Hybrid heuristics for a maritime short sea inventory routing problem

Agostinho Agra[a], Marielle Christiansen[b], Alexandrino Delgado[c], Luidi Simonetti[d]

[a]*Department of Mathematics and CIDMA, University of Aveiro*
[b]*Department of Industrial Economics and Technology Management, Norwegian University of Science and Technology*
[c]*Cape Verde University and CIDMA*
[d]*Institute for Computing, Universidade Federal Fluminense*

Abstract

We consider a short sea fuel oil distribution problem where an oil company is responsible for the routing and scheduling of ships between ports such that the demand for various fuel oil products is satisfied during the planning horizon. The inventory management has to be considered at the demand side only, and the consumption rates are given and assumed to be constant within the planning horizon. The objective is to determine distribution policies that minimize the routing and operating costs, while the inventory levels are maintained within their limits. We propose an arc-load flow formulation for the problem which is tightened with valid inequalities. In order to obtain good feasible solutions for planning horizons of several months, we compare different hybridization strategies. Computational results are reported for real small-size instances.

Keywords: Maritime Transportation; Hybrid heuristics; Inventory Routing; Mixed Integer Programming

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1. Introduction

Maritime transportation is the major mode of transportation of goods worldwide. The importance of this mode of transportation is obvious for the long distance transportation of cargoes but it is also crucial in local economies where the sea is the natural link between the local developed regions, such as countries formed by archipelagoes. When a company has the responsibility of coordinating the transportation of goods

Email addresses: `aagra@ua.pt` (Agostinho Agra), `marielle.christiansen@iot.ntnu.no` (Marielle Christiansen), `alexandrino.delgado@docente.unicv.edu.cv` (Alexandrino Delgado), `luidi@ic.uff.br` (Luidi Simonetti)

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with the inventories at the ports, the underlying planning problem is a maritime inventory routing problem. Such problems are very complex. Usually modest improvements in the supply chain planning can translate into significant cost savings.

In this paper we consider a real maritime Short Sea Inventory Routing Problem (SSIRP) occurring in the archipelago of Cape Verde. An oil company is responsible for the inventory management of different oil products in several tanks located in the main islands. Fuel oil products are imported and delivered to specific islands and stored in large supply storage tanks, so the inventory management does not need to be considered in these tanks. From these islands, fuel oil products are distributed among all the inhabited islands using a small heterogeneous fleet of ships with dedicated tanks. These products are stored in consumption storage tanks with limited capacity. Consumption rates are assumed to be given and constant over a time horizon of several months. Some ports have both supply tanks for some products and consumption tanks of other products.

We have witnessed an increased interest in studying optimization problems within maritime transportation [14, 15, 16] and, in particular, in the last fifteen years, problems combining routing and inventory management [8, 12]. These problems are often called Maritime Inventory Routing Problems (MIRPs). Most of the published MIRP contributions are based on real cases from the industry, see for the single product case [11, 21, 22, 24] and for the multiple products case [7, 13, 28, 30, 33, 35].

This SSIRP is addressed in a companion paper [4] where different mathematical formulations are discussed and compared for the SSIRP considering a shorter time horizon. There, two main approaches to model the problem are considered. One uses a continuous time model where an index indicating the visit number to a particular port is added to most of the variables. This approach was used in [7], [11] and [33] for MIRPs where the production and/or consumption rates are considered given and fixed during the planning horizon. The other approach consists of using a model that combines a discrete and continuous time where the discrete time corresponds to an artificial discretization of the continuous time. Discrete time models have been developed in [2, 22, 23, 24, 28, 30, 34] to overcome the complicating factors with time varying production and consumption rates. In addition, for each approach two new extended formulations are tested in [4].

In [3], the SSIRP for short-term planning is considered. For the short-term plans demand orders are considered, that is, fixed amounts of oil products that must be delivered at each port within a fixed period of time. These orders are determined from the initial stock levels and the consumption rates and lead to a problem with varying demands (corresponding to the demand orders). Several key issues taken into account in the short-term problem are relaxed here or incorporated indirectly in the data. For instance, port operating time windows that are essential in the short-term plan are ignored here. Otherwise, the problems considered originate from the same company in the same region. These problems are solved using the same commercial solver we use here, considering a formulation which is improved by the strengthening of defining inequalities and the inclusion (through separation) of valid inequalities. In [7] a problem similar to the SSIRP is considered with constant consumption rates and dedicated compartments in the ships.

In this paper we develop and compare different hybrid heuristics for the SSIRP. As discussed in [8, 34], most combined maritime routing and inventory management problems described in the literature have particular features and characteristics, and tailor-made methods are developed to solve the problems [12]. These methods are often based on heuristics or decomposition techniques. Recent hybrid heuristics that use MIP solvers as a black-box tool have been proposed. Here we consider and combine three hybrid heuristics: Rolling Horizon (RH), Local Branching (LB) and Feasibility Pump (FP). In RH heuristics the planning horizon is split into smaller sub-horizons. Then, each limited and tractable mixed integer problem is solved to optimality. Within maritime transportation RH heuristics have been used in [25, 28, 32, 33, 34]. Local Branching (LB) was introduced by Fichetti and Lodi [19] to improve feasible solutions. LB heuristics search for local optimal solutions by restricting the number of binary variables that are allowed to change their value in the current solution. Feasibility Pump (FP) was introduced by Fischetti, Glover and Lodi [18] to find initial feasible solutions for MIP problems.

Computational experiments reported in Section 6 show that a combined heuristic using the three approaches outperformed the other tested heuristics and, in particular, outperformed the most used approach within MIRPs, the RH heuristic.

To solve each subproblem we consider the arc-load flow (ALF) formulation introduced in [4], since this was the model with the best performance among all the tested models for this problem with short time horizons. The ALF formulation is improved by a pre-computation of estimates for the number of visits to each port, and with the inclusion of valid inequalities. In particular, we introduce a new family of clique inequalities for MIRPs when continuous time models are used.

The main contributions of this paper, the heuristic strategies and the valid inequalities, can easily be used in other MIRPs.

The remainder of this paper is organized as follows. In Section 2, we describe the real problem. The arc-load flow formulation is presented in Section 3 and strategies to tighten the formulation are discussed in Section 4. In Section 5 we describe several hybrid heuristics. The computational experimentations are reported in Section 6. Final conclusions are given in Section 7.

2. Problem description

In Cape Verde, fuel oil products are imported and delivered to specific islands and stored in large supply storage tanks. From these islands, fuel oil products are distributed among all the inhabited islands using a small heterogeneous fleet of ships. The products are stored in consumption storage tanks. Two ports have both supply tanks for some products and consumption tanks for other products, while the remaining ports have only consumption tanks. Not all islands consume all products. The consumptions (which are usually forecasted) are assumed to be constant over the time horizon. It is assumed that each port can receive at most one ship at a time and a minimum interval between the departure of a ship and the arrival of the next one must be considered. Waiting times are allowed.

Each ship has a specified load capacity, fixed speed and cost structure. The cargo hold of each ship is separated into several cargo tanks. The products can not be mixed, so we assume that the ships have dedicated tanks to particular products.

The traveling times between two consecutive ship visits are an estimation based on practical experience. Additionally, we consider set-up times for the coupling and decoupling of pipes, and operating times.

To prevent a ship from delivering small quantities, minimum delivery quantities are considered. The maximum delivered quantity is imposed by the capacity of the consumption storage tank. Safety stocks are considered at consumption tanks. As the capacity of the supply tanks is very large when compared to the total demand over the horizon, we omit the inventory aspects for these tanks.

In each problem instance we are given the initial stock levels at the consumption tanks, initial ship positions (which can be a point at sea) and quantities on board each ship. The inter-island distribution plan consists of designing routes and schedules for the fleet of ships including determining the number of visits to each port and the (un)loading quantity of each product at each port visit. The plan must satisfy the safety stocks of each product at each island and the capacities of the ship tanks. The transportation and operation costs of the distribution plan must be minimized over a finite planning horizon.

3. Mathematical Model

In [4] a comparison of six different formulations for the SSIRP for a shorter time horizon is given. Three of those formulations consider a time discretization and the other three consider continuous time. For each time option the following formulations are considered: an arc-load formulation, where the model keeps only track of the information of the load on board each ship compartment in each port visit; an arc-load flow formulation, where new variables are used to keep the information about the quantity of each product in each compartment when a ship leaves a port en route to the next one; and a multi-commodity formulation, where the flow on each arc is disaggregated accordingly to its destination. That comparison led to the choice of the continuous time arc-load flow formulation. In this section we present that arc-load flow formulation.

