Problems of optimal transportation on the circle and their mechanical applications

Alexander Plakhov Tatiana Tchemisova*

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Abstract

We consider a mechanical problem concerning a 2D axisymmetric body moving forward on the plane and making slow turns of fixed magnitude about its axis of symmetry. The body moves through a medium of non-interacting particles at rest, and collisions of particles with the body's boundary are perfectly elastic (billiard-like). The body has a blunt nose: a line segment orthogonal to the symmetry axis. It is required to make small cavities with special shape on the nose so as to minimize its aerodynamic resistance. This problem of optimizing the shape of the cavities amounts to a special case of the optimal mass transfer problem on the circle with the transportation cost being the squared Euclidean distance. We find the exact solution for this problem when the amplitude of rotation is smaller than a fixed critical value, and give a numerical solution otherwise. As a by-product, we get explicit description of the solution for a class of optimal transfer problems on the circle.

Mathematics subject classifications: Primary 49Q10, 49K30, 37N05

Key words and phrases: problems of minimal resistance, billiards, Monge-Kantorovich problem, optimal mass transportation, shape optimization.

1 Introduction

Consider a body with piecewise smooth boundary moving through a rarefied medium of point particles at rest. When colliding with the body surface, the particles are reflected in the perfectly elastic way. Since the medium is highly diluted, mutual interactions of

*Center for R&D in Mathematics and Applications, Department of Mathematics, University of Aveiro, 3810-193 Aveiro, Portugal

particles can be neglected. On the other hand, multiple body-particle interactions are possible; that is, (in a reference system connected with the body) one can observe a particle approaching the body, hitting it one or several times, and then going away.

This description defines a simple aerodynamical model first introduced by I. Newton in his *Principia* [26]. It is actually very special; one can imagine a spacecraft moving through a sparse medium (say, thin atmosphere near a planet or a cloud of interstellar gas). The surface of the spacecraft is very well polished, so that collisions with the gas particles are perfectly elastic.

This mechanical model, in spite of being extremely simple, is a vast source of interesting mathematical problems. As a result of collisions of the body with the particles, the force of resistance is created, which acts on the body and slows down its velocity. The problem of resistance minimization for a translating body gives rise to various interesting problems related to calculus of variations [4]-[9],[17, 18] and theory of billiards [30]-[33]. The first problem of this kind (for classes of convex axially symmetric bodies) was stated by Newton in [26] and is now quoted as a starting point for calculus of variations (see, e.g., [1]).

It is proved that the resistance of a 3-dimensional translating body can be made arbitrarily small just by making hollows on its surface [33]. (A similar result is valid in the 2D case, if disconnected bodies are also taken into consideration.) This result (too good to be physically correct) suggests that the underlying mathematical model is poor and needs to be enriched, perhaps by introducing some rotation of the body.

Very interesting problems arise in the study of bodies performing both translational and rotational motion. Even more intriguing, they turn out to be closely connected to the theory of Monge-Kantorovich optimal mass transfer [27, 28, 31, 32]. One such problem reads as follows. A convex body moves forward and rotates (somersaults) very slowly and uniformly. One needs to roughen (that is, make small hollows on) its surface so as to minimize its (time averaged) resistance. Surprisingly, the resistance can really be diminished by roughening, but not much. In the 2D setting the maximal decrease (independently of the original convex shape) is approximately 1.22%, in the 3D setting about 3.05%, and in the limit of infinite dimension the minimum value of resistance is approximately 79% of the original one [31, 32, 27]. This is an exact mathematical result. Unfortunately, the method of solution gives no idea what an applicable shape of minimal resistance should look like.

The solution of this problem amounts to optimal transportation. Indeed, in a reference system attached to the body one observes flows falling on a cavity of the body surface with all possible velocities of incidence. Each incident and each reflected particle are identified with the points of the unit sphere corresponding to the initial velocity of the particle and to its final velocity. The cavity transforms the incident flow into the reflected flow, thus generating a mapping of a part of the sphere onto another one (transportation of a mass distributed on the sphere's surface onto another mass). The specific resistance created by the cavity can be expressed as the cost of the transportation, and the resistance

(a) (b)

Figure 1: (a) A body with a flat blunt nose makes rotations of amplitude T in the course of motion. (b) The same body with its nose roughened.

minimization amounts to minimization of this cost; that is, we come to a special case of the Monge-Kantorovich problem.

Problems of optimal resistance related to quickly rotating bodies can also be reduced to optimal mass transportation [34]. In this case the resistance is essentially a vector, and the corresponding Monge-Kantorovich problem is vector-valued.

Thus, some basic problems concerning both *no rotation* and *full rotation* cases are mainly solved. There naturally appear actual and interesting problems concerning the intermediate case, where the body rotates non-uniformly or makes turns of fixed magnitude around its axis. The present paper is devoted to such an intermediate problem.

We consider a 2-dimensional axisymmetric body with blunt nose that moves forward in the plane and makes very slow rotations — the symmetry axis of the body oscillates around the direction of motion, with the oscillation angle varying between $-T$ and T. Here T is a positive constant. The rotation with a small constant angular velocity ω proceeds until the angle reaches the value T. At that moment the angular velocity changes to $-\omega$ and remains constant until the angle reaches the value $-T$, etc. One can think that the body is equipped with a protecting device that makes the angular velocity switch between ω to $-\omega$.

Blunt nose means that the front part of the body's boundary is a line segment orthogonal to the symmetry axis. We are going to make small hollows on the segment, our goal being to reduce the resulting (time-averaged) resistance of the body (see Fig. 1). The problem is to find the shape of the hollows that minimizes the resistance.

An equivalent setting of this problem is as follows. The body turned by a fixed angle φ translates through the medium. The angle is chosen randomly and uniformly from $[-T, T]$. The body has a blunt nose; one needs to minimize the mathematical expectation of the resistance of the nose by making hollows on it.

Remarkably, as will be seen later, this mechanical problem amounts to solving a special problem of optimal transportation. Again, the problem concerns moving a mass distribution on the circumference into another, with the transportation cost being equal to the

squared Euclidean distance.

We shall actually consider two problems of optimal transportation, with and without a certain symmetry condition, while only the problem *with* the condition is related to our mechanical model. The solutions to these problems coincide, if $0 < T \leq \pi/6$ or $T = \pi/2$, and are different, if $\pi/6 < T < \pi/2$. The problem *without* the symmetry condition is solved analytically. On the other hand, the problem *with* the condition is too difficult to be treated analytically, and therefore is solved by numerical simulation. Thus, the solution for the underlying mechanical problem has an analytical component (for $0 < T \leq \pi/6$ and $T = \pi/2$) and a numerical one (for $\pi/6 < T < \pi/2$). Notice that the case $T = \pi/2$ is equivalent to the case of constant uniform rotation already studied in [28, 31, 32].

The problem of optimal transportation has a long history. It was born with the publication [25] of G. Monge in 1781. An important turning point in the development of this theory is connected with the works of L. Kantorovich [14, 15], where a relaxation of the problem and the dual problem were introduced. The new rise of interest to the theory in 1980s continues until now. There are several excellent monographs [35, 36, 37] and lecture notes [11, 2] on the subject. Many questions of the theory, including regularity of solutions and their properties, are now a subject of deep study. The theory of optimal transportation nowadays has numerous applications, including probability [35], weather prediction [10], hydrodynamics [3], and even structure of the early Universe [12]. A more detailed exposition of the current state of the theory can be found in the recent lecture notes [24].

Several ingredients of the specific conditions of the problem under study can be encountered in various papers. The important paper by Brenier [3] introduces the polar factorization of the solution in the case when the cost equals the squared distance, $c(x, y) = \frac{1}{2}|x - y|^2$, and the initial and final distributions are supported on sufficiently regular subsets of positive measure. Further, the paper [13] treats the case with the same cost and with the initial and final distributions concentrated on hypersurfaces (curves in the 2D case). The paper [16] studies the case of the transportation from the sphere onto itself with the quadratic cost; under some assumptions on the distributions it is proved that the optimal transport plan is a diffeomorphism. The paper [23] contains an explicit solution of a one-dimensional transportation problem with economic applications, where the cost is a function of distance, $c(x, y) = f(|x - y|)$, with f being a concave function. Moreover, this paper contains a review of optimal transfer problems that have been exactly solved by that time. In the paper [22] regularity of the optimal transport plan for a problem on the sphere is studied.

The most part of the literature on the subject is devoted to qualitative description of solutions. At the same time examples of exactly solvable optimal transportation problems are not so numerous by the moment (see, however, the papers [19, 20, 21, 23, 29]). No wonder: like, for instance, exactly solvable PDEs or completely integrable Hamiltonian dynamical systems, these examples represent exception from the rule rather than the rule. In our opinion, finding new exactly solvable problems is an important task for the future.

In this paper we provide a new exactly solvable problem that should be added to the list. It is essentially one-dimensional, with the initial and final mass distributions having monotone decreasing densities and being supported on segments, and with the cost of transportation being $f(x+y)$, where f is an odd function and is convex (concave) on $\mathbb{R}_+$ ($\mathbb{R}_-$). A particular case of this problem reduces to a mass transportation on the circle with the transportation cost equal to the squared distance. The study of the problem is based on finding all cost-monotone transport plans. We were lucky enough to discover that the set of all such plans is either a singleton or forms a 1-parameter or 2-parameter family. In the former case this singleton represents the solution, and in the latter case the solution is extracted from the family by minimizing a function of one or two variables.

Of course it would be natural to generalize our mechanical problem in several ways. First, minimize the resistance by making hollows on all the surface (not only on the nose). Second, consider 3-dimensional (and in general multidimensional) analogues of the problem. We are going to work on them in the future. Note, however, that these problems (and especially the multidimensional one) lead to much more difficult and intriguing problems of optimal transportation.

The remainder of the paper is organized as follows. In Section 2 we give an exposition of some known facts on billiard scattering on rough obstacles, including the theorem of characterization of scattering in terms of measures with fixed marginals. In Section 3 we introduce the mechanical system, compare the resistances of the smooth nose and the rough one, and reduce the problem of minimizing the resistance of rough nose to a problem of optimal transportation. This problem is then studied analytically in Section 4. The results of numerical study, including figures and graphs related to optimal shapes are presented in Section 5.

2 Billiard scattering and optimal transportation

The exposition in this section closely follows Chapter 4 of the book [27]. Here we describe the billiard scattering by a curve γ and state (without proof) theorems characterizing the scattering[1].

Take $l > 0$ and $\varepsilon > 0$. Consider a rectangle of the size $l \times \varepsilon$, fix a side of the rectangle with length l (the segment AB in Fig. 2), and define the natural parameter $\xi \in [-l/2, l/2]$ on this side. We shall consider piecewise smooth, non self-intersecting curves γ of finite length that are contained in the rectangle and join the endpoints A and B of the side (see Fig. 2).

Consider the billiard in $\mathbb{R}^2 \setminus \gamma$. Take an incident particle that comes from outside, intersects the segment AB, makes several (a finite number of) reflections from γ, and then intersects the segment AB again and goes away. Denote by n the outer normal to AB. Let ξ and ξ^+ denote the points of first and second intersection with AB, and let

[1]The curve γ will be interpreted later on as the one forming the shape of the nose.

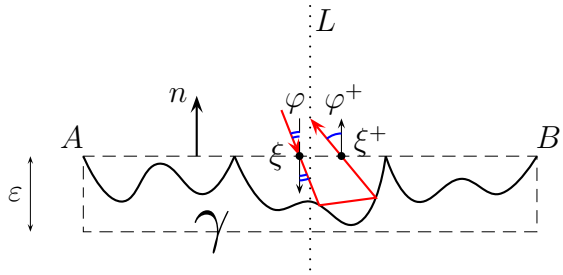


Figure 2: Billiard scattering by a curve.

φ, $\varphi^+ \in (-\pi/2,\, \pi/2)$ be the angles formed by the initial and final velocities, respectively, with $-n$ and n (see Fig. 2). The angles are counted counterclockwise from $-n$ and n, respectively.

This description defines a one-to-one map

$$T_\gamma : (\varphi, \xi) \mapsto (\varphi^+, \xi^+) = (\varphi^+_\gamma(\varphi, \xi),\ \xi^+_\gamma(\varphi, \xi))$$

from a full-measure subset of $[-\pi/2,\, \pi/2] \times [-l/2,\, l/2]$ onto itself. The map T_γ preserves the measure μ on $[-\pi/2,\, \pi/2] \times [-l/2,\, l/2]$ defined by $d\mu(\varphi, \xi) = \frac{1}{l} \cos\varphi \, d\varphi \, d\xi$. Let us additionally define the map $\mathcal{T}_\gamma : (\varphi, \xi) \mapsto (\varphi, \varphi^+_\gamma(\varphi, \xi))$ and introduce the push-forward measure η_γ on $\mathbb{R}^2$ by

$$\mathcal{T}_\gamma^{\#} \mu = \eta_\gamma.$$

The measure η_γ is supported in the square $[-\pi/2,\, \pi/2] \times [-\pi/2,\, \pi/2]$.

Remark 1. *The measure η_γ describes the billiard scattering by the curve γ and can be interpreted in the following way. Place a body with the front boundary γ in an ether, a uniform and isotropic medium of point particles moving at unit velocities in all possible directions and reflecting elastically from γ, and for each particle falling from above on γ register the pair (φ, φ^+) of its angles of incidence and reflection. The resulting probability distribution generated by these pairs coincides with $\frac{1}{2}\eta_\gamma$.*

Below we use the following notation. Let μ_1 and μ_2 be Borel measures in $\mathbb{R}$; by Γ_{μ_1,μ_2} we denote the set of measures η in $\mathbb{R}^2$ whose projections on the first and second coordinate axes coincide, respectively, with μ_1 and μ_2. The set of measures η in $\mathbb{R}^2_{\{x,y\}}$ symmetric with respect to the line $x = y$ is denoted by Γ^E. The set of measures η centrally symmetric with respect to the origin is denoted by Γ^C. Further, we denote

$$\Gamma_{\mu_1,\mu_2} \cap \Gamma^E =: \Gamma^E_{\mu_1,\mu_2}, \quad \Gamma_{\mu_1,\mu_2} \cap \Gamma^C =: \Gamma^C_{\mu_1,\mu_2} \quad \text{and} \quad \Gamma^E_{\mu_1,\mu_2} \cap \Gamma^C_{\mu_1,\mu_2} =: \Gamma^{E,C}_{\mu_1,\mu_2}.$$

More precisely, let P_1 and P_2 be the projections on the first and second coordinate axes, $P_1(x, y) = x,\ P_2(x, y) = y$, let the map E exchange the coordinates, $E(x, y) = (y, x)$, and the map C be the central symmetry with respect to the origin, $C(x, y) = -(x, y)$. Introduce the following conditions on a measure η in $\mathbb{R}^2$,

P. $P_1^{\#}\eta = \mu_1,\ P_2^{\#}\eta = \mu_2$.

E. $E^{\#}\eta = \eta$.
C. $C^{\#}\eta = \eta$.
That is, Condition P states that the marginal measures of η coincide with μ_1 and μ_2. Condition E means that the measure η is symmetric with respect to the line $x = y$, and Condition C means that η is centrally symmetric with respect to the origin.

Definition 1. The set of measures η in $\mathbb{R}^2$ satisfying P is denoted by Γ_{μ_1,μ_2}. The set of measures $\eta \in \Gamma_{\mu_1,\mu_2}$ satisfying Condition E or C is denoted by $\Gamma^E_{\mu_1,\mu_2}$ or $\Gamma^C_{\mu_1,\mu_2}$, respectively. The set of measures $\eta \in \Gamma_{\mu_1,\mu_2}$ satisfying both Conditions E and C is denoted by $\Gamma^{E,C}_{\mu_1,\mu_2}$.

In what follows we shall use either the pair of variables x, y (mainly in the abstract setting), or the pair of variables φ, φ^+ (when the problem is interpreted in terms of billiard scattering or resistance of moving bodies).

The following theorem characterizes the set of measures $\{\eta_\gamma\}$ generated by the curves γ. This characterization does not depend on the choice of ε and l.

Define the measure λ on $\mathbb{R}$ by $d\lambda(\varphi) = \chi_{[-\pi/2,\,\pi/2]}(\varphi)\,\cos\varphi\,d\varphi$.

Theorem 1. *The weak closure of the set $\{\eta_\gamma\}$ coincides with $\Gamma^E_{\lambda,\lambda}$.*

This theorem is a slight modification of Theorem 4.1 in Chapter 4 of the book [27]. Its proof can be found there.

Remark 2. *Theorem 1 implies that (a) for each curve γ the measure η_γ belongs to $\Gamma^E_{\lambda,\lambda}$ and (b) each measure $\eta \in \Gamma^E_{\lambda,\lambda}$ is the weak limit of a sequence η_{γ_i} of measures generated by curves; that is, for any continuous function f in $\mathbb{R}^2$,*

$$\lim_{i\to\infty} \iint_{\mathbb{R}^2} f(x,y)\,d\eta_{\gamma_i}(x,y) = \iint_{\mathbb{R}^2} f(x,y)\,d\eta(x,y)$$

(notice that all measures η, η_{γ_i} are supported in the square $[-\pi/2,\,\pi/2]^2$).

