -n, o\};\ \ \mathcal{P}_4 = \{-j\};\ \ \mathcal{P}_5 = \{-o\}. \tag{6.3}$$

Having in view Proposition 25 and the partition of $\mathcal{P}$, next result, proved in [4], imposes conditions on the elements $\alpha \in \mathcal{I}\backslash\{i, -i, j\}$ which satisfy $|\mathcal{G}_{i\alpha}| = 3$.

**Proposition 26.** *There are, at least, two elements $\alpha, \beta \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ satisfying $|\mathcal{G}_{i\alpha}| = |\mathcal{G}_{i\beta}| = 3$.*

By Proposition 26, let us consider $\alpha \in \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ such that $|\mathcal{G}_{i\alpha}| = 3$. Analyzing the partition $\mathcal{P}$ of $\mathcal{I}\backslash\{i, -i, j\}$, see (6.3), we distinguish, without loss of generality, the hypotheses: $\alpha = k$; $\alpha = m$; $\alpha = -m$; $\alpha = o$. Our aim is to characterize all possible index distributions for the codewords of $\mathcal{G}_i$. For that, we analyze each one of the referred hypotheses. This analysis is presented in [4] and follows the same idea described in the last part of the Section 3. For any index distribution of the codewords of $\mathcal{G}_i$, the description of other codewords of the set $\mathcal{G} \cup \mathcal{F}$ implies always superposition between codewords, contradicting the definition of PL(7, 2) code. Thus, we establish the main theorem:

**Theorem 2.** *There exist no PL(7, 2) code.*

## 7 Conclusion

The Golomb-Welch conjecture states that there is no PL$(n, r)$ code for $n \geq 3$ and $r \geq 2$. Here, we reinforce the conjecture proving the non-existence of PL(7, 2) codes. The way how the proof was built reveals how difficult was to solve the case. We have focused our attention on words which dist three units from $O = (0, ..., 0)$. Actually, there exist many ways to try to cover all these words by codewords, and although we have obtained many results which restrict the number of such hypotheses, in many cases, to achieve contradictions we had to apply exhaustion methods to study a large number of cases. This was the major hard work of the proof. In some cases we have tried to use computational methods having in view a quick analysis of the many cases we had to deal with, however, it would be necessary to implement an algorithm requiring a lot of information, not being easy to do it, at least, with our knowledge.

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