Routing constraints

Let V denote the set of ships. Each ship $v \in V$ must depart from its initial position in the beginning of the planning horizon. The set of ports is denoted by N. For each port we consider an ordering of the visits accordingly to the time of the visit. The ship paths are defined on a network where the nodes are represented by a pair (i, m), where i is the port and m represents the m^{th} visit to port i. Direct ship movements (arcs) from port arrival (i, m) to port arrival (j, n) are represented by (i, m, j, n).

We define S^A as the set of possible port arrivals (i, m), S^A_v as the set of ports that may be visited by ship v, and set S^X_v as the set of all possible movements (i, m, j, n) of ship v.

For the routing we define the following binary variables: x_{imjnv} is 1 if ship v sails from port arrival (i, m) directly to port arrival (j, n), and 0 otherwise; x_{oimv} indicates whether ship v sails directly from its initial position to port arrival (i, m) or not; w_{imv} is 1 if ship v visits port i at arrival (i, m), and 0 otherwise; z_{imv} is equal to 1 if ship v ends its route at port arrival (i, m), and 0 otherwise; z_{ov} is equal to 1 if ship v is not used and 0 otherwise; y_{im} indicates whether a ship is visiting port arrival (i, m) or not.

$$\sum_{(i,m)\in S^A_v} x_{oimv} + z_{ov} = 1, \qquad \forall v \in V, \tag{1}$$

$$w_{imv} - \sum_{(j,n)\in S^A_v} x_{jnimv} - x_{oimv} = 0, \qquad \forall v \in V, (i, m) \in S^A_v, \tag{2}$$

$$w_{imv} - \sum_{(j,n)\in S^A_v} x_{imjnv} - z_{imv} = 0, \qquad \forall v \in V, (i, m) \in S^A_v, \tag{3}$$

$$\sum_{v\in V} w_{imv} = y_{im}, \qquad \forall (i, m) \in S^A, \tag{4}$$

$$y_{i(m-1)} - y_{im} \geq 0, \qquad \forall (i, m) \in S^A : m > 1, \tag{5}$$

$$x_{oimv}, w_{imv}, z_{imv} \in \{0, 1\}, \qquad \forall v \in V, (i, m) \in S^A_v, \tag{6}$$

$$x_{imjnv} \in \{0, 1\}, \qquad \forall v \in V, (i, m, j, n) \in S^X_v, \tag{7}$$

$$z_{ov} \in \{0, 1\}, \qquad \forall v \in V, \tag{8}$$

$$y_{im} \in \{0, 1\}, \qquad \forall (i, m) \in S^A. \tag{9}$$

Equations (1) ensure that each ship either departs from its initial position and sails towards another port or the ship is not used. Equations (2) and (3) are the flow conservation constraints, ensuring that a ship arriving at a port also leaves that port or ends its route. Constraints (4) ensure that one ship only visits port (i, m) if y_{im} is equal to one. Constraints (5) state that if port i is visited m times, then it must also have been visited $m - 1$ times. Constraints (6)-(9) define the variables as binary.

Load and unload constraints

Let K represent the set of products and K_v represent the set of products that ship v can transport. Not all ports consume all products. Parameter J_{ik} is 1 if port i is a supplier of product k; -1 if port i is a consumer of product k, and 0 if i is neither a consumer nor a supplier of product k. The quantity of product k on board ship v at the beginning of the planning horizon is given by Q_{vk}, and C_{vk} is the capacity of the compartment of ship v dedicated for product k. The minimum and the maximum discharge quantities of product k at port i are given by $\underline{Q}_{ik}$ and $\overline{Q}_{ik}$, respectively.

In order to model the loading and unloading constraints, we define the following binary variables: o_{imvk} is equal to 1 if product k is loaded onto or unloaded from ship v at port visit (i, m), and 0 otherwise. In addition, we define the following continuous variables: q_{imvk} is the amount of product k loaded onto or unloaded from ship v at port visit (i, m), f_{imjnvk} denotes the amount of product k that ship v transports from port visit (i, m) to port visit (j, n), and f_{oimvk} gives the amount of product k that ship v transports from its initial position to port visit (i, m).

The loading and unloading constraints are given by:

$$f_{oimvk} + \sum_{(j,n)\in S_v^A} f_{jnimvk} + J_{ik} q_{imvk} = \sum_{(j,n)\in S_v^A} f_{imjnvk}, \qquad \forall v \in V, (i, m) \in S_v^A, k \in K_v \tag{10}$$

$$f_{oimvk} = Q_{vk} x_{oimv}, \qquad \forall v \in V, (i, m) \in S_v^A, k \in K_v, \tag{11}$$

$$f_{imjnvk} \leq C_{vk} x_{imjnv}, \qquad \forall\ v \in V, (i, m, j, n) \in S_v^X, k \in K_v, \tag{12}$$

$$0 \leq q_{imvk} \leq C_{vk} o_{imvk}, \qquad \forall v \in V, (i, m) \in S_v^A, k \in K_v : J_{ik} = 1, \tag{13}$$

$$\underline{Q}_{ik} o_{imvk} \leq q_{imvk} \leq \overline{Q}_{ik} o_{imvk}, \qquad \forall v \in V, (i, m) \in S_v^A, k \in K_v : J_{ik} = -1, \tag{14}$$

$$\sum_{k\in K_v} o_{imvk} \geq w_{imv}, \qquad \forall v \in V, (i, m) \in S_v^A, \tag{15}$$

$$o_{imvk} \leq w_{imv}, \qquad \forall v \in V, (i, m) \in S_v^A, k \in K_v, \tag{16}$$

$$f_{imjnvk} \geq 0, \qquad \forall v \in V, (i, m, j, n) \in S_v^A, k \in K_v, \tag{17}$$

$$f_{oimvk}, q_{imvk} \geq 0, \qquad \forall v \in V, (i, m) \in S_v^A, k \in K_v, \tag{18}$$

$$o_{imvk} \in \{0, 1\}, \qquad \forall v \in V, (i, m) \in S_v^A, k \in K_v. \tag{19}$$

Equations (10) are the flow conservation constraints. Equations (11) determine the quantity on board when ship v sails from its initial port position to port arrival (i, m). Constraints (12) require that the vehicle capacity is obeyed. Constraints (13) impose an upper bound on the quantity loaded at a supply port. Constraints (14) impose lower and upper limits on the unloaded quantities. Constraints (15) ensure that if ship v visits port arrival (i, m), then at least one product must be (un)loaded. Constraints (16) ensure that if ship v (un)loads one product at visit (i, m), then w_{imv} must be one. Constraints (17)-(19) are the non-negativity and integrality constraints.

Time constraints

In order to keep track of the inventory level it is necessary to determine the start and the end times at each port arrival. We define the following parameters: T^Q_{ik} is the time required to load/unload one unit of product k at port i; T^S_{ik} is the set-up time required to operate product k at port i. T_{ijv} is the traveling time between port i and j by ship v; T^O_{iv} indicates the traveling time required by ship v to sail from its initial position to port i; T^B_i is the minimum interval between the departure of one ship and the next arrival at port i. T is the length of the time horizon. Given the start time t_{im} and end time t^E_{im} variables for port arrival (i, m), the time constraints can be written as:

$$t^E_{im} \geq t_{im} + \sum_{v \in V} \sum_{k \in K_v} T^Q_{ik} q_{imvk} + \sum_{v \in V} \sum_{k \in K_v} T^S_{ik} o_{imvk}, \qquad \forall (i, m) \in S^A, \tag{20}$$

$$t_{im} - t^E_{i(m-1)} - T^B_i y_{im} \geq 0, \qquad \forall (i, m) \in S^A : m > 1, \tag{21}$$

$$t^E_{im} + T_{ijv} - t_{jn} \leq T(1 - x_{imjnv}), \qquad \forall v \in V, (i, m, j, n) \in S^X_v, \tag{22}$$

$$\sum_{v \in V} T^O_{iv} x_{oimv} \leq t_{im}, \qquad \forall (i, m) \in S^A, \tag{23}$$

$$t_{im}, t^E_{im} \geq 0, \qquad \forall (i, m) \in S^A. \tag{24}$$

Constraints (20) define the end time of service at port visit (i, m). Constraints (21) impose a minimum interval between two consecutive visits at port i. Constraints (22) relate the end time of port visit (i, m) to the start time of port visit (j, n) when ship v sails directly from port visit (i, m) to (j, n). Constraints (23) ensure that if ship v travels from its initial position directly to port visit (i, m), then the start time is at least the traveling time between the two positions. Constraints (24) define the continuous time variables.