Now consider curves γ that are symmetric with respect to the perpendicular bisector of AB (the line L in Fig. 2). In what follows, by symmetry of a curve we always mean symmetry with respect to this bisector.

Corollary 1. *The weak closure of the set $\{\eta_\gamma,\ \gamma \text{ is symmetric}\}$ coincides with $\Gamma^{E,C}_{\lambda,\lambda}$.*

Before starting the proof, let us define an auxiliary notion of operations with curves. Introduce the coordinates ξ and ζ on the plane by identifying the straight line AB with the ξ-axis and L with the ζ-axis. Define the scaling transformations σ_- and σ_+ on the plane by

$$\sigma_- : (\xi,\zeta) \mapsto (\xi/2 - l/4,\ \zeta/2), \quad \sigma_+ : (\xi,\zeta) \mapsto (\xi/2 + l/4,\ \zeta/2).$$

For two curves γ_1 and γ_2 let $\gamma_1 + \gamma_2 := \sigma_-(\gamma_1) + \sigma_+(\gamma_2)$. That is, the curve $\gamma_1 + \gamma_2$ is obtained by putting together small copies of γ_1 and γ_2 (see Fig. 3 (a)). For a curve γ, let γ^* be the curve symmetric to γ with respect to L (see Fig. 3 (b)).

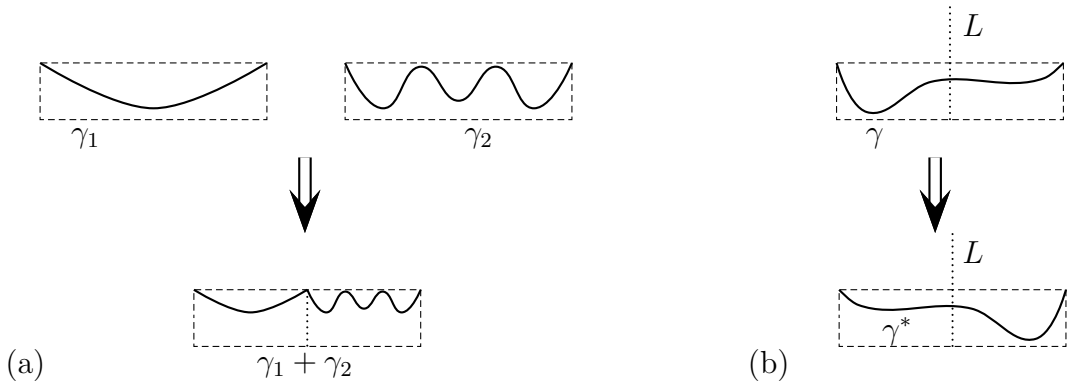


Figure 3: (a) The sum of two curves; (b) the symmetry of curves.

Lemma 1. (a) $\eta_{\gamma_1+\gamma_2} = (\eta_{\gamma_1} + \eta_{\gamma_2})/2$; (b) $\eta_{\gamma^*} = C^{\#}\eta_\gamma$.

Proof. By similarity considerations, the σ_--image of a billiard trajectory in $\mathbb{R}^2 \setminus \gamma_1$ is a billiard trajectory in $\mathbb{R}^2 \setminus \sigma_-(\gamma_1)$, the σ_+-image of a billiard trajectory in $\mathbb{R}^2 \setminus \gamma_2$ is a billiard trajectory in $\mathbb{R}^2 \setminus \sigma_+(\gamma_2)$, and the line symmetric to a billiard trajectory in $\mathbb{R}^2 \setminus \gamma$ is a billiard trajectory in $\mathbb{R}^2 \setminus \gamma^*$. One easily derives from this that

$$\varphi^+_{\gamma_1+\gamma_2}(\varphi,\xi) = \begin{cases} \varphi^+_{\gamma_1}(\varphi, 2\xi + l/2), & \text{if } -l/2 < \xi < 0 \\ \varphi^+_{\gamma_2}(\varphi, 2\xi - l/2), & \text{if } 0 < \xi < l/2 \end{cases}$$

and $\varphi^+_{\gamma^*}(\varphi,\xi) = -\varphi^+_\gamma(-\varphi,-\xi)$.

For a Borel set $A \subset [-\pi/2,\, \pi/2]^2$ one has

$$\eta_{\gamma_1+\gamma_2}(A) = \mu(\{(\varphi,\xi) : (\varphi,\, \varphi^+_{\gamma_1+\gamma_2}(\varphi,\xi)) \in A\})$$

$$= \mu(\{(\varphi,\xi) : (\varphi,\, \varphi^+_{\gamma_1+\gamma_2}(\varphi,\xi)) \in A,\ \xi < 0\}) + \mu(\{(\varphi,\xi) : (\varphi,\, \varphi^+_{\gamma_1+\gamma_2}(\varphi,\xi)) \in A,\ \xi > 0\})$$

$$= \mu(\{(\varphi,\xi) : (\varphi,\, \varphi^+_{\gamma_1}(\varphi, 2\xi + l/2)) \in A\}) + \mu(\{(\varphi,\xi) : (\varphi,\, \varphi^+_{\gamma_2}(\varphi, 2\xi - l/2)) \in A\})$$

$$= \frac{1}{2}\,\mu(\{(\varphi,\hat\xi) : (\varphi,\, \varphi^+_{\gamma_1}(\varphi,\hat\xi)) \in A\}) + \frac{1}{2}\,\mu(\{(\varphi,\tilde\xi) : (\varphi,\, \varphi^+_{\gamma_2}(\varphi,\tilde\xi)) \in A\})$$

$$= \frac{1}{2}\,\eta_{\gamma_1}(A) + \frac{1}{2}\,\eta_{\gamma_2}(A).$$

Further,

$$\eta_{\gamma^*}(A) = \mu(\{(\varphi,\xi) : (\varphi,\, \varphi^+_{\gamma^*}(\varphi,\xi)) \in A\}) = \mu(\{(\varphi,\xi) : (\varphi,\, -\varphi^+_\gamma(-\varphi,-\xi)) \in A\})$$

$$= \mu(\{(\hat\varphi,\hat\xi) : (-\hat\varphi,\, -\varphi^+_\gamma(\hat\varphi,\hat\xi)) \in A\}) = \mu(\{(\hat\varphi,\hat\xi) : (\hat\varphi,\, \varphi^+_\gamma(\hat\varphi,\hat\xi)) \in C(A)\}) = \eta_\gamma(C(A)).$$

□

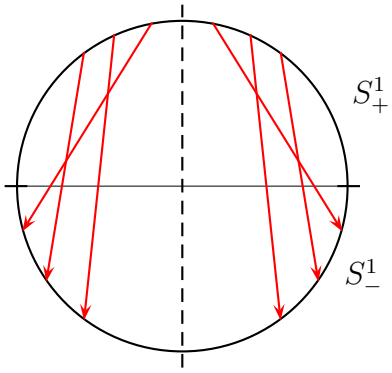


Figure 4: A schematic representation of a transport plan on the circle symmetric with respect to the vertical (imaginary) axis.

Let us now prove Corollary 1.

Proof. Let γ be symmetric; then the line symmetric to a billiard trajectory is also a billiard trajectory. This implies that $\varphi^+_\gamma(-\varphi,-\xi) = -\varphi^+_\gamma(\varphi,\xi)$, and thus,

$$\mathcal{T}_\gamma(-\varphi,-\xi) = -\mathcal{T}_\gamma(\varphi,\xi).$$

Since μ is invariant with respect to the mapping $(\varphi,\xi) \mapsto (-\varphi,-\xi)$, one easily concludes that η_γ is invariant with respect to C, and therefore satisfies Condition C. Since by Theorem 1 $\eta_\gamma \in \Gamma^E_{\lambda,\lambda}$, one finds that $\eta_\gamma \in \Gamma^{E,C}_{\lambda,\lambda}$.

Let now $\eta \in \Gamma^{E,C}_{\lambda,\lambda}$. By Theorem 1 there exists a sequence of curves γ_i such that η_{γ_i} weakly converges to η. Further, the curves $\gamma_i + \gamma_i^*$ are symmetric, and by Lemma 1

$$\eta_{\gamma_i+\gamma_i^*} = (\eta_{\gamma_i} + C^\# \eta_{\gamma_i})/2.$$

Taking into account that $C^\#\eta_{\gamma_i}$ weakly converge to $C^\#\eta$ and $C^\#\eta = \eta$, one concludes that $\eta_{\gamma_i+\gamma_i^*}$ weakly converges to η. □

Remark 3. *The class $\Gamma_{\lambda,\lambda}$ can be represented in the following, perhaps more visual, way. Introduce the notation $z = z_1 + iz_2 = e^{i\varphi+i\pi/2}$, $w = e^{i\varphi^+-i\pi/2}$; $S^1 = \{|z| = 1\} \subset \mathbb{C}$; $S^1_+ = \{z_2 \geq 0\} \cap S^1$, $S^1_- = \{z_2 \leq 0\} \cap S^1$. Let the measures λ_u and λ_d be the push-forward measures of λ by the maps $\varphi \mapsto e^{i\varphi+i\pi/2}$ and $\varphi^+ \mapsto e^{i\varphi^+-i\pi/2}$, respectively. Otherwise they can be defined by the formulas $\lambda_u = (\pi\rfloor_{S^1_+})^{-1}_\# \mathfrak{l}$, $\lambda_d = (\pi\rfloor_{S^1_-})^{-1}_\# \mathfrak{l}$, where $\pi : z_1 + iz_2 \mapsto z_1$ is the projection on the real axis and $\mathfrak{l}$ is the Lebesgue measure on $[-1, 1]$.*

That is, λ_u is supported on the upper semi-circumference S^1_+ and λ_d on the lower semi-circumference S^1_-, and the correspondence between $\lambda_{u,d}$ and the Lebesgue measure on the horizontal diameter of the circle can be realized by vertical translation. Conditions E and

C are transformed into the conditions of symmetry under the maps $\varsigma_1 : (z, w) \mapsto (-w, -z)$ and $\varsigma_2 : (z, w) \mapsto (-z^*, -w^*)$. The set of measures on $S^1 \times S^1$ that have the marginals λ_u and λ_d and are symmetric with respect to σ_1 and σ_2 is denoted by $\Gamma^{\varsigma_1,\varsigma_2}_{\lambda_u,\lambda_d}$.

Define the map $\phi : \mathbb{R}^2 \to S^1 \times S^1$ by

$$\phi(\varphi, \varphi^+) = (e^{i\varphi + i\pi/2},\ e^{i\varphi^+ - i\pi/2});$$

then the push-forward mapping $\phi^\#$ establishes a one-to correspondence between the classes $\Gamma^{E,C}_{\lambda,\lambda}$ and $\Gamma^{\varsigma_1,\varsigma_2}_{\lambda_u,\lambda_d}$. Fig. 4 illustrates a transport plan on the circle symmetric with respect to ς_2.

3 The mechanical system

Consider an axially symmetric body in $\mathbb{R}^2$ with a flat nose of length l perpendicular to the symmetry axis. In Fig. 5 (a) the nose is the segment AB. Fix an $\varepsilon > 0$ and take the rectangle of size $l \times \varepsilon$ such that one of its sides coincides with AB. As described in the previous section (see also Fig. 2), we consider curves γ joining the points A and B and lying in the rectangle. It is additionally assumed that γ is symmetric with respect to the symmetry axis of the body. When the segment AB is substituted with a curve γ, one obtains a body with a rough nose (see Fig. 5 (b)).

The body moves in a fixed direction with unit velocity, and the symmetry axis makes slow rotations about this direction. The angle φ between the axis and the direction varies between $-T$ and T, and correspondingly the angular velocity takes small values ω and $-\omega$. The time-averaged resistance depends on the curve γ. Our aim is to find the minimal ratio

$$m(T) = \inf \frac{\text{resistance of a } \textit{rough} \text{ nose}}{\text{resistance of the } \textit{flat} \text{ nose}}.$$

Indeed, as will be seen later, the infimum depends only on the amplitude of oscillation T, and does not depend on the other parameters l, ε, and ω (provided that ω is small).

Like in the previous section, introduce the natural parameter $\xi \in [-l/2, l/2]$ on the segment AB. In the reference system translating with the body one observes a parallel flow of incident particles with unit velocity. Let a body with a rough surface be turned at a given moment by an angle φ, $-T \le \varphi \le T$. For a particle that intersects AB at a point ξ, makes several reflections and then goes away, the angle between the directions of coming in and going away equals $\varphi - \varphi^+_\gamma(\varphi, \xi)$ (see Fig. 5 (b)).

The momentum imparted by the particle to the body can be decomposed into two components; the component along the direction of motion is proportional to $1 + \cos(\varphi - \varphi^+_\gamma(\varphi, \xi))$, and the transversal component is proportional to $\sin(\varphi - \varphi^+_\gamma(\varphi, \xi))$. The proportionality ratio is equal to the mass of the particle.

The force of resistance is the sum of all momenta imparted to the body during a small time interval divided by the length of the interval. Assume that the flow density equals 1;

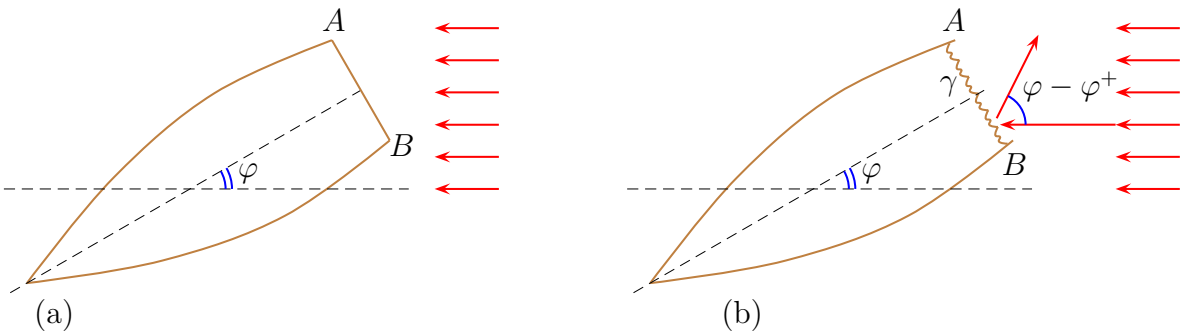


Figure 5: The body turned by an angle φ and the flow of incident particles; (a) the case of a flat nose and (b) the case of a rough nose.

then the summation amounts to integration over $\cos\varphi\, d\xi$. Thus, the longitudinal $R_\gamma(T,\varphi)$ and transversal $R_\gamma^\perp(T,\varphi)$ components of the resistance are equal to

$$R_\gamma(T,\varphi) = \int_{-l/2}^{l/2} [1+\cos(\varphi-\varphi_\gamma^+(\varphi,\xi))]\,\cos\varphi\, d\xi,$$
$$R_\gamma^\perp(T,\varphi) = \int_{-l/2}^{l/2} \sin(\varphi-\varphi_\gamma^+(\varphi,\xi))\,\cos\varphi\, d\xi.$$

Integrating these values over $\varphi \in [-T,\, T]$, one obtains the longitudinal and transversal components of the time-averaged resistance,

$$R_\gamma(T) = \frac{1}{2T}\int_T^T R_\gamma(T,\varphi)\, d\varphi = \frac{l}{2T}\iint\limits_{[-T,T]\times[-l/2,l/2]} [1+\cos(\varphi-\varphi_\gamma^+(\varphi,\xi))]\, d\mu(\varphi,\xi), \quad (1)$$
$$R_\gamma^\perp(T) = \frac{1}{2T}\int_T^T R_\gamma^\perp(T,\varphi)\, d\varphi = \frac{l}{2T}\iint\limits_{[-T,T]\times[-l/2,l/2]} \sin(\varphi-\varphi_\gamma^+(\varphi,\xi))\, d\mu(\varphi,\xi). \quad (2)$$

Recall that the measure μ on $[-\pi/2,\, \pi/2] \times [-l/2,\, l/2]$ is defined by $d\mu(\varphi,\xi) = \frac{1}{l}\cos\varphi\, d\varphi\, d\xi$.

Since γ is symmetric, we have $\varphi_\gamma^+(-\varphi,-\xi) = -\varphi_\gamma^+(\varphi,\xi)$; therefore the integrand in (2) is antisymmetric with respect to the origin. On the other hand, the domain of integration in (2) is symmetric with respect to the origin; it follows that

$$R_\gamma^\perp(T) = 0.$$

Thus, the time-averaged resistance of the nose is a vector whose magnitude equals $R_\gamma(T)$ and direction is opposite to the direction of motion of the body.

