Fundamental solutions of the time fractional diffusion-wave and parabolic Dirac operators[*]

M. Ferreira[†,‡]   N. Vieira[‡]

[†] School of Technology and Management,

Polytechnic Institute of Leiria

P-2411-901, Leiria, Portugal.

[‡] CIDMA - Center for Research and Development in Mathematics and Applications

Department of Mathematics, University of Aveiro

Campus Universitário de Santiago, 3810-193 Aveiro, Portugal.

E-mails: milton.ferreira@ipleiria.pt,nloureirovieira@gmail.com

Abstract

In this paper we study the multidimensional time fractional diffusion-wave equation where the time fractional derivative is in the Caputo sense with order $\beta \in \,]0,2]$. Applying operational techniques via Fourier and Mellin transforms we obtain an integral representation of the fundamental solution (FS) of the time fractional diffusion-wave operator. Series representations of the FS are explicitly obtained for any dimension. From these we derive the FS for the time fractional parabolic Dirac operator in the form of integral and series representation. Fractional moments of arbitrary order $\gamma > 0$ are also computed. To illustrate our results we present and discuss some plots of the FS for some particular values of the dimension and of the fractional parameter.

Keywords: Time fractional diffusion-wave operator; Time fractional parabolic Dirac operator; Fundamental solutions; Caputo fractional derivative; Fractional moments.

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1   Introduction

Fractional diffusion-wave equations are obtained from the standard diffusion and wave equations by replacing time and/or space derivatives by a fractional derivative of order $\beta \in \,]0,2]$, for example, Riemann-Liouville, Weyl, Caputo, Riesz, or Riesz-Feller, just to mention some of the most used types of fractional derivatives. The introduction of fractional derivatives allows to represent the physical reality more accurately by introducing a memory mechanism in the process (see [2]). These equations represent anomalous diffusion $(0 < \beta < 1)$ or anomalous wave propagation $(1 < \beta < 2)$ and have been studied over the last years by several authors. Just to clarify, an anomalous diffusion propagation process corresponds to a propagation process that does not follow Gaussian statistics on long time intervals.

In the one dimensional case, solutions of time and/or space fractional diffusion-wave equations have been constructed and studied comprehensively in several papers (see, for example, [7, 19, 20, 23, 25–27, 30, 34] and the references therein indicated). The fundamental solution (FS) of the time fractional diffusion-wave equation represents a slow diffusion process for $0 < \beta < 1$ whereas for $1 < \beta < 2$ it represents a wave diffusion faster than the Gaussian diffusion. This gives a unification of diffusion and wave propagation phenomena. One of the

first works in this direction was made by Wyss in [34]. Here the author obtained the FS of the one dimensional time fractional diffusion equation in the form of Fox H-functions. In [7] the author studied independently the fractional diffusion-wave equation, obtaining not only a representation for the FS but also additional properties of it. In [27] the FS for the Cauchy and Signalling problems associated to the time fractional diffusion-wave equation were expressed in terms of entire functions of Wright type. In [25] it was showed that the FS obtained in [27] can be interpreted as a spatial probability density function evolving in time with similarity properties. In [19,20,23,26] the authors studied and obtained the FS for more general equations where both space and time derivatives are fractional.

For the multidimensional case there are some works in this direction (see e.g. [11,12,18,30]). In [30] a closed form of the FS in terms of Fox H-functions was obtained and some of their properties were studied. In [11,12,18] multidimensional time-fractional and space-time fractional diffusion-wave equations were investigated. However, in these works a generic series representation for the FS in an arbitrary dimension was not obtained. In [11,12] there are only integral representations for the FS and in [18] series representations were obtained up to dimension 3 for the neutral fractional wave equation. Furthermore, a representation of the FS in the form of an absolute convergent series enables us to handle these functions in an easier way and to apply them in approximations.

In this paper we obtain explicitly integral and series representations for the FS of the diffusion-wave equation, for an arbitrary dimension. The computations are much more involved and the series representations obtained depend on the parity of the dimension. In connection with the time fractional diffusion-wave operator we also consider the time fractional parabolic Dirac operator. This is a first-order differential operator in space combined with a fractional derivative of order $\beta \in \,]0,2]$ in time, written using a Witt basis. This operator factorizes the time fractional diffusion-wave operator. Hence, its solutions can be seen as a refinement of the solutions of the time fractional diffusion-wave operator. We also obtain the FS of the time fractional parabolic Dirac operator for an arbitrary dimension. This opens new possibilities for the development of a fractional function theory for this operator in the context of Clifford analysis and the study, e.g., of the fractional Schrödinger equation. For the integer case $\beta = 1$ the parabolic Dirac operator was proposed in [4] using a Clifford algebra approach to study the time-dependent Navier-Stokes equation. This allowed a successful adaptation of already existent techniques in elliptic function theory (see [10]) to non-stationary problems in time-varying domains (see for example [3,17,33]). The geometric nature of Clifford algebras allows the resolution of PDEs using the geometric properties of the domain where the differential operator acts (see [4]). Connections between Clifford analysis and fractional calculus were recently established in [6,14,32] in the study of the stationary fractional Dirac operator.

The structure of the papers reads as follows: in the preliminaries section we recall some basic concepts about Clifford analysis, Witt basis, fractional calculus, special functions and integral transforms. In Sections 3 and 4 we construct integral and series representations for the FS of the time fractional diffusion-wave operator and the time fractional parabolic Dirac operator in $\mathbb{R}^n \times \mathbb{R}^+$, respectively. These representations depend on the parity of the space dimension. The particular cases of $\beta = 0, 1, 2$ are discussed in Section 5. Fractional moments of arbitrary order $\gamma > 0$ are computed in Section 6 for the FS of the time fractional diffusion-wave operator. Finally, in Sections 7 and 8 we present and discuss some plots of the FS obtained in Sections 3 and 4.

2 Preliminaries

2.1 Clifford analysis

We consider the n-dimensional vector space $\mathbb{R}^n$ endowed with an orthonormal basis $\{e_1, \cdots, e_n\}$. The universal real Clifford algebra $C\ell_{0,n}$ is defined as the 2^n-dimensional associative algebra which obeys the multiplication rule

$$e_i e_j + e_j e_i = -2\delta_{ij}, \quad i,j = 1,\ldots,n. \tag{1}$$

A vector space basis for $C\ell_{0,n}$ is generated by the elements $e_0 = 1$ and $e_A = e_{h_1} \cdots e_{h_k}$, where $A = \{h_1,\ldots,h_k\} \subset M = \{1,\ldots,n\}$, for $1 \leq h_1 < \cdots < h_k \leq n$. Each element $x \in C\ell_{0,n}$ can be represented by $x = \sum_A x_A e_A$, with $x_A \in \mathbb{R}$. The Clifford conjugation is defined by $\overline{x} = \sum_A x_A \overline{e}_A$, where $\overline{e}_A = \overline{e}_{h_k} \cdots \overline{e}_{h_1}$, and $\overline{e}_j = -e_j$, for $j = 1,\ldots,n$, and $\overline{e}_0 = e_0 = 1$. We introduce the complexified Clifford algebra $\mathbb{C}_n$ as the tensor

product

$$\mathbb{C}_n := \mathbb{C} \otimes C\ell_{0,n} = \left\{ w = \sum_A w_A e_A, \ w_A \in \mathbb{C}, A \subset M \right\},$$

where the imaginary unit i of $\mathbb{C}$ commutes with the basis elements, i.e., $ie_j = e_j i$ for all $j = 1, \ldots, n$. To avoid ambiguities with the Clifford conjugation, we denote the complex conjugation by $\sharp$, in the sense that for a complex scalar $w_A = a_A + ib_A$ we have that $w_A^\sharp = a_A - ib_A$. The complex conjugation can be extended linearly to whole of the Clifford algebra and leaves the elements e_j invariant, i.e., $e_j^\sharp = e_j$ for all $j = 1, \ldots, n$. We also have a pseudonorm on $\mathbb{C}_n$ defined by $|w| := \sum_A |w_A|$ where $w = \sum_A w_A e_A$. Notice also that for $a, b \in \mathbb{C}_n$ we only have $|ab| \le 2^n |a||b|$.

A $\mathbb{C}_n$-valued function defined on an open set $U \subseteq \mathbb{R}^n$ has the representation $f = \sum_A f_A e_A$ with $\mathbb{C}$-valued components f_A. Properties such as continuity and differentiability need to be understood componentwise. Next, we introduce the Euclidean Dirac operator $D_x = \sum_{i=1}^n e_i \partial_{x_i}$, which factorizes the n-dimensional Euclidean Laplacian, i.e., $D_x^2 = -\Delta = -\sum_{i=1}^n \partial_{x_i}^2$. A Clifford valued C^1-function f is called *left-monogenic* if it satisfies $D_x f = 0$ on U (resp. *right-monogenic* if it satisfies $f D_x = 0$ on U).

In order to define the parabolic Dirac operator we need to introduce a Witt basis. We start considering the embedding of $\mathbb{R}^n$ into $\mathbb{R}^{n+2}$ and two new elements e_+ and e_- such that $e_+^2 = +1$, $e_-^2 = -1$, and $e_+ e_- = -e_- e_+$. Moreover, e_+ and e_- anticommute with all the basis elements e_i, $i = 1, \ldots, n$. Hence, $\{e_1, \ldots, e_n, e_+, e_-\}$ spans $\mathbb{R}^{n+1,1}$. With the elements e_+ and e_- we construct two nilpotent elements $\mathfrak{f}$ and $\mathfrak{f}^\dagger$ given by

$$\mathfrak{f} = \frac{e_+ - e_-}{2} \quad \text{and} \quad \mathfrak{f}^\dagger = \frac{e_+ + e_-}{2}.$$

These elements satisfy the following relations

$$(\mathfrak{f})^2 = (\mathfrak{f}^\dagger)^2 = 0, \qquad \mathfrak{f}\mathfrak{f}^\dagger + \mathfrak{f}^\dagger \mathfrak{f} = 1, \qquad \mathfrak{f}e_j + e_j \mathfrak{f} = \mathfrak{f}^\dagger e_j + e_j \mathfrak{f}^\dagger = 0, \ j = 1, \ldots, n. \tag{2}$$

The extended basis $\{e_1, \ldots, e_n, \mathfrak{f}, \mathfrak{f}^\dagger\}$ allow us to define the parabolic Dirac operator as $D_{x,t} := D_x + \mathfrak{f}\partial_t + \mathfrak{f}^\dagger$, where D_x stands for the Dirac operator in $\mathbb{R}^n$. The operator $D_{x,t}$ acts on $\mathbb{C}_n$-valued functions defined on time dependent domains $\Omega \times I \subseteq \mathbb{R}^n \times \mathbb{R}^+$, i.e., functions in the variables $(x_1, x_2, \ldots, x_n, t)$ where $x_i \in \mathbb{R}$ for $i = 1, \ldots, n$, and $t \in \mathbb{R}^+$. For the sake of readability, we abbreviate the space-time tuple $(x_1, x_2, \ldots, x_n, t)$ simply by (x, t), assigning $x = x_1 e_1 + \cdots + x_n e_n$. For additional details on Clifford analysis, we refer the interested reader for instance to [4, 5, 10, 33].

2.2 Integral and Integro/differential operators and special functions

2.2.1 Integral and integro/differential operators

For a locally integrable function f on $\mathbb{R}^n$, the multidimensional Fourier transform of f is the function defined by the integral

$$(\mathcal{F}f)(\kappa) = \widehat{f}(k) = \int_{\mathbb{R}^n} e^{i\kappa \cdot x} f(x) \, dx, \tag{3}$$

where $x, \kappa \in \mathbb{R}^n$ and $\kappa \cdot x$ denotes the usual inner product in $\mathbb{R}^n$. For a differential operator $\frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}}$ of order $\alpha = |\alpha| := \alpha_1 + \ldots + \alpha_n$ we have the following relation between differentiation and the Fourier transform (3):

$$\frac{\widehat{\partial^{|\alpha|} f}}{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}} = (i)^{|\alpha|} \, \kappa^\alpha \, \widehat{f}(\kappa),$$

where $\kappa^\alpha := \kappa_1^{\alpha_1} \cdot \ldots \cdot \kappa_n^{\alpha_n}$. In particular, for the Laplace operator $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial_{x_i}^2}$ we have

$$\widehat{\Delta f}(\kappa) = (i)^2 \, (\kappa_1^2 + \ldots + \kappa_n^2) \, \widehat{f}(\kappa) = -|\kappa|^2 \, \widehat{f}(\kappa). \tag{4}$$

Another integral transform that we use in this work is the Mellin transform. For f locally integrable on $]0, \infty[$ it is defined by

$$\mathcal{M}\{f\}(s) = \int_0^{+\infty} x^{s-1} f(x) \, dx, \quad s = \gamma + i\eta, \tag{5}$$

and the inverse Mellin transform is given by

$$f(x) = \mathcal{M}^{-1}\{f(s)\}(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} f(s)\ x^{-s}\ ds, \quad t > 0, \quad \gamma = \mathrm{Re}(s). \tag{6}$$

The condition for the existence of (6) is that $-p < \gamma < -q$ (called the fundamental strip), where p, q, are the order of f at the origin and ∞, respectively. The integration in (6) is performed along the imaginary axis and the result does not depend on the choice of γ inside the fundamental strip. For more information about this transform and its properties, see e.g. [28]. The Mellin convolution between two functions is defined by

$$(f *_{\mathcal{M}} g)(x) = \int_0^{+\infty} f\left(\frac{x}{u}\right)\ g(u)\ \frac{du}{u}, \tag{7}$$

and satisfies the Mellin convolution Theorem (see formula (8.4.1.2) in [28])

$$\mathcal{M}\{f *_{\mathcal{M}} g\}(s) = \mathcal{M}\{f\}(s)\ \mathcal{M}\{g\}(s). \tag{8}$$