Inventory constraints

The inventory constraints are considered for each unloading port. They ensure that the stock levels are within the corresponding bounds and link the stock levels to the (un)loaded quantities.

For each consumption port i, and for each product k, the consumption rate, R_{ik}, the minimum $\underline{S}_{ik}$, the maximum $\overline{S}_{ik}$ and the initial stock S^0_{ik} levels, are given. The parameter $\overline{\mu}_i$ denotes the maximum number of visits at port i.

We define the nonnegative continuous variables s_{imk} and s^E_{imk} indicating the stock levels at the start and at the end of port visit (i, m) for product k, respectively. The inventory constraints are as follows:

$$s_{i1k} = S^0_{ik} - R_{ik}t_{i1}, \qquad \forall i \in N, k \in K : J_{ik} = -1, \tag{25}$$

$$s^E_{imk} = s_{imk} + \sum_{v \in V} q_{imvk} - R_{ik}(t^E_{im} - t_{im}), \qquad \forall (i,m) \in S^A, k \in K : J_{ik} = -1, \tag{26}$$

$$s_{imk} = s^E_{i(m-1)k} - R_{ik}(t_{im} - t^E_{i(m-1)}), \qquad \forall (i,m) \in S^A : m > 1, k \in K : J_{ik} = -1, \tag{27}$$

$$\underline{S}_{ik} \leq s_{imk}, s^E_{imk} \leq \overline{S}_{ik}, \qquad \forall (i,m) \in S^A, k \in K : J_{ik} = -1, \tag{28}$$

$$\underline{S}_{ik} \leq s^E_{i\overline{\mu}_i k} - R_{ik}(T - t^E_{i\overline{\mu}_i}) \leq \overline{S}_{ik}, \qquad \forall i \in N, k \in K : J_{ik} = -1. \tag{29}$$

Equations (25) calculate the stock level of each product at the first visit. Equations (26) calculate the stock level of each product when the service ends at port visit (i, m). Equations (27) relate the stock level at the start of port visit (i, m) to the stock level at the end of port visit $(i, m-1)$. The upper and lower bounds on the stock levels are ensured by constraints (28)-(29).

Objective function

The objective is to minimize the total routing costs including traveling, operating and set-up costs. The traveling cost of ship v from port i to port j is denoted by C^T_{ijv}, while C^T_{oiv} represents the traveling cost of ship v from its initial port positions to port i. The set-up cost of product k at port i is denoted by C^O_{ik}. The objective function is as follow:

$$\sum_{v \in V} \sum_{(i,m,j,n) \in S^X_v} C^T_{ijv} x_{imjnv} + \sum_{v \in V} \sum_{(i,m) \in S^A_v} C^T_{oiv} x_{oimv} + \sum_{v \in V} \sum_{(i,m) \in S^A_v} \sum_{k \in K_v} C^O_{ik} o_{imvk}. \tag{30}$$

The formulation defined by (1)-(30) is denoted by F-SSIRP, and the feasible set will be denoted by X.

4. Tightening the formulation

Tightening the formulation provided in the previous section is essential to speed up the solution approaches (Branch and Bound and hybrid heuristics), and to provide tighter bounds that will be used in Section 6 to evaluate the quality of the tested heuristics. The tightening is done by including new inequalities. Many families of inequalities were tested. Here we present only the ones that provided best results from a preliminary study.

4.1. Tightening time constraints

Time constraints (22) linking the time variables with the routing variables are very weak. Parameter T works as a big M constant. An approach to tighten such constraints is to establish time windows to the time events.

$$A_{im} \leq t_{im} \leq B_{im}, \qquad \forall (i,m) \in S^A, \tag{31}$$

$$A^E_{im} \leq t^E_{im} \leq B^E_{im}, \qquad \forall (i,m) \in S^A. \tag{32}$$

Then, constraints (22) can be replaced by the stronger inequalities

$$t^E_{im} - t_{jn} + (B^E_{im} + T_{ijv} - A_{jn})x_{imjnv} \leq B^E_{im} - A_{jn}.$$

These inequalities can be further strengthened as follows (see Proposition 1 in [5]):

$$t^E_{im} - t_{jn} + \sum_{v \in V|(i,m,j,n) \in S^X_v} max\{0, B^E_{im} + T_{ijv} - A_{jn}\}x_{imjnv} \leq B^E_{im} - A_{jn}, \forall (i,m),(j,n) \in S^A. \tag{33}$$

One can take $A_{im} = A^E_{im} = 0$ and $B_{im} = B^E_{im} = T$. However, by reducing the widths of the time windows we strengthen inequalities (33). In this SSIRP we are dealing with multiple ships, multiple products, and all supply ports also act as demand ports of other products. Because of this characteristics it is hard to derive tight time windows.

For simplicity, we provide only those time windows formulas that proved to be most effective for our case. Other rules can be derived adapting the ones given in [10] for the single item case. Since inventory aspects are only relevant for consumption tanks, and since all the loading ports of certain products are also consumption ports of other products, time windows are established based on the unloading products only.

The start of time windows are computed as follows:

$$A_{im} = min_{v \in V}\{T^O_{iv}\} + (m-1) * \left(T^B_i + min_{k \in K|J_{ik}=-1}\left\{T^Q_{ik}\underline{Q}_{ik} + T^S_{ik}\right\}\right),$$

$$A^E_{im} = min_{v \in V}\{T^O_{iv}\} + (m-1) * T^B_i + m * min_{k \in K|J_{ik}=-1}\left\{T^Q_{ik}\underline{Q}_{ik} + T^S_{ik}\right\},$$

and the end of time windows are computed as follows:

$$B_{im} = min\left\{T, min_{k \in K|J_{ik}=-1}\left\{\left(S^0_{ik} + (m-1) * \overline{S}_{ik} - \underline{S}_{ik}\right)/R_{ik} - T^S_{ik}\right\}\right\},$$

$$B^E_{im} = min\left\{T, min_{k \in K|J_{ik}=-1}\left\{\left(S^0_{ik} + m * \overline{S}_{ik} - \underline{S}_{ik}\right)/R_{ik} - T^S_{ik}\right\} - T^B_i\right\}.$$

The end of time windows can be further strengthened. Let $\underline{\mu_i}$ denote a lower bound on the number of visits to port i, $i \in N$ (see in Section 4.2 how to compute these parameters). If $m \leq \underline{\mu_i}$, then T in the B_{im} formula given above can be replaced by

$$T - (\underline{\mu_i} - m) * T^B_i - (\underline{\mu_i} - m + 1) * min_{k \in K|J_{ik}=-1}\left\{T^Q_{ik}\underline{Q}_{ik} + T^S_{ik}\right\},$$

and, if $m < \underline{\mu_i}$, then T in the B^E_{im} formula can be replaced by

$$T - (\underline{\mu_i} - m) * \left\{ T^B_i + min_{k \in K | J_{ik} = -1} \left\{ \underline{Q}_{ik} T^Q_{ik} + T^S_{ik} \right\} \right\}.$$

4.2. Lower bounds on the number of visits

A common approach to tighten formulations for routing problems is to include constraints imposing a minimum number of visits to each node. The impact on the reduction of the integrality gap is usually high. Equations

$$y_{i\underline{\mu_i}} = 1, \forall i \in N \tag{34}$$

can be added to each model. These parameters $\underline{\mu_i}$ can be computed from the inventory information and traveling times. However, since the traveling times between islands are small, the number of visits is better estimated through the inventory information and storage capacities (at ships and ports).

For each port $i \in N$ where product k is unloaded, $J_{ik} = -1$, let

$$D^N_{ik} = max\{T \times R_{ik} - S^0_{ik} + \underline{S}_{ik}, \quad \underline{Q}_{ik}\}$$

denote the net consumption over the time horizon. The minimum number of visits to port i for unloading product k is given by

$$\underline{\lambda}_{ik} = \left\lceil \frac{D^N_{ik}}{\overline{Q}_{ik}} \right\rceil.$$

In the real problem, each product has a single origin. As inventory management at supply tanks is disregarded, the minimum number of visits to load a product can be estimated using the total consumption supplied by that origin. The consumption of that product must be satisfied either from that port or from the quantity in the ship tanks at the beginning of the planning horizon.

For each product $k \in K$, loaded at port $i \in N$ ($J_{ik} = 1$) let

$$D^N_{ik} = \sum_{j \in N | J_{jk} = -1} (T \times R_{jk} - S^0_{jk} + \underline{S}_{jk}),$$

denote the net consumption of this product over the time horizon. The minimum number of loadings of product k at port i is given by

$$\underline{\lambda}_{ik} = \left\lceil \frac{D^N_{ik} - \sum_{v \in V} Q_{vk}}{max\{C_{vk} : v \in V\}} \right\rceil.$$

A lower bound on the total number of visits to port $i \in N$ can be given by the following equation:

$$\underline{\mu}_i = max\{\underline{\lambda}_{ik} : k \in K\}. \tag{35}$$

Better bounds can be obtained by solving subproblems for each port. A subproblem is solved for the consumption products at the port and, if the port is also a supplier of other products, another subproblem is solved for the supply products.