By making the change of variables $(\varphi, \xi) \mapsto (\varphi, \varphi^+)$, one can rewrite (1) in the form

$$R_\gamma(T) = \frac{l}{2T} \iint\limits_{[-T,T]\times\mathbb{R}} [1 + \cos(\varphi - \varphi^+)]\, d\eta_\gamma(\varphi, \varphi^+). \tag{3}$$

Note that the resistance $R_{AB}(T)$ of the flat nose can easily be calculated. Indeed, in this case γ coincides with the segment AB, and the density of the corresponding measure η_{AB} equals $\cos\varphi\, \delta(\varphi + \varphi^+)$; therefore

$$R_{AB}(T) = \frac{l}{2T} \int_{-T}^{T} (1 + \cos 2\varphi)\, \cos\varphi\, d\varphi = \frac{2l}{T}\Big(\sin T - \frac{1}{3}\, \sin^3 T\Big),$$

and so,

$$\frac{R_\gamma(T)}{R_{AB}(T)} = \frac{1}{4(\sin T - \frac{1}{3}\, \sin^3 T)} \iint\limits_{[-T,T]\times\mathbb{R}} [1 + \cos(\varphi - \varphi^+)]\, d\eta_\gamma(\varphi, \varphi^+).$$

By Corollary 1, the set of measures η_γ belongs to the class $\Gamma^{E,C}_{\lambda,\lambda}$ and is weakly dense in it. Therefore the minimal ratio

$$m(T) = \inf_{\gamma \text{ symmetric}} \frac{R_\gamma(T)}{R_{AB}(T)}$$

equals

$$m(T) = \frac{1}{2(\sin T - \frac{1}{3}\, \sin^3 T)}\, \mathcal{C}(T), \tag{4}$$

where

$$\mathcal{C}(T) = \inf_{\eta \in \Gamma^{E,C}_{\lambda,\lambda}} \mathcal{F}_T(\eta), \quad \text{with } \mathcal{F}_T(\eta) = \frac{1}{2} \iint\limits_{[-T,T]\times\mathbb{R}} [1 + \cos(\varphi - \varphi^+)]\, d\eta(\varphi, \varphi^+). \tag{5}$$

Remark 4. *The problem of finding $m(T)$ has already been solved in the extreme cases $T = 0$ and $T = \pi/2$. In the case of translation, $T = 0$, one has $m(0) = 1/2$; that is, the resistance of the nose can be halved by roughening [30]. Loosely speaking, the corresponding scattering of the flow particles approaches the scenario when the particles are reflected in the orthogonal directions.*

The case $T = \pi/2$ is equivalent to slow uniform rotation of the body. In this case one has $m(\pi/2) \approx 0.987820$ (see [31] or [27]). In other words, the maximum decrease of resistance is slightly greater than 1%. Thus, one can a priori expect that $m(T)$ takes intermediate values for intermediate values of the argument $0 < T < \pi/2$ and approaches 0.5 and 0.987820 when T goes, respectively, to 0 and $\pi/2$.

Remark 5. *Using the representation introduced in Remark 3 and denoting $\nu = \phi^{\#}\eta$, one can rewrite the integrand in (5) in the form $1 + \cos(\varphi - \varphi^+) = \frac{1}{2}|z - w|^2$, and so, the problem of finding $m(T)$ amounts to a special Monge-Kantorovich problem on the circle with appropriate marginal measures, where the cost is the squared distance and $S^1_{+,T} = \{e^{i\varphi + i\pi/2},\ \varphi \in [-T,\, T]\}$,*

$$\inf_{\nu \in \Gamma^{\varsigma_1,\varsigma_2}_{\lambda_u,\lambda_d}} \iint_{S^1_{+,T}\times S^1} |z - w|^2 \, d\nu(z, w).$$

Note that the measures in Problem (5) are supported in the square $[-\pi/2,\, \pi/2]^2$. The following lemma allows one to effectively reduce the support to the smaller square $[0,\, \pi/2]^2$.

Let λ_+ be defined by $\lambda_+(x) = \chi_{[0,\pi/2]}(x) \cos x \, dx$; that is, λ_+ is the restriction of λ on the positive semi-axis. Let R be the reflection with respect to the horizontal axis, $R(x, y) = (x, -y)$.

Lemma 2. *(a) We have $\mathcal{C}(T) = \inf_{\eta \in \Gamma^E_{\lambda^+,\lambda^+}} F^+_T(\eta)$, where*

$$F^+_T(\eta) = \iint\limits_{[0,\,T]\times[0,\,\pi/2]} [1 + \cos(x + y)] \, d\eta(x, y). \tag{6}$$

In other words, the minima in (5) and (6) coincide.

(b) If η minimizes (6), then $R^{\#}(\eta + C^{\#}\eta)$ is a solution of (5).

The proof of Lemma 2 is given in Appendix A.

Take $K = K(T) \in [0,\, \pi/2)$ such that $\sin T + \sin K = 1$ and define the measures $\mu_1 = \mu_1(T)$ and $\mu_2 = \mu_2(T)$ on $\mathbb{R}$ by $d\mu_1(x) = \chi_{[0,\,T]}(x) \cos x \, dx$ and $d\mu_2(x) = \chi_{[K,\,\pi/2]}(x) \cos x \, dx$. Consider the problem

$$\inf_{\nu \in \Gamma_{\mu_1,\mu_2}} \Phi(\nu), \quad \text{where} \quad \Phi(\nu) = \iint_{\mathbb{R}^2} [1 + \cos(x + y)] \, d\nu(x, y). \tag{7}$$

The following lemma states that if $0 < T \le \pi/6$ or $T = \pi/2$, problems (6) and (7) are equivalent.

Lemma 3. *Let $0 < T \le \pi/6$ or $T = \pi/2$. Then*

(a) *The minimal value in (6) is equal to the minimal value in (7).*

(b) *If η is a solution of (6) then $\nu = \eta\rfloor_{[0,\,T]\times[K,\,\pi/2]}$ is a solution of (7).*

(c) *If ν is a solution of (7) then there exists a measure η (non-unique in the case $0 < T < \pi/6$) solving (6) and such that $\eta\rfloor_{[0,\,T]\times[K,\,\pi/2]} = \nu$ (see Fig. 6).*

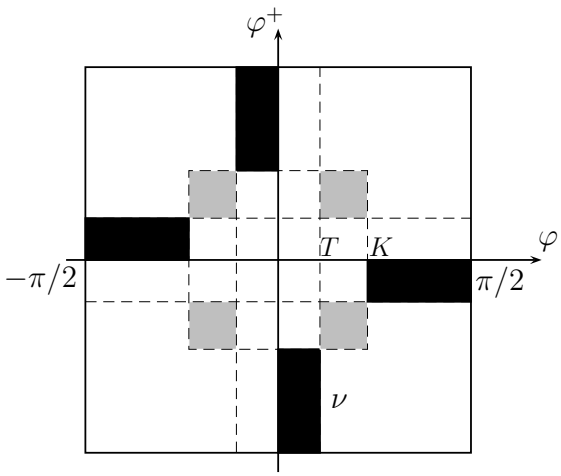


Figure 6: An illustration to the correspondence between the solutions of (5) and (7) for $0 < T < \pi/6$. If the measure ν supported on $[0, T] \times [K, \pi/2]$ solves (7), then the corresponding measure η, whose support is schematically indicated in the figure, solves (5). The restriction of η on the rectangle $[0, T] \times [-\pi/2, -K]$ (shown black in the figure) is symmetric to ν with respect to the x-axis, and the restrictions of η on the other black rectangles are symmetric to each other with respect to the diagonals $\varphi^+ = \varphi$ and $\varphi^+ = -\varphi$. The domains shown white carry zero η-measure. The domains shown light gray carry a measure whose projections on the coordinate axes have the density $\cos\varphi$. There is a large variety of such measures; this implies that the optimal measure η is not unique.

The proof of Lemma 3 is also given in Appendix A.

Notice that the intervals $(0, T)$ and $(K, \pi/2)$ are disjoint, if $0 < T \leq \pi/6$, and coincide, if $T = \pi/2$. Basically, Lemma 3 means that for $0 < T \leq \pi/6$ a solution of (7) (which is defined on the rectangle $\Pi = [0, T] \times [K, \pi/2]$) can be symmetrically extended to a solution of (6) (which is supported in the square $[0, \pi/2]^2$). To that end, one makes the reflection of the rectangle Π with respect to the diagonal $x = y$. This extension cannot be done in the case $\pi/6 < T < \pi/2$, when the images of Π overlap. In the case $T = \pi/2$, however, the rectangle coincides with the square $[0, \pi/2]^2$, and the resulting solution is symmetric with respect to the diagonal $x = y$.

That is, we have two problems of optimal transportation: the *symmetric problem* (6) with a direct mechanical meaning and the *non-symmetric problem* (7). If $0 < T \leq \pi/6$ or $T = \pi/2$, these problems are equivalent. Problem (6) (in the complementary case $\pi/6 < T < \pi/2$) seems to be much more complicated than problem (7). We give an analytical solution to problem (7) in the next Section 4, and solve problem (6) numerically in Section 5.

4 The problem of optimal transportation

In this section we study a more general problem of optimal transportation, containing problem (7) as a special case. The main Theorem 2 in Subsection 4.1 gives a characterization of the optimal measure. In Subsection 4.2 some special cases of the problem are considered, which allow one to further specify the solution. This specification is given in Theorem 4. Finally, in Subsection 4.3 the obtained results are used to solve Problem (7).

4.1 Main theorem

Consider two measures, μ_x and μ_y, on $\mathbb{R}$. They are supported on segments $\mathcal{X} = [X_1,\, X_2]$ and $\mathcal{Y} = [Y_1,\, Y_2]$, respectively, and are defined by their densities, non-negative monotone decreasing functions $\rho_x : \mathcal{X} \to \mathbb{R}$ and $\rho_y : \mathcal{Y} \to \mathbb{R}$. Let Γ_{μ_x,μ_y} be the set of measures ν in $\mathbb{R}^2$ with the marginal measures equal to μ_x and μ_y. In other words,

$$\Gamma_{\mu_x,\mu_y} = \{\nu :\, \pi_1^{\#}\nu = \mu_x \text{ and } \pi_2^{\#}\nu = \mu_y\},$$

where P_1 and P_2 are the projections of $\mathbb{R}^2$ onto $\mathbb{R}$, $P_1(x,y) = x$, $P_2(x,y) = y$.

We also consider a function $f : \mathbb{R} \to \mathbb{R}$ such that

(a) f is odd;

(b) f is convex on $\mathbb{R}_+ = (0,\, +\infty)$ (and therefore concave on $\mathbb{R}_- = (-\infty,\, 0)$);

(c) f is continuous at 0 (and therefore everywhere).

Consider the following problem of optimal mass transportation

$$\inf_{\nu\in\Gamma_{\mu_x,\mu_y}} F(\nu), \quad \text{where } F(\nu) = \iint_{\mathbb{R}^2} f(x+y)\, d\nu(x,y). \tag{8}$$

For a measure ν on $\mathcal{X}\times\mathcal{Y}$ we define the sets

$$S_- = \overline{\operatorname{spt}\nu \cap \{x+y<0\}} \quad \text{and} \quad S_+ = \operatorname{spt}\nu \cap \{x+y\geq 0\}, \tag{9}$$

where $\overline{(\cdots)}$ means closure. One obviously has

$$\operatorname{spt}\nu = S_- \cup S_+.$$

Our aim in this subsection is to prove the following theorem.

Theorem 2. *Let ν be an optimal measure for Problem (8). Then*

(a) the corresponding set S_+ is either empty or the graph of a continuous monotone decreasing function defined on a segment.

(b) The set S_- is either empty or the graph of a continuous monotone increasing function defined on a segment (let it be $[x^i,\, x^f]$).

(c) Take a point $(x_0, y_0) \in S_-$ with $x_0 \neq x^f$ and consider the vertical line $x = x_0$ and the horizontal line $y = y_0$.

(i) If $2x_0 + y_0 < -Y_2$ then $\{x = x_0\} \cap S_+ = \emptyset$.

(ii) If $2x_0 + y_0 \geq -Y_2$ then $\{x = x_0\} \cap S_+ = \{(x_0, -2x_0 - y_0)\}$.

(iii) If $x_0 + 2y_0 < -X_2$ then $\{y = y_0\} \cap S_+ = \emptyset$.

(iv) If $x_0 + 2y_0 \geq -X_2$ then $\{y = y_0\} \cap S_+ = \{(-x_0 - 2y_0, y_0)\}$.

If we are given particular measures μ_x and μ_y, this theorem allows one to specify a 1-parameter or 2-parameter family of measures containing the optimal measure (or, if there are more than one optimal measure, containing all of them). This family does not depend on f. Then one compares the values of $F(\nu)$ for ν in the family and chooses the minimal one.

We remind the following well-known definition (in a slightly adapted form) and the theorem.

Definition 2. A set $A \subset \mathbb{R}^2$ is called to be *f-monotone*, if for each pair of points $(x, y) \in A$ and $(x_1, y_1) \in A$,

$$f(x + y) + f(x_1 + y_1) \leq f(x_1 + y) + f(x + y_1). \tag{10}$$

Theorem 3. *If a measure ν solves Problem (8), then its support $\mathrm{spt}\,\nu$ is f-monotone.*

It is well known from the general theory (see, e.g., [36]) that a solution to Problem (8) always exists.

In the following we, step by step, specify all measures $\nu \in \Gamma_{\mu_x,\mu_y}$ with the f-monotone support.

Lemma 4. *A set $A \subset \mathbb{R}^2$ is f-monotone, if and only if for each pair of points $(x, y) \in A$ and $(x_1, y_1) \in A$,*

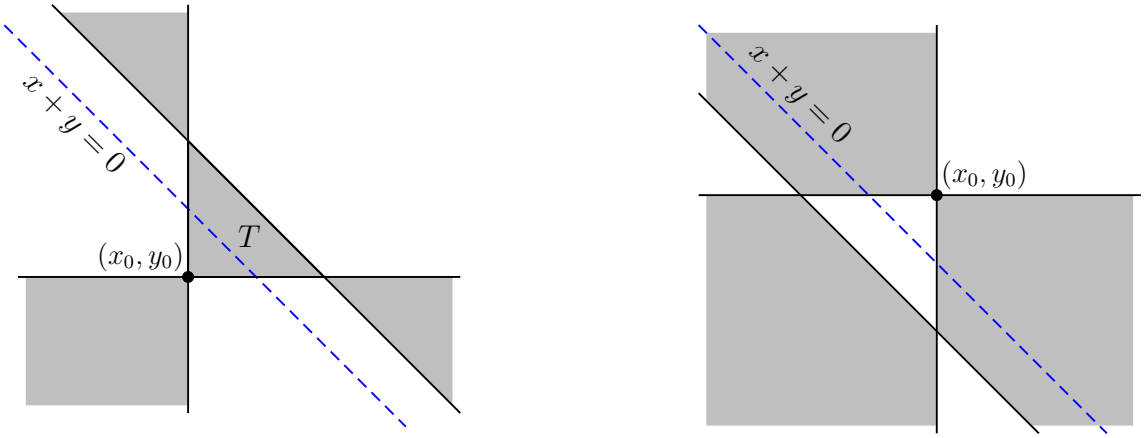
$$(x - x_1)(y - y_1)(x + y + x_1 + y_1) \leq 0. \tag{11}$$

The proof of the lemma is given in [29]. For the reader's convenience, we reproduce it here.

Proof. For $x = x_1$ we have both relations (10) and (11), therefore it suffices to consider the case $x \neq x_1$.

Since f is continuous, odd, and convex on $\mathbb{R}_+$, we see that f' is defined everywhere except possibly at countably many points, is even and increases on $\mathbb{R}_+$, and f is its primitive. Therefore inequality (10) can be written as follows:

$$\int_y^{y_1} (f'(x_1 + t) - f'(x + t))\, dt \leq 0. \tag{12}$$



(a) $x_0 + y_0 < 0$ (b) $x_0 + y_0 > 0$

Figure 7: The sets $G(x_0, y_0)$ in the cases (a) and (b) are shown gray.

Changing the variable $s = t + (x + x_1)/2$, setting $\Delta x = (x_1 - x)/2$, $\Delta y = (y_1 - y)/2$, $\sigma = (x + x_1 + y + y_1)/2$, and introducing the function

$$h(s) = \frac{f'(s + \Delta x) - f'(s - \Delta x)}{\Delta x},$$

we can rewrite (12) in the form

$$\Delta x \cdot \int_{\sigma - \Delta y}^{\sigma + \Delta y} h(s)\, ds \le 0. \tag{13}$$

Since the function $h(s)$ is odd and positive for $s > 0$, one easily sees that (13) holds if and only if $\Delta x \Delta y\, \sigma \le 0$. □

Denote by $G(x, y)$ the set of points (x_1, y_1) satisfying (11). Obviously, a set A is f-monotone if and only if $A \subset G(x, y)$ for any $(x, y) \in A$. It is easy to determine the set $G(x, y)$. It is depicted in Fig. 7 in the cases $x + y < 0$ and $x + y > 0$. In the case $x + y = 0$ (not shown), $G(x, y)$ is just the union of three angles with angular sizes 90^0, 45^0, 45^0 with the common vertex (x, y).

