Bisimulation for reactive frames

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Abstract. Reactive frames are those whose structure is not fixed but can vary according to the path chosen. This kind of frame has been study and both a logic and an axiomatization for it were already developed. In this paper we take this study further and define a notion of bisimulation for reactive models. We show that the logic introduced by Marcelino ([7]) for these frames is invariant under our notion of bisimulation. Finally, we proof the Hennessy-Milner theorem for a class of reactive models.

1 Introduction and motivation.

The idea of a frame whose structure can vary is not new. In fact, in 2005, Johan van Benthem publishes a seminal paper where introduces the sabotage logic (see [4]). In this logic, it is possible to delete an edge from the Kripke frame. In the opposite way, we can also consider the bridge logic where we can have new edges being created. Also, in [1], C. Areces *et al.* introduce the swap logic where the direction of an edge is swapped after cross it. More information about these logics and other generalizations can be found in [2].

In this paper, we focus our attention in reactiveness of frames and in the logic proposed by Gabbay and Marcelino in [7] for reactive frames.

Our motivation to study such models is due to its ability of modelling complex biochemical phenomena. Biological processes as those occuring in vaccination sugest that biological systems present, in some sense, "memory". To present and example of a system where reactiveness occurs we introduce the directed enzyme prodrug therapy. In this procedure, the prodrug (usually a chemical compound) has not effect by itself when administrated, however, some other enzyme can interact with the prodrug in order to convert it in an active form which will have therapeutic effect. This kind of procedure has been experimented in the treatment of cancer: enzymes are addministrated and accumulate near the cancerous cells. When the prodrug is administrated, it becomes extremely toxic by reacting with the enzyme. Because of this, we can destroy the cancerous cells marked by enzymes and still avoid the damage to general cells (see [3]).

In Figures 1. and 2., the existence of a double arrows from an edge a to to an edge b indicates that the edge b must be deleted when the edge a is crossed.

Example 1. We call the prodrug system referred in [8] where the authors propose and test three enzymes to bind tumor cells and administrate a prodrug afterwards. This experiment had promising results and a reactive model of it is shown in Figure 1.

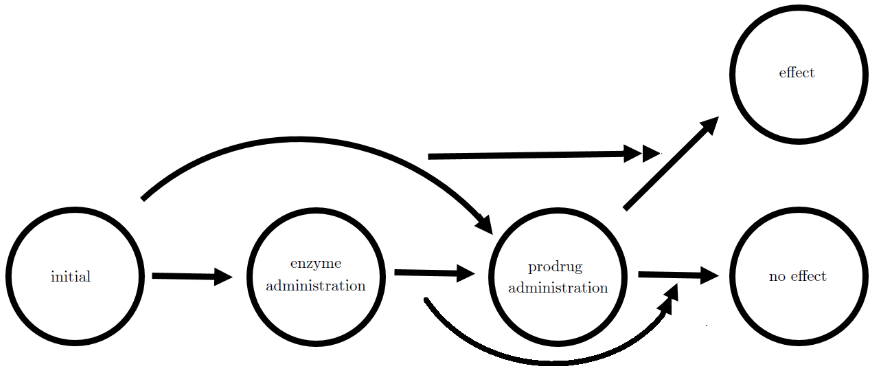

Fig. 1. Prodrug system.

This example shows the importance of these models and why is it necessary to have theoretical tools to study them. In this paper we study the concept of bisimulation which is an important tool to study and reduce models.

2 A Logic to reactive frames.

We assume that the reader in familiarized with the concepts of modal logic, Kripke models and bisimulation. Otherwise, see [6] for more information.

Definition 1. *Let W be a set (of vertices). W^* denotes the set of all finite sequences* (paths) *over elements from W.*

Let $\Delta \subseteq W^*$ be a nonempty set of paths. (W, Δ) *is a* reactive frame *if* $(w) \in \Delta$ *for any $w \in W$ and if $(w_1, ..., w_n, w_{n+1}) \in \Delta$ implies $(w_1, ..., w_n) \in \Delta$.*

Let $\lambda = (w_1, ..., w_n)$, be a path. We simply write λw when referring to the path $(w_1, ..., w_n, w)$. Also, if no ambiguity arises, we denote by w_1 the path (w_1).