Although the subproblems are NP-hard, they can be solved very quickly using a commercial software.

First we state the subproblem for consumption products. All the routing constraints are ignored in the subproblems. For these subproblems associated to each port the inventory and time constraints are the same as for the original model. The ship capacity for each product is overestimated by the maximum of the ship capacities for that product.

Let $\overline{C}_k = max\{C_{vk} : v \in V, k \in K_v\}$. For each port i let $M_i = \{1, 2, \cdots, \overline{\mu}_i\}$. The subproblem is defined as follows:

$$NV^D(i) : min \sum_{m \in M_i} y_{im} \tag{36}$$

$$s.t.$$

$$\overline{q}_{imk} \leq \overline{C}_k \overline{o}_{imk}, \quad \forall m \in M_i, k \in K, J_{ik} = -1 \tag{37}$$

$$\underline{Q}_{ik} \overline{o}_{imk} \leq \overline{q}_{imk} \leq \overline{Q}_{ik} \overline{o}_{imk}, \quad \forall m \in M_i, k \in K : J_{ik} = -1, \tag{38}$$

$$\overline{o}_{imk} \leq y_{im}, \quad \forall m \in M_i, \forall k \in K : J_{ik} = -1, \tag{39}$$

$$\text{Constraints } (25) - (29) \text{ for node } i$$

$$\text{Constraints } (20), (21), (24) \text{ for node } i$$

$$y_{im} \in \{0, 1\}, \quad \forall m \in M_i, \tag{40}$$

$$\overline{o}_{imk} \in \{0, 1\}, \quad \forall m \in M_i, k \in K : J_{ik} = -1, \tag{41}$$

$$\overline{q}_{imk} \geq 0, \quad \forall m \in M_i, k \in K : J_{ik} = -1, \tag{42}$$

where $\overline{o}_{imk} = \sum_{v \in V} o_{imkv}$, $\overline{q}_{imk} = \sum_{v \in V} q_{imkv}$.

The objective function (36) minimizes the number of visits at port i. Constraints (37) - (39) have a similar meaning as constraints (13), (14), (16), only now the ship is ignored and an overestimation of the ship capacities is used.

If port i is also a supplier, we define the following subproblem, $NV^S(i)$, where only the ship tank capacities are considered.

$$min\{\sum_{v \in V} u_{iv} : \sum_{v \in V} C_{vk} u_{iv} \geq \sum_{j \in N : J_{jk} = -1} D^N_{jk} - \sum_{v \in V} Q_{vk}, \forall k \in K : J_{ik} = 1, u_{iv} \in \mathbb{Z}_+, \forall v \in V\},$$

where u_{iv} indicates the number of visits of ship v to port i.

If port i is simultaneously a consumption and a supply port, the minimum number of visits is the maximum between $NV^D(i)$ and $NV^S(i)$. These two subproblems will be called port subproblems.

4.3. Integer knapsack inequalities

Inequalities from knapsack relaxations have previously been used for MIRPs, see for instance [24, 27, 34].

Let $D_k(S)$ denote the total demand of product k, from ports in S during the planning horizon, where $S \subseteq N$ and $J_{ik} = -1$ for all $i \in S$. Hence, $D_k(S) = \sum_{i \in S} T \times R_{ik}$. Let $ND_k(S)$ denote the amount of demand $D_k(S)$ that must be transported from ports in $N \setminus S$. That is, $ND_k(S) = D_k(S) - \sum_{v \in V} Q_{vk} - \sum_{i \in S}(S^0_{ik} - \underline{S}_{ik})$. Then, the following integer set is a relaxation of X :

$$RX = \left\{ \chi \in \mathbb{Z}^{|V|}_+ : \sum_{v \in V} C_{vk}\chi_v \geq ND_k(S) \right\}.$$

where

$$\chi_v = \sum_{(i,m) \in S^A_v | i \in N \setminus S} \; \sum_{(j,n) \in S^A_v | j \in S} x_{imjnv},$$

denotes the number of times ship v visits a port in S coming from a port not in S during the planning horizon T.

Valid inequalities for RX are valid for X. A particular case of these inequalities is the following Gomory cut

$$\sum_{v \in V} \; \sum_{(i,m) \in S^A_v | i \in N \setminus S} \; \sum_{(j,n) \in S^A_v | j \in S} \left\lceil \frac{C_{vk}}{Q} \right\rceil x_{imjnv} \geq \left\lceil \frac{ND_k(S)}{Q} \right\rceil, \tag{43}$$

where Q can be any positive number. We take $Q = \overline{C}_k$.

However, when $| \; V \; | = 2$ the convex hull of RX can be completely described in polynomial time, see [6]. When $| \; V \; | > 2$ facet defining inequalities for restrictions of RX to two variables χ_v can be lifted using the lifting function ω_3 presented in [6]. This approach was used in [3]. Here we provide an example.

Example 4.1. *Let $N = \{1, 2, \cdots, 7\}$, $V = \{1, 2, 3, 4\}$, $K = \{1, 2, 3, 4\}$. Fix port $i = 6$, and consider the capacities of the compartments dedicated to product $k = 1 : C_{11} = 900$, $C_{21} = 600$, $C_{31} = 920$, and $C_{41} = 700$. Suppose that for $i = 6$ and $k = 1$ with $J_{61} = -1$, we have $ND_{61} = 3675$. The following relaxation is derived*

$$RX = \{\chi_v \in \mathbb{Z}_+ : 900\chi_1 + 600\chi_2 + 920\chi_3 + 700\chi_4 \geq 3675\}.$$

Inequality $3\chi_1 + 2\chi_2 \geq 13$ *is a facet-defining inequality for RX restricted to* $\chi_3 = \chi_4 = 0$*. The lifting function associated with this inequality is:*

$$\begin{aligned} \varphi(z) \quad &= \max \quad 13 - 3\chi_1 - 2\chi_2 \\ s.\ t. \quad & \qquad 900\chi_1 + 600\chi_2 \geq 3675 - z, \\ & \qquad \chi_1, \chi_2 \in \mathbb{Z}_+. \end{aligned}$$

In order to lift simultaneously the coefficients of χ_3 *and* χ_4*, the lifting function* $\varphi(z)$ *can be overestimated by the subadditive lifting function* ω_3 *described in [6]. Both functions are depicted in Figure 1. Then the lifted inequality* $3\chi_1 + 2\chi_2 + \omega_3(920)\chi_3 + \omega_3(700)\chi_4 \geq 13$ $\Leftrightarrow 3\chi_1 + 2\chi_2 + 3.26667\chi_3 + 3\chi_4 \geq 13$ *is valid for RX.*

Notice that if only three variables are considered then one can use $\varphi(z)$ *instead of* ω_3 *which gives a better coefficient for* χ_3 *since* $\varphi(920) = 3$*.*

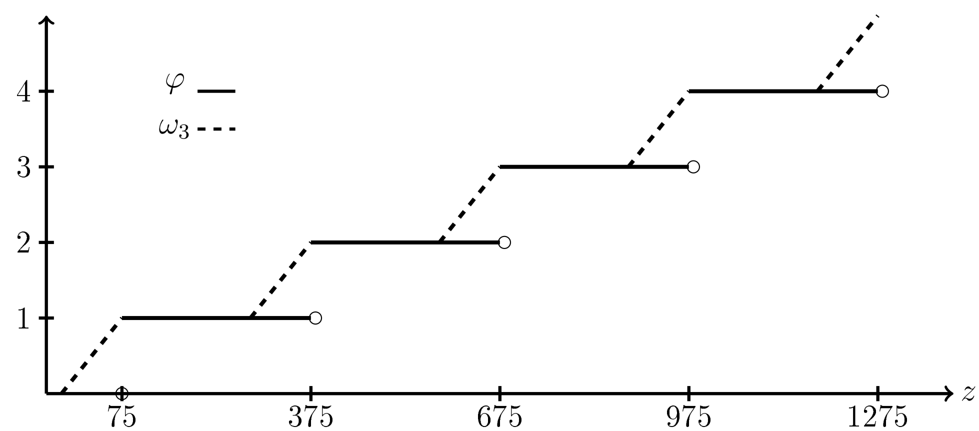

Figure 1: Lifting function φ and subadditive function ω_3.