Each set $G(x_0, y_0)$ is bounded by three straight lines $x = x_0$, $y = y_0$, and $x + y = -x - y$. The points of their pairwise intersection (x_0, y_0), $(x_0, -2x_0 - y_0)$, $(-x_0 - 2y_0, y_0)$ form a triangle (shown in Fig. 7). The triangle shrinks to a point when $x_0 + y_0 = 0$.

In what follows we assume that $\nu \in \Gamma_{\mu_x, \mu_y}$ and $\operatorname{spt} \nu$ is f-monotone. The following lemmas specify properties of the sets S_- and S_+ defined by (9). In the proofs below we shall widely use intuitively appealing graphical representation of the sets $G(x, y)$. These proofs can easily be transformed to the (rigorous but cumbersome) analytical form.

Lemma 5. *(a) Each vertical $\{x\} \times \mathbb{R}$ or horizontal $\mathbb{R} \times \{y\}$ straight line, with $x \in \mathcal{X}$ and $y \in \mathcal{Y}$, has nonempty intersection with $\operatorname{spt} \nu$.*

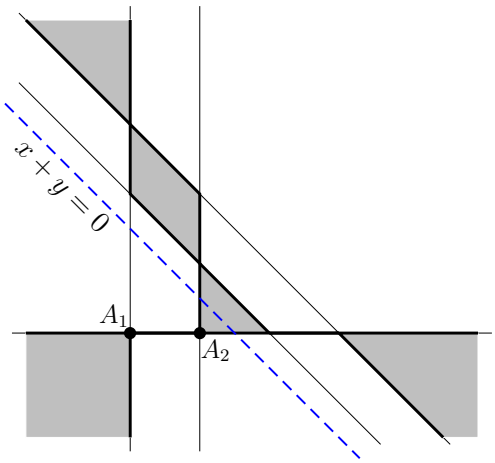


Figure 8: The set $G(A_1) \cap G(A_2)$ for the points $A_1 \in S_-$ and $A_2 \in S_-$ consists of 5 domains shown gray and the horizontal line through A_1 and A_2.

(b) No horizontal or vertical straight line carries positive measure ν; therefore sptν contains no isolated points.

(c) The intersection of any horizontal or vertical straight line with S_- contains at most two points.

Proof. (a) This is a direct consequence of the fact that both $\mathcal{X}$ and $\mathcal{Y}$ are compact and coincide with the supports of the marginal measures μ_x and μ_y, respectively.

(b) The measures μ_x and μ_y contain no atoms, therefore any horizontal or vertical straight line carries zero measure ν.

(c) Consider two points $A_1 = (x_1, y) \in S_-$ and $A_2 = (x_2, y) \in S_-$ on the same horizontal line; we are going to prove that no interior point of the segment A_1A_2 belongs to S_-.

Consider the set $G(x_1, y) \cap G(x_2, y)$ (see Fig. 8); we know that it contains spt ν. As seen in the figure, a neighborhood of any point on the line interval (A_1, A_2) minus this interval has measure zero. On the other hand, the interval also has measure zero. This implies that the interval (A_1, A_2) contains no points of spt ν.

The argument for a vertical line is the same. □

Lemma 6. *For any two points (x, y), (x_1, y_1) of S_- we have*

$$(x - x_1)(y - y_1) \geq 0, \tag{14}$$

and for any two points (x, y), (x_1, y_1) of S_+ we have

$$(x - x_1)(y - y_1) \leq 0. \tag{15}$$

Proof. If (x, y) and (x_1, y_1) belong to the set spt $\nu \cap \{x + y < 0\}$ then $x + y + x_1 + y_1 < 0$, and therefore, according to (11), one has (14). If (x, y) and/or (x_1, y_1) are limit points of this set, the required property is obtained by the limiting process.

If (x, y) and (x_1, y_1) belong to S_+, consider two cases.

(a) $x + y > 0$ or $x_1 + y_1 > 0$. In this case we have $x + y + x_1 + y_1 > 0$, and using (11) again, one comes to (15).

(b) $x + y = 0$ and $x_1 + y_1 = 0$. In this case we have $x - x_1 = -(y - y_1)$, and therefore, $(x - x_1)(y - y_1) \leq 0$. □

This lemma implies that the intersection $S_- \cap \{x + y = 0\}$ contains at most one point, and therefore $S_- \cap S_+$ is either empty, or a singleton lying on the diagonal $\{x + y = 0\}$.

Lemma 7. *The set S_+ is either empty, or the graph of a continuous strictly monotone decreasing function g defined on a segment (let it be $[a, b]$). We do not exclude the degenerate case $a = b$.*

Proof. Let us prove that for two different points (x_1, y_1) and (x_2, y_2) in S_+

(a) the strict inequality

$$(x_1 - x_2)(y_1 - y_2) < 0$$

actually takes place, and

(b) there is a point $(x, y) \in S_+$ between them satisfying $(x - x_1)(x_2 - x) > 0$ and $(y - y_1)(y_2 - y) > 0$. The proof is based on Figure 9.

(a) It suffices to prove that $x_1 \neq x_2$ and $y_1 \neq y_2$. Let us assume, for example, that $x_1 = x_2\,(= x)$; we are going to come to a contradiction. (The second inequality is treated in a completely similar way.) The set $G(x, y_1) \cap G(x, y_2)$ containing $\operatorname{spt} \nu$ is shown in Fig. 9 (a); it consists of 3 domains shown gray and the vertical line through (x, y_1) and (x, y_2). Let $y_1 < y_2$. Since $\nu \in \Gamma_{\mu_x, \mu_y}$, we see that $\mu_y((y_1, y_2)) = \nu(\mathbb{R} \times (y_1, y_2))$. On the other hand, we have $\nu(\mathbb{R} \times (y_1, y_2)) = \nu(\{x\} \times (y_1, y_2))$, and since the vertical interval $\{x\} \times (y_1, y_2)$ has zero ν-measure, we come to the impossible equality $\mu_y((y_1, y_2)) = 0$.

(b) Let $x_1 < x_2$ and $y_1 > y_2$ and suppose that there are no points of S_+ between $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$. This means that the open rectangle with two opposite vertices at A_1 and A_2 shown in Fig. 9 (b) does not contain points of S_+. We are going to come to a contradiction.

Consider two cases: (i) at least one of the values $x_1 + y_1$ and $x_2 + y_2$ is nonzero; (ii) $x_1 + y_1 = x_2 + y_2 = 0$.

(i) Without loss of generality assume that $x_1 + y_1 \geq x_2 + y_2 \geq 0$ and $x_1 + y_1 > 0$. Then there exists $y' \in [y_2, y_1)$ such that $x_1 + y' > 0$ and therefore the (smaller) rectangle $(x_1, x_2) \times (y', y_1)$ is contained in the half-plane $x + y > 0$.

Let us show that the horizontal strip $y' < y < y_1$ has zero ν-measure. It is the union of subsets $\{y' < y < y_1, x > x_2\} \cup \{y' < y < y_1, x_1 < x < x_2\} \cup \{y' < y < y_1, x = x_1 \text{ or } x_2\} \cup \{y' < y < y_1, x < x_1\}$.

The set $y' < y < y_1$, $x > x_2$ does not intersect $G(A_2)$ and therefore has measure zero. The rectangle $y' < y < y_1$, $x_1 < x < x_2$ contains no points of S_+, and therefore no points of $\operatorname{spt} \nu$.The vertical segments $y' < y < y_1$, $x = x_1$ or x_2 have measure zero.

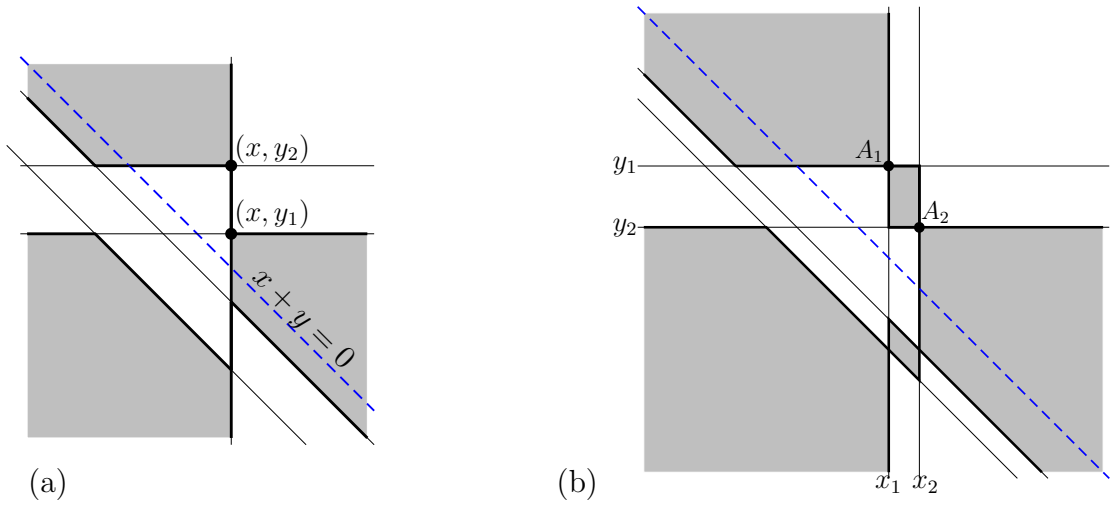


Figure 9: Illustration to the proof of Lemma 7: the sets $G(x, y_1) \cap G(x, y_2)$ and $G(A_1) \cap G(A_2)$ are shown gray in figures (a) and (b).

Suppose there is a point $(x, y) \in \operatorname{spt} \nu$ in the set $y' < y < y_1$, $x < x_1$. Applying Lemma 4 to the pairs of points (x, y), (x_1, y_1) and (x, y), (x_2, y_2), one obtains

$$(x - x_1)(y - y_1)(x + y + x_1 + y_1) \leq 0, \tag{16}$$

$$(x - x_2)(y - y_2)(x + y + x_2 + y_2) \leq 0. \tag{17}$$

One has $x - x_1 < 0$, $y - y_1 < 0$, $x - x_2 < 0$, $y - y_2 > 0$, hence one obtains the inequalities $x + y + x_1 + y_1 \leq 0$ and $x + y + x_2 + y_2 \geq 0$. Using also the inequality $x_1 + y_1 \geq x_2 + y_2$, one concludes that $x_1 + y_1 = x_2 + y_2 > 0$ and

$$x + y = -x_1 - y_1. \tag{18}$$

Thus, the intersection $\operatorname{spt} \nu \cap \cup\{y' < y < y_1, x < x_1\}$ is contained in the line (18). Therefore it is also contained in S_- and by Lemma 5 (c) cannot contain more than two points. The ν-measure of each point is zero.

Thus, we have proved that the strip $y' < y < y_1$ has zero ν-measure, and we come to the impossible equality $\mu_y((y', y_1)) = 0$.

(ii) Let $x_1 + y_1 = x_2 + y_2 = 0$. The set $G(A_1) \cap G(A_2)$ with $A_1 = (x_1, y_1)$ and $A_2 = (x_2, y_2)$ is shown in Fig. 10.

The open square with the opposite vertices A_1 and A_2 contains no points of S_+; one can see from the figure that it also contains no points of S_-. It follows that both strips $x_1 < x < x_2$ and $y_2 < y < y_1$ have zero ν-measure, and one comes to the impossible equations $\mu_x((x_1, x_2)) = 0$ and $\mu_y((y_1, y_2)) = 0$. Thus, (b) is proved.

Since S_+ is closed, we conclude from (a) that it is the graph of a continuous strictly monotone decreasing function defined on a closed set. From (b) we conclude that this set is a segment. $\square$

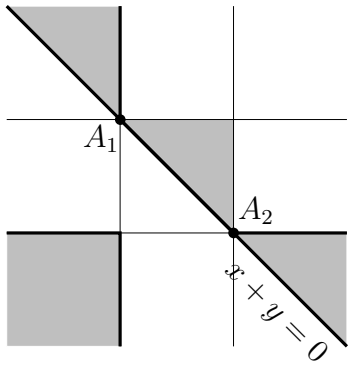


Figure 10: The set $G(A_1) \cap G(A_2)$ in the case when both A_1 and A_2 lie on the line $x+y=0$ is shown gray.

Lemma 8. *(a) Let $(x_0, y_0) \in S_-$; then*

$$\mu_x([x_0,\ -x_0 - 2y_0]) = \mu_y([y_0,\ -2x_0 - y_0]). \tag{19}$$

(b) Let (x_1, y_1) and (x_2, y_2) be two different points in S_-. Assume that $x_1 \le x_2$ and $y_1 \le y_2$; then

$$\mu_x([x_1,\ x_2]) \ge \mu_y([-2x_2 - y_2,\ -2x_1 - y_1]), \tag{20}$$

$$\mu_y([y_1,\ y_2]) \ge \mu_x([-x_2 - 2y_2,\ -x_1 - 2y_1]). \tag{21}$$

If the rectangle $[x_1,\ x_2] \times [y_1,\ y_2]$ has zero ν-measure, then the inequalities in (20) and (22) become equalities.

(c)

$$\mu_x([x_1,\ x_2]) + \mu_x([-x_2 - 2y_2,\ -x_1 - 2y_1]) = \mu_y([y_1,\ y_2]) + \mu_y([-2x_2 - y_2,\ -2x_1 - y_1]). \tag{22}$$

Proof. (a) Let T be the triangle with the vertices (x_0, y_0), $(x_0, -2x_0 - y_0)$, $(-x_0 - 2y_0, y_0)$ (shown gray in Fig. 7). The set $G(x_0, y_0)$ contains $\operatorname{spt} \nu$; therefore we have

$$\nu(T) = \nu([x_0,\ -x_0 - 2y_0] \times \mathbb{R}) = \mu_x([x_0,\ -x_0 - 2y_0])$$

and

$$\nu(T) = \nu(\mathbb{R} \times [y_0,\ -2x_0 - y_0]) = \mu_y([y_0,\ -2x_0 - y_0]).$$

Thus, (a) is proved.

(b) The support of ν is contained in the set $G(x_1, y_1) \cap G(x_2, y_2)$ (the union of 7 domains shown gray in Fig. 11). Some of these domains are: the rectangle $R = [x_1,\ x_2] \times [y_1,\ y_2]$, the parallelogram P_{v} with the lower vertex $(x_2, -2x_2 - y_2)$ and the upper vertex

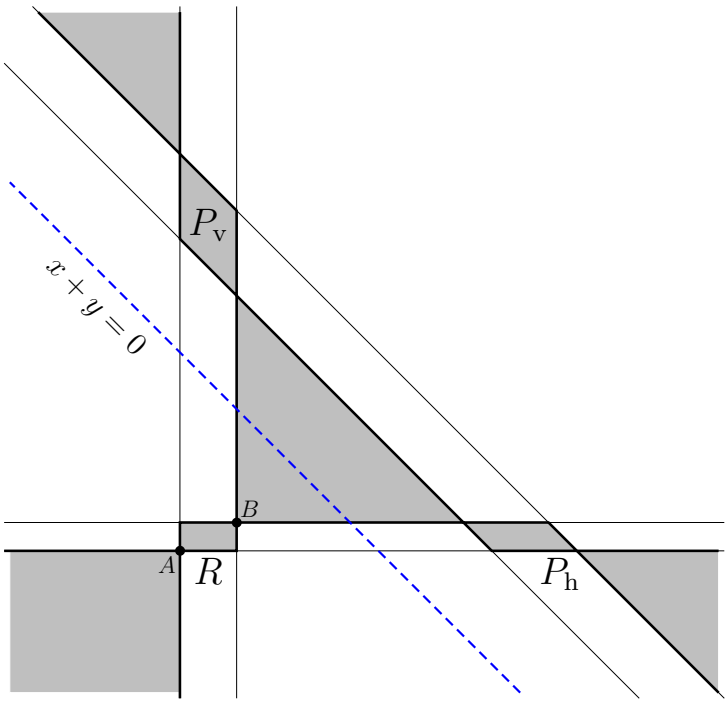


Figure 11: The set $G(x_1, y_1) \cap G(x_2, y_2)$ with two different points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ in S_-.

$(x_1, -2x_1 - y_1)$, and the parallelogram P_{h} with the left vertex $(-x_2 - 2y_2, y_2)$ and the right vertex $(-x_1 - 2y_1, y_1)$. We have

$$\mu_x([x_1,\, x_2]) = \nu([x_1,\, x_2] \times \mathbb{R}) = \nu(R) + \nu(P_{\mathrm{v}}) \geq \nu(P_{\mathrm{v}})$$

$$= \nu(\mathbb{R} \times [-2x_2 - y_2,\ -2x_1 - y_1]) = \mu_y([-2x_2 - y_2,\ -2x_1 - y_1]),$$

$$\mu_y([y_1,\, y_2]) = \nu(\mathbb{R} \times [y_1,\, y_2]) = \nu(R) + \nu(P_{\mathrm{h}}) \geq \nu(P_{\mathrm{h}})$$

$$= \nu([-x_2 - 2y_2,\ -x_1 - 2y_1] \times \mathbb{R}) = \mu_x([-x_2 - 2y_2,\ -x_1 - 2y_1]).$$

If $\nu(R) = 0$, these inequalities become equalities. Thus, (b) is also proved.