Finally, we recall the definition of the Caputo fractional derivative ${}^{C}\partial_t^{\beta}$ of order $\beta > 0$ (see [15])

$$\left({}^{C}\partial_t^{\beta} f\right)(t) = \frac{1}{\Gamma(n-\beta)}\ \int_0^t \frac{f^{(n)}(w)}{(t-w)^{\beta-n+1}}\ dw, \qquad n = [\beta] + 1, \quad t > 0, \tag{9}$$

where $[\beta]$ means the integer part of β. For $\beta = n \in \mathbb{N}$, the Caputo fractional derivative coincides with the standard derivative of order n. Of importance use in this paper is the Caputo fractional derivative of the power function which is given by

$$
{}^{C}\partial_t^{\beta}[t^k] = \begin{cases} \dfrac{\Gamma(k+1)}{\Gamma(1+k-\beta)} t^{k-\beta}, & n-1 < \beta < n, \quad k > n-1, \quad k \in \mathbb{R} \\[2mm] 0, & n-1 < \beta < n, \quad k \le n-1, \quad k \in \mathbb{R} \end{cases}. \tag{10}
$$

2.2.2 Special Functions

In this subsection we present some special functions used in this paper and some of their properties. We start with the Gamma function (see [1]), which is defined by the following integral

$$\Gamma(z) = \int_0^{+\infty} t^{z-1}\ e^{-t}\ dt, \qquad \mathrm{Re}(z) > 0,$$

and satisfies the following equalities:

$$\Gamma(z+n) = (z)_n\ \Gamma(z), \quad n \in \mathbb{N}_0 \tag{11}$$

$$\Gamma(z-n) = \frac{(-1)^n\ \Gamma(z)}{(1-z)_n}, \quad n \in \mathbb{N}_0 \tag{12}$$

$$\Gamma\left(z+\frac{1}{2}\right) = \frac{2^{1-2z}\sqrt{\pi}\ \Gamma(2z)}{\Gamma(z)} \tag{13}$$

$$\Gamma(z)\ \Gamma(-z) = \frac{-\pi}{z\ \sin(\pi z)} \tag{14}$$

$$\Gamma\left(\frac{1}{2}-z\right) = \frac{\pi}{\cos(\pi z)\ \Gamma\left(\frac{1}{2}+z\right)}. \tag{15}$$

For the poles of the Gamma function we have the following relation

$$\mathrm{res}_{s=-k}\Gamma(s) = \frac{(-1)^k}{k!}, \quad k \in \mathbb{Z}_0^+. \tag{16}$$

One important special function in fractional calculus is the one-parametric Mittag-Leffler function E_{α} (see [8]), which is defined in terms of the power series by

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha \in \mathbb{C}, \quad z \in \mathbb{C}. \tag{17}$$

In particular, for $\alpha \in \mathbb{R}^+$ it is an entire function. For some particular values of α we obtain some elementary functions:

$$E_1(\pm z) = e^{\pm z}, \qquad E_2(-z^2) = \cos(z), \qquad E_2(z^2) = \cosh(z), \qquad E_0(\pm z) = \frac{1}{1 \mp z}, \quad |z| < 1.$$

For $0 < \alpha < 2$, $p \in \mathbb{Z}^+$, and $\theta \in \left] \frac{\pi\alpha}{2}, \min\{\pi, \pi\alpha\} \right[$ we have the following asymptotic formula

$$E_\alpha(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(1 - \alpha k)} + O(|z|^{-1-p}), \qquad |z| \to +\infty, \quad \theta \leq |\arg(z)| \leq \pi. \tag{18}$$

The Mellin transform of E_α is given by (see formula (8.4.51.7) in [28] with $\mu = 1$ and $\rho = \frac{1}{\alpha}$)

$$\mathcal{M}\{E_\alpha(-z)\}(s) = \frac{\Gamma(s)\,\Gamma(1-s)}{\Gamma(1-\alpha s)}, \qquad 0 < \alpha < 2, \quad 0 < \Re(s) < 1. \tag{19}$$

Another special function with an important role in fractional calculus is the Wright function $W_{\alpha,\beta}$ (see [9]) which is defined as the convergent series

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{+\infty} \frac{z^n}{\Gamma(\alpha n + \beta)\, n!}, \qquad \alpha > -1, \quad \beta \in \mathbb{R}.$$

For some particular values of α and β we get some elementary functions:

$$W_{-\frac{1}{2},0}(-z) = \frac{z}{2\sqrt{\pi}}\, \exp\left(-\frac{z^2}{4}\right), \qquad W_{-\frac{1}{2},-\frac{1}{2}}(-z) = \frac{z^2 - 2}{4\sqrt{\pi}}\, \exp\left(-\frac{z^2}{4}\right), \tag{20}$$

$$W_{-\frac{1}{2},\frac{1}{2}}(-z) = \frac{1}{\sqrt{\pi}}\, \exp\left(-\frac{z^2}{4}\right), \qquad W_{-\frac{1}{2},-1}(-z) = \frac{(z^2 - 6)z}{8\sqrt{\pi}}\, \exp\left(-\frac{z^2}{4}\right). \tag{21}$$

The Wright function is related with the Bessel function of first kind with index ν by the formula

$$J_\nu(z) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{\Gamma(n + \nu + 1)\, k!} \left(\frac{z}{2}\right)^{2k+\nu} = \left(\frac{z}{2}\right)^{\nu} W_{1,\nu+1}\left(-\frac{z^2}{4}\right). \tag{22}$$

The Mellin transform of the Bessel function is given by (see formula (8.4.19.2) in [28])

$$\mathcal{M}\left\{J_\nu\left(\frac{2}{\sqrt{x}}\right)\right\}(s) = \frac{\Gamma\left(\frac{\nu}{2} - s\right)}{\Gamma\left(s + \frac{\nu}{2} + 1\right)}, \qquad -\frac{3}{4} < \Re(s) < \frac{\nu}{2}. \tag{23}$$

The FS obtained in this paper can be represented in terms of Fox H-functions and Fox-Wright functions. The Fox H-function $H_{p,q}^{m,n}$ is defined via a Mellin-Barnes type integral in the form (see [16])

$$H_{p,q}^{m,n}\left[z \,\left|\, \begin{array}{c} (a_1,\, \alpha_1), \ldots, (a_p,\, \alpha_p) \\ (b_1,\, \beta_1), \ldots, (b_q,\, \beta_q) \end{array} \right. \right] = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{\prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)}{\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s)} z^{-s} \, ds, \tag{24}$$

where $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}^{\mathsf{l}}$, for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, and $\mathcal{L}$ is a suitable contour in the complex plane separating the poles of the two factors in the numerator (see [16]). The Fox-Wright function is a generalization of the generalized hypergeometric function $_pF_q$ based on the idea of E. Wright (1953) (see [31])

$$_p\Psi_q\left[\begin{array}{c} (a_1,\, \alpha_1), \ldots, (a_p,\, \alpha_p) \\ (b_1,\, \beta_1), \ldots, (b_q,\, \beta_q) \end{array} \,\left|\, z \right. \right] = \sum_{n=0}^{+\infty} \frac{\Gamma(a_1 + \alpha_1 n) \cdots \Gamma(a_p + \alpha_p n)}{\Gamma(b_1 + \beta_1 n) \cdots \Gamma(b_q + \beta_q n)} \frac{z^n}{n!}, \tag{25}$$

where $z \in \mathbb{C}$, $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}$, for $i = 1, \ldots, p$, $j = 1, \ldots, q$ (see [24]).

3 Multidimensional time fractional diffusion-wave equation

3.1 Problem formulation

In this paper, we consider the multidimensional time fractional diffusion-wave equation

$$\left({}^C\partial_t^\beta - c^2 \Delta \right) u(x,t) = 0, \qquad x \in \mathbb{R}^n, \quad t \in \mathbb{R}^+, \quad c \in \mathbb{R}^+, \quad 0 < \beta \leq 2, \tag{26}$$

where $^{C}\partial_{t}^{\beta}$ is the Caputo time fractional derivative of order $\beta \in]0,2]$ defined by (see [15]):

$$\left(^{C}\partial_{t}^{\beta}u\right)(x,t) = \begin{cases} \dfrac{1}{\Gamma(1-\beta)} \displaystyle\int_{0}^{t} (t-w)^{-\beta} \dfrac{\partial u}{\partial w}(x,w) \ dw, & 0 < \beta \leq 1 \\[4mm] \dfrac{1}{\Gamma(2-\beta)} \displaystyle\int_{0}^{t} (t-w)^{1-\beta} \dfrac{\partial^{2}u}{\partial w^{2}}(x,w) \ dw, & 1 < \beta < 2 \end{cases}. \tag{27}$$

Equation (26) interpolates between the Helmholtz equation (when β tends to zero), the heat equation (when $\beta = 1$) and the wave equation (when $\beta = 2$). Here we want to study the behaviour and properties of the first FS (Green function) G_{n}^{β} of equation (26), i.e., its solution that satisfies the initial conditions

$$u(x,0) = \prod_{i=1}^{n} \delta(x_{i}) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0, \tag{28}$$

where $x \in \mathbb{R}^{n}$, and δ is the delta Dirac function. We remark that the second initial condition is necessary for the case when $1 < \beta \leq 2$.

3.2 Fundamental solution of the time fractional diffusion-wave equation

Applying the multidimensional Fourier transform (3) to the equation (26) and the initial conditions (28), we get the following initial-value problem in Fourier domain

$$\begin{cases} \left(^{C}\partial_{t}^{\beta} + c^{2}\,|\kappa|^{2}\right) \widehat{G}_{n}^{\beta}(\kappa,t) = 0 \\[3mm] \widehat{G}_{n}^{\beta}(\kappa,0) = 1 \\[3mm] \dfrac{\partial \widehat{G}_{n}^{\beta}}{\partial t}(\kappa,0) = 0 \end{cases}. \tag{29}$$

The unique solution of the problem (29) is given in terms of the Mittag-Leffler function by (see [22])

$$\widehat{G}_{n}^{\beta}(\kappa,t) = E_{\beta}\left(-c^{2}|\kappa|^{2}t^{\beta}\right). \tag{30}$$

Taking into account the asymptotic formula (18) we conclude that $\widehat{G}_{n}^{\beta}$ belongs to the functional space $L_{1}(\mathbb{R}^{n})$ with respect to κ for $0 < \beta \leq 2$ and $p \geq n+1$. Hence, we can apply the inverse Fourier transform and get the following integral representation of G_{n}^{β}

$$G_{n}^{\beta}(x,t) = \frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} e^{-i\kappa\cdot x} \, E_{\beta}\left(-c^{2}|\kappa|^{2}t^{\beta}\right) \ d\kappa, \qquad x \in \mathbb{R}^{n}, \qquad t > 0. \tag{31}$$

Taking into account the following formula presented in [29]:

$$\frac{1}{(2\pi)^{n}} \int_{\mathbb{R}^{n}} e^{-i\kappa\cdot x} \, \phi(|\kappa|) \ d\kappa = \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{0}^{+\infty} \phi(\tau) \, \tau^{\frac{n}{2}} \, J_{\frac{n}{2}-1}(\tau|x|) \ d\tau, \tag{32}$$

and the fact that $E_{\beta}\left(-c^{2}|\kappa|^{2}t^{\beta}\right)$ is a radial function in κ, (31) can be rewritten as

$$G_{n}^{\beta}(x,t) = \frac{|x|^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_{0}^{+\infty} \tau^{\frac{n}{2}} \, E_{\beta}\left(-c^{2}\tau^{2}t^{\beta}\right) \, J_{\frac{n}{2}-1}\left(\tau|x|\right) \ d\tau. \tag{33}$$

Taking into account the relation (22) we get another integral representation of (31):

$$G_{n}^{\beta}(x,t) = \frac{2}{(4\pi)^{\frac{n}{2}}} \int_{0}^{+\infty} \tau^{n-1} \, E_{\beta}\left(-c^{2}\tau^{2}t^{\beta}\right) \, W_{1,\frac{n}{2}}\left(-\frac{\tau^{2}|x|^{2}}{4}\right) \ d\tau. \tag{34}$$

To compute explicitly the integral (34) we are going to use the Mellin transform. First, we rewrite (33) as a Mellin convolution (7). In fact, considering the functions

$$g(\tau) = E_{\beta}\left(-c^{2}\tau^{2}t^{\beta}\right) \qquad \text{and} \qquad f(\tau) = \frac{1}{(2\pi)^{\frac{n}{2}} \, |x|^{n} \, \tau^{\frac{n}{2}+1}} \, J_{\frac{n}{2}-1}\left(\frac{1}{\tau}\right)$$

its Mellin convolution at the point $\frac{1}{|x|}$ gives

$$
\begin{aligned}
(f *_{\mathcal{M}} g)\left(\frac{1}{|x|}\right) &= \int_0^{+\infty} f\left(\frac{1}{\tau|x|}\right) g(\tau) \frac{d\tau}{\tau} \\
&= \int_0^{+\infty} \frac{\tau^{\frac{n}{2}+1} |x|^{\frac{n}{2}+1}}{(2\pi)^{\frac{n}{2}} |x|^n} J_{\frac{n}{2}-1}(\tau|x|) E_\beta\left(-c^2\tau^2 t^\beta\right) \frac{d\tau}{\tau} \\
&= G_n^\beta(x,t).
\end{aligned}
$$

By the following relation (see (8.4.1.7) in [28])

$$
\mathcal{M}\left\{f\left(\frac{1}{x}\right)\right\}(s) = \mathcal{M}\{f\}(-s), \tag{35}
$$

and the Mellin convolution Theorem (8) we have that

$$
\begin{aligned}
\mathcal{M}\left\{G_n^\beta\right\}(s) &= \mathcal{M}\left\{(f *_{\mathcal{M}} g)\left(\frac{1}{|x|}\right)\right\}(s) \\
&= \mathcal{M}\{f\}(-s)\,\mathcal{M}\{g\}(-s),
\end{aligned}
$$

which is equivalent to

$$
\mathcal{M}\left\{G_n^\beta\right\}(-s) = \mathcal{M}\{f\}(s)\,\mathcal{M}\{g\}(s). \tag{36}
$$