Similar knapsack inequalities can be derived for loading ports and for relaxations using the operating variables o_{imvk} instead of the traveling variables. For brevity we omit those inequalities.

4.4. Clique inequalities

The name *clique inequalities* has been used for different families of valid inequalities for vehicle routing problems. Here we introduce a family of clique inequalities which can be regarded as a generalization of the subtour elimination constraints (SEC):

$$x_{imjnv} + x_{jnimv} \leq 1$$

Although subtour elimination constraints including more than two variables can be useful to improve the integrality gap, our experience showed that good computational results can be obtained using SEC including only two variables. These inequalities can be regarded a particular case of clique inequalities on a given conflict graph. Consider the conflict graph $G = (\mathcal{N}, E)$, where each node in $\mathcal{N}$, denoted by (i, m, j, n, v), corresponds to a variable x_{imjnv}, and there is an edge in E between two nodes if the corresponding variables cannot be set simultaneously to one (the two nodes are in conflict).

Definition 4.2. *Let $G = (\mathcal{N}, E)$ be a conflict graph. Then we define the following pairs of incompatible variables:*

(i) x_{imjnv} *and* x_{jnimv}, $\forall v \in V, (i, m, j, n) \in S_{X_v}$.

(ii) x_{imjnv_1} *and* x_{imlwv_2}, $\forall v_1, v_2 \in V, (i, m, j, n) \in S_{X_{v_1}}, (i, m, l, w) \in S_{X_{v_2}}$.

(iii) x_{lwjnv_1} *and* x_{imjnv_2}, $\forall v_1, v_2 \in V, (l, w, j, n) \in S_{X_{v_1}}, (i, m, j, n) \in S_{X_{v_2}}$.

(iv) x_{lwjnv_1} *and* x_{jnimv_2}, $\forall v_1, v_2 \in V : v_1 \neq v_2, (l, w, j, n) \in S_{X_{v_1}}, (j, n, i, m) \in S_{X_{v_2}}$.

As consequence of the above discussion we have the following result:

Proposition 4.1. *If $C \subset \mathcal{N}$ is a clique in the conflict graph G, then the inequality*

$$\sum_{(i,m,j,n,v) \in C} x_{imjnv} \leq 1 \tag{44}$$

is valid for X.

Remark 4.3. *An inequality based on a pair of incompatible inequalities of type (i) is a SEC.*

In order to separate clique inequalities we need to consider weights on the nodes. The weight of node (i, m, j, n, v) is given by the value of the variable x_{imjnv} in the linear solution. Finding the most violated clique inequality implies to solve the maximum weight clique problem, which is known to be strongly NP-hard. Here we use a simple greedy separation heuristic. First, find the maximum weight clique with two nodes and update C accordingly. Then augment set C in a greedy fashion. In each iteration add to C the maximum weight node that forms a clique with the nodes in C, that is, $C \leftarrow C \cup \{v^*\}$ where

$$v^* = argmax\{w_v : \forall u \in C, \{u, v\} \in E\}.$$

and w_v is the weight of node v. The process stops when a maximal clique is found. If the resulting clique inequality (44) is violated then it is added as a cut, otherwise no new inequality is added.

Figure 2 shows an example of a linear relaxation solution and the respective conflict graph. Starting with the maximum weight clique with two nodes

$$C = \{(1,1,2,1,2),(1,1,2,2,2)\}.$$

C is further expanded. First with $(2,2,1,1,2)$ and then with $(3,1,1,1,1)$. Hence, $C = \{(1,1,2,1,2),(1,1,2,2,2),(2,2,1,1,2),(3,1,1,1,1)\}$. The (violated) maximal clique inequality is

$$x_{11212} + x_{11222} + x_{31111} + x_{22112} \leq 1$$

.

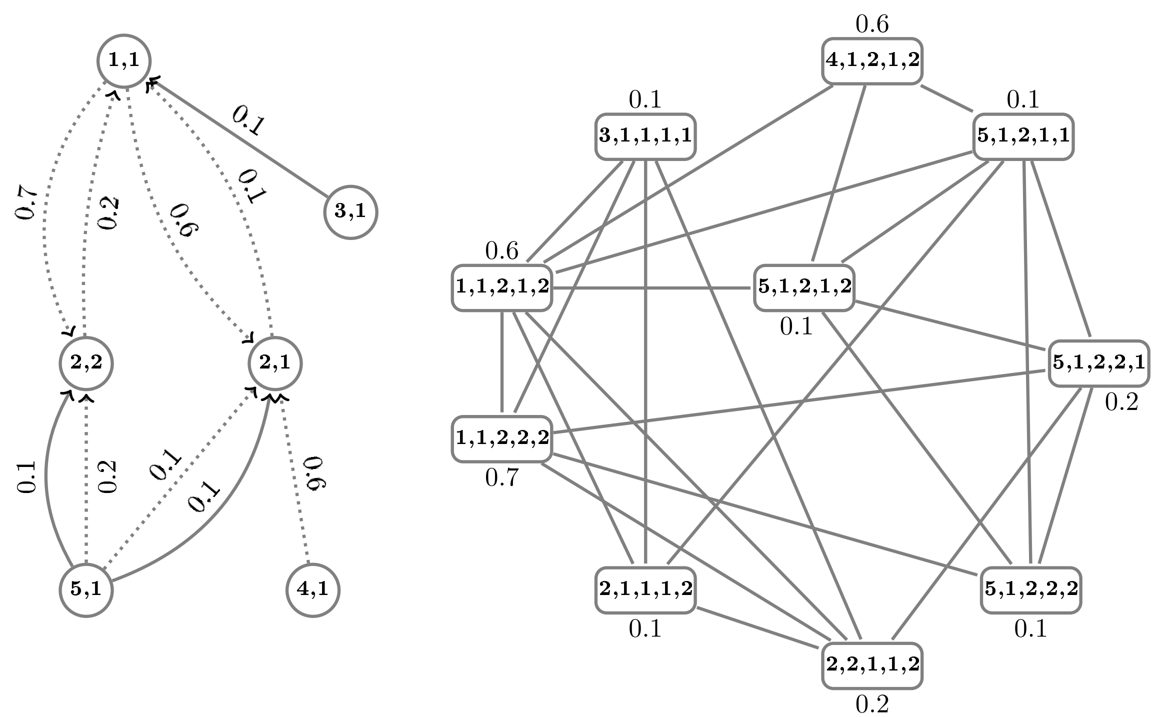

Figure 2: Example of a partial linear relaxation on the left. The two types of arcs represent different ships. The corresponding conflict graph is given on the right.

5. Hybrid heuristics

The formulation F-SSIRP tightened with the strategies discussed in the previous section can hardly be used to solve real instances using a generic MIP solver. However, recent hybrid heuristics have been proposed that use MIP solvers as a black-box tool. Here we consider and combine three such heuristic procedures: rolling horizon, local branching and feasibility pump.

5.1. Rolling Horizon heuristic

When considering a planning horizon of several months, the tested instances become too large to be handled by commercial software. To provide feasible solutions we have developed a Rolling Horizon (RH) heuristic. The main idea of the RH heuristic is to split the planning horizon into smaller sub-horizons, and then repeatedly solve limited and tractable mixed integer problem for the shorter sub-horizons. In transportation problems, RH heuristics have been used in several related works [9, 31, 28, 32].

In each iteration k of the RH heuristic (except the first and last one), the sub-horizon considered is divided into three parts: (i) a frozen part where binary variables are fixed; (ii) a central part (CP_k) where no restriction or relaxation is considered, and (iii) a forecasting period (FP_k) where binary variables are relaxed. The central period in iteration k becomes a frozen period in iteration $k+1$, and the forecasting period from iteration k becomes the central period in iteration $k+1$, see Figure 3. The process is repeated until the whole planning horizon is covered. In each iteration the limited mixed integer problem is solved. When moving from iteration k to iteration $k+1$ we (a) fix the values of the binary variables, (b) update the initial stock level of each product at each port, (c) calculate the quantity of each product on board each ship, and (d) update, for each ship, the initial position and the travel time and cost from that position to every port, see Algorithm 1. Based on preliminary tests we set $CP_k = FP_k = 5$ days.