(c) We have

$$\mu_x([x_1,\, x_2]) + \mu_x([-x_2 - 2y_2,\ -x_1 - 2y_1]) = \nu(R) + \nu(P_{\mathrm{v}}) + \nu(P_{\mathrm{h}})$$

$$= \mu_y([y_1,\, y_2]) + \mu_y([-2x_2 - y_2,\ -2x_1 - y_1]).$$

□

Lemma 9. *The set S_- is either empty, or the graph of a continuous strictly monotone increasing function h defined on a segment (say, $[x^{\imath},\, x^{\jmath}]$ with $x^{\imath} < x^{\jmath}$).*

Proof. First we prove that for any two different points (x_1, y_1) and (x_2, y_2) in S_-,

(a) the strict inequality $(x_1 - x_2)(y_1 - y_2) > 0$ takes place, and

(b) there is a point $(x, y) \in S_-$ between them satisfying $(x - x_1)(x_2 - x) > 0$ and $(y - y_1)(y_2 - y) > 0$.

(a) We need to prove that the cases $x_1 = x_2$ and $y_1 = y_2$ are impossible. We prove the impossibility of the latter case; the former one is considered analogously.

Let $(x_1, y_0),\ (x_2, y_0) \in S_-$ and $x_1 < x_2$. One can again use Fig. 11 with $A = (x_1, y_0)$ and $B = (x_2, y_0)$. The rectangle R in Fig. 11 degenerates to a segment, so its ν-measure equals zero. By Lemma 5 (c) the line segment (A, B) contains no points of S_-. By Lemma 8 one has $\mu_y([-2x_2 - y_0,\ -2x_1 - y_0]) = \mu_x([x_1,\ x_2])$. Since the functions ρ_x and ρ_y are strictly monotone decreasing, we obtain the chain of (in)equalities

$$\rho_y(-2x_2 - y_0) \cdot (2x_2 - 2x_1) \geq \mu_y([-2x_2 - y_0,\ -2x_1 - y_0]) = \mu_x([x_1,\ x_2]) > \rho_x(x_2) \cdot (x_2 - x_1).$$

It follows that

$$2\rho_y(-2x_2 - y_0) > \rho_x(x_2). \tag{23}$$

The point (x_2, y_0) does not lie on the line $x + y = 0$. Indeed, otherwise the set $S_- \cap \{x + y < 0\}$ lies in the half-plane $x \leq x_1$, and we come to the impossible conclusion that (x_2, y_0) is an isolated point of S_-. Therefore we have $x_2 + y_0 < 0$.

The set $\operatorname{spt} \nu$ has no isolated points, therefore there exists a sequence of points $(x_i, y_i) \in \operatorname{spt} \nu \setminus \{(x_2, y_0)\},\ i \geq 3$ converging to (x_2, y_0). Since $x_2 + y_0 < 0$, the strip $x_1 < x < x_2$ contains no points of S_-, and each of the lines $x = x_2,\ y = y_0$ contains no more than two points of S_-, we conclude that all points of the sequence, except possibly for a single point, satisfy $(x_i, y_i) \in S_-,\ x_i > x_2$, and $y_i > y_0$. Take a point (x_i, y_i) satisfying these relations. Applying Lemma 8 (b) to the pair of points $(x_2, y_0),\ (x_i, y_i)$, one obtains

$$\mu_x([x_2,\ x_i]) \geq \mu_y([-2x_i - y_i,\ -2x_2 - y_0]),$$

therefore

$$\rho_x(x_2) \cdot (x_i - x_2) > \mu_x([x_2,\ x_i]) \geq \mu_y([-2x_i - y_i,\ -2x_2 - y_0])$$
$$\geq \rho_y(-2x_2 - y_0) \cdot (2x_i - 2x_2 + y_i - y_0) \geq 2\rho_y(-2x_2 - y_0) \cdot (x_i - x_2).$$

Thus, one has the inequality

$$\rho_x(x_2) > 2\rho_y(-2x_2 - y_0),$$

in contradiction with (23). Thus, (a) is proved.

(b) Now prove that for each pair of points $(x_1, y_1),\ (x_2, y_2) \in S_-$ with $x_1 < x_2$ and $y_1 < y_2$ there exists a point $(x, y) \in S_-$ such that $x_1 < x < x_2$ and $y_1 < y < y_2$.

Assume the contrary; then the ν-measure of the rectangle $R = [x_1,\ x_2] \times [y_1,\ y_2]$ is zero (see again Fig. 11 for an illustration). We have $x_2 + y_2 < 0$, since otherwise the point (x_2, y_2) lies on the line $x + y = 0$ and is an isolated point of S_-.

Applying again Lemma 8 (b), we obtain

$$\rho_x(x_2)\,(x_2-x_1) < \mu_x([x_1,\,x_2]) = \mu_y([-2x_2-y_2,\,-2x_1-y_1])$$
$$< \rho_y(-2x_2-y_2)\,(2x_2-2x_1+y_2-y_1),$$

hence

$$\frac{\rho_x(x_2)}{\rho_y(-2x_2-y_2)} - 2 < \frac{y_2-y_1}{x_2-x_1};$$

and

$$\rho_y(y_2)\,(y_2-y_1) < \mu_y([y_1,\,y_2]) = \mu_x([-x_2-2y_2,\,-x_1-2y_1])$$
$$< \rho_x(-x_2-2y_2)\,(x_2-x_1+2y_2-2y_1),$$

hence

$$\frac{\rho_y(y_2)}{\rho_x(-x_2-2y_2)} - 2 < \frac{x_2-x_1}{y_2-y_1}.$$

We conclude that either

$$\left(\frac{\rho_x(x_2)}{\rho_y(-2x_2-y_2)} - 2\right)\left(\frac{\rho_y(y_2)}{\rho_x(-x_2-2y_2)} - 2\right) < 1 \tag{24}$$

and both factors in (24) are positive, or at least one of these factors is non-positive.

Now, like in the case (a), find a sequence $(x_i, y_i) \in S_-,\ i \geq 3$ converging to (x_2, y_2). Since the rectangle $R = [x_1,\,x_2] \times [y_1,\,y_2]$ contains no points of S_- except for (x_1, y_1) and (x_2, y_2), we have the inequalities $x_i > x_2$ and $y_i > y_2$ for i sufficiently large. Take a point (x_i, y_i) satisfying these inequalities and and apply Lemma 8 (b) to (x_2, y_2) and (x_i, y_i). We have

$$\rho_x(x_2)\,(x_i-x_2) > \mu_x([x_2,\,x_i]) \geq \mu_y([-2x_i-y_i,\,-2x_2-y_2])$$
$$> \rho_y(-2x_2-y_2)\,(2x_i-2x_2+y_i-y_2),$$

hence either $\rho_y(-2x_2-y_2) = 0$, or

$$\frac{\rho_x(x_2)}{\rho_y(-2x_2-y_2)} - 2 > \frac{y_i-y_2}{x_i-x_2}, \tag{25}$$

and

$$\rho_y(y_2)\,(y_i-y_2) > \mu_y([y_2,\,y_i]) \geq \mu_x([-x_i-2y_i,\,-x_2-2y_2])$$
$$> \rho_x(-x_2-2y_2)\,(x_i-x_2+2y_i-2y_2),$$

hence either $\rho_x(-x_2-2y_2) = 0$, or

$$\frac{\rho_y(y_2)}{\rho_x(-x_2-2y_2)} - 2 > \frac{x_i-x_2}{y_i-y_2}. \tag{26}$$

Thus we obtain that either

$$\left(\frac{\rho_x(x_2)}{\rho_y(-2x_2-y_2)}-2\right)\left(\frac{\rho_y(y_2)}{\rho_x(-x_2-2y_2)}-2\right)>1,$$

with both factors being positive, or one of the values $\rho_y(-2x_2-y_2),\ \rho_x(-x_2-2y_2)$ equals 0. Here we come to a contradiction with inequality (24) and the conclusion therein, so (b) is proved.

Suppose that S_- is non-empty. It cannot be a singleton, since each point of S_- is a limiting point of S_-. Since S_- is closed, we conclude from (a) that it is the graph of a continuous monotone increasing function h defined on a closed set. From (b) we conclude that this set is segment, say $[x^{\mathrm{i}},\, x^{\mathrm{f}}]$ with $x^{\mathrm{i}} < x^{\mathrm{f}}$. Lemma 9 is proved. □

Lemma 10. *Let* $(x_0, y_0) \in S_- \setminus \{(x^f, h(x^f))\}$.

(a) If $2x_0 + y_0 < -Y_2$ *then* $\{x = x_0\} \cap S_+ = \emptyset$.

(b) If $2x_0 + y_0 \geq -Y_2$ *then* $\{x = x_0\} \cap S_+ = \{(x_0, -2x_0 - y_0)\}$.

(c) If $x_0 + 2y_0 < -X_2$ *then* $\{y = y_0\} \cap S_+ = \emptyset$.

(d) If $x_0 + 2y_0 \geq -X_2$ *then* $\{y = y_0\} \cap S_+ = \{(-x_0 - 2y_0, y_0)\}$.

Proof. We shall prove (a) and (b); the proof of (c) and (d) is completely analogous.

(a) Since S_- is the graph of a continuous function defined on $[x^{\mathrm{i}},\, x^{\mathrm{f}}]$ and $x^{\mathrm{i}} \leq x_0 < x^{\mathrm{f}}$, there exists a point $(x_1, y_1) \in S_-$ such that $x_1 > x_0$ and $x_0 + x_1 + y_1 < -Y_2$.

The intersection of the vertical line $x = x_0$ with $G(x_0, y_0) \cap G(x_1, y_1)$ is the union of two half-lines, $(\{x_0\} \times (-\infty,\, y_1]) \cup (\{x_0\} \times [-x_0 - x_1 - y_1,\ +\infty))$ (see again Fig. 11 for an illustration, where this time $A = (x_0, y_0)$ and $B = (x_1, y_1)$). The former half-line lies in the half-plane $x + y < 0$ and the latter one does not intersect the rectangle $\mathcal{X} \times \mathcal{Y}$ (since $-x_0 - x_1 - y_1 > Y_2$), therefore the line $x = x_0$ does not contain points of S_+.

(b) Consider a sequence of points $(x_i, y_i) \in S_-$ converging to (x_0, y_0) as $i \to \infty$ and such that $x_i > x_0,\ y_i > y_0$. It is helpful to use again Fig. 11, where $A = (x_0, y_0),\ B = (x_i, y_i)$, and the parallelogram P_{v} has the left upper vertex $(x_0, -2x_0 - y_0)$ and the right lower vertex $(x_i, -2x_i - y_i)$.

Since $Y_1 \leq y_i \leq -2x_i - y_i < -2x_0 - y_0 \leq Y_2$, we have $[-2x_i - y_i,\ -2x_0 - y_0] \subset \mathcal{Y}$, and therefore

$$\nu(P_{\mathrm{h}}) = \mu_y([-2x_i - y_i,\ -2x_0 - y_0]) > 0.$$

As $i \to \infty$, the sequence of these parallelograms shrinks to the point $(x_0, -2x_0 - y_0)$, therefore this point belongs to $\operatorname{spt}\nu$. Taking into account that $x_0 + (-2x_0 - y_0) > 0$, one concludes that

$$(x_0, -2x_0 - y_0) \subset \{x = x_0\} \cap S_+. \tag{27}$$

Since S_+ is the graph of a (strictly monotone) function, the set in the right hand side of (27) contains at most one point. (b) is proved. □

The proof of Theorem 2 is complete.

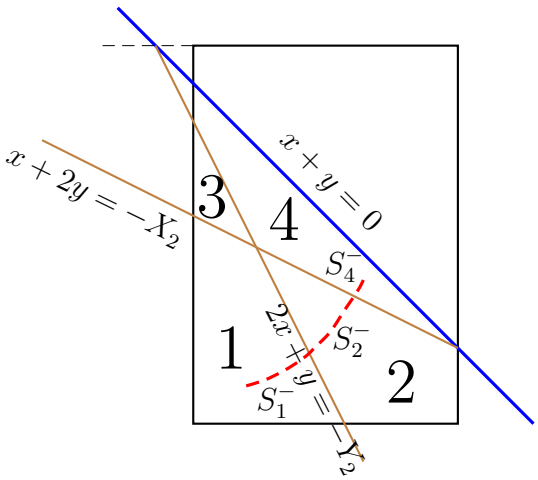


Figure 12: The domain $(\mathcal{X} \times \mathcal{Y}) \cap \{x + y \leq 0\}$ is divided by the lines $2x + y = -Y_2$ and $x+2y = -X_2$ into four sets marked by "1", "2", "3", "4". The dashed curve schematically indicates S_-.

4.2 Special cases

Making additional assumptions, one can further specify the optimal measure.

Divide the half-plane $x + y \leq 0$ into 4 domains defined by

1) $x + 2y \leq -X_2,\ 2x + y \leq -Y_2$;

2) $x + 2y \leq -X_2,\ 2x + y \geq -Y_2$;

3) $x + 2y \geq -X_2,\ 2x + y \leq -Y_2$;

4) $x + 2y \geq -X_2,\ 2x + y \geq -Y_2$.

These domains are marked by the digits "1", "2", "3", "4" in Fig. 12. The intersection of S_- with the kth domain is denoted by S_k^-, $k = 1, 2, 3, 4$. That is,

$S_1^- = S_- \cap \{(x, y) :\ x + 2y \leq -X_2,\ 2x + y \leq -Y_2\}$;

$S_2^- = S_- \cap \{(x, y) :\ x + 2y \leq -X_2,\ 2x + y \geq -Y_2\}$;

$S_3^- = S_- \cap \{(x, y) :\ x + 2y \geq -X_2,\ 2x + y \leq -Y_2\}$;

$S_4^- = S_- \cap \{(x, y) :\ x + 2y \geq -X_2,\ 2x + y \geq -Y_2\}$.

Since S_- is the graph of a continuous monotone increasing function and the straight lines $x + 2y = -X_2$ and $2x + y = -Y_2$ are the graphs of decreasing functions, S_- has at most one point of intersection with each of these lines.

We shall consider 6 different cases: $\langle 1 \rangle$, $\langle 12 \rangle$, $\langle 13 \rangle$, $\langle 124 \rangle$, $\langle 134 \rangle$, $\langle 14 \rangle$, where a digit k in the angle brackets indicates that the corresponding set S_k^- is more than a singleton. For example, the case $\langle 12 \rangle$ means that the sets S_1^- and S_2^- are more than singletons, and S_3^- and S_4^- are singletons or empty.

We additionally assume that

$$X_1 + Y_1 + X_2 + Y_2 \leq 0. \tag{28}$$

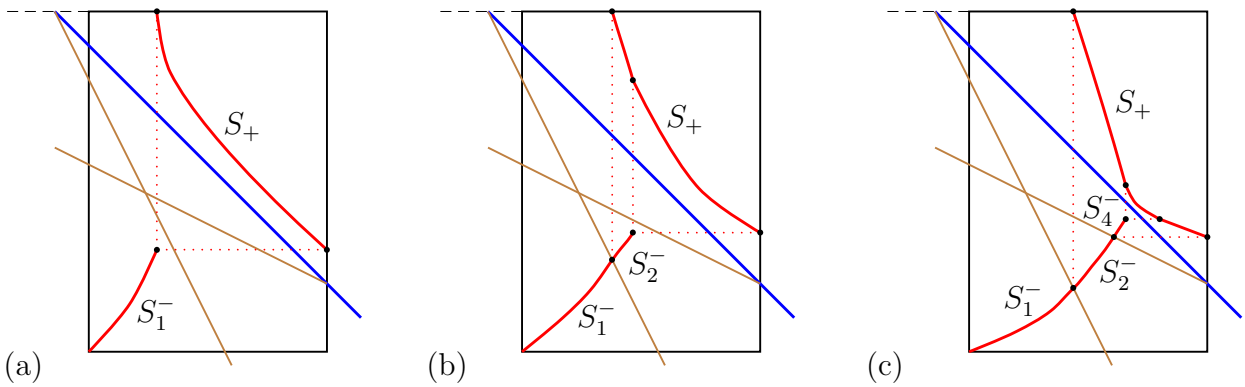


Figure 13: The cases $\langle 1\rangle$, $\langle 12\rangle$, and $\langle 124\rangle$ are schematically represented in figures (a), (b), and (c), respectively.

The following theorem specifies the optimal measure ν.