Using now the definition of the Mellin transform (5) and the formulas (19) and (23) we get

$$
\mathcal{M}\{f\}(s) = \frac{\Gamma\left(\frac{n-s}{2}\right)}{\pi^{\frac{n}{2}} |x|^n\, 2^s\, \Gamma\left(\frac{s}{2}\right)}, \qquad \mathcal{M}\{g\}(s) = \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(1-\frac{s}{2}\right)}{2\left(c^2 t^\beta\right)^{\frac{s}{2}}\Gamma\left(1-\frac{\beta s}{2}\right)}.
$$

Finally, using the inverse Mellin transform (6) applied to (36) we obtain the representation of G_n^β as a Mellin-Barnes integral and, consequently, as a Fox H-function:

$$
\begin{aligned}
G_n^\beta(x,t) &= \frac{1}{2\pi^{\frac{n}{2}} |x|^n}\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{n-s}{2}\right)}{\Gamma\left(1-\frac{\beta s}{2}\right)}\left(\frac{2c\,t^{\frac{\beta}{2}}}{|x|}\right)^{-s} ds \tag{37}
\end{aligned}
$$

$$
= \frac{1}{2\pi^{\frac{n}{2}} |x|^n} H_{2,1}^{0,2}\left[\frac{2c\,t^{\frac{\beta}{2}}}{|x|}\,\middle|\,\begin{array}{c}\left(0,\frac{1}{2}\right),\left(1-\frac{n}{2},\frac{1}{2}\right)\\[4pt]\left(0,\frac{\beta}{2}\right)\end{array}\right]. \tag{38}
$$

To compute explicitly the integral (37) we need to apply the Residue Theorem. Applying general conditions of the Mellin-Barnes integrals to (37) (see [16]) we conclude that for $0 < \beta \le 2$ the integral is convergent and the contour of integration must be transformed to the loop $\mathcal{L}_{+\infty}$ starting and ending at infinity and encircling all the poles of the functions $\Gamma\left(1-\frac{s}{2}\right)$ and $\Gamma\left(\frac{n-s}{2}\right)$ in the numerator. Since the gamma function $\Gamma(s)$ has simple poles at $s = -n$, with $n \in \mathbb{N}_0$, we have that $\Gamma\left(1-\frac{s}{2}\right)$ has poles at $s = 2k + 2$, with $k \in \mathbb{N}_0$, and $\Gamma\left(\frac{n-s}{2}\right)$ has poles at $s = 2k+n$, with $k \in \mathbb{N}_0$. From this it is evident that we have different types of poles accordingly with the parity of the dimension n.

3.2.1 The case of odd dimension

Since n is odd we have two non-coincident sequences of simple poles, $s = 2k+2$, $k \in \mathbb{N}_0$, and $s = 2k+n$, $k \in \mathbb{N}_0$. Applying the Residue Theorem for each gamma function and taking into account (16) we get the following series representation for G_n^β:

$$
G_n^\beta(x,t) = \frac{1}{\pi^{\frac{n}{2}} |x|^n}\left[\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\frac{\Gamma\left(-1-k+\frac{n}{2}\right)}{\Gamma(1-\beta(k+1))}\left(\frac{2c\,t^{\frac{\beta}{2}}}{|x|}\right)^{-2k-2} + \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!}\frac{\Gamma\left(1-k-\frac{n}{2}\right)}{\Gamma\left(1-\beta\left(k+\frac{n}{2}\right)\right)}\left(\frac{2c\,t^{\frac{\beta}{2}}}{|x|}\right)^{-2k-n}\right].
$$

Both series can be combined in one single series. For obtaining this we start by considering the change of variables $m = 2k + 1$ and $m = 2k$ in the first and the second series, respectively. Hence, we get

$$
G_n^\beta(x,t) = \frac{1}{\pi^{\frac{n}{2}} |x|^n} \left[\sum_{\substack{m=1 \\ m \text{ odd}}}^{+\infty} \frac{(-1)^{\frac{m-1}{2}}}{\left(\frac{m-1}{2}\right)!} \frac{\Gamma\left(-1 - \frac{m-1}{2} + \frac{n}{2}\right)}{\Gamma\left(1 - \beta\left(1 + \frac{m-1}{2}\right)\right)} \left(\frac{|x|}{2c\,t^{\frac{\beta}{2}}}\right)^{m+1} \right.
$$

$$
\left. + \sum_{\substack{m=0 \\ m \text{ even}}}^{+\infty} \frac{(-1)^{\frac{m}{2}}}{\left(\frac{m}{2}\right)!} \frac{\Gamma\left(1 - \frac{m}{2} - \frac{n}{2}\right)}{\Gamma\left(1 - \frac{\beta(m+n)}{2}\right)} \left(\frac{|x|}{2c\,t^{\frac{\beta}{2}}}\right)^{m+n} \right].
$$

To have the same exponent in both series we consider in the first series the change of the order of summation $m = p + n - 1$ and in the second series $m = p$, yielding

$$
G_n^\beta(x,t) = \frac{1}{\pi^{\frac{n}{2}} |x|^n} \left[\sum_{\substack{p=2-n \\ p \text{ odd}}}^{-1} \frac{(-1)^{\frac{p+n-2}{2}}}{\Gamma\left(\frac{p+n}{2}\right)} \frac{\Gamma\left(-\frac{p}{2}\right)}{\Gamma\left(1 - \frac{\beta(p+n)}{2}\right)} \left(\frac{|x|}{2c\,t^{\frac{\beta}{2}}}\right)^{p+n} \right.
$$

$$
+ \sum_{\substack{p=1 \\ p \text{ odd}}}^{+\infty} \frac{(-1)^{\frac{p+n-2}{2}}}{\Gamma\left(\frac{p+n}{2}\right)} \frac{\Gamma\left(-\frac{p}{2}\right)}{\Gamma\left(1 - \frac{\beta(p+n)}{2}\right)} \left(\frac{|x|}{2\,c\,t^{\frac{\beta}{2}}}\right)^{p+n}
$$

$$
\left. + \sum_{\substack{p=0 \\ p \text{ even}}}^{+\infty} \frac{(-1)^{\frac{p}{2}}}{\Gamma\left(\frac{p}{2}+1\right)} \frac{\Gamma\left(1 - \frac{p}{2} - \frac{n}{2}\right)}{\Gamma\left(1 - \frac{\beta(p+n)}{2}\right)} \left(\frac{|x|}{2c\,t^{\frac{\beta}{2}}}\right)^{p+n} \right]. \tag{39}
$$

Now we analyse the coefficients of the odd and the even series. For p odd, using (11) and (13) we have

$$
\Gamma\left(\frac{p+n}{2}\right) = \Gamma\left(\frac{p+1}{2} + \frac{n-1}{2}\right) = \left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} \Gamma\left(\frac{p}{2} + \frac{1}{2}\right) = \frac{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} 2^{1-p} \sqrt{\pi}\,\Gamma(p)}{\Gamma\left(\frac{p}{2}\right)}. \tag{40}
$$

Using (40) and (14) we get

$$
\frac{(-1)^{\frac{p+n-2}{2}} \Gamma\left(-\frac{p}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)} = \frac{(-1)^{\frac{p+n-2}{2}} \Gamma\left(-\frac{p}{2}\right) \Gamma\left(\frac{p}{2}\right)}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} 2^{1-p} \sqrt{\pi}\,\Gamma(p)} = -\frac{(-1)^{\frac{n-1}{2}} \sqrt{\pi}\,2^p}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} p!}. \tag{41}
$$

On the other hand, for p even, using (12), (15), and (13) we obtain

$$
\Gamma\left(1 - \frac{p}{2} - \frac{n}{2}\right) = \Gamma\left(\frac{1}{2} - \frac{p}{2} - \frac{n-1}{2}\right) = \frac{(-1)^{\frac{n-1}{2}} \Gamma\left(\frac{1}{2} - \frac{p}{2}\right)}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}}
$$

$$
= \frac{(-1)^{\frac{n-1}{2}} \pi}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} \cos\left(\frac{p\pi}{2}\right) \Gamma\left(\frac{p}{2} + \frac{1}{2}\right)} = \frac{(-1)^{\frac{n-p-1}{2}} \sqrt{\pi}\,\Gamma\left(\frac{p}{2}\right)}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} 2^{1-p}\,\Gamma(p)}. \tag{42}
$$

Hence, using (42) and (11) we get

$$
\frac{(-1)^{\frac{p}{2}} \Gamma\left(1 - \frac{p}{2} - \frac{n}{2}\right)}{\Gamma\left(\frac{p}{2} + 1\right)} = \frac{(-1)^{\frac{n-1}{2}} \sqrt{\pi}\,2^p}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} p!}. \tag{43}
$$

From (41) and (43) we conclude that the coefficients of the series that appear in (39) are equal up to a minus sign in the odd series, which can be included as $(-1)^p$ for p odd and even. Hence, summing the odd and the even series and considering the change $p = 2k + 2 - n$ in the finit sum, we get the simplified series representation of G_n^β given by

$$
G_n^\beta(x,t) = \frac{1}{4c^2\,\pi^{\frac{n}{2}} |x|^{n-2}\,t^\beta} \sum_{k=0}^{\frac{n-3}{2}} \frac{\Gamma\left(-1 - k + \frac{n}{2}\right)}{\Gamma\left(1 - \beta(k+1)\right) k!} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^k
$$

$$
+ \frac{(-1)^{\frac{n-1}{2}} \sqrt{\pi}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \sum_{p=0}^{+\infty} \frac{1}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}} \Gamma\left(1 - \frac{\beta(p+n)}{2}\right) p!} \left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^p. \tag{44}
$$

Taking into account (11) the series in (44) can be represented as a Fox-Wright function (cf. (25)):

$$
\begin{aligned}
G_n^\beta(x,t) \;=\;& \frac{1}{4\,c^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}\,t^\beta} \sum_{k=0}^{\frac{n-3}{2}} \frac{\Gamma\left(-1-k+\frac{n}{2}\right)}{\Gamma\left(1-\beta(k+1)\right)\,k!} \left(-\frac{|x|^2}{4c^2t^\beta}\right)^k \\
& + \frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \;{}_1\Psi_2 \left[\begin{array}{c} \left(\frac{1}{2},\frac{1}{2}\right) \\ \left(\frac{n}{2},\frac{1}{2}\right),\,\left(1-\frac{\beta n}{2},-\frac{\beta}{2}\right) \end{array} \Bigg| -\frac{|x|}{c\,t^{\frac{\beta}{2}}} \right].
\end{aligned}
\tag{45}
$$

3.2.2 The case of even dimension

Since now n is even we have a finite sequence of simple poles coming from $\Gamma\left(1-\frac{s}{2}\right)$ at the points $s = 2k+2$, for $k = 0,1,\ldots,\frac{n}{2}-2$, and an infinite sequence of double poles coming from $\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{n-s}{2}\right)$ at the points $s = 2k+2$, for $k \geq \frac{n}{2}-1$. Applying the Residue Theorem we get the following series representation for G_n^β:

$$
\begin{aligned}
G_n^\beta(x,t) \;=\;& \frac{1}{4\,c^2\,\pi^{\frac{n}{2}}\,t^\beta\,|x|^{n-2}} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma\left(\frac{n}{2}-k-1\right)}{\Gamma(1-\beta(k+1))\,k!} \left(-\frac{|x|^2}{4c^2t^\beta}\right)^k \\
& + \frac{(-1)^{\frac{n}{2}+1}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \sum_{k=0}^{+\infty} \frac{\psi\left(k+\frac{n}{2}\right)-\beta\,\psi\left(1-\beta\left(k+\frac{n}{2}\right)\right)+\psi(k+1)+\ln\left(\frac{4c^2t^\beta}{|x|^2}\right)}{\Gamma\left(k+\frac{n}{2}\right)\,\Gamma\left(1-\beta\left(k+\frac{n}{2}\right)\right)\,k!} \left(\frac{|x|^2}{4c^2t^\beta}\right)^k,
\end{aligned}
\tag{46}
$$

where $\psi(z)$ denotes the digamma function. When $\beta(k+\frac{n}{2}) \in \mathbb{N}$ we have an indetermination in the series coefficients due to the terms $\beta\psi\left(1-\beta\left(k+\frac{n}{2}\right)\right)$ and $\Gamma\left(1-\beta\left(k+\frac{n}{2}\right)\right)$, although the expression is well defined since for all $y_0 \in \mathbb{N}$ the following limit exists:

$$
\lim_{y \to y_0} \frac{\psi(1-y)}{\Gamma(1-y)} \in \mathbb{R}.
$$

Nevertheless, we can remove the indetermination applying (11) with $n=1$, (14), (15), and the relation $\psi(1-z) = \pi\cot(\pi z) + \psi(z)$ for the digamma function (see [1]). After straightforward calculations we get the new series representation given by