Algorithm 1 Rolling Horizon heuristic

k $\leftarrow$ 1
U $\leftarrow$ number of iterations to cover the planning horizon $[1, \cdots, T]$
while $k \leq U$ **do**
  Relax binary variables in forecasting period FP_k
  Solve a limited mixed integer problem defined by CP_k and FP_k
  Freeze the variables $x_{imjnv}, x_{oimv}, o_{imvk}, w_{imv}, z_{imv}$ and y_{im} in CP_k
  if $k < U$ **then**
    Update the initial stock level of product k at port i
    Calculate the quantity of each product on board each ship v
    Update, for each ship v, the initial position and the travel time and cost from that position to every port i
  end if
  $k \leftarrow k + 1$
end while

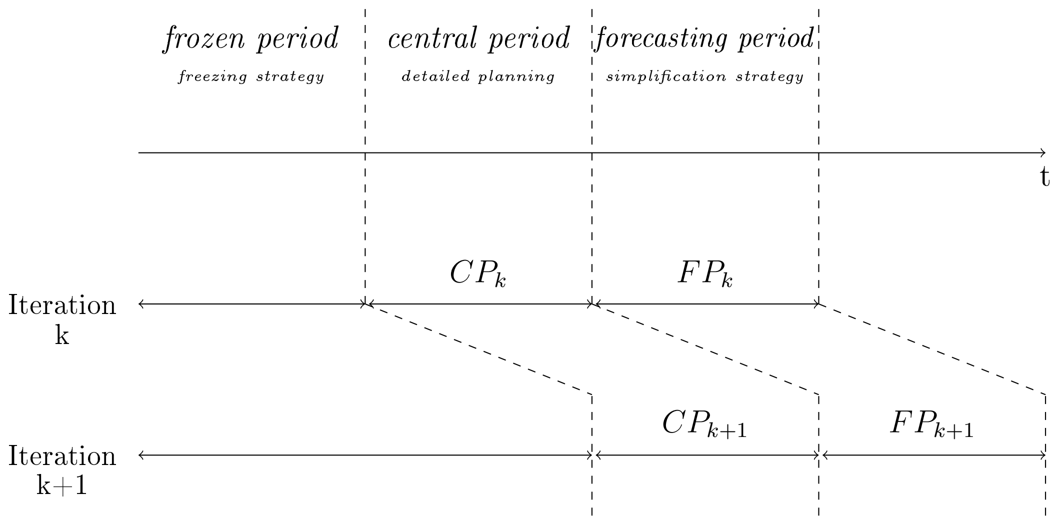

Figure 3: *The rolling horizon heuristic*

5.2. Local Branching heuristic

Local Branching (LB) was introduced in [19] to improve a given feasible solution. The LB heuristic searches for a local optimum by restricting the number of variables that can change their value in the current feasible solution.

More formally, consider a feasible set of the form $\{(u, v) \in \{0, 1\}^n \times \mathbb{R}^m \cap P\}$ where P is a polyhedron. Given a feasible solution $(\overline{u}, \overline{v})$, let $\overline{S} = \{j \in \{1, \cdots, n\} : \overline{u}_j = 1\}$ denote the set of indices of the binary variables that are set to 1. The extra constraint

$$\sum_{j \in \overline{S}} (1 - u_j) \leq \Delta, \tag{45}$$

is considered, where Δ is a given positive integer parameter, indicating the number of variables $u_j, j \in \overline{S}$ that are allowed to flip from one to zero.

Many strategies were tested to combine the two heuristic approaches RH and LB. Here we present only three such strategies. In the RH, the problem is decomposed into subproblems. In each iteration the subproblem is solved to optimality. For the combined heuristics we used the same decomposition as for the RH. For all three combined strategies, for each subproblem, a constraint (45) with $\Delta = 0$ is added on the variables of the frozen period. Doing so, we allow the continuous variables to change their value within the frozen period. The strategies differ in the solution approach for each subproblem, and on whether they perform a local search in the neighborhood of the final solution or not.

LB1: For each subproblem, the solver is interrupted when the first feasible solution is reached.

LB2: Solve each subproblem twice. First the solver is run until either an integrality gap ($gap = 100 \times (UB - LR)/LR$ where UB is the best known upper bound and LR is the best known lower bound) less than or equal to 10% is achieved or a maximum time limit is reached. Then a constraint (45) with $\Delta = 2$ is added over the variables in the central period, and the subproblem is solved again until a gap of 5% is reached or the time limit is attained.

LB3: Obtain a feasible solution with LB2. For a t, $0 < t < T$, impose a constraint (45) with $\Delta = 0$ for the period $[0, T-t]$, and a constraint (45) with $\Delta = 6$ for the period $[T-t, T]$. Solve the new problem. Using the new solution impose new constraints on periods $[0, T-2t]$, with $\Delta = 0$, and $[T-2t, T]$, with $\Delta = 6$, and solve the problem again. This procedure is repeated until at least one of the following stopping criterion is reached: (i) time limit; (ii) maximum number of iterations without improvement; (iii) a maximum number of iterations. This algorithm is detailed in Algorithm 2. In our experiments we used $t = 5$ days, and a maximum number of 5 iterations.

5.3. Feasibility Pump heuristic

Feasibility Pump (FP) was introduced by Fischetti, Glover and Lodi [18] as a heuristic scheme to find a feasible solution for a given mixed integer program. Such a procedure can be useful for those problems where finding an initial solution can be an hard task. FP is a rounding scheme that generates a sequence of fractional solutions from the linear relaxation which are rounded. The heuristic stops when a feasible solution is found or other stopping criteria is reached.

Here we use FP to speed-up the finding of an initial feasible solution. Although we followed the underlying ideas of FP, it was necessary to adjust this heuristic scheme to our MIRP. We focus on the problem at hand and not on the general FP scheme.

In this section, and for simplicity, we denote the points in the space of variables of F-SSIRP by x. First the linear relaxation of F-SSIRP is solved and a linear solution x^* is obtained. Then the binary variables with fractional values are rounded, and a solution $\overline{x}$ is obtained. If $\overline{x}$ is feasible ($\overline{x} \in X$) we stop. Otherwise, a new fractional solution is derived by finding the linear solution in the linear relaxation of X that minimizes a distance function to $\overline{x}$. The process is repeated until a feasible solution is found or a predefined maximal number of iterations is reached. If the rounding procedure stops without a feasible solution, then we run the solver.

Next we address the main steps of the FP algorithm in more detail.

Rounding scheme

For the rounding scheme we first consider the routing variables, x_{imjnv}. We set $\overline{x}_{imjnv} = 1$ whenever $x_{imjnv} > 0.5$ and $\overline{x}_{imjnv} = 0$ whenever $x_{imjnv} < \epsilon$, for small ϵ.

Algorithm 2 LB3 heuristic

// first part (obtain a feasible solution for a planning horizon T, $\overline{x}_{imjnv}$)

T $\leftarrow$ length of the planning horizon
$T_1 \leftarrow$ length of the sub-horizon
Solve the problem for a time horizon of $T_1 = 2t$ periods
Save the feasible solution, $\overline{x}_{imjnv}$, and compute $\overline{S}$
$T_1 \leftarrow T_1 + t$
$\Delta_1 \leftarrow 0$
$\Delta_2 \leftarrow 6$
Bin $\leftarrow$ 0
while $T_1 \leq T$ **do**
    Using the port subproblem $NV^D(i)$, determine the minimum number of visits at each port i for time horizon $[0, T_1]$
    Add constraints $\sum_{j \in \overline{S}} (1 - \overline{x}_j) \leq \Delta_1$ for time horizon $[0; T_1 - 3t]$
    if $Bin = 0$ **then**
        Solve the problem until gap $\leq$ 10% or time limit is reached
        Bin $\leftarrow$ 1
    else
        Add constraints $\sum_{j \in \overline{S}} (1 - \overline{x}_j) \leq \Delta_2$ for time horizon $[T_1 - 3t; T_1]$
        Solve the problem until gap $\leq$ 5% or time limit is reached
        Bin $\leftarrow$ 0
        $T_1 \leftarrow T_1 + t$
        Remove all added constraints and update the model
    end if
    Update the solution, $\overline{x}_{imjnv}$ and $\overline{S}$
end while

// second part (improve the feasible solution , $\overline{x}_{imjnv}$)

number of iterations $\leftarrow 1$
while *number of iterations* $\leq$ *max number of iterations* and *solution improves* **do**
    Reduce the fixed period of variables with t days: $T_1 \leftarrow T_1 - t$
    Add constraints $\sum_{j \in \overline{S}} (1 - \overline{x}_j) \leq \Delta_2$
    Update the solution, $\overline{x}_{imjnv}$ and $\overline{S}$
    number of iterations $\leftarrow$ *number of iterations*$+1$
end while

Using the routing flow conservation constraints we fix the value of the remaining routing variables. Then the remaining binary variables x_{oimv}, w_{imv}, z_{imv}, y_{im} o_{im} are trivially fixed. This guided rounding scheme provided better results than rounding all binary variables simultaneously or rounding all the routing variables simultaneously first. Sophisticated rounding schemes are discussed in [20]. In our experiments we use $\epsilon = 0.1$.