Theorem 4. *The set S_1^- contains the point (X_1, Y_1) (and therefore is nonempty). Further,*

(a) if $\mu_x([X_1,\ (X_2-2Y_2)/3]) < \mu_y([Y_1,\ (Y_2-2X_2)/3])$, only the cases $\langle 1\rangle$, $\langle 12\rangle$, or $\langle 124\rangle$ can be realized;

(b) if $\mu_x([X_1,\ (X_2-2Y_2)/3]) > \mu_y([Y_1,\ (Y_2-2X_2)/3])$, only the cases $\langle 1\rangle$, $\langle 13\rangle$, or $\langle 134\rangle$ can be realized;

(c) if $\mu_x([X_1,\ (X_2-2Y_2)/3]) = \mu_y([Y_1,\ (Y_2-2X_2)/3])$, only the cases $\langle 1\rangle$ or $\langle 14\rangle$ can be realized.

The points $(x, y) \in S_1^-$ satisfy the equation

$$\mu_x([X_1,\ x]) = \mu_y([Y_1,\ y]), \tag{29}$$

the points $(x, y) \in S_2^-$ satisfy the equation

$$\mu_x([X_1,\ x]) = \mu_y([Y_1,\ y]) + \mu_y([-2x-y,\ Y_2]), \tag{30}$$

the points $(x, y) \in S_3^-$ satisfy the equation

$$\mu_x([X_1,\ x]) + \mu_x([-x-2y,\ X_2]) = \mu_y([Y_1,\ y]), \tag{31}$$

and the points $(x, y) \in S_4^-$ satisfy the equation

$$\mu_x([X_1,\ x]) + \mu_x([-x-2y,\ X_2]) = \mu_y([Y_1,\ y]) + \mu_y([-2x-y,\ Y_2]). \tag{32}$$

The cases $\langle 1\rangle$, $\langle 12\rangle$, and $\langle 124\rangle$ are illustrated in Fig. 13.

Lemma 11. $(X_1, Y_1) \in spt\,\nu$.

Proof. Assume that $(X_1, Y_1) \notin \operatorname{spt}\nu$; we are going to come to a contradiction.

By Lemma 5, there exist points (x, Y_1) and (X_1, y), on the lower and left sides of the rectangle $\mathcal{X} \times \mathcal{Y}$, respectively, that belong to $\operatorname{spt}\nu$. By the assumption, $x > X_1$ and $y > Y_1$. Inequality (11) for these two points takes the form

$$(x - X_1)(Y_1 - y)(x + Y_1 + X_1 + y) \le 0.$$

Since $x - X_1 > 0$ and $Y_1 - y < 0$, it follows that $X_1 + Y_1 + x + y \ge 0$. On the other hand, by (28) one has

$$X_1 + Y_1 + x + y \le X_1 + Y_1 + X_2 + Y_2 \le 0,$$

hence

$$x = X_2, \quad y = Y_2, \quad \text{and} \quad X_1 + Y_1 + X_2 + Y_2 = 0.$$

Thus, the two vertices (X_2, Y_1) and (X_1, Y_2) belong to $\operatorname{spt}\nu$, and $X_2 + Y_1 = -(X_1 + Y_2)$. Assume without loss of generality that $X_2 + Y_1 \le 0$ and $X_1 + Y_2 \ge 0$; then $G(X_2, Y_1) \cap G(X_1, Y_2) \cap (\mathcal{X} \times \mathcal{Y})$ is the union of the right side of the rectangle $\mathcal{X} \times \mathcal{Y}$ and the part of the rectangle lying in the half-plane $x + y \ge X_1 + Y_2$,

$$G(X_2, Y_1) \cap G(X_1, Y_2) \cap (\mathcal{X} \times \mathcal{Y}) = (\{X_2\} \times \mathcal{Y}) \cup ((\mathcal{X} \times \mathcal{Y}) \cap \{x + y \ge X_1 + Y_2\}).$$

By Lemma 5, each vertical line has zero ν-measure; therefore

$$\operatorname{spt}\nu \subset (\mathcal{X} \times \mathcal{Y}) \cap \{x + y \ge X_1 + Y_2\}. \tag{33}$$

In particular, the point (X_2, Y_1) belongs to the set in the right hand side of (33); hence $X_2 + Y_1 = 0 = X_1 + Y_2$. Thus, $\mathcal{X} \times \mathcal{Y}$ is a square.

Take a point $(x_0, y_0) \in \operatorname{spt}\nu$ that does not coincide with the points (X_2, Y_1) and (X_1, Y_2), and therefore is intermediate between them. We have $X_1 < x_0 < X_2$, $Y_1 < y_0 < Y_2$, and $x_0 + y_0 \ge 0$. The intersection of $G(x_0, y_0)$ with the triangle $(\mathcal{X} \times \mathcal{Y}) \cap \{x + y \ge 0\}$ is contained in the set $(x - x_0)(y - y_0) \le 0$ (the union of two right angles), therefore $\operatorname{spt}\nu$ is also contained in this set. It follows that

$$\mu_x([X_1,\, x_0]) = \nu([X_1,\, x_0] \times [y_0,\, Y_2]) = \mu_y([y_0,\, Y_2]) \le \mu_y([-x_0,\, Y_2]),$$

hence

$$\frac{x_0 - X_1}{X_2 - X_1}\,\mu_x(\mathcal{X}) < \mu_x([X_1,\, x_0]) \le \mu_y([-x_0,\, Y_2]) < \frac{Y_2 + x_0}{Y_2 - Y_1}\,\mu_y(\mathcal{Y}) = \frac{x_0 - X_1}{X_2 - X_1}\,\mu_x(\mathcal{X}).$$

We have arrived to the contradiction: $\mu_x(\mathcal{X}) < \mu_x(\mathcal{X})$; thus, Lemma 11 is proved. □

It follows that S_1^- is nonempty, and therefore one of the cases: $\langle 1\rangle$, $\langle 12\rangle$, $\langle 13\rangle$, $\langle 124\rangle$, $\langle 134\rangle$, $\langle 14\rangle$ should be realized. Let us show that if

$$\mu_x([X_1,\, (X_2 - 2Y_2)/3]) > \mu_y([Y_1,\, (Y_2 - 2X_2)/3]), \tag{34}$$

the cases $\langle 12\rangle$, $\langle 124\rangle$, and $\langle 14\rangle$ cannot be realized.

Indeed, in each of these three cases the point of intersection of the curve S_- and the line $2x+y=-Y_2$ lies below and to the right of the point $((X_2-2Y_2)/3,\ (Y_2-2X_2)/3)$ of intersection of the lines $2x+y=-Y_2$ and $x+2y=-X_2$ (and may coincide with this point). This means that $x\ge (X_2-2Y_2)/3$ and $y\le (Y_2-2X_2)/3$. We have therefore

$$\mu_x((X_1,\ (X_2-2Y_2)/3))\le \mu_x((X_1,\ x))=\nu((X_1,\ x)\times\mathbb{R})$$

$$=\nu(\mathbb{R}\times(Y_1,\ y))=\mu_y((Y_1,\ y))\le \mu_y((Y_1,\ (Y_2-2X_2)/3)),$$

in contradiction with (34). Thus, the claim (b) of Theorem 4 is proved. The claims (a) and (c) can be proved in a completely similar manner.

Substituting $X_1,\ Y_1,\ x,\ y$, respectively, for $x_1,\ y_1,\ x_2,\ y_2$ in (22) and taking into account that $-X_1-2Y_1\ge -X_1-Y_1-Y_2\ge X_2$ and $-2X_1-Y_1\ge -X_1-X_2-Y_1\ge Y_2$, we obtain

$$\mu_x([X_1,\ x])+\mu_x([-x-2y,\ X_2])=\mu_y([Y_1,\ y])+\mu_y([-2x-y,\ Y_2]).$$

If $(x,y)\in S_1^-$, we have $-x-2y\ge X_2$ and $-2x-y\ge Y_2$, therefore $\mu_x([-x-2y,\ X_2])=0$ and $\mu_y([-2x-y,\ Y_2])=0$, and we come to (29). If $(x,y)\in S_2^-$, we have $-x-2y\ge X_2$ and $-2x-y\le Y_2$, therefore $\mu_x([-x-2y,\ X_2])=0$ and $\mu_y([-2x-y,\ Y_2])$ is generally nonzero, and we come to (30). The equalities (31) and (44) are verified in a completely similar way. The proof of Theorem 4 is complete.

4.3 Application to the mechanical problem

Let us now reproduce problem (7) stated at the end of Section 3:

$$\inf_{\eta\in\Gamma_{\mu_1,\mu_2}}\iint_{\mathbb{R}^2}(1+\cos(x+y))\,d\eta(x,y).$$

Recall that $d\mu_1(x)=\cos x\cdot\chi_{[0,T]}(x)\,dx$ and $d\mu_2(y)=\cos y\cdot\chi_{[K,\pi/2]}(y)\,dy$, with $0<T\le\pi/2$ and $0\le K<\pi/2$, and $\sin T+\sin K=1$. Thus, μ_1 is supported on $[0,\ T]$ and μ_2 on $[K,\ \pi/2]$, and $\mu_1(\mathbb{R})=\sin T=1-\sin K=\mu_2(\mathbb{R})$. One easily checks that $T+K\le\pi/2$, and the equality is attained when $T=\pi/2$.

We also recall that for $0<T\le\pi/6$ or $T=\pi/2$ problem (7) is equivalent to the symmetric (mechanical) problem (6), and the minimum value in (7) equals the minimum value in (6) times $2(\sin T-\frac{1}{3}\sin^3 T)$.

Make the change of variables $x=\tilde x+\pi/4,\ y=\tilde y+\pi/4$; then the problem takes the form

$$\inf_{\nu\in\Gamma_{\mu_x,\mu_y}}F(\nu),\quad \text{with } F(\nu)=\iint_{\mathbb{R}^2}f(\tilde x+\tilde y)\,d\nu(\tilde x,\tilde y),$$

where $f(\tilde x)=1-\sin\tilde x,\ \mu_x$ is the push-forward measure of μ_1 under the mapping $x\mapsto x-\pi/4$, and μ_y is the push-forward measure of μ_2 under the mapping $y\mapsto y-\pi/4$. Thus,

μ_x and μ_y are supported on the segments $\mathcal{X} = [-\pi/4,\ T-\pi/4]$ and $\mathcal{Y} = [K-\pi/4,\ \pi/4]$, respectively, and their densities are

$$\rho_x(\tilde{x}) = \cos(\tilde{x}+\pi/4)\cdot\chi_{[-\pi/4,\,T-\pi/4]}(\tilde{x}) \quad \text{and} \quad \rho_y(\tilde{y}) = \cos(\tilde{y}+\pi/4)\cdot\chi_{[K-\pi/4,\,\pi/4]}(\tilde{y}).$$

That is, in our case the endpoints of the segments are $X_1 = -\pi/4$, $X_2 = T-\pi/4$, $Y_1 = K-\pi/4$, $Y_2 = \pi/4$, and $X_1+X_2+Y_1+Y_2 = T+K-\pi/2 \le 0$.

The function f and the measures μ_x and μ_y satisfy the assumptions adopted in Theorems 2 and 4; therefore the optimal measure satisfies the claims of these Theorems. Further, one has

$$\mu_x([X_1,\ (X_2-2Y_2)/3]) = \mu_x([-\pi/4,\, T/3-\pi/4]) = \mu_1([0,\, T/3]) = \sin(T/3),$$

$$\mu_y([Y_1,\ (Y_2-2X_2)/3]) = \mu_y([K-\pi/4,\, -2T/3+\pi/4]) = \mu_2([K,\, -2T/3+\pi/2])$$
$$= \cos(2T/3) - \sin K = \cos(2T/3) - 1 + \sin T = -2\sin^2(T/3) + 3\sin(T/3) - 4\sin^3(T/3)$$
$$= \sin(T/3) + 2\sin(T/3)(1+\sin(T/3))(1-2\sin(T/3)) \ge \sin(T/3),$$

and the equality is attained when $T = \pi/2$. That is, for $0 < T < \pi/2$ one has $\mu_x([X_1,\ (X_2-2Y_2)/3]) < \mu_y([Y_1,\ (Y_2-2X_2)/3])$, and so, claim (a) of Theorem 4 is applicable. This means that one of the cases $\langle 1\rangle$, $\langle 12\rangle$, or $\langle 124\rangle$ can be realized. If, otherwise, $T = \pi/2$ then $\mu_x([X_1,\ (X_2-2Y_2)/3]) = \mu_y([Y_1,\ (Y_2-2X_2)/3])$, and claim (c) of Theorem 4 is applicable. This means that one of the cases $\langle 1\rangle$ and $\langle 14\rangle$ can be realized. In all cases the optimal measure contains the point $(-\pi/4,\ T-\pi/4)$.

Applying Theorem 4 and making the inverse change of variables $x = \tilde{x}+\pi/4,\ y = \tilde{y}+\pi/4$, one concludes that the points $(x,y) \in S_1^-$ satisfy the relations

$$\sin x - \sin y + 1 - \sin T = 0, \quad 2x + y \le \pi/2.$$

The points $(x,y) \in S_2^-$ satisfy the relations

$$\sin x - \sin y + \sin(2x+y) - \sin T = 0, \quad 2x+y \ge \pi/2, \quad x+2y \le \pi - T.$$

The points $(x,y) \in S_4^-$ satisfy the relations

$$\sin x - \sin(x+2y) = \sin y - \sin(2x+y), \quad x+2y \ge \pi - T, \quad x+y \le \pi/2. \tag{35}$$

Using these relations, for each value of T one defines a one-parameter family of measures containing the optimal one, and finding the optimal measure amounts to minimizing a function of one variable. The description of these families is, however, quite cumbersome.

Introduce the values

$$\varphi_* = \varphi_*(T) = \arcsin(-1/4 + \sqrt{1/16 + (\sin T)/2}) \tag{36}$$

and

$$T_1 = \arcsin(1/\sqrt{5}) \approx 0.464, \quad T_2 = \arcsin(5-3\sqrt{2}) \approx 0.859, \quad T_3 = \arcsin\sqrt{27/32} \approx 1.164.$$

One easily sees that $0 < \varphi_*(T) \le \pi/6$, and the equality is attained when $T = \pi/2$.

Denote by $\Psi(x)$ the function defined on $[\varphi_*,\, T]$ by $\sin x + \sin\Psi - \sin(2x+\Psi) - \sin T = 0,\ \varphi_* \le x \le \tau,\ 0 \le \Psi \le \pi/2$ and define the sets $\mathcal{S}_i^{\pm}(\tau) = \mathcal{S}_i^{\pm}(\tau, T),\ i = 1,\, 2,\, 3$ by

$$\begin{aligned}
&\mathcal{S}_1^-(\tau) = \{(x,y):\ \sin x - \sin y + 1 - \sin T = 0,\ \ 0 \le x \le \tau,\ 0 \le y \le \pi/2\},\\
&\mathcal{S}_2^-(\tau) = \{(x,y):\ \sin x - \sin y + \sin(2x+y) - \sin T = 0,\ \varphi_* \le x \le \tau,\ 0 \le y \le \pi/2\},\\
&\mathcal{S}_3^-(\tau) = \{(x,x):\ (\pi - T)/3 \le x \le \tau\},\\
&\mathcal{S}_1^+(\tau) = \{(x,y):\ \sin x + \sin y - \sin(2x+y) - \sin T = 0,\ \varphi_* \le x \le \tau,\ 0 \le y \le \pi/2\},\\
&\mathcal{S}_2^+(\tau) = \{(x,\pi-3x):\ (\pi - T)/3 \le x \le \tau\} \cup \{(\pi-3x, x):\ (\pi - T)/3 \le x \le \tau\},\\
&\mathcal{S}_3^+(\tau,\sigma,\tau^+) = \{(x,y):\ \ \sin x + \sin y = \sin\tau + \sin\tau^+,\ \ \tau \le x \le \sigma,\ 0 \le y \le \pi/2\}.
\end{aligned}$$

One comes to the following theorem. Its proof is quite involved and is placed in Appendix B.

Theorem 5. *(a) If $0 < T \le T_1$, the optimal measure is uniquely defined by its support $\operatorname{spt}\nu = \mathcal{S}_1^-(\varphi_*) \cup \mathcal{S}_2^-(T) \cup \mathcal{S}_1^+(T)$.*

(b) If $T_1 < T \le T_2$, the optimal measure belongs to a 1-parameter family with the parameter τ; each measure in the family is defined by its support $\operatorname{spt}\nu = \mathcal{S}_1^-(\varphi_*) \cup \mathcal{S}_2^-(\tau) \cup \mathcal{S}_1^+(\tau) \cup \mathcal{S}_3^+(\tau, T, \Psi(\tau)),\ \varphi_* < \tau < \min\{T,\, (\pi - T)/3\}$.