$$
\begin{aligned}
G_n^\beta(x,t) \;=\;& \frac{1}{4\,c^2\,\pi^{\frac{n}{2}}\,t^\beta\,|x|^{n-2}} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma\left(\frac{n}{2}-k-1\right)}{\Gamma(1-\beta(k+1))\,k!} \left(-\frac{|x|^2}{4c^2t^\beta}\right)^k \\
& + \frac{(-1)^{\frac{n}{2}}\,\beta\,\pi}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \sum_{k=0}^{+\infty} \frac{\Gamma\left(\beta\left(k+\frac{n}{2}\right)\right)}{\Gamma\left(k+\frac{n}{2}\right)\,\Gamma\left(\frac{1}{2}+\beta\left(k+\frac{n}{2}\right)\right)\,\Gamma\left(\frac{1}{2}-\beta\left(k+\frac{n}{2}\right)\right)\,k!} \left(\frac{|x|^2}{4c^2\,t^\beta}\right)^k \\
& + \frac{(-1)^{\frac{n}{2}+1}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}\,\pi} \sum_{k=0}^{+\infty} \left\{ \left[\psi\left(k+\frac{n}{2}\right) - \beta\psi\left(\beta\left(k+\frac{n}{2}\right)\right) + \psi(k+1) + \ln\left(\frac{4c^2t^\beta}{|x|^2}\right)\right] \right. \\
& \left. \times \frac{\Gamma\left(\beta\left(k+\frac{n}{2}\right)\right)\,\sin\left(\pi\beta\left(k+\frac{n}{2}\right)\right)}{\Gamma\left(k+\frac{n}{2}\right)\,k!} \left(\frac{|x|^2}{4c^2t^\beta}\right)^k \right\}.
\end{aligned}
\tag{47}
$$

in (47) we split the solution in two convergent series in order to facilitate future work and to make the structure more clear. For instance, when $\beta = 1$ the finite sum and the second series vanishes, while the first series gives the classical FS (cf. Section 5). Taking into account (25) we can rewrite the previous expression as

$$
\begin{aligned}
G_n^\beta(x,t) \;=\;& \frac{1}{4c^2\,\pi^{\frac{n}{2}}\,t^\beta\,|x|^{n-2}} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma\left(\frac{n}{2}-k-1\right)}{\Gamma(1-\beta(k+1))\,k!} \left(-\frac{|x|^2}{4c^2t^\beta}\right)^k \\
& + \frac{(-1)^{\frac{n}{2}}\,\beta\,\pi}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \;{}_1\Psi_3 \left[\begin{array}{c} \left(\frac{\beta n}{2},\beta\right) \\ \left(\frac{n}{2},1\right),\,\left(\frac{1+\beta n}{2},\beta\right),\,\left(\frac{1-\beta n}{2},-\beta\right) \end{array} \Bigg| \frac{|x|^2}{4c^2t^\beta} \right] \\
& + \frac{(-1)^{\frac{n}{2}+1}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}\,\pi} \sum_{k=0}^{+\infty} \left\{ \left[\psi\left(k+\frac{n}{2}\right) - \beta\,\psi\left(\beta\left(k+\frac{n}{2}\right)\right) + \psi(k+1) + \ln\left(\frac{4c^2t^\beta}{|x|^2}\right)\right] \right. \\
& \left. \times \frac{\Gamma\left(\beta\left(k+\frac{n}{2}\right)\right)\,\sin\left(\pi\beta\left(k+\frac{n}{2}\right)\right)}{\Gamma\left(k+\frac{n}{2}\right)\,k!} \left(\frac{|x|^2}{4c^2t^\beta}\right)^k \right\}.
\end{aligned}
\tag{48}
$$

We summarize the results of this section in the next theorem.

Theorem 3.1 *For n odd and $0 < \beta \leq 2$ the FS of the time fractional diffusion-wave operator is given by*

$$
G_n^\beta(x,t) = \frac{1}{4c^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}\,t^\beta} \sum_{k=0}^{\frac{n-3}{2}} \frac{\Gamma\left(-1-k+\frac{n}{2}\right)}{\Gamma\left(1-\beta(k+1)\right)\,k!} \left(-\frac{|x|^2}{4c^2t^\beta}\right)^k
$$
$$
+ \frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \sum_{p=0}^{+\infty} \frac{1}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}\Gamma\left(1-\frac{\beta(p+n)}{2}\right)\,p!} \left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^p. \tag{49}
$$

For n even and $0 < \beta \leq 2$ the FS of the time fractional diffusion-wave operator is given by

$$
G_n^\beta(x,t) = \frac{1}{4c^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}\,t^\beta} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma\left(\frac{n}{2}-k-1\right)}{\Gamma(1-\beta(k+1))\,k!} \left(-\frac{|x|^2}{4c^2t^\beta}\right)^k
$$
$$
+ \frac{(-1)^{\frac{n}{2}+1}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \sum_{k=0}^{+\infty} \frac{\psi\left(k+\frac{n}{2}\right) - \beta\,\psi\left(1-\beta\left(k+\frac{n}{2}\right)\right) + \psi(k+1) + \ln\left(\frac{4c^2t^\beta}{|x|^2}\right)}{\Gamma\left(k+\frac{n}{2}\right)\,\Gamma\left(1-\beta\left(k+\frac{n}{2}\right)\right)\,k!} \left(\frac{|x|^2}{4c^2t^\beta}\right)^k. \tag{50}
$$

4 Fundamental solution of the time fractional parabolic Dirac operator

In this section we compute the FS for the time fractional parabolic Dirac operator defined by

$$
D_{x,t}^\beta := c\,D_x + \mathfrak{f}\,{}^C\partial_t^\beta + \mathfrak{f}^\dagger, \tag{51}
$$

where D_x is the Dirac operator in $\mathbb{R}^n$ (see Subsection 2.1), ${}^C\partial_t^\beta$ is the Caputo time fractional derivative of order $0 < \beta \leq 2$ (see (27)), and $c \in \mathbb{R}^+$. This operator factorizes the time fractional diffusion-wave operator $-c^2\Delta + {}^C\partial_t^\beta$ for Clifford valued functions f given by $f = \sum_A e_A f_A$, where $f_A \in C^2(\Omega \times I)$, with $\Omega \times I \subseteq \mathbb{R}^n \times \mathbb{R}^+$. In fact, taking into account the multiplication rules (1) and (2) we have that

$$
\begin{aligned}
\left(D_{x,t}^\beta\right)^2 &= \left(c\,D_x + \mathfrak{f}\,{}^C\partial_t^\beta + \mathfrak{f}^\dagger\right)\left(c\,D_x + \mathfrak{f}\,{}^C\partial_t^\beta + \mathfrak{f}^\dagger\right) \\
&= c^2\,D_x^2 - \mathfrak{f}\,c\,D_x\,{}^C\partial_t^\beta - \mathfrak{f}^\dagger\,c\,D_x + \mathfrak{f}\,c\,D_x\,{}^C\partial_t^\beta + (\mathfrak{f})^2\,{}^C\partial_t^\beta\,{}^C\partial_t^\beta + \mathfrak{f}\mathfrak{f}^\dagger\,{}^C\partial_t^\beta + \mathfrak{f}^\dagger\,c\,D_x + \mathfrak{f}^\dagger\mathfrak{f}\,{}^C\partial_t^\beta + (\mathfrak{f}^\dagger)^2 \\
&= c^2\,D_x^2 + (\mathfrak{f}\mathfrak{f}^\dagger + \mathfrak{f}^\dagger\mathfrak{f})\,{}^C\partial_t^\beta \\
&= -c^2\Delta + {}^C\partial_t^\beta.
\end{aligned}
$$

The FS of the time fractional parabolic Dirac operator will be denoted by $\mathfrak{G}_n^\beta$ and is obtained by the application of the operator $D_{x,t}^\beta$ to G_n^β, i.e.,

$$
\mathfrak{G}_n^\beta(x,t) = D_{x,t}^\beta G_n^\beta(x,t) = c\,D_x\,G_n^\beta(x,t) + \mathfrak{f}\,{}^C\partial_t^\beta\,G_n^\beta(x,t) + \mathfrak{f}^\dagger\,G_n^\beta(x,t). \tag{52}
$$

Indeed, by the above factorization we have $D_{x,t}^\beta\mathfrak{G}_n^\beta(x,t) = (-c^2\Delta + {}^C\partial_t^\beta)G_n^\beta(x,t) = 0$, which shows that $\mathfrak{G}_n^\beta$ is the FS of $D_{x,t}^\beta$. Thereby, applying $D_{x,t}^\beta$ to (34), using the Leibniz rule for improper integrals and the series representation of the Mittag-Leffler and the Wright functions, together with (10) and the relation

$$
D_x\left[|x|^k\right] = \sum_{i=1}^n e_i\,\partial_{x_i}\left(\left(\sum_{j=1}^n x_j^2\right)^{\frac{k}{2}}\right) = k\,|x|^{k-2}\,x, \qquad k \in \mathbb{N}_0,
$$

we obtain

$$
\begin{aligned}
\mathfrak{G}_n^\beta(x,t) &= \left(c\,D_x + \mathfrak{f}\,{}^C\partial_t^\beta + \mathfrak{f}^\dagger\right)G_n^\beta(x,t) \\
&= -\frac{c\,x}{(4\pi)^{\frac{n}{2}}}\int_0^{+\infty} \tau^{n+1}\,E_\beta\left(-c^2\tau^2t^\beta\right)\,W_{1,\frac{n}{2}+1}\left(-\frac{\tau^2|x|^2}{4}\right)\,d\tau \\
&\quad - \mathfrak{f}\,\frac{2c^2}{(4\pi)^{\frac{n}{2}}}\int_0^{+\infty} \tau^{n+1}\,E_\beta\left(-c^2\tau^2t^\beta\right)\,W_{1,\frac{n}{2}}\left(-\frac{\tau^2|x|^2}{4}\right)\,d\tau \\
&\quad + \mathfrak{f}^\dagger\,\frac{2}{(4\pi)^{\frac{n}{2}}}\int_0^{+\infty} \tau^{n-1}\,E_\beta\left(-c^2\tau^2t^\beta\right)\,W_{1,\frac{n}{2}}\left(-\frac{\tau^2|x|^2}{4}\right)\,d\tau. \tag{53}
\end{aligned}
$$

Using (22) we can write (53) in the following way:

$$
\begin{aligned}
\mathfrak{G}_n^\beta(x,t) &= -\frac{c\,x}{(2\pi)^{\frac{n}{2}}\,|x|^{\frac{n}{2}}} \int_0^{+\infty} \tau^{\frac{n}{2}+1}\, E_\beta\left(-c^2\tau^2 t^\beta\right)\, J_{\frac{n}{2}}\left(\tau|x|\right)\, d\tau \\
&\quad -\mathfrak{f}\,\frac{c^2}{(2\pi)^{\frac{n}{2}}\,|x|^{\frac{n}{2}-1}} \int_0^{+\infty} \tau^{\frac{n}{2}+2}\, E_\beta\left(-c^2\tau^2 t^\beta\right)\, J_{\frac{n}{2}-1}\left(\tau|x|\right)\, d\tau \\
&\quad +\mathfrak{f}^\dagger\,\frac{1}{(2\pi)^{\frac{n}{2}}\,|x|^{\frac{n}{2}-1}} \int_0^{+\infty} \tau^{\frac{n}{2}}\, E_\beta\left(-c^2\tau^2 t^\beta\right)\, J_{\frac{n}{2}-1}\left(\tau|x|\right)\, d\tau.
\end{aligned} \tag{54}
$$

The integral representation (54) will be useful to make plots of the components of $\mathfrak{G}_n^\beta$ in Section 8. It is possible to write the components of $\mathfrak{G}_n^\beta$ as Fox H-functions. For doing that we apply the operator $D_{x,t}^\beta$ to (37) and we obtain:

$$
\begin{aligned}
\mathfrak{G}_n^\beta(x,t) &= \frac{c\,x}{2\pi^{\frac{n}{2}}\,|x|^{n+2}}\, H_{3,2}^{1,2}\left[\frac{2c\,t^{\frac{\beta}{2}}}{|x|}\,\middle|\,
\begin{array}{c}
\left(0,\frac{1}{2}\right),\,\left(1-\frac{n}{2},\frac{1}{2}\right),\,(-n,1) \\[4pt]
(1-n,1),\,\left(0,\frac{\beta}{2}\right)
\end{array}\right] \\
&\quad +\mathfrak{f}\,\frac{1}{2\pi^{\frac{n}{2}}\,|x|^n\,t^\beta}\, H_{2,1}^{0,2}\left[\frac{2c\,t^{\frac{\beta}{2}}}{|x|}\,\middle|\,
\begin{array}{c}
\left(0,\frac{1}{2}\right),\,\left(1-\frac{n}{2},\frac{1}{2}\right) \\[4pt]
\left(\beta,\frac{\beta}{2}\right)
\end{array}\right] \\
&\quad +\mathfrak{f}^\dagger\,\frac{1}{2\pi^{\frac{n}{2}}\,|x|^n}\, H_{2,1}^{0,2}\left[\frac{2c\,t^{\frac{\beta}{2}}}{|x|}\,\middle|\,
\begin{array}{c}
\left(0,\frac{1}{2}\right),\,\left(1-\frac{n}{2},\frac{1}{2}\right) \\[4pt]
\left(0,\frac{\beta}{2}\right)
\end{array}\right].
\end{aligned}
$$

In order to get a series representation for $\mathfrak{G}_n^\beta$ we apply $D_{x,t}^\beta$ to (49) and (50). The following formula concerning the Dirac operator is necessary to compute $\mathfrak{G}_n^\beta$:

$$
\begin{aligned}
D_x\left[\ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)|x|^{2k}\right] &= \sum_{i=1}^n e_i\left[-2x_i\,|x|^{2k-2} + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right) 2k\,x_i\,|x|^{2k-2}\right] \\
&= 2|x|^{2k-2}x\left(k\ln\left(\frac{4c^2 t^\beta}{|x|^2}\right) - 1\right).
\end{aligned} \tag{55}
$$

From (55) we easily conclude that

$$
D_x\left[\left(C+\ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)\right)|x|^{2k}\right] = 2x\,|x|^{2k-2}\left(k\left(C+\ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)\right)-1\right),\quad C\in\mathbb{R}. \tag{56}
$$