The distance function

Given a 0-1 MIP solution obtained by rounding $\overline{x}$ we define the following distance function

$$\begin{aligned}\phi(x_{imjnv}, \overline{x}_{imjnv}) &= \sum_{v \in V} \sum_{(i,m,j,n) \in S_v^X} |x_{imjnv} - \overline{x}_{imjnv}| \\ &= \sum_{v \in V} \sum_{(i,m,j,n) \in S_v^X | \overline{x}_{imjnv}=1} (1 - x_{imjnv}) \\ &\quad + \sum_{v \in V} \sum_{(i,m,j,n) \in S_v^X | \overline{x}_{imjnv}=0} x_{imjnv} \end{aligned} \tag{46}$$

If $\phi(x_{imjnv}, \overline{x}_{imjnv}) = 0$, then a feasible solution can be derived. Otherwise a new linear solution x^* is obtained by solving the problem:

$$min\{\phi(x_{imjnv}, \overline{x}_{imjnv}) : x \in X_L\}$$

where X_L denotes the linear relaxation of the feasible set X of $F - SSIRP$.

Random perturbation

During the execution of the procedure two problems may arise: (i) the algorithm can be caught in a cycle, i.e., the same sequence is visited consecutively; and (ii) the convergence to a feasible solution is very slow.

Both problems (i) and (ii) are solved by performing a restart, that is, a new 0-1 MIP solution is derived by performing a random perturbation step. This step is similar to the one given in [1] and it is applied to the routing variables on the rounding scheme, that is, $\overline{x}_{imjnv} = \lfloor x^*_{imjnv} + \rho(z) \rfloor$ where $z \in [0, 1]$ is a uniform random variable and $\rho(z) = 2z(1 - z)$ if $z \leq 0.5$ and $\rho(z) = 1 - 2z(1 - z)$ if $z > 0.5$.

To measure the convergence speed we compute the difference between the value of the distance function in two consecutive solutions. When this difference is very small (smaller than a given δ) we perform the random perturbation.

Algorithm 3 describes the FP heuristic. In the computational results we set $\delta = 0.1$ and a maximum number of 50 iterations.

Algorithm 3 Feasibility Pump heuristic

```
Relax binary variables
Solve LP-relaxation of F-SSIRP. Let x* denote its optimal solution
Obtain x̄ by rounding x*
number of iterations ← 1
while number of iterations ≤ max number of iterations and φ(x_imjnv, x̄_imjnv) > 0 do
    Solve the LP: x* ← argmin{φ(x_imjnv, x̄_imjnv) : x ∈ X_L}
    Obtain x̄ by rounding x*
    if φ(x̄_imjnv, x*_imjnv) < δ then
        Apply the random perturbation step
    end if
    number of iterations ← number of iterations+1
end while
```

6. Computational experimentation

In this section we report the computational results when testing different hybrid heuristic approaches.

All computations were performed using the optimization software Xpress Optimizer Version 20.00.05 with Xpress Mosel Version 3.0.0, on a computer with processor Intel Core 2 Duo 2.2GHz and with 4GB of RAM.

We tested 12 real instances from a company in Cape Verde with 2 different ships, 7 ports and 4 products.

First we report a summary of results that testify the model choices. These tests were run for periods of 15 days. Then we report the results from the tests conducted to compare several hybrid strategies for periods of 2 and 6 months.

6.1. Model tuning

First we consider the use of port subproblems to estimate the minimum number of port visits. Figure 4, on the left, shows the minimum number of visits calculated using the formula (35), calculated using the subproblems, and the number of visits in the optimal solution for the 12 instances tested. On the right, the figure depicts the integrality gap (GAP), given by $GAP = 100 \times (OPT - LR)/OPT$ where OPT is the optimal value, obtained using the Xpress optimizer, and LR is the value of the linear relaxation. We consider the cases: "initial" when no minimum number of visits is imposed, "formula" when the minimum is obtained using (35), "subproblem" when the minimum is obtained using port subproblems and "exact" when we consider the minimum equal to the number of visits in the optimal solution.

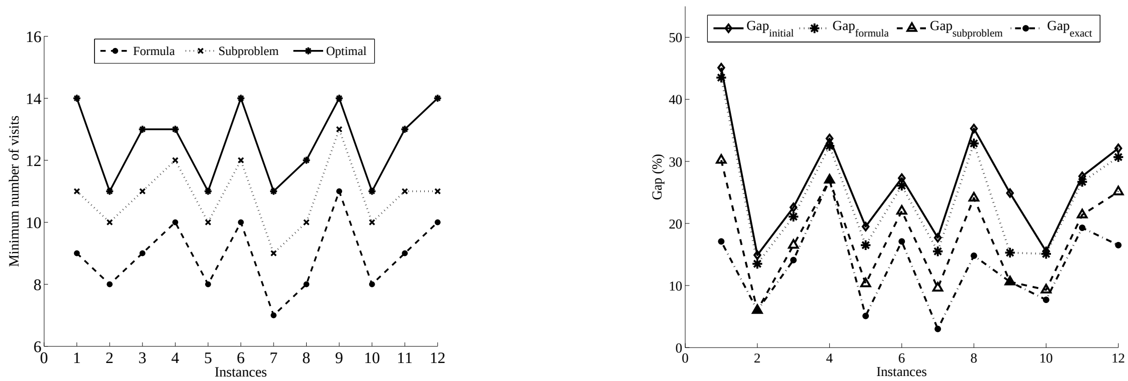

Figure 4: Estimation of the minimum number of visits (on the left) and its impact on the integrality gap (on the right).

In average, the initial integrality gap is 26.7%, drops to 24.1% using equations (35), and drops to 17.7% using subproblems. If the exact value in the optimal solution is used, the average gap is 13.2%.

Table 1 summarizes the integrality gaps when model F-SSIRP is used. TT means that the time constraints were tightened, SP means that the minimum number of visits was estimated using the port subproblem. IK indicates that the Integer Knapsack inequalities are added, and C means that the clique inequalities are added.

Table 1: Evolution of the average integrality gap with model tightening.

F-SSIRP + TT	F-SSIRP + TT + SP	F-SSIRP + TT + SP + IK	F-SSIRP+TT+SP+IK+C
26.7	17.7	10.9	10.9

In Table 2 we present the average solutions times, the number of B&B nodes, and the number of cuts added in each case. We can see that although the clique inequalities do not improve the integrality gap significantly, they are important with regard to the reduction in number of B&B nodes and running time.

6.2. Hybrid heuristics

In this section we report experiments carried out for comparing the hybrid heuristics in terms of running time, integrality gap and number of B&B nodes over two planning horizons: 2 and 6 months. Since the optimal solutions could not be obtained for these time horizons, the integrality gap (GAP) is computed as $GAP = 100 \times (UB - LR)/LR$ where UB is the value obtained by the heuristic and LR is the value of the linear

Table 2: Comparison of time (in seconds), and B&B nodes using valid inequalities.

	F-SSIRP+TT		F-SSIRP+TT+SP+IK			F-SSIRP+TT+SP+IK+C		
Inst.	Time	Nodes	Time	Nodes	Cuts	Time	Node	Cuts
1	288	23788	38	1017	12	36	1015	16
2	11	19	25	1491	5	9	7	6
3	31	1377	51	3451	9	55	5678	16
4	63	3970	26	919	9	17	575	10
5	19	2777	15	2307	7	16	533	11
6	69	6188	23	2433	9	23	2433	9
7	15	754	8	379	5	6	327	6
8	20	8785	18	2917	10	10	622	11
9	40	8071	23	1423	7	24	603	9
10	40	1551	23	3535	9	9	3	13
11	58	16729	111	5383	9	73	2509	11
12	71	9299	41	8003	8	41	8003	8
Average	60.4	6942.3	33.5	2771.5	8.3	26.6	1859.0	10.5

relaxation. The value LR is obtained using the port subproblems to estimate the number of visits, and including IK and C inequalities. These model strengthening techniques are used whenever the optimization of the model F-SSIRP occurs as a subproblem embedded in a hybrid heuristic. The valid inequalities are added only at the root node.

For a time horizon of 2 months, Table 3 shows the performance of the RH heuristic, LB1 and LB1 combined with FP. It reports the time in seconds, the number of B&B and the integrality gap for each heuristic. The performance of LB2 and LB2 combined with FP is given in Table 4, and the performance of LB3 and LB3 combined with FP is given in Table 5.

Table 3: Computational results using RH, LB1 and LB1+FP for $T = 2$ months.