(c) If $T_2 \le T \le T_3$, the optimal measure belongs to the union of two 1-parameter families with the parameter τ. The former one is as in the case (b), and the latter one is given by $\operatorname{spt}\nu = \mathcal{S}_1^-(\tau) \cup \mathcal{S}_3^+(\tau, T, \pi/2),\ \arcsin(\sqrt{2}-1) \le \tau \le \varphi_*$.

(d) If $T_3 < T < \pi/2$, the optimal measure belongs to the union of three 1-parameter families. The first two ones are as in (c), and the last one is given by $\operatorname{spt}\nu = \mathcal{S}_1^-(\varphi_*) \cup \mathcal{S}_2^-((\pi-T)/3) \cup \mathcal{S}_3^-(\tau) \cup \mathcal{S}_1^+((\pi-T)/3) \cup \mathcal{S}_2^+(\tau) \cup \mathcal{S}_3^+(\tau, \pi-3\tau, 3\tau),\ (\pi-T)/3 \le \tau \le \pi/4$.

(e) Finally, if $T = \pi/2$, the optimal measure belongs to the union of two 1-parameter families. The former one is given by $\operatorname{spt}\nu = \{(x,x):\ 0 \le x \le \tau\} \cup \mathcal{S}_3^+(\tau, \pi/2, \pi/2)$, $\arcsin(\sqrt{2}-1) \le \tau \le \pi/6$, and the latter one is given by $\operatorname{spt}\nu = \{(x,x):\ 0 \le x \le \tau\} \cup \mathcal{S}_2^+(\tau) \cup \mathcal{S}_3^+(\tau, \pi-3\tau, 3\tau),\ \pi/6 \le \tau \le \pi/4$.

That is, in (a) and (b) the case $\langle 12\rangle$ is realized, in (c) the cases $\langle 1\rangle$ and $\langle 12\rangle$ can be realized, in (d) the cases $\langle 1\rangle$, $\langle 12\rangle$, and $\langle 124\rangle$ can be realized, and in (e) the cases $\langle 1\rangle$ and $\langle 14\rangle$ can be realized.

Remark 6. *In each case (a)–(e) the 1-parameter family (or the union of families) of sets indicated in the theorem contains the family of monotone sets generating measures*

from Γ_{μ_1,μ_2}, but actually does not coincide with it. An additional effort can be made to diminish the range of variation of the parameter τ, so that the coincidence of the families is attained. However, this work is quite laborious and results in a complicated description of the range of τ, and therefore is omitted here.

Remark 7. *The theorem remains true, if the function $1+\cos(x+y)$ in (4.3) is substituted with const $+f(x+y-\pi/2)$, where f is an odd continuous function defined on $[-\pi/2,\, -\pi/2]$ and convex on $[0,\, -\pi/2]$. One can take, for example, $\sqrt{\cos(x+y)}$.*

5 Minimal resistance of oscillating bodies

In this section we obtain graphical and numerical solutions for the problem of optimal transportation (7) and its mechanical prototype, the problem with symmetry (6). (Recall that by Lemma 2 problems (6) and (5) are equivalent.) For problem (7), we use analytical results of the previous section to find the optimal transport plans for several values of T and calculate the cost of the optimal transport plan (which is a function of T). Namely, for each value of T we first use Theorem 5 to determine the family of f-monotone measures, and then extract from this family the measure with the smallest cost. On the other hand, the optimal transport plans and the costs for the symmetric problem (6) are calculated numerically.

Our numerical approach is based on discretization of problems (6) and (7). The segment $[0,\, \pi/2]$ is divided into N small intervals with equal λ^+-measure, $I_i = [\arcsin\frac{i-1}{N},\, \arcsin\frac{i}{N}]$, $i = 1, \ldots, N$; we have $\lambda^+(I_i) = 1/N$ for all i. The center of mass of each I_i is at the point $x_i = \arcsin\frac{2i-1}{2N}$. Respectively, the square $[0,\, \pi/2]^2$ is divided into N^2 small rectangles $I_i \times I_j$ centered at (x_i, x_j).

Further, we substitute the integrand $1 + \cos(x + y)$ in (7) and (6) with the piecewise constant function equal to

$$c_{ij} = 1 + \cos(x_i + x_j) = 1 + \cos\Big(\arcsin\frac{2i-1}{2N} + \arcsin\frac{2j-1}{2N}\Big)$$

in each rectangle $I_i \times I_j$, and denote $\eta(I_i \times I_j) = \eta_{ij}$. Next we fix an integer $1 \le M \le N$ and take $T = \arcsin(M/N)$, so that the segment $[0,\, T]$ contains an integer number of small segments, $[0,\, T] = I_1 \cup \ldots \cup I_M$. Correspondingly, we have $K = \arcsin((N - M)/N)$, so that the interval $[K,\, \pi/2]$ contains an integer number of small segments, $[K,\, \pi/2] = I_{N-M+1} \cup \ldots \cup I_N$.

As a result, the value of the integral in (6) is substituted with the approximate value

$$\sum_{i=1}^{M}\sum_{j=1}^{N} c_{ij}\eta_{ij}$$

and the condition $\eta \in \Gamma^E_{\lambda^+,\lambda^+}$ is transformed into the conditions (P) $\sum_{i=1}^N \eta_{ij} = 1/N$ for $j = 1, \ldots, N$, $\sum_{j=1}^N \eta_{ij}$ for $i = 1, \ldots, N$ and (E) $\eta_{ij} = \eta_{ji}$. Finally, taking $x_{ij} = N\eta_{ij}$, the (symmetric, mechanical) problem (6) is reduced to the following problem of linear programming:

$$\begin{aligned} &\min \sum_{i=1}^M \sum_{j=1}^N c_{ij}\, x_{ij} \\ &\text{s.t.} \ \sum_{i=1}^N x_{ij} = 1 \quad \text{for all } \ j = 1, \ldots, N; \\ &\qquad \sum_{j=1}^N x_{ij} = 1 \quad \text{for all } \ i = 1, \ldots, N; \\ &\qquad x_{ij} = x_{ji} \geq 0 \quad \text{for all } \ 1 \leq i,\, j \leq N. \end{aligned} \tag{37}$$

Note that the constraints in the second and third lines of (37) imply the constraint in the fourth line; this means that the problem is abundant.

In order to verify our theoretical results, we also consider the discretization of the (nonsymmetric) problem (7), which takes the form

$$\begin{aligned} &\min \sum_{i=1}^M \sum_{j=N-M+1}^N c_{ij}\, x_{ij} \\ &\text{s.t.} \ \sum_{i=1}^N x_{ij} = 1 \quad \text{for all } \ j = 1, \ldots, N; \\ &\qquad \sum_{j=1}^N x_{ij} = 1 \quad \text{for all } \ i = 1, \ldots, N; \\ &\qquad x_{ij} \geq 0 \quad \text{for all } \ 1 \leq i,\, j \leq N. \end{aligned} \tag{38}$$

Recall that the problems (37) and (38) are equivalent for $1 \leq M \leq N/2$ and $M = N$ (these values correspond to $0 < T \leq \pi/6$ and $T = \pi/2$, respectively).

In the simulation we take $N = 800$. The computational tests were performed on a PC Pentium IV, 2.0Ghz and 512 Mb RAM and using the optimization package Xpress-IVE, Version 1.19.00 with the modeler MOSEL.

The results of the numerical simulations are given in Figures 14 and 15. The figures in the left column represent the supports of the optimal measures (the optimal transport plans) for the mechanical problem (6) for the values $T = \arcsin(1/8)$, $\arcsin(3/8)$, $\arcsin(5/8)$, and $\arcsin(7/8)$. The figures in the right column represent the optimal transport plans for the non-symmetric problem (7), for the same values of T. The solid lines

there are obtained using Theorem 5, while the dotted lines are obtained from a numerical solution of the discretized problem. One can observe a good agreement between the theoretical and numerical results.

The first two values $T = \arcsin(1/8) \approx 0.125$ and $T = \arcsin(3/8) \approx 0.384$ (Fig. 14 (b), (d)) correspond to the case (a) of Theorem 5. In both cases the optimal transport plan is the union of the graphs of a monotone increasing and a monotone decreasing functions contained in the rectangle $[0,\, T] \times [K,\, \pi/2]$. The graphs meet at the point $(T,\, \pi/2 - T)$ on the right vertical side of the rectangle, and therefore their union is a single curve with a singularity at that point.

The two cases $T = \arcsin(5/8) \approx 0.675$ and $T = \arcsin(7/8) \approx 1.065$ (Fig. 15 (b), (d)) correspond to the cases (b) and (c) of Theorem 5. The optimal transport plan is again the union of graphs of monotone increasing and decreasing functions, but this time these graphs are disjoint. The upper point of the former graph and the lower point of the latter graph lie on the same horizontal line.

One can observe that the restriction of the transport plan to the smaller rectangle $[0,\, T] \times [K,\, \pi/2]$ in the left column is identical to the transport plan in the right column in the cases $T = \arcsin(1/8)$ and $T = \arcsin(3/8)$, and is not identical in the cases $T = \arcsin(5/8)$ and $T = \arcsin(7/8)$.

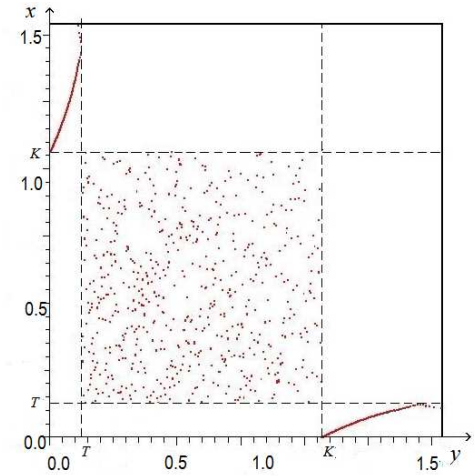
For the reader's convenience, the support of the optimal measure for $T = \pi/2$ is presented in Fig. 16. This measure was found analytically earlier (see, e.g., section 5.4 in the book [27]).

Recall that the minimal relative resistance $m(T)$ is given by formula (4). It is calculated numerically, and its graph is presented in Fig. 17 (dotted line). The solid curve in this figure represents the minimal value in the non-symmetric problem (7) calculated using Theorem 5 and then divided by $2(\sin T - \frac{1}{3}\sin^3 T)$. One can see that the dotted and solid lines coincide for the values $0 < T \le \pi/6$ and $T = \pi/2$.

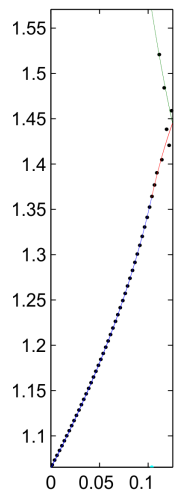
The optimal transport plan for the mechanical problem (6) determines the optimal scattering of the particles by the nose, and thus, ensures the minimal resistance. In the limiting cases $T = 0$ and $\pi/2$ one has $m(0^+) = 0.5$ and $m(\pi/2) \approx 0.987820$.

In Fig. 17 the graph of the resistance $m(T)$ as a function of $s = \sin T$ is depicted. Looking at the graph one can conjecture that the derivative is infinity as $s = 0$, and this is really so. To prove it, let us calculate the asymptotic behavior of $m(s)$ as $s \to 0$.

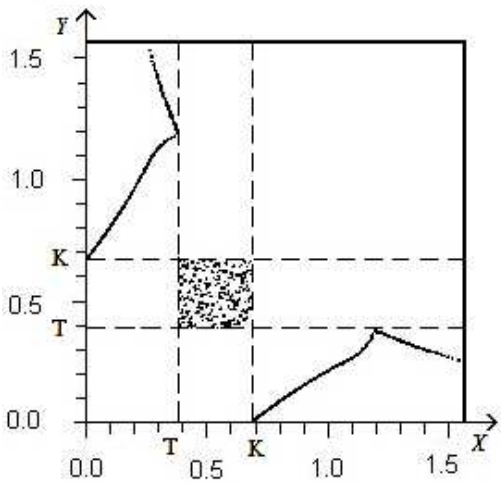
Notice first that the small interval $[\varphi_*,\, T]$ has the length $O(s^2)$, and therefore the relative length $O(s)$ in the interval $[0,\, T]$. Therefore, if the measure is modified on the small rectangle $[\varphi_*,\, T] \times [K,\, \pi/2]$ (still remaining in Γ_{μ_1,μ_2}), the value of the corresponding integral will change by $O(s)$. Let us substitute the optimal measure by the one supported on the curve $\sin y - \arcsin(\sin x + 1 - s)$. The corresponding modified value $\tilde{m}(s) =$



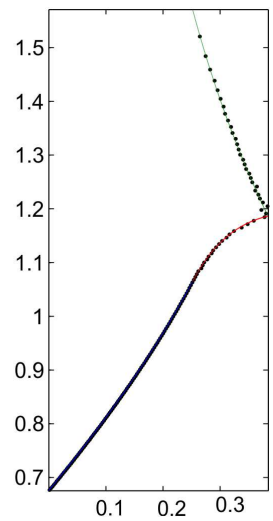
(a) $T = \arcsin(1/8)$, symmetric case



(b) $T = \arcsin(1/8)$, non-symmetric case

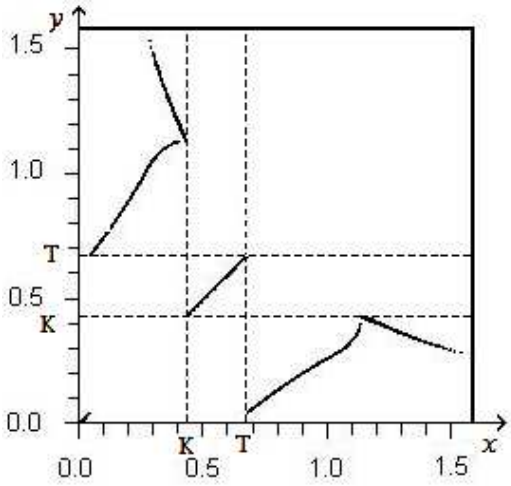


(c) $T = \arcsin(3/8)$, symmetric case

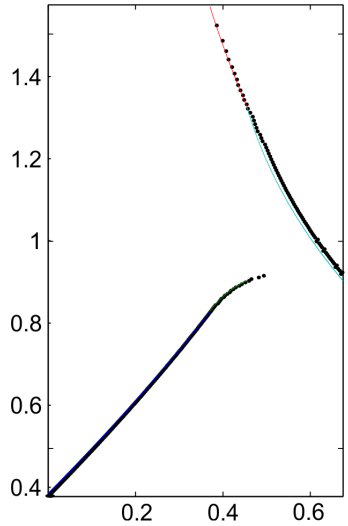


(d) $T = \arcsin(3/8)$, non-symmetric case

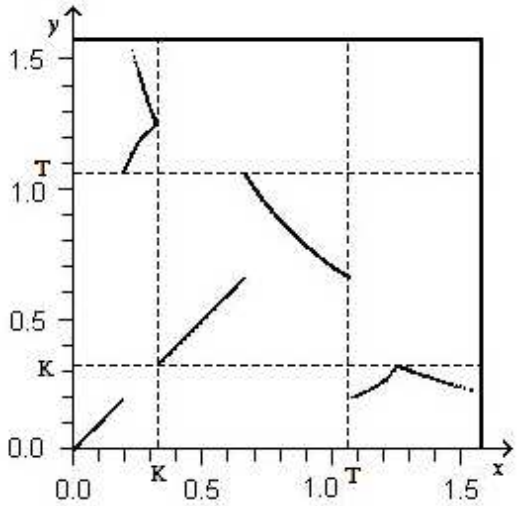
Figure 14: The optimal measures in the cases $T = \arcsin(1/8)$ and $T = \arcsin(3/8)$ are shown, for the symmetric (left) and non-symmetric (right) problems. Recall that the measures in the left column are contained in the square $[0, \pi/]^2$, while the measures in the right column are contained in the rectangle $[0, T] \times [K, \pi/2]$. The optimal measures in the cases (a) and (c) are not unique; they are arbitrary in the square $[T, K]^2$, provided that the standard conditions of symmetry and projections on the coordinate axes are satisfied.



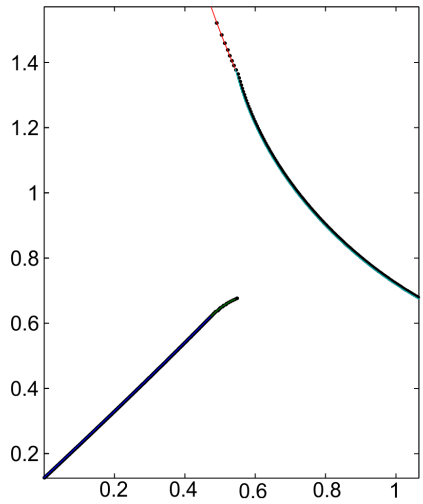
(a) $T = \arcsin(5/8)$, symmetric case



(b) $T = \arcsin(5/8)$, non-symmetric case



(c) $T = \arcsin(7/8)$, symmetric case



(d) $T = \arcsin(7/8)$, non-symmetric case

Figure 15: The optimal measures in the cases $T = \arcsin(5/8)$ and $T = \arcsin(7/8)$ are shown, for the symmetric and the non-symmetric problems.