Concerning the Caputo fractional derivative we use the symbolic calculus software Mathematica. When $0<\beta\le 1$ we obtain

$$
\begin{aligned}
{}^C\partial_t^\beta&\left[\left(C+\ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)\right)t^{-\beta\left(k+\frac{n}{2}\right)}\right] \\
&= \frac{\Gamma\left(1-\beta\left(k+\frac{n}{2}\right)\right)}{t^{\beta\left(k+\frac{n}{2}+1\right)}\,\Gamma\left(1-\beta\left(k+\frac{n}{2}+1\right)\right)}\times\left[C+\beta\,\pi\cot\left(\beta\pi\left(k+\frac{n}{2}\right)\right)-\beta\,H\left(-\beta\left(k+\frac{n}{2}+1\right)\right)\right. \\
&\qquad\left. +\beta\,H\left(-1+\beta\left(k+\frac{n}{2}\right)\right)+\ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)\right] \tag{57} \\
&= \frac{\Gamma\left(1-\beta\left(k+\frac{n}{2}\right)\right)}{t^{\beta\left(k+\frac{n}{2}+1\right)}\,\Gamma\left(1-\beta\left(k+\frac{n}{2}+1\right)\right)}\times\left[C+\beta\,\psi\left(1-\beta\left(k+\frac{n}{2}\right)\right)-\beta\,\psi\left(1-\beta\left(k+\frac{n}{2}+1\right)\right)+\ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)\right],
\end{aligned}
$$

$$
\tag{58}
$$

with $C\in\mathbb{R}$. In (57) H denotes the generalized Harmonic number. To obtain (58) we used the identities $H(z)=\psi(z+1)+\gamma$ and $\psi(1-z)=\pi\cot(\pi z)+\psi(z)$, where γ is the Euler-Mascheroni constant. When

$1 < \beta \le 2$ we obtain

$$
{}^C\partial_t^\beta \left[\left(C + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right) \right) t^{-\beta\left(k+\frac{n}{2}\right)} \right]
$$

$$
= \frac{\Gamma\left(1 - \beta\left(k+\frac{n}{2}\right)\right)}{t^{\beta\left(k+\frac{n}{2}+1\right)}\, \Gamma\left(1 - \beta\left(k+\frac{n}{2}+1\right)\right)} \times \left[C - \frac{1}{k+\frac{n}{2}} - \frac{\beta}{1+\beta\left(k+\frac{n}{2}\right)} - \beta\, H\left(-\beta\left(k+\frac{n}{2}+1\right)\right) \right.
$$

$$
\left. + \beta\, H\left(-2 - \beta\left(k+\frac{n}{2}\right)\right) + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right) \right], \quad C \in \mathbb{R}. \tag{59}
$$

By using the identity $H(-1+z) = H(z) - \dfrac{1}{z}$ twice we get

$$
\beta\, H\left(-2 - \beta\left(k+\frac{n}{2}\right)\right) = \beta\, H\left(-\beta\left(k+\frac{n}{2}\right)\right) + \frac{\beta}{1+\beta\left(k+\frac{n}{2}\right)} + \frac{1}{k+\frac{n}{2}}. \tag{60}
$$

Therefore, considering (60) and applying the identity $H(z) = \psi(z+1) + \gamma$ we can conclude that when $1 < \beta \le 2$ the expression (59) is equal to (58). Finally, applying (10), (56), and (58) to (49) and (50), and after straightforward computations we get the following result.

Theorem 4.1 *For n odd and $0 < \beta \le 2$ the FS of the time fractional parabolic Dirac operator is given by*

$$
\mathfrak{G}_n^\beta(x,t) = \frac{-x}{8c\,\pi^{\frac{n}{2}}\,|x|^n\,t^\beta} \sum_{k=0}^{\frac{n-3}{2}} \frac{\Gamma\left(\frac{n}{2}-k\right)}{\Gamma(1-\beta(k+1))\,k!} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^k
$$

$$
- \frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}\,x}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}\,t^{\frac{\beta}{2}}\,|x|} \sum_{p=0}^{+\infty} \frac{1}{\left(\frac{p+2}{2}\right)_{\frac{n-1}{2}}\Gamma\left(1 - \frac{\beta(p+n+1)}{2}\right)p!} \left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^p
$$

$$
+ \mathfrak{f}\left(\frac{1}{4c^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}\,t^{2\beta}} \sum_{k=0}^{\frac{n-3}{2}} \frac{\Gamma\left(-1-k+\frac{n}{2}\right)}{\Gamma(1-\beta(k+2))\,k!} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^k \right.
$$

$$
\left. + \frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}\,t^\beta} \sum_{p=0}^{+\infty} \frac{1}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}\Gamma\left(1 - \frac{\beta(p+n+2)}{2}\right)p!} \left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^p \right)
$$

$$
+ \mathfrak{f}^\dagger \left(\frac{1}{4c^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}\,t^\beta} \sum_{k=0}^{\frac{n-3}{2}} \frac{\Gamma\left(-1-k+\frac{n}{2}\right)}{\Gamma\left(1-\beta(k+1)\right)\,k!} \left(-\frac{|x|^2}{4c^2\,t^\beta}\right)^k \right.
$$

$$
\left. + \frac{(-1)^{\frac{n-1}{2}}\sqrt{\pi}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \sum_{p=0}^{+\infty} \frac{1}{\left(\frac{p+1}{2}\right)_{\frac{n-1}{2}}\Gamma\left(1 - \frac{\beta(p+n)}{2}\right)p!} \left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^p \right). \tag{61}
$$

For n even and $0 < \beta \le 2$ the FS of the time fractional parabolic Dirac operator is given by

$$
\mathfrak{G}_n^\beta(x,t) = \frac{-x}{2c\,\pi^{\frac{n}{2}}\,t^\beta\,|x|^n} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma\left(\frac{n}{2}-k\right)}{\Gamma(1-\beta(k+1))\,k!} \left(-\frac{|x|^2}{4c^2\,t^\beta}\right)^k
$$

$$
+ \frac{(-1)^{\frac{n}{2}+1}\,2c\,x}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}\,|x|^2} \sum_{k=0}^{+\infty} \frac{k\left[\psi\left(k+\frac{n}{2}\right) - \beta\,\psi\left(1 - \beta\left(k+\frac{n}{2}\right)\right) + \psi(k+1) + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)\right] - 1}{\Gamma\left(k+\frac{n}{2}\right)\Gamma\left(1 - \beta\left(k+\frac{n}{2}\right)\right)k!} \left(\frac{|x|^2}{4c^2 t^\beta}\right)^k
$$

$$
+ \mathfrak{f}\left(\frac{1}{4c^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}\,t^{2\beta}} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma\left(\frac{n}{2}-k-1\right)}{\Gamma(1-\beta(k+2))\,k!} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^k \right.
$$

$$
\left. + \frac{(-1)^{\frac{n}{2}+1}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}\,t^\beta} \sum_{k=0}^{+\infty} \frac{\psi\left(k+\frac{n}{2}\right) - \beta\,\psi\left(1 - \beta\left(k+\frac{n}{2}+1\right)\right) + \psi(k+1) + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)}{\Gamma\left(k+\frac{n}{2}\right)\Gamma\left(1 - \beta\left(k+\frac{n}{2}+1\right)\right)k!} \left(\frac{|x|^2}{4c^2 t^\beta}\right)^k \right)
$$

$$
+ \mathfrak{f}^\dagger \left(\frac{1}{4c^2\,\pi^{\frac{n}{2}}\,|x|^{n-2}\,t^\beta} \sum_{k=0}^{\frac{n}{2}-2} \frac{\Gamma\left(\frac{n}{2}-k-1\right)}{\Gamma\left(1-\beta(k+1)\right)\,k!} \left(-\frac{|x|^2}{4c^2 t^\beta}\right)^k \right.
$$

$$
\left. + \frac{(-1)^{\frac{n}{2}+1}}{(4\pi\,c^2\,t^\beta)^{\frac{n}{2}}} \sum_{k=0}^{+\infty} \frac{\psi\left(k+\frac{n}{2}\right) - \beta\,\psi\left(1 - \beta\left(k+\frac{n}{2}\right)\right) + \psi(k+1) + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)}{\Gamma\left(k+\frac{n}{2}\right)\Gamma\left(1 - \beta\left(k+\frac{n}{2}\right)\right)k!} \left(\frac{|x|^2}{4c^2 t^\beta}\right)^k \right). \tag{62}
$$

Remark 4.2 *For some rational values of β the series coefficients in (62) have an indetermination due to the terms $\beta\psi\left(1 - \beta\left(k + \frac{n}{2}\right)\right)$ and $\Gamma\left(1 - \beta\left(k + \frac{n}{2}\right)\right)$ in the first and the third series, while in the second series the indetermination is due to the terms $\beta\psi\left(1 - \beta\left(k + \frac{n}{2} + 1\right)\right)$ and $\Gamma\left(1 - \beta\left(k + \frac{n}{2} + 1\right)\right)$. To remove these indeterminations we can proceed as was done in Subsection 3.2.2 applying (11) with $n = 1$, (14), (15), and the relation $\psi(1 - z) = \pi\cot(\pi z) + \psi(z)$ for the digamma function.*

5 Particular cases of β

In this section we present the FS for the time fractional diffusion-wave operator and the time fractional parabolic Dirac operator for the special cases $\beta = 0, 1, 2$.

5.1 Case of $\beta = 0$

For $\beta = 0$ the time fractional diffusion-wave operator simplifies to the Helmholtz operator $-c^2\Delta + I$ and the FS reduces in the even and odd cases to

$$G_n^0(x, t) = \frac{1}{(2\pi c^2)^{\frac{n}{2}}}\left(\frac{|x|}{c}\right)^{1 - \frac{n}{2}} K_{\frac{n}{2} - 1}\left(\frac{|x|}{c}\right), \tag{63}$$

where K_ν is the modified Bessel function of second kind with parameter ν. We would like to remark that (63) is a radial eigenfunction of the operator $c^2\Delta$. In the case of the time fractional parabolic Dirac operator, when $\beta = 0$ it simplifies to the operator $c\,D_x + \mathfrak{f}\,I + \mathfrak{f}^\dagger I$, and the FS reduces in the even and odd cases to

$$\mathfrak{G}_n^0(x, t) = \frac{1}{(2\pi c^2)^{\frac{n}{2}}}\left(\frac{|x|}{c}\right)^{-\frac{n}{2}}\left(-\frac{x}{c}\,K_{\frac{n}{2}}\left(\frac{|x|}{c}\right) + (\mathfrak{f} + \mathfrak{f}^\dagger)\frac{|x|}{c}\,K_{\frac{n}{2} - 1}\left(\frac{|x|}{c}\right)\right).$$

5.2 Case of $\beta = 1$

For $\beta = 1$ the time fractional diffusion-wave operator reduces to the Heat operator $-c^2\Delta + \partial_t$ and the FS reduces in the even and odd cases to the classical FS given by

$$G_n^1(x, t) = \frac{1}{(4\pi\,c^2 t)^{\frac{n}{2}}}\exp\left(-\frac{|x|^2}{4c^2 t}\right).$$

In the case of the time fractional parabolic Dirac operator, when $\beta = 1$ it simplifies to the operator $c\,D_x + \mathfrak{f}\,\partial_t + \mathfrak{f}^\dagger$ and the FS reduces in the even and odd cases to

$$\mathfrak{G}_n^1(x, t) = \frac{1}{(4\pi\,c^2\,t)^{\frac{n}{2}}}\exp\left(-\frac{|x|^2}{4c^2 t}\right)\left(-\frac{x}{2c\,t} + \mathfrak{f}\left(\frac{|x|^2}{4c^2 t^2} - \frac{n}{2t}\right) + \mathfrak{f}^\dagger\right), \tag{64}$$

which is in accordance with the FS considered in [33] with $c = 1$.

5.3 Case of $\beta = 2$

For $\beta = 2$ the time fractional diffusion-wave operator simplifies to the wave operator $-c^2\Delta + \partial_{tt}^2$ and the FS reduces in the even case to

$$G_n^2(x, t) = \frac{(-1)^{\frac{n}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}}\,(c\,t)^n}\left(1 - \frac{|x|^2}{c^2 t^2}\right)^{-\frac{n+1}{2}}. \tag{65}$$

We would like to remark that (65) coincides up to a function in the kernel of the wave operator to the classical FS of the wave operator. For n odd it is known that the FS of the wave operator contains delta functions and derivatives of delta functions, these last appearing only for $n \geq 5$ (for more details see [13]). For instance, we have the following limits in distributional sense

$$\lim_{\beta \to 2^-} G_1^\beta(x, t) = \frac{1}{2}\left(\delta\left(|x| - c\,t\right) + \delta\left(|x| + c\,t\right)\right)$$

$$\lim_{\beta \to 2^-} G_3^\beta(x, t) = \frac{1}{4\pi\,|x|}\left(\delta\left(|x| - c\,t\right) - \delta\left(|x| + c\,t\right)\right) \tag{66}$$

$$\lim_{\beta \to 2^-} G_5^\beta(x, t) = \frac{1}{16\pi^2\,c\,|x|^3}\left(\delta\left(|x| - c\,t\right) - \delta\left(|x| + c\,t\right) + |x|\,\delta'\left(|x| - c\,t\right) - |x|\,\delta'\left(|x| + c\,t\right)\right).$$

In the case of the time fractional parabolic Dirac operator, when $\beta = 2$ it simplifies to the operator $c\,D_x + \mathfrak{f}\,\partial_{tt}^2 + \mathfrak{f}^\dagger$ and the FS reduces when n is even to

$$\mathfrak{G}_n^2(x,t) = \frac{(-1)^{\frac{n}{2}}\,\Gamma\left(\frac{n+3}{2}\right)\,2x}{(c^2\pi)^{\frac{n+1}{2}}\,t^{n+2}}\left(1 - \frac{|x|^2}{c^2t^2}\right)^{-\frac{n+3}{2}} + \mathfrak{f}\,\frac{(-1)^{\frac{n}{2}}\,\Gamma\left(\frac{n+3}{2}\right)}{c^n\,\pi^{\frac{n+1}{2}}\,t^{n+2}}\left(2n + \frac{6|x|^2}{c^2t^2}\right)\left(1 - \frac{|x|^2}{c^2t^2}\right)^{-\frac{n+5}{2}}$$
$$+ \mathfrak{f}^\dagger\,\frac{(-1)^{\frac{n}{2}}\,\Gamma\left(\frac{n+1}{2}\right)}{c^n\,\pi^{\frac{n+1}{2}}\,t^n}\left(1 - \frac{|x|^2}{c^2t^2}\right)^{-\frac{n+1}{2}}.$$

For n odd the FS of the time fractional parabolic Dirac operator contains delta functions and derivatives of delta functions.