	RH			**LB1**			**LB1+FP**		
Inst.	**Time**	**Nodes**	**Gap**	**Time**	**Nodes**	**Gap**	**Time**	**Nodes**	**Gap**
1	1409	141380	37,1	45	1631	24,8	62	1753	27,7
2	951	148330	26,0	31	692	18,1	88	3229	31,2
3	1421	119833	12,4	365	30027	30,2	401	12420	16,8
4	4908	349909	41,1	51	2118	22,0	110	1700	28,2
5	649	105135	33,5	81	2829	30,8	126	2744	36,2
6	711	106265	33,0	598	53813	38,3	405	29366	30,9
7	362	47432	29,5	384	24356	28,2	321	22785	18,7
8	1285	160392	28,0	225	17487	29,1	256	16439	23,4
9	1107	122907	31,5	684	60289	33,6	322	13265	22,1
10	865	105245	25,8	97	3706	27,0	108	11027	27,1
11	985	143251	28,5	97	3706	28,1	64	2023	26,9
12	1106	167755	30,2	3	13	24,3	74	2838	32,9
Av.	**1313,3**	**143152,8**	**29,7**	**221,8**	**16722,3**	**27,9**	**194,8**	**9965,8**	**26,8**

Table 4: Computational results for LB2 and LB2+FP for $T = 2$ months.

	LB2			**LB2+FP**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	277	19887	23,2	106	4014	16,1
2	104	7982	11,8	72	3859	12,4
3	817	54236	21,8	780	48717	20,7
4	155	10214	22,6	192	12692	18,6
5	552	31737	15,2	252	10013	17,8
6	1755	122197	20,4	940	78983	20,4
7	1066	79101	21,3	481	26912	16,2
8	734	63262	20,0	672	28244	25,4
9	846	54919	16,7	1083	41811	21,7
10	1047	52706	17,5	397	7660	14,1
11	285	10004	20,6	423	11650	18,4
12	744	27989	11,2	456	12493	14,7
Average	**698,5**	**44519,5**	**18,5**	**487,8**	**23920,7**	**18,1**

Table 5: Computational results for LB3 and LB3 + FP for $T = 2$ months.

	LB3			**LB3+FP**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	301	20561	20,5	107	4014	12,9
2	105	7982	8,6	144	7718	12,4
3	951	64918	18,8	781	48717	18,1
4	185	15624	18,2	384	25384	18,6
5	573	33366	11,9	504	20026	17,8
6	2018	131345	20,4	1211	86043	20,4
7	1079	79303	18,5	485	26943	12,9
8	760	64206	17,0	686	28353	17,0
9	850	54919	13,7	1088	41811	18,7
10	1050	52706	14,5	399	7660	11,0
11	312	10770	17,9	425	11650	15,7
12	753	28264	7,8	461	12494	11,5
Average	**744,8**	**46997,0**	**15,7**	**556,3**	**26734,4**	**15,6**

We can see that LB heuristics combined with FP are, in average, faster than the LB heuristics which are in turn faster than the RH heuristic. The use of FP is more relevant on those harder instances, where the solver is not able to find good initial feasible solutions quickly. As expected, LB1 is faster than LB2, and LB2 is faster than LB3. However, the quality of the solutions obtained varies in the opposite direction. The most sophisticated heuristic, LB3 combined with FP, provides solutions with an integrality gap which is, in average, half of the integrality gap of the usual RH heuristic. The running time is almost a third of the running time of the RH heuristic.

Tables 6 and 7 give the computational results for 6 months for heuristics RH, LB1, and LB2 and LB3 combined FP. The behavior of these algorithms is similar to the case of 2 months. Only the gaps are higher. However, as this gap is computed by use of the linear relaxation value we do not know whether this increase results from a deterioration of the upper bound, the lower bound, or both.

Table 6: Computational results for RH and LB1 for $T = 6$ months.

	RH			**LB1+FP**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	3324	107998	42,6	2816	25114	24,3
2	10258	207125	44,8	1937	23517	28,7
3	3451	62775	45,6	2872	57014	26,1
4	4631	115802	41,6	1040	14311	26,5
5	6149	103324	47,7	3689	48353	32,8
6	10288	139427	42,5	3977	77989	31,5
7	7219	105059	42,4	1468	35739	27,8
8	3776	166414	46,2	1213	34326	32,5
9	4196	209323	47,2	7792	102636	29,7
10	2658	113510	45,1	4854	39172	30,5
11	13244	208361	44,8	569	12772	27,9
12	2079	93102	45,1	3042	35513	29,4
Average	**5939,4**	**136018,3**	**44,6**	**2939,1**	**42204,7**	**29,0**

Table 7: Computational results for LB2 and LB3 for $T = 6$ months

	LB2+FP			**LB3+FP**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	4404	166993	23,1	4551	167148	21,1
2	1260	78999	20,7	1300	79060	18,6
3	2469	83566	23,8	2507	83647	22,0
4	1736	83330	20,3	1819	83457	18,2
5	2917	99785	28,2	3142	100031	26,6
6	3109	114450	28,7	3125	114455	27,1
7	2899	102661	31,9	3004	102776	30,4
8	2349	113899	28,7	2480	114137	27,1
9	3894	142451	21,1	4109	142606	19,2
10	1392	53626	20,7	1598	53742	18,7
11	2308	110136	24,4	2454	110286	22,6
12	1607	67245	24,5	1881	67355	22,8
Average	**2528,7**	**101428,4**	**24,7**	**2664,1**	**101558,5**	**22,9**

To test the heuristic approaches that performed best on the larger instances, we created two artificial future scenarios where the demands as well as the number of ships are increased. One scenario with three ships and demands that are 1.5 times the current demands, and another scenario with four ships and double demands. Each scenario is identified by the number of ships ($| V |= 3$ and $| V |= 4$). We opted not to reduce the length of each sub horizon. All the tested heuristics run within a reasonable computational time effort for 2 months. For 6 months, RH, LB2 and LB3 heuristics were too time consuming.

In Table 8 we give the computational results. For $| V |= 3$ we used a variant of LB2, where only the first run (until a gap of 10%) is performed, combined with FP. For $| V |= 4$ we used LB1 combined with FP. We could not solve most of the linear relaxations within 1 day time limit. To compute the lower bound we computed the linear relaxation of the model obtained from F-SSIRP by removing all time and inventory constraints, and with the additional cuts discussed in Section 4. Additionally we imposed, for each port i and each product k such that $J_{ik} = -1$, the constraint $\sum_{v \in V} \sum_{m=1}^{\mu_i} q_{imvk} \geq T \times R_{ik} + \underline{S}_{ik} - S^0_{ik}$.

Table 8: Computational results for larger instances with 3 and 4 ships

	|V|=3			**|V|=4**		
Instance	**Time (sec.)**	**Nodes**	**Gap**	**Time (sec.)**	**Nodes**	**Gap**
1	988	10154	27,0	5218	45921	31,0
2	1096	20695	29,7	5017	44186	35,1
3	924	30403	29,8	4633	51406	24,5
4	2120	34692	30,3	6804	47798	28,2
5	2120	49307	32,7	5706	49415	35,9
6	2199	25836	36,9	10988	55062	40,8
7	1158	32612	33,7	3338	48450	31,2
8	2340	62303	33,3	4173	54671	30,7
9	1486	51884	29,9	6813	52666	35,2
10	1857	51934	35,0	9958	47864	34,3
11	2275	25875	31,1	4581	49583	36,6
12	2628	30691	31,1	5064	47717	31,2
Average	**1765,9**	**35532,2**	**31,7**	**6024,4**	**49561,6**	**32,9**

7. Conclusions

We have presented a mathematical model for the short sea inventory routing problem. This model is tightened with valid inequalities and an estimation of the minimum number of visits to each port by solving some port subproblems. In particular we introduced new clique inequalities that can be used to tighten continuous time maritime inventory routing models.

Given the long time horizons, we propose and compare different strategies of combining three well-known heuristics that use the mathematical model as a black-box. The Rolling Horizon heuristic is used to decompose the original problem into smaller and more tractable problems, the Feasibility Pump heuristic is used to find initial solutions for MIP problems, and the Local Branching heuristic is used to improve feasible solutions.

The best strategy tested combines all the three heuristics, and allowed us to obtain solutions whose integrality gap is in average half of the integrality gap obtained using the rolling horizon heuristic alone. We provided computational results for time horizons up to 6 months.

In order to evaluate the quality of the solutions obtained by the hybrid procedures, an important future direction of research is to investigate approaches to derive tight lower bounds, specially for long time horizons where the size of the linear relaxation model is quite large.

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