Given a set W and a set of paths $\Delta \subseteq W^*$ (of finite length, always), we define the function $t : \Delta \to W$ by $t(\lambda) = t(w_1, ..., w_n) = w_n$.

Definition 2. *Let Π be a set of propositions and $X \subseteq \Pi$. A X-*reactive model *is triple (W, Δ, V) where (W, Δ) is a reactive frame and $V : \Pi \to 2^\Delta$ is a function such that $V(p)(\lambda) = V(p)(t(\lambda))$ for any $p \in X, \lambda \in \Delta$.*

The set X contains the variables whose validity depend exclusively on the last world of the path. Next, we introduce the syntax and the semantics of the logic for reactive frames and some concepts which will be needed further.

Definition 3. *Let (W, Δ) be a reactive frame. Let $R \subseteq \Delta \times \Delta$ be a relation of paths defined as $R = \{(\lambda, \lambda w) : w \in W\}$. Also, we define the relation $P \subseteq \Delta \times \Delta$ as $P = \{(\lambda, \gamma) : t(\lambda) = t(\gamma)\}$. Given a set Π of proposition, we define the set of* $\mathcal{L}_r$-formulas *by the least set containing Π and such that, for any formulas φ and ψ, $\neg\varphi$, $\varphi \vee \psi$, $\Diamond_R \varphi$ and $\Diamond_P \varphi$ are formulas.*

Definition 4. *We define the validity of a $\mathcal{L}_r$-formula φ for a X-reactive model M at $\lambda \in \Delta$ and denote it by $M, \lambda \vDash_X \varphi$ recursively as:*

- $M, \lambda \vDash_X p$ *iff* $\lambda \in V(p)$
- $M, \lambda \vDash_X \neg\varphi$ *iff* $M, \lambda \nvDash_X \varphi$
- $M, \lambda \vDash_X \varphi \vee \psi$ *iff* $M, \lambda \vDash_X \varphi$ *or* $M, \lambda \vDash_X \psi$
- $M, \lambda \vDash_X \Diamond_R \varphi$ *iff* $\exists w \in W\ (M, \lambda w \vDash_X \varphi$ *and* $\lambda w \in \Delta)$
- $M, \lambda \vDash_X \Diamond_P \varphi$ *iff* $\exists \gamma \in \Delta\ (M, \gamma \vDash_X \varphi$ *and* $t(\lambda) = t(\gamma))$

We note that, if $\varphi \in X$, then $M, \lambda \vDash_X \varphi \Leftrightarrow M, (t(\lambda)) \vDash_X \varphi$

3 Bisimulation for reactive-models

In this section, we define the bisimulation for reactive models and prove the semantical equivalence between bisimular states.

Definition 5. *Let (W, Δ, V) and (W', Δ', V') be reactive models. We say that a relation $\mathcal{S} \subseteq \Delta \times \Delta'$ is a bisimulation if and only if $\forall \lambda \in \Delta, \forall \lambda' \in \Delta'$ such that $(\lambda, \lambda') \in \mathcal{S}$:*

atom: $V(p)(\lambda) = V'(p)(\lambda')$, *for all* $p \in \Pi$
R-zig: $\forall w \in W(\lambda w \in \Delta \Rightarrow \exists w' \in W', \lambda' w' \in \Delta'$ *such that* $(\lambda w, \lambda' w') \in \mathcal{S})$
R-zag: $\forall w' \in W'(\lambda' w' \in \Delta' \Rightarrow \exists w \in W, \lambda w \in \Delta$ *such that* $(\lambda w, \lambda' w') \in \mathcal{S})$
P-zig: $\forall \gamma \in \Delta(t(\lambda) = t(\gamma) \Rightarrow \exists \gamma' \in \Delta'(t(\lambda') = t(\gamma')$ *and* $(\gamma, \gamma') \in \mathcal{S}))$
P-zag: $\forall \gamma' \in \Delta'(t(\lambda') = t(\gamma') \Rightarrow \exists \gamma \in \Delta(t(\lambda) = t(\gamma)$ *and* $(\gamma, \gamma') \in \mathcal{S}))$