6 Fractional Moments

In this section we compute the fractional moments of order $\gamma > 0$ associated to G_n^β. It is well known that the Mellin transform (5) can be interpreted as the fractional moment of order $s - 1$ of the function f (see [18,19]). Therefore, we can calculate the moments of order $\gamma > 0$ of G_n^β. From the definition of the Mellin transform we have that

$$\mathbf{M}_n^{\beta,\gamma}(t) = \int_0^{+\infty} r^\gamma\,G_n^\beta(r,t)\,dr$$
$$= \int_0^{+\infty} r^{\gamma-n+1-1}\,r^n\,G_n^\beta(r,t)\,dr$$
$$= \mathcal{M}\{r^n\,G_n^\beta(r,t)\}(\gamma - n + 1).$$

Taking into account (36) and the integral representation (37) we obtain:

$$\mathcal{M}\{r^n\,G_n^\beta(r,t)\}(\gamma - n + 1) = \left.\frac{(4c^2\,t^\beta)^{\frac{s}{2}}\,\Gamma\left(1 + \frac{s}{2}\right)\,\Gamma\left(\frac{n+s}{2}\right)}{2\,\pi^{\frac{n}{2}}\,\Gamma\left(1 + \frac{\beta s}{2}\right)}\right|_{s=\gamma-n+1}$$
$$= \frac{(4c^2\,t^\beta)^{\frac{\gamma-n+1}{2}}\,\Gamma\left(\frac{3+\gamma-n}{2}\right)\,\Gamma\left(\frac{\gamma+1}{2}\right)}{2\,\pi^{\frac{n}{2}}\,\Gamma\left(1 + \frac{\beta(\gamma-n+1)}{2}\right)}.$$

Therefore, the fractional moments are given for $0 < \beta \leq 2$, $\gamma > 0$, and $n \in \mathbb{N}$ by

$$\mathbf{M}_n^{\beta,\gamma}(t) = \frac{(4c^2\,t^\beta)^{\frac{\gamma-n+1}{2}}\,\Gamma\left(\frac{3+\gamma-n}{2}\right)\,\Gamma\left(\frac{\gamma+1}{2}\right)}{2\,\pi^{\frac{n}{2}}\,\Gamma\left(1 + \frac{\beta(\gamma-n+1)}{2}\right)}. \tag{67}$$

We present in the following table the expression of the moments for some particular values of n and γ. From the expression (67) there are two special cases that we analyse now:

- When $\gamma = n - 1$, with $n \in \mathbb{N}$, the moment is independent of the time variable, and it is given by

$$\mathbf{M}_n^{\beta,n-1}(t) = \frac{\Gamma\left(\frac{n}{2}\right)}{2\pi^{\frac{n}{2}}}.$$

- When $\gamma = n - 2k - 3$, with $n > 2k + 3$ and $k \in \mathbb{N}_0$, the moment becomes infinite, for all $\beta \in]0,2[\backslash\{1\}$. For example, when $k = 0$ and $n = 4$ the moment $\mathbf{M}_4^{\beta,1}$ is infinite for all $\beta \in]0,2[\backslash\{1\}$.

We would like to remark that in the case of $n = 1$ the moments were calculated only for positive values of x. The true moments of the FS should be calculated over the whole real line. However, in that case it is not possible to compute the fractional moments for every order $\gamma > 0$ since the power function r^γ is not always well defined. Nevertheless, two cases are of special importante: the odd integer moments ($\gamma = 2k + 1, k \in \mathbb{N}$) vanish, while the even integer moments are given by

$$\mathbf{M}_1^{\beta,2k}(t) = \int_{-\infty}^{+\infty} r^{2k}\,G_1^\beta(r,t)\,dr = 2\int_0^{+\infty} r^{2k}\,G_1^\beta(r,t)\,dr$$
$$= \frac{(4c^2\,t^\beta)^k\,\Gamma\left(1 + k\right)\,\Gamma\left(k + \frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(1 + \beta k\right)} = \frac{\Gamma\left(1 + 2k\right)\,(c^2\,t^\beta)^k}{\Gamma\left(1 + \beta k\right)}.$$

	$n = 1$	$n = 2$	$n = 3$	$n \in \mathbb{N}$
$\gamma = 1$ (mean value)	$\dfrac{c\,t^{\frac{\beta}{2}}}{2\,\Gamma\left(1+\frac{\beta}{2}\right)}$	$\dfrac{1}{2\pi}$	$\dfrac{1}{4\pi\,c\,t^{\frac{\beta}{2}}\,\Gamma\left(1-\frac{\beta}{2}\right)}$	$\dfrac{\left(4c^2\,t^{\beta}\right)^{\frac{2-n}{2}}\,\Gamma\left(2-\frac{n}{2}\right)}{2\,\pi^{\frac{n}{2}}\,\Gamma\left(1+\frac{\beta(2-n)}{2}\right)}$
$\gamma = 2$ (variance)	$\dfrac{c^2\,t^{\beta}}{\Gamma(1+\beta)}$	$\dfrac{c\,t^{\frac{\beta}{2}}}{4\,\Gamma\left(1+\frac{\beta}{2}\right)}$	$\dfrac{1}{4\pi}$	$\dfrac{\left(4c^2\,t^{\beta}\right)^{\frac{3-n}{2}}\,\Gamma\left(\frac{5-n}{2}\right)}{4\,\pi^{\frac{n-1}{2}}\,\Gamma\left(1+\frac{\beta(3-n)}{2}\right)}$
$\gamma = 3$ (3rd moment)	$\dfrac{3\left(c^2\,t^{\beta}\right)^{\frac{3}{2}}}{\Gamma\left(1+\frac{3\beta}{2}\right)}$	$\dfrac{2c^2\,t^{\beta}}{\pi\,\Gamma(1+\beta)}$	$\dfrac{c\,t^{\frac{\beta}{2}}}{\pi\,\Gamma\left(1+\frac{\beta}{2}\right)}$	$\dfrac{\left(4c^2\,t^{\beta}\right)^{\frac{4-n}{2}}\,\Gamma\left(\frac{6-n}{2}\right)}{2\,\pi^{\frac{n}{2}}\,\Gamma\left(1+\frac{\beta(4-n)}{2}\right)}$

Table 1: Particular moments of G_n^{β}.

These moments agree with the ones calculated in [26] when $c = 1$. For the case of the time fractional parabolic Dirac operator it is also possible to compute the fractional moments of $\mathfrak{G}_n^{\beta}$. However, due to the extension of the formulas we do not present them here.

7 Graphical representations of G_n^{β}

In this section we present and discuss some plots of G_n^{β}, for $c = 1$, $n = 1, 2, 3, 4$, and some values of the fractional parameter β. For the one-dimensional case we use the series representation of G_1^{β} since the FS reduces to a Wright function and we can use the algorithm developed by Luchko in [21] to numerically evaluate the Wright function with a good accuracy. For $n = 2, 3, 4$ we use the integral representation (33) to make the plots since the series for theses cases doesn't involve exclusively Wright functions. Moreover, the integral representation (33) allow us to evaluate the FS with a better accuracy. For using the series representation it would be necessary to perform an analysis of its asymptotic behaviour, similarly to the study made for the one-dimensional case (see [21]).

7.1 Case $n = 1$

For $n = 1$, the FS can be written as a Wright function (see [25]). In fact, putting $n = 1$ in (49) we obtain

$$
\begin{aligned}
G_1^{\beta}(x,t) & = \frac{1}{(4c^2\,t^{\beta})^{\frac{1}{2}}} \sum_{p=0}^{+\infty} \frac{1}{\Gamma\left(1-\frac{\beta(p+1)}{2}\right)\,p!}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^p \\
& = \frac{1}{2c\,t^{\frac{\beta}{2}}}\,W_{-\frac{\beta}{2},1-\frac{\beta}{2}}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right).
\end{aligned}
$$

Since the variable $X = |x| / \left(c\,t^{\beta/2}\right)$ acts as a similarity variable we show in Figure 1 the graphical representation of the reduced Green function $U(x) = G_1^{\beta}(x,1)$, for $c = 1$ and some values of the fractional parameter β. In Figure 2 we show some plots of G_1^{β} for some fixed values of β and t.
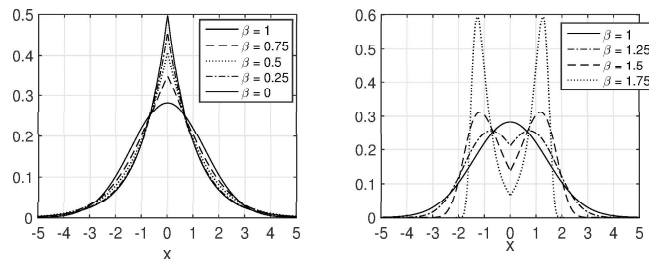

Figure 1: Plots of $G_1^{\beta}(x,1)$ for $c = 1$, $\beta = 0$, 0.25, 0.5, 0.75, 1 (left), and $\beta = 1$, 1.25, 1.5, 1.75 (right).
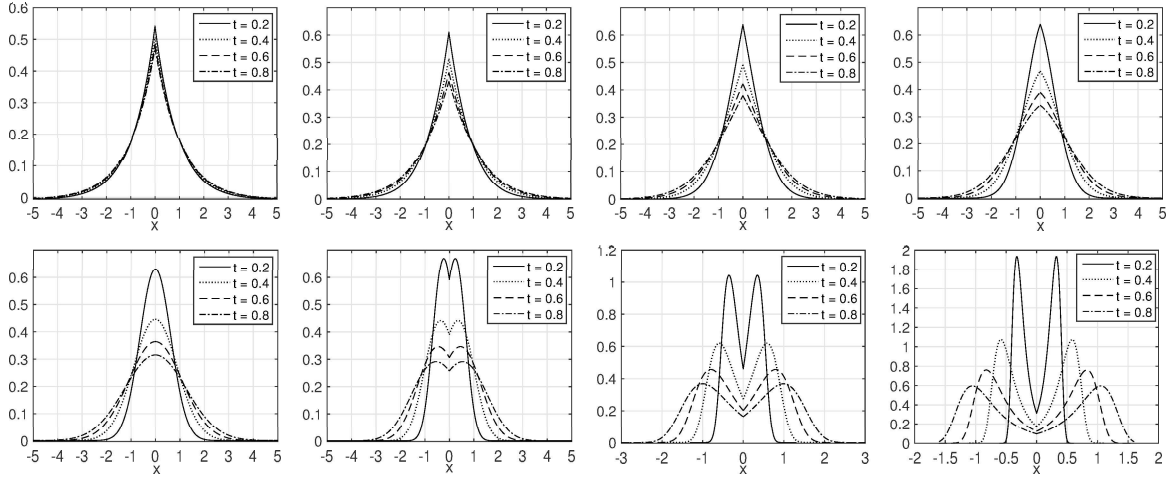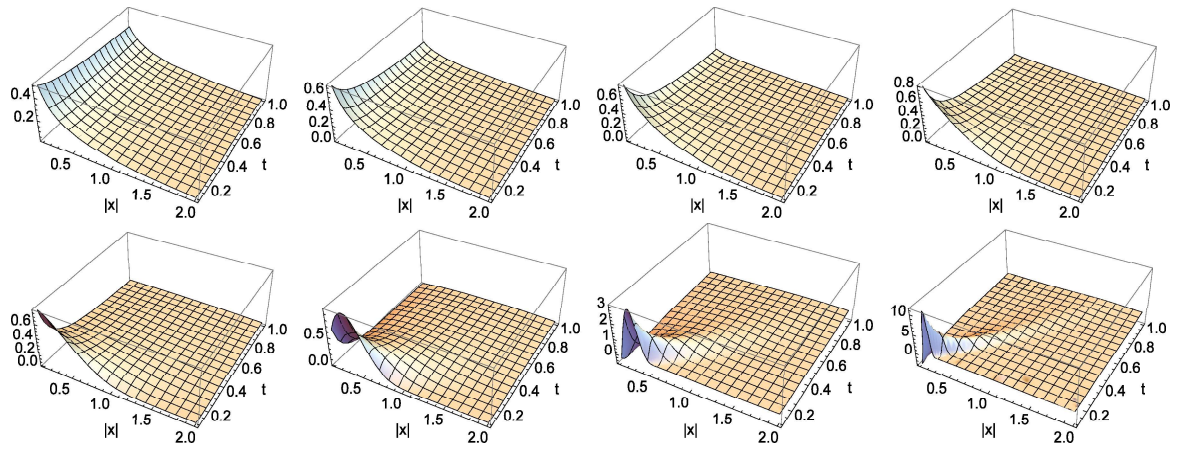
Figure 2: Plots of G_1^β for $c = 1$, $\beta = 0.2$, 0.5, 0.75, 0.9 (1st line, from left to right), $\beta = 1.0, 1.2, 1.5, 1.7$ (2nd line, from left to right), and $t = 0.2, 0.4, 0.6, 0.8$.

The plots obtained are similar to those presented in [26]. It is known that G_1^β is a probability density function corresponding to a slow diffusion in the case $0 < \beta < 1$ and a fast diffusion when $1 < \beta < 2$ (see [26]). In the slow diffusion case the FS attains its maximum value at $x = 0$, its first derivative is discontinuous in this point, and it exhibits exponential slow decay similar to the Gaussian (case of $\beta = 1$). In the fast diffusion case the FS attains two symmetric maxima that move apart from the origin with time and the fractional parameter β, exhibiting exponential decay faster than the Gaussian. For more details about the analysis of the one dimensional case see [19, 25, 26].

7.2 Case $n = 2$

Considering $n = 2$ in (50) the FS simplifies to

$$G_2^\beta(x,t) = \frac{1}{4\pi\, c^2\, t^\beta} \sum_{k=0}^{+\infty} \frac{2\psi\,(k+1) - \beta\,\psi\,(1 - \beta\,(k+1)) + \ln\left(\frac{4c^2 t^\beta}{|x|^2}\right)}{\Gamma\,(k+1)\,\Gamma\,(1 - \beta\,(k+1))\,k!} \left(\frac{|x|^2}{4c^2 t^\beta}\right)^k.$$

In Figure 3 we show some plots of G_2^β for $c = 1$ and some fixed values of β and t.

Figure 3: Plots of G_2^β for $c = 1$, $\beta = 0.2$, 0.5, 0.75, 0.9 (1st line, from left to right), and $\beta = 1.0, 1.2, 1.5, 1.7$ (2nd line, from left to right), with $|x| \in [0.1, 2]$ and $t \in [0.1, 1]$.

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From the plots we observe that for $\beta \in]0,1[$ the FS represents a slow diffusion process both in space and time. The decay is more pronounced in space than in time due to the term $|x|^2$ that appears in the series representation of the FS. For $\beta \in]1,2[$ the FS represents a fast diffusion process and its behaviour can be interpreted as propagation of damped waves whose amplitude decreases with time. For $\beta = 1$ the plot represents the classical solution of the heat equation. In the limit case $\beta \to 2^-$, G_2^β reduces to the FS of the wave equation, which has support for $|x| < t$, accordingly to (65).

7.3 Case $n = 3$

For $n = 3$ we obtain from (49) the following expression:

$$G_3^\beta(x,t) \;=\; \frac{1}{4c^2\,\pi\,|x|\,t^\beta\,\Gamma(1-\beta)} - \frac{1}{4\pi(c^2\,t^\beta)^{\frac{3}{2}}} \sum_{p=0}^{+\infty} \frac{1}{\Gamma\left(1 - \frac{\beta(p+3)}{2}\right)\,(p+1)!}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^p. \tag{68}$$

Making the change of variable $p = k - 1$ and joining the terms we can write G_3^β as a Wright function:

$$
\begin{aligned}
G_3^\beta(x,t) \;&=\; \frac{1}{4c^2\,\pi\,|x|\,t^\beta\,\Gamma(1-\beta)} + \frac{1}{4c^2\,\pi\,|x|\,t^\beta} \sum_{k=1}^{+\infty} \frac{1}{\Gamma\left(1 - \frac{\beta(k+2)}{2}\right)\,k!}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^k \\[2mm]
&=\; \frac{1}{4c^2\,\pi\,|x|\,t^\beta} \sum_{k=0}^{+\infty} \frac{1}{\Gamma\left(1 - \frac{\beta(k+2)}{2}\right)\,k!}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right)^k \\[2mm]
&=\; \frac{1}{4c^2\,\pi\,|x|\,t^\beta}\; W_{-\frac{\beta}{2},1-\beta}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right).
\end{aligned}
\tag{69}
$$

In Figure 4 we show some plots of G_3^β for $c = 1$, and some fixed values of β and t.
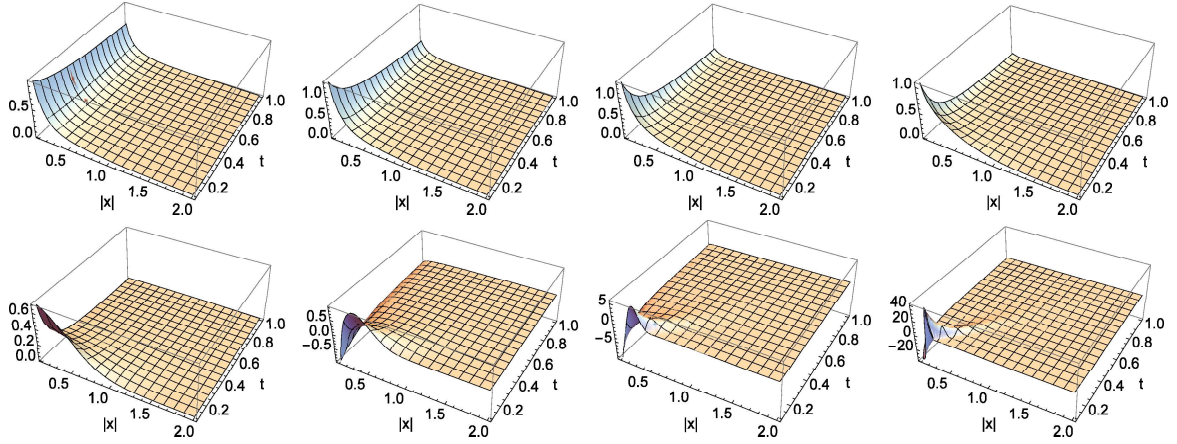
Figure 4: Plots of G_3^β for $c = 1$, $\beta = 0.2$, 0.5, 0.75, 0.9 (1st line, from left to right), and $\beta = 1.0$, 1.2, 1.5, 1.7 (2nd line, from left to right), with $|x| \in [0.1, 2]$ and $t \in [0.1, 1]$.

For the three dimensional case we observe similar behaviours of slow and fast diffusion of the FS as in the case $n = 2$, but for $1 < \beta < 2$ the solutions are no longer non-negative. Moreover, the range of values of G_3^β increases. There is a difference from the case $n = 2$, since in the limit case $\beta \to 2^-$, the FS of the wave equation has support only for $|x| = t$, accordingly to (66). This difference can be observed between even and odd dimensions.

7.4 Case $n = 4$

In Figure 5 we show some plots of G_4^β for some fixed values of β and t. Although the range of values assumed by G_4^β increases we can observe the same behaviours of the FS previously described for $n = 2$ and $n = 3$.
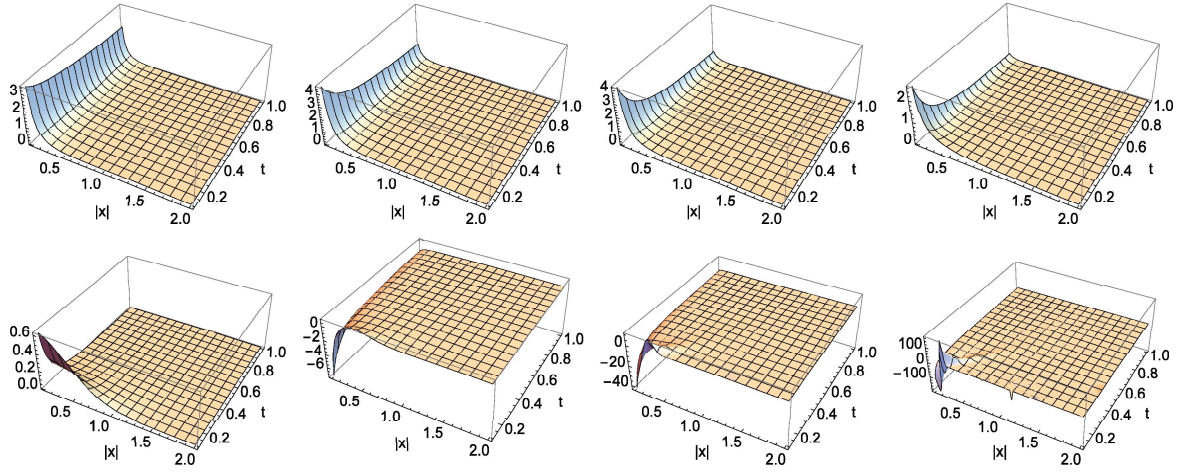
Figure 5: Plots of G_4^β for $c = 1$, $\beta = 0.2$, 0.5, 0.75, 0.9 (1st line, from left to right), and $\beta = 1.0$, 1.2, 1.5, 1.7 (2nd line, from left to right), with $|x| \in [0.1, 2]$ and $t \in [0.1, 1]$.

8 Graphical representations of $\mathfrak{G}_n^\beta$

In this section we present and discuss some plots of $\mathfrak{G}_n^\beta$, for $c = 1$, $n = 1, 2$ and some values of the fractional parameter β. By the same reasons as in the case of G_n^β we use the series representation in the one-dimensional case and for $n = 2$ we use the integral representation (54) to make the plots. We represent only the vectorial and $\mathfrak{f}$−components since the $\mathfrak{f}^\dagger$−component coincides with G_n^β.

8.1 Case $n = 1$

Considering $n = 1$ in (61) we obtain the following expression:

$$
\mathfrak{G}_1^\beta(x, t) \quad = \quad -\frac{x}{2c\,t^\beta\,|x|}\, W_{-\frac{\beta}{2}, 1-\beta}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right) + \mathfrak{f}\,\frac{1}{2c\,t^{\frac{3\beta}{2}}}\, W_{-\frac{\beta}{2}, 1-\frac{3\beta}{2}}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right) + \mathfrak{f}^\dagger\,\frac{1}{2c\,t^{\frac{\beta}{2}}}\, W_{-\frac{\beta}{2}, 1-\frac{\beta}{2}}\left(-\frac{|x|}{c\,t^{\frac{\beta}{2}}}\right).
$$

In Figure 6 we show the plots of the real part of the reduced Green function $\mathfrak{G}_1^\beta(x, 1)$ for some values of the fractional parameter β. In Figure 7 we show some plots of the real part of $\mathfrak{G}_1^\beta$ for different values of β and t.
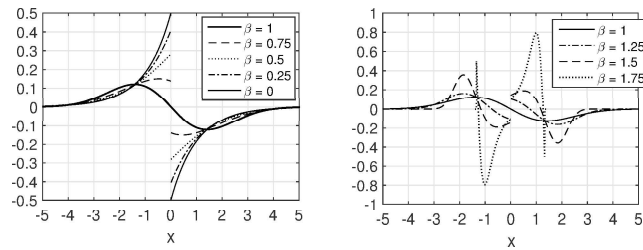

Figure 6: Plots of the real part of $\mathfrak{G}_1^\beta(x, 1)$ for $c = 1$, $\beta = 0$, 0.25, 0.5, 0.75, 1 (left), and $\beta = 1$, 1.25, 1.5, 1.75 (right).
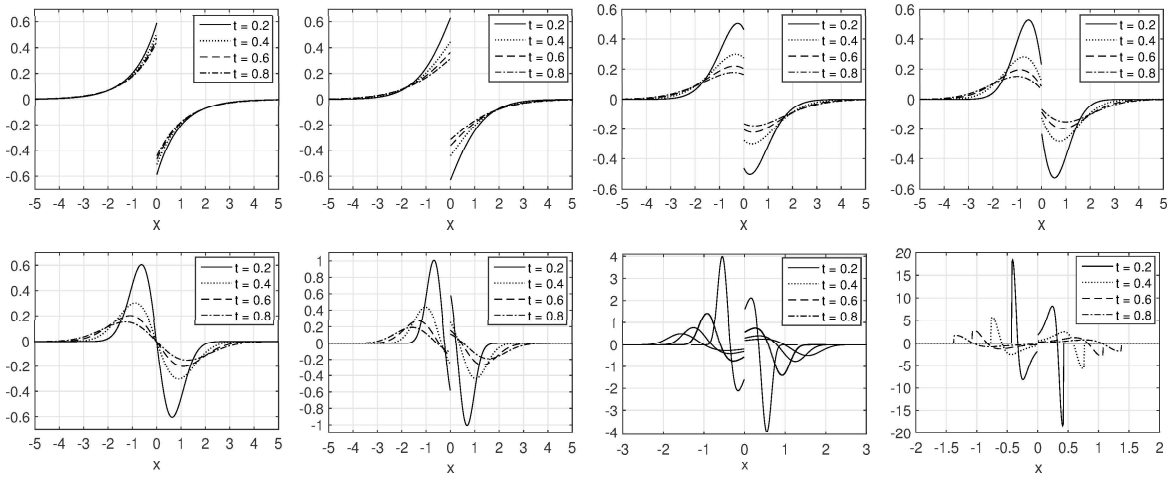
Figure 7: Plots of the real part of $\mathfrak{G}_1^\beta$ for $c = 1$, $\beta = 0.2, 0.5, 0.75, 0.9$ (1st line, from left to right), $\beta = 1.0, 1.2, 1.5, 1.7$ (2nd line, from left to right), and $t = 0.2, 0.4, 0.6, 0.8$.

From the plots of Figures 6 and 7 we see that the real component its an odd function and is discontinuous at $x = 0$. In the range of $\beta \in]0, 1[$ we observe two different behaviours: initially the function behaves like the hyperbola curve and then the curve starts to have extrema points, which its absolute values decrease with time. When $\beta = 1$ we have a odd continuous function that corresponds to the real component in (64). For $\beta \in]1, 2[$ we observe again a discontinuity at $x = 0$ and the function starts to have more extrema points and the function shrinks horizontally when $\beta \to 2^-$. This behaviour is in accordance to the expected wave propagation phenomena.