Example 2. In Figure 6, two bisimilar reactive models are shown. In fact, we can verify that the following relation is a bisimulation:

$\{((w_1),(v_1)), ((w_1,w_2),(v_1,v_2)), ((w_2),(v_2)), ((w_2,w_3),(v_2,v_2)), ((w_4),(v_1)), ((w_4,w_3),(v_1,v_2)), ((w_3),(v_2)), ((w_3,w_2),(v_2,v_2))\}$

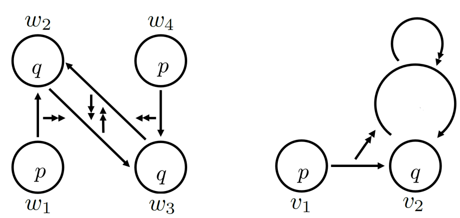

Fig. 2. Two bisimilar models.

A basic property of bisimulation is that it preserves validity. This is stated in the following theorem.

Theorem 1. *Let $\lambda \in \Delta, \lambda' \in \Delta'$ and let $\mathcal{S} \subseteq \Delta \times \Delta'$ be a bisimulation. Then $(\lambda, \lambda') \in \mathcal{S}$ implies $M, \lambda \vDash_X \varphi \Leftrightarrow M', \lambda' \vDash_X \varphi$ for any formula φ.*

Proof. The proof is done via induction over formulas.

If $\varphi \in \Pi$, then $M, \lambda \vDash_X \varphi \Leftrightarrow M', \lambda' \vDash_X \varphi$ by definition of bisimulation.

$M, \lambda \vDash_X \neg\varphi \Leftrightarrow M, \lambda \nvDash_X \varphi \Leftrightarrow M', \lambda' \nvDash_X \varphi \Leftrightarrow M', \lambda' \vDash_X \neg\varphi$.

$M, \lambda \vDash_X \varphi \wedge \psi \Leftrightarrow M, \lambda \vDash_X \varphi$ and $M, \lambda \vDash_X \psi$
$\Leftrightarrow M', \lambda' \vDash_X \varphi$ and $M', \lambda' \vDash_X \psi \Leftrightarrow M', \lambda' \vDash_X \varphi \wedge \psi$.

$M, \lambda \vDash_X \Diamond_R\varphi \Leftrightarrow \exists w \in W, \lambda w \in \Delta$ and $M, \lambda w \vDash_X \varphi$
$\Leftrightarrow \exists w' \in W', \lambda' w' \in \Delta'$ and $M', \lambda' w' \vDash_X \varphi \Leftrightarrow M', \lambda' \vDash_X \Diamond_R\varphi$.

$M, \lambda \vDash_X \Diamond_P\varphi \Leftrightarrow \exists \gamma \in \Delta,\ t(\gamma) = t(\lambda)$ and $M, \gamma \vDash_X \varphi$
$\Leftrightarrow \exists \gamma' \in \Delta',\ t(\gamma') = t(\lambda')$ and $M', \gamma' \vDash_X \varphi \Leftrightarrow M', \lambda' \vDash_X \Diamond_P\varphi$.

□

For the reciprocal of Theorem 1, we follow the idea of [2] in order to construct a bisimulation which relates paths indistinguishable by $\mathcal{L}_r$-formulas.

Definition 6. *Let Σ be a set of formulas and (W, Δ, V) a X-reactive model.*

Σ is satisfiable over a set of paths $\Lambda \subseteq \Delta$ if there is a path $\lambda \in \Lambda$ such that $\lambda \vDash_X \varphi$ for any $\varphi \in \Sigma$.

Σ is finitely satisfiable over a set of paths $\Lambda \subseteq \Delta$ if, for any finite subset $\bar{\Sigma} \subseteq \Sigma$, there is a path $\lambda \in \Lambda$ such that $\lambda \vDash_X \varphi$ for any $\varphi \in \bar{\Sigma}$.

Consider a relation Z and let $Z_\lambda = \{\gamma : \lambda Z \gamma\}$. A model is Z-saturated *if, for all λ, any set Σ is satisfiable over Z_λ whenever Σ is finitely satisfiable over Z_λ.*

Theorem 2. *Let M and M' be two P-saturated and R-saturated cs-model. If we define the relation $\mathcal{S} \subseteq \Delta \times \Delta'$ such that $(\lambda, \lambda') \in \mathcal{S}$ iff for any formula φ, $M, \lambda \vDash_X \varphi \Leftrightarrow M', \lambda' \vDash_X \varphi$. Then $\mathcal{S}$ is a bisimulation.*

Proof. Let us now suppose that $(\lambda, \gamma) \in R$, for some $\gamma \in \Delta$ and consider $Sat(\gamma) = \{\varphi : M, \gamma \vDash_X \varphi\}$. Then, for each finite subset $\Sigma' \subseteq Sat(\gamma)$, we know that $M, \lambda \vDash_X \Diamond_R \bigwedge_{\varphi \in \Sigma'} \varphi$ and, therefore, $M', \lambda' \vDash_X \Diamond_R \bigwedge_{\varphi \in \Sigma'} \varphi$. This means that $Sat(\gamma)$ is finitely satisfiable over $R_{\lambda'}$ and since M' is R-saturated, $Sat(\gamma)$ is satisfied over $R_{\lambda'}$. Thus, exists a state γ' such that $(\lambda', \gamma') \in R$ and $(\gamma, \gamma') \in S$.

Analogously, if $(\lambda, \lambda') \in \mathcal{S}$ and $(\lambda', \gamma') \in R$, then there exists some $w \in W$ such that $(\lambda w, \lambda' w') \in \mathcal{S}$.

Let us now suppose that $(\lambda, \gamma) \in P$, for some $\gamma \in \Delta$ and consider $Sat(\gamma) = \{\varphi : M, \gamma \vDash_X \varphi\}$. Then, for each finite subset $\Sigma' \subseteq Sat(\gamma)$, we know that $M, \lambda \vDash_X \Diamond_P \bigwedge_{\varphi \in \Sigma'} \varphi$ and, therefore, $M', \lambda' \vDash_X \Diamond_P \bigwedge_{\varphi \in \Sigma'} \varphi$. This means that $Sat(\gamma)$ is finitely satisfiable over $P_{\lambda'}$ and since M' is P-saturated, $Sat(\gamma)$ is satisfied over $P_{\lambda'}$. Thus, exists a state γ' such that $(\lambda', \gamma') \in P$ and $(\gamma, \gamma') \in S$.

Analogously, if $(\lambda, \lambda') \in \mathcal{S}$ and $(\lambda', \gamma') \in P$, then there exists some $\gamma \in \Delta$ such that $(\lambda, \gamma) \in P$ and $(\gamma, \gamma') \in \mathcal{S}$.

Finally, if $(\lambda, \lambda') \in \mathcal{S}$, then we can trivially verify that, $\forall p \in \Pi$, $M, \lambda \vDash_X p \Leftrightarrow M', \lambda' \vDash_X p$ by definition.

□

We note that, without the condition that the models are R-saturated, this theorem would be, in general, false. This result can be obtained from the counterexample for the analogous theorem about standard Kripke frames which can be found in [5].

4 Conclusion and future work

In this paper we continue the study of reactive frames in order to formalize and provide tools to describe and characterize model with reactiveness. In particular, we focused in the notion of bisimulation. Furthermore, we establish, under few natural conditions, the complete Hennessy-Milner theorem.

This work gains importance due to the possibility of application in biochemical systems as, for instance, those concerning Health and therapeutics.

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