In Figure 8 we show the plots of the $\mathfrak{f}-$component of the reduced Green function $\mathfrak{G}_1^\beta(x, 1)$ for some values of the fractional parameter β. In Figure 9 we show some plots of $\mathfrak{f}-$component of $\mathfrak{G}_1^\beta$ for different values of β and t.
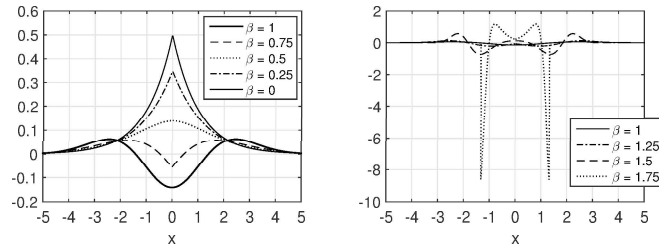

Figure 8: Plots of the $\mathfrak{f}-$component of $\mathfrak{G}_1^\beta(x, 1)$ for $c = 1$, $\beta = 0, 0.25, 0.5, 0.75, 1$ (left), and $\beta = 1.25, 1.5, 1.75$ (right).
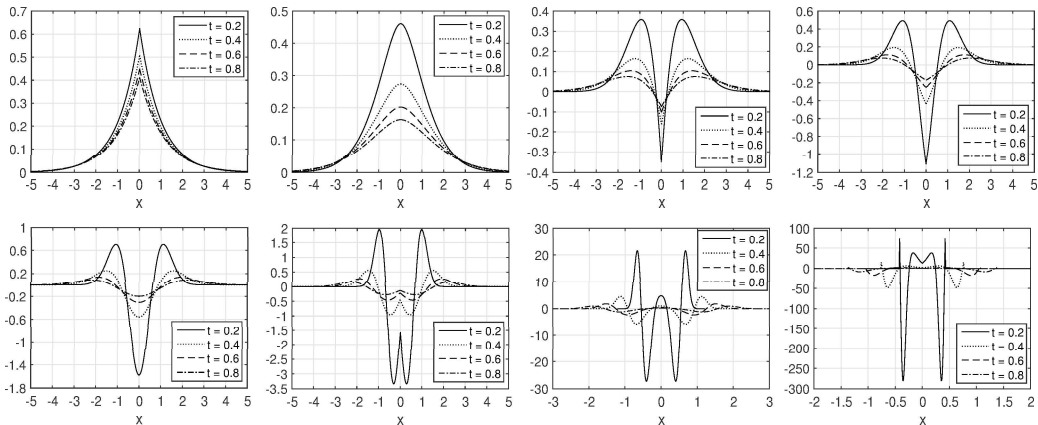

Figure 9: Plots of the $\mathfrak{f}-$component of $\mathfrak{G}_1^\beta$ for $c = 1$, $\beta = 0.2, 0.5, 0.75, 0.9$ (1st line, from left to right), $\beta = 1.0, 1.2, 1.5, 1.7$ (2nd line, from left to right), and $t = 0.2, 0.4, 0.6, 0.8$.

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From the plots of Figures 8 and 9 we observe that the function is even and has different behaviours. Initially, for small values of β the function has only one maxima at $x = 0$ and as far as we approach $\beta = 1$ the function has two symmetric maximum points and one minimum point at $x = 0$. When $\beta = 1$ the plot corresponds to $\mathfrak{f}-$component of $\mathfrak{G}_1^1$ (see (64)). For $\beta \in]1, 2[$ the function starts to increase the number of extrema points, passing to two symmetric maximum points, and two symmetric minimum points, and a local maximum at $x = 0$. Again, the function shrinks horizontally when $\beta \to 2^-$, which is in accordance to the expected wave propagation phenomena.

8.2 Case $n = 2$

In Figures 10 and 11 we show the plots of the vectorial part and the $\mathfrak{f}-$component of $\mathfrak{G}_2^\beta$ for different values of β and t.
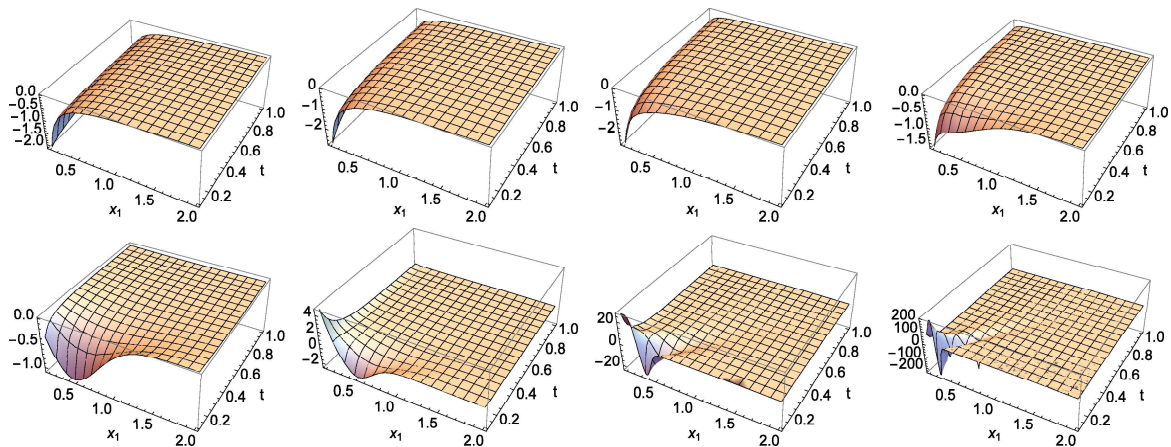

Figure 10: Plots of the vectorial part of $\mathfrak{G}_2^\beta$ for $c = 1$, $\beta = 0.2$, 0.5, 0.75, 0.9 (1st line, from left to right), and $\beta = 1.0$, 1.2, 1.5, 1.7 (2nd line, from left to right), with $x_1 \in [0.1, 2]$, $x_2 = 0$, and $t \in [0.1, 1]$.
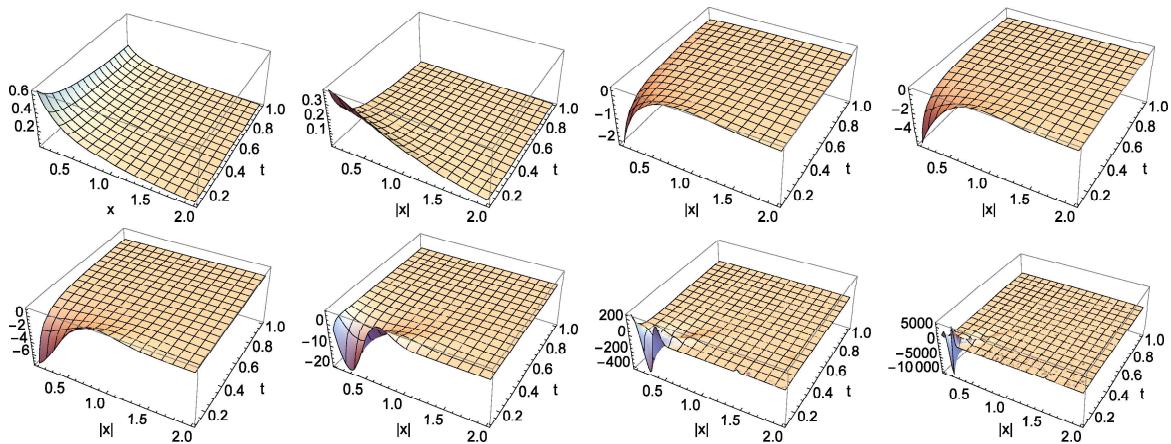

Figure 11: Plots of the $\mathfrak{f}-$component of $\mathfrak{G}_2^\beta$ for $c = 1$, $\beta = 0.2$, 0.5, 0.75, 0.9 (1st line, from left to right), and $\beta = 1.0$, 1.2, 1.5, 1.7 (2nd line, from left to right), with $|x| \in [0.1, 2]$ and $t \in [0.1, 1]$.

The plots presented in Figures 10 and 11 show that $\mathfrak{G}_2^\beta$ has a similar behaviour as described in the one dimensional case for $\mathfrak{G}_1^\beta$. The main difference is the increase of the range os the values of each component of the FS. Similar plots can be obtained for $n > 2$.

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables. 10th printing*, National Bureau of Standards, Wiley-Interscience Publication, John Wiley & Sons, New York etc., 1972.

[2] M. Caputo, *The Green function of the diffusion of fluids in porous media with memory*, Rend. Fis. Acc. Lincei (Ser. 9), **7**, (1996), 243–250.

[3] P. Cerejeiras, U. Kähler, M.M. Rodrigues and N. Vieira, *Hodge decomposition in variable exponent spaces for the time-dependent operators: the Schrödinger case*, Commun. Pure Appl. Anal., **13**-No.6, (2014), 2253–2272.

[4] P. Cerejeiras, U. Kähler and F. Sommen, *Parabolic Dirac operators and the Navier-Stokes equations over time-varying domains*, Math. Meth. in Appl. Sc., **28**-No.14, (2005), 1715–1724.

[5] R. Delanghe, F. Sommen and V. Souček, *Clifford algebras and spinor-valued functions. A function theory for the Dirac operator*, Mathematics and its Applications-Vol.53, Kluwer Academic Publishers, Dordrecht etc., 1992.

[6] M. Ferreira, and N. Vieira, *Eigenfunctions and fundamental solutions of the fractional Laplace and Dirac operators: the Riemann-Liouville case*, Complex Anal. Oper. Theory, **10**-No.5, (2016), 1081-1100.

[7] Y. Fujita, *Integrodifferential equations which interpolate the heat equation and the wave equation*, Osaka J. Math., **27**-No.2, (1990), 309–321

[8] R. Gorenflo, A.A. Kilbas, F. Mainardi and S. Rogosin, *Mittag-Leffler functions. Theory and applications*, Springer Monographs in Mathematics, Springer, Berlin, 2014.

[9] R. Gorenflo, Y. Luchko and F. Mainardi, *Analytical properties and applications of the Wright function*, Fract. Calc. Appl. Anal., 2-No.4, (1999), 383–414.

[10] K. Gürlebeck and W. Sprößig, *Quaternionic and Clifford calculus for physicists and engineers*, Mathematical Methods in Practice, Wiley, Chichester, 1997.

[11] A. Hanyga, *Multidimensional solutions of time-fractional diffusion-wave equations*, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci., **458**-No.2020,(2002), 933–957.

[12] A. Hanyga, *Multi-dimensional solutions of space-time-fractional diffusion equations*, Proc. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **458**-No.2018, (2002), 429–450.

[13] T. Jonsson and J. Yngvason, *Waves and distributions*, World Scientific, Singapore, 1995.

[14] U. Kähler and N. Vieira, *Fractional Clifford analysis*, In: Hypercomplex Analysis: New perspectives and applications, Trends in Mathematics, S. Bernstein, U. Kähler, I. Sabadini, and F. Sommen (eds.), Birkhäuser, Basel, (2014), 191–201.

[15] A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and applications of fractional differential equations*, North-Holland Mathematics Studies-Vol.204, Elsevier, Amsterdam, 2006.

[16] A. Kilbas and M. Saigo, *H-transforms. Theory and applications*, Analytical Methods and Special Functions-Vol.9, Chapman & Hall/CRC, Boca Raton, FL, 2004.

[17] R. Kraußhar, M.M. Rodrigues and N. Vieira, *Time-dependent operators on some non-orientable projective orbifolds*, Math. Methods Appl. Sci., **38**-No.18, (2015), 5305–5319.

[18] Y. Luchko, *Multi-dimensional fractional wave equation and some properties of its fundamental solution*, Commun. Appl. Ind. Math., **6**-No.1, (2014), Article ID 485, 21p.

[19] Y. Luchko, *Fractional wave equation and damped waves*, J. Math. Phys., **54**-No.3, (2013), Article ID: 031505, 16p.

[20] Y. Luchko and A. Punzi, *Modeling anomalous heat transport in geothermal reservoirs via fractional diffusion equations*, GEM. Int. J. Geomath., **1**-No.2, (2011), 257–276.

[21] Y. Luchko, *Algorithms for evaluation of the Wright function for the real arguments values*, Fract. Calc. Appl. Anal., **11**-No.1, (2008), 57–75.

[22] Y. Luchko, *Operational method in fractional calculus*, Frac. Calc. Appl. Anal., **2**-No.4, (1999), 463–489.

[23] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity. An introduction to mathematical models*, World Scientific, Hackensack, NJ, 2010.

[24] F. Mainardi and G. Pagnini, *The role of the Fox-Wright functions in fractional sub-diffusion of distributed order*, J. Comput. Appl. Math., **207**-No.2, (2007), 245–257.

[25] F. Mainardi and G. Pagnini, *The Wright functions as solutions of the time-fractional diffusion equation*, Appl. Math. Comput., **141**-No.1, (2003), 51–62.

[26] F. Mainardi, Y. Luchko, and G. Pagnini, *The fundamental solution of the space-time fractional diffusion equation*, Fract. Calc. Appl. Anal., **4**-No.2, (2001), 153–192.

[27] F. Mainardi, *The fundamental solutions for the fractional diffusion-wave equation*, Appl. Math. Lett, **9**-No.6, (1996), 23–28.

[28] A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and series. Volume 3: More special functions. Transl. from the Russian by G.G. Gould*, Gordon and Breach Science Publishers, New York, 1990.

[29] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional integrals and derivatives: theory and appllications*, Gordon and Breach, New York, NY, 1993.

[30] W.R. Schneider, and W. Wyss, *Fractional diffusion and wave equations*, J. Math. Phys., **30**-No.1, (1989), 134–144.

[31] H.M. Srivastava, and H.L. Manocha, *A treatise on generating functions*, Ellis Horwood Series in Mathematics and Its Applications, Halsted Press (division of John Wiley & Sons), Chichester (Ellis Horwood Limited), New York etc., 1984.

[32] N. Vieira, *Fischer Decomposition and Cauchy-Kovalevskaya extension in fractional Clifford analysis: the Riemann-Liouville case*, Proc. Edinb. Math. Soc., II. Ser., in press, DOI: 10.1017/S0013091516000109.

[33] N. Vieira, *Theory of the parabolic Dirac operators and its applications to non-linear differential equations*, PhD Thesis, University of Aveiro, Portugal, 2009.

[34] W. Wyss, *The fractional diffusion equation*, J. Math. Phys., **27**-No.11, (1986), 2782–2785.