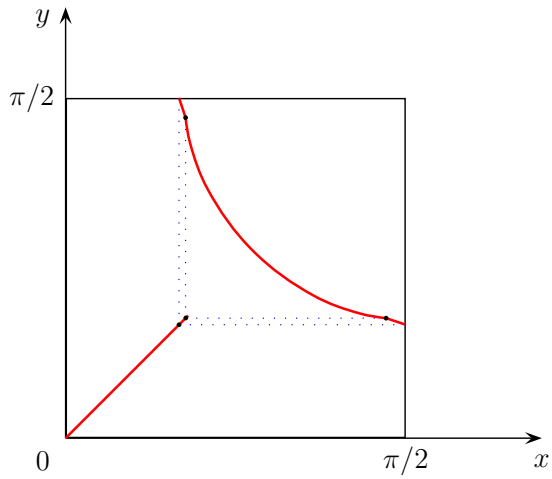


Figure 16: The support of the optimal measure in the case $T = \pi/2$ is the disjoint union of two curves. It is not, however, the graph of a function, since the projections of the curves on the x-axis overlap.

$m(\arcsin s) + O(s)$ is then

$$\tilde{m}(s) = \frac{1}{2(s - s^2/3)} \int_0^T (1 + \cos(x + \arcsin(\sin x + 1 - s))) \cos x \, dx.$$

Notice that substituting $\cos(x + \arcsin(\sin x + 1 - s))$ with $\cos(\arcsin(\sin x + 1 - s))$ will change the integrand by $O(s)$. So, changing the variable $\xi = \sin x$ in the integral, one obtains

$$\tilde{m}(s) = \frac{1 + O(s)}{2s} \int_0^s (1 + \cos(\arcsin(\xi + 1 - s))) \, d\xi$$

$$= \frac{1 + O(s)}{2s} \int_0^s (1 + \sqrt{1 - (\xi + 1 - s)^2}) \, d\xi \approx \frac{1}{2} + \frac{\sqrt{2}}{3} \sqrt{s}.$$

We came to an unfortunate conclusion: even a small-amplitude oscillation leads to a relatively large increase of minimal resistance. More precisely, the minimal resistance is

$$1/2 + \sqrt{2}/3 \sqrt{T} + O(T) \approx 0.5 + 0.47\sqrt{T} \quad \text{as} \quad T \to 0;$$

that is, the increase of the resistance is proportional to $\sqrt{T}$.

Appendix A

Here we prove Lemmas 2 and 3.

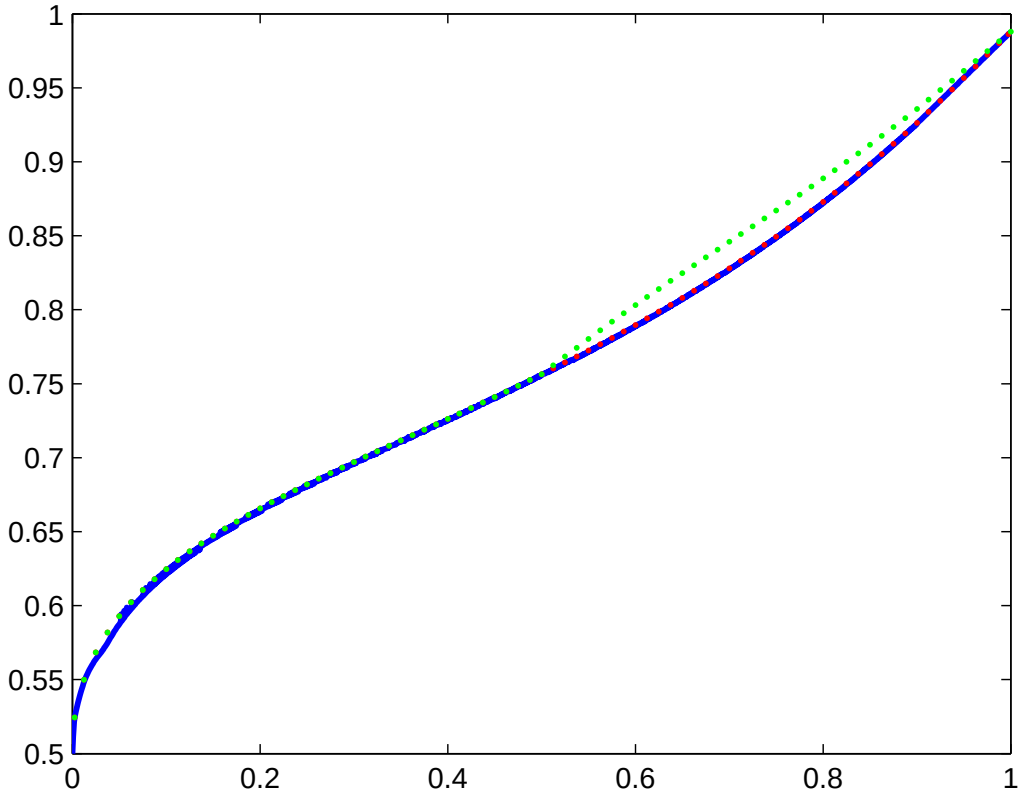


Figure 17: The graph of $m(T)$ (dotted line) and the graph of the normalized minimal value for problem (7) (solid line). These graphs coincide for $0 < \sin T \leq 1/2$ and for $\sin T = 1$.

Since both the integrand in (5) and the measures from $\Gamma^{E,C}_{\lambda,\lambda}$ are centrally symmetric with respect to origin, one has

$$\mathcal{F}_T(\eta) = \iint\limits_{[0,\,T]\times\mathbb{R}} [1+\cos(\varphi-\varphi^+)]\,d\eta(\varphi,\varphi^+). \tag{39}$$

Further, changing the variables $x=\varphi,\ y=-\varphi^+$, recalling that R is the reflection with respect to the x-axis, and taking into account that each $\eta\in\Gamma^{E,C}_{\lambda,\lambda}$ is supported in $[-\pi/2,\,\pi/2]^2$, one concludes that η minimizes (39) iff $R^{\#}\eta$ solves the problem

$$\inf_{\eta\in\Gamma^{E,C}_{\lambda,\lambda}} F_T(\eta), \qquad \text{with} \quad F_T(\eta) = \iint\limits_{[0,\,T]\times[-\pi/2,\,\pi/2]} [1+\cos(x+y)]\,d\eta(x,y). \tag{40}$$

We shall successively prove that

(i) if $\eta\in\Gamma^{E,C}_{\lambda,\lambda}$, then there exists $\chi\in\Gamma^{E}_{\lambda^+,\lambda^+}$ such that $F^+_T(\chi)\le F_T(\eta)$;

(ii) if $\chi\in\Gamma^{E}_{\lambda^+,\lambda^+}$, then there exists $\nu\in\Gamma_{\mu_1,\mu_2}$ such that $\Phi(\nu)\le F^+_T(\chi)$, and if $\chi([0,\,T]\times[0,\,K])>0$, the inequality is strict.

The proof is based on rearrangement arguments. In the proof of (i) we use that for $x>0,\ y>0$ the transportation $x\mapsto y,\ -x\mapsto -y$ is cheaper than the transportation $x\mapsto -y,\ -x\mapsto y$ (see Fig. 18 (a)). In the proof of (ii) we use that a monotone upward shift in the y-space diminishes the cost of transportation (in Fig. 18 (b) we see that the transportation $x\mapsto y_2$ is cheaper than $x\mapsto y_1$).

(i) Define the measure χ by $\chi(A):=\eta(A\cup R(A))$ for any Borel set $A\subset[0,\,\pi/2]^2$ and by $\chi(A)=0$ in the case $A\cap[0,\,\pi/2]^2=\emptyset$. It is straightforward to verify that $\chi\in\Gamma^{E}_{\lambda^+,\lambda^+}$.

One has

$$F^+_T(\chi) = \iint\limits_{[0,\,T]\times[0,\,\pi/2]} [1+\cos(x+y)]\,d\chi(x,y) = \iint\limits_{[0,\,T]\times\mathbb{R}} [1+\cos(x+|y|)]\,d\eta(x,y).$$

Taking into account that $0\le x\le T$ and η is supported in the set $-\pi/2\le y\le\pi/2$, one easily sees that $\cos(x+|y|)\le\cos(x+y)$. Therefore

$$F^+_T(\chi) \le \iint\limits_{[0,\,T]\times[-\pi/2,\,\pi/2]} [1+\cos(x+y)]\,d\eta(x,y) = F_T(\eta).$$

Thus, (i) is proved.

(ii) We have $\chi([0,\,T]\times[0,\,\pi/2])=\lambda^+([0,\,T])=\sin T$. The quantity $\chi([0,\,T]\times[y,\,\pi/2])$ is a continuous monotone nonincreasing function of y taking the value $\sin T$ when $y=0$

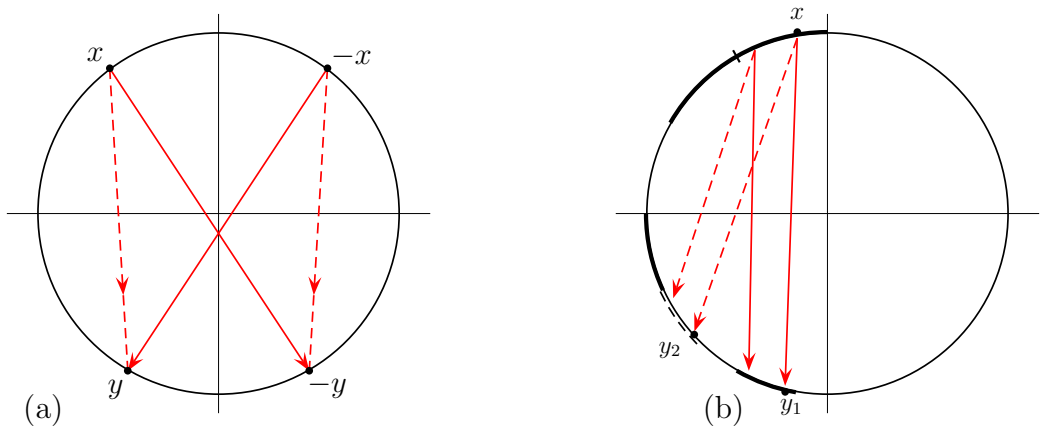


Figure 18: Complex representation is used to illustrate rearrangement arguments in the proof of (i) and (ii). The values $\pm x$, $\pm y$ are represented by the points $e^{i\pi/2\pm ix}$, $-e^{-i\pi/2\pm iy}$ on the complex circle. In both figures, the transportation indicated by dashed lines is cheaper than that indicated by solid lines.

and 0 when $y = \pi/2$. Therefore one can find a continuous monotone nondecreasing function g such that

$$\chi([0,\, T] \times [y,\, \pi/2]) = 1 - \sin g(y),$$

with $g(0) = K$ (so that $1 - \sin g(0) = 1 - \sin K = \sin T$) and $g(\pi/2) = \pi/2$. Further, for $0 \le y \le \pi/2$ one has

$$1 - \sin g(y) \le \chi(\mathbb{R} \times [y,\, \pi/2]) = \lambda^+([y,\, \pi/2]) = 1 - \sin y,$$

hence $g(y) \ge y$.

Now define the map $G : [0,\, T] \times [0,\, \pi/2] \to [0,\, T] \times [K,\, \pi/2]$ by $G(x, y) = (x, g(y))$ and define the measure ν supported in $[0,\, T] \times [K,\, \pi/2]$ by $\nu := G^{\#}\chi$. In other words, for any Borel set $A \subset [0,\, T] \times [K,\, \pi/2]$ we have

$$\nu(A) = \chi(G^{-1}(A)).$$

For a Borel set $I \subset [0,\, T]$ one obviously has $\nu(I \times \mathbb{R}) = \mu_1(I)$. Now take a line segment $J \subset [K,\, \pi/2]$ and find y_1 and y_2 so as $J = [g(y_1),\, g(y_2)]$. One has

$$\nu(\mathbb{R} \times J) = \nu([0,\, T] \times [g(y_1),\, g(y_2)]) = \chi([0,\, T] \times [y_1,\, y_2])$$

$$= [1 - \sin g(y_1)] - [1 - \sin g(y_2)] = \sin g(y_2) - \sin g(y_1) = \mu_2(J).$$

Thus, $\nu \in \Gamma_{\mu_1,\mu_2}$.

Further, since $g(y) \geq y$, we have $\cos(x+g(y)) \leq \cos(x+y)$, and so,

$$\Phi(\nu) = \iint_{\mathbb{R}^2} [1+\cos(x+y)]\, d\nu(x,y) = \iint\limits_{[0,T]\times[0,\pi/2]} [1+\cos(x+g(y))]\, d\chi(x,y)$$

$$\leq \iint\limits_{[0,T]\times[0,\pi/2]} [1+\cos(x+y)]\, d\chi(x,y) = F_T^+(\chi),$$

and if $\chi([0,\, T]\times[0,\, K]) > 0$, the inequality here is strict. Thus, (ii) is proved.

It is straightforward to check that if $\eta \in \Gamma^E_{\lambda^+,\lambda^+}$, then $\eta + C^\#\eta \in \Gamma^{E,C}_{\lambda,\lambda}$ and

$$F_T^+(\eta) = F_T(\eta + C^\#\eta) \tag{41}$$

Therefore

$$\inf_{\eta\in\Gamma^E_{\lambda^+,\lambda^+}} F_T^+(\eta) \geq \inf_{\eta\in\Gamma^{E,C}_{\lambda,\lambda}} F_T(\eta).$$

On the other hand, by (i) we have

$$\inf_{\eta\in\Gamma^{E,C}_{\lambda,\lambda}} F_T(\eta) \geq \inf_{\eta\in\Gamma^E_{\lambda^+,\lambda^+}} F_T^+(\eta).$$

Thus, these infima coincide. This proves claim (a) of Lemma 2.

If η minimizes (6), then by (41) and by claim (a) of Lemma 2, $\eta + C^\#\eta$ solves (40), and therefore $R^\#(\eta + C^\#\eta)$ solves (5). This proves claim (b) of Lemma 2.

Let $\nu \in \Gamma_{\mu_1,\mu_2}$. If $0 < T \leq \pi/6$, then the (mutually symmetric) rectangles $\Pi = [0,\, T]\times[K,\, \pi/2]$ and $E(\Pi) = [K,\, \pi/2]\times[0,\, T]$ do not intersect, and we define the measure η as follows. First, $\eta\rfloor_\Pi = \nu$ and $\eta\rfloor_{E(\Pi)} = E^\#\nu$. Then we define the restriction of η on the square $[T,\, K]^2$ in the symmetric way (that is, $E^\#\eta\rfloor_{[T,\,K]^2} = \eta\rfloor_{[T,\,K]^2}$) and also in such a way that both its projections on the coordinate axes have the density $\cos x$. Additionally, require that η is zero on the complement $\mathbb{R}^2 \setminus (\Pi \cup E(\Pi) \cup [T,\, K]^2)$. If $T = \pi/2$, we just set $\eta = \nu$. One easily sees that $\eta \in \Gamma^E_{\lambda^+,\lambda^+}$ and $F_T^+(\eta) = \Phi(\nu)$; therefore

$$\inf_{\eta\in\Gamma^E_{\lambda^+,\lambda^+}} F_T^+(\eta) \leq \inf_{\nu\in\Gamma_{\mu_1,\mu_2}} \Phi(\nu).$$

On the other hand, from (ii) we deduce that

$$\inf_{\nu\in\Gamma_{\mu_1,\mu_2}} \Phi(\nu) \leq \inf_{\eta\in\Gamma^E_{\lambda^+,\lambda^+}} F_T^+(\eta).$$

Thus, these infima coincide, and claim (a) of Lemma 3 is proved.

Let ν be a solution of (7); the above construction defines a (non-unique) measure η such that $F_T^+(\eta) = \Phi(\nu)$ (and therefore η minimizes (6)) and $\eta\rfloor_{[0,T]\times[K,\pi/2]} = \nu$. This proves claim (c) of Lemma 3.

Let $\chi \in \Gamma^E_{\lambda^+,\lambda^+}$ minimize (6); by (ii) and by claim (a) of Lemma 3 there exists $\nu \in \Gamma_{\mu_1,\mu_2}$ such that $\Phi(\nu) = F_T^+(\chi)$. Using (ii) again, one concludes that $\chi\rfloor_{[0,T]\times[0,K]} = 0$, and therefore the restriction of χ on $[0,T] \times [K, \pi/2]$ lies in Γ_{μ_1,μ_2} and minimizes (7). Claim (b) of Lemma 3 is proved.

Appendix B

In this appendix we prove Theorem 5.

Theorems 2 and 4 can be reformulated in our case as follows.

For a measure ν on $[0,\, T] \times [K,\, \pi/2]$ define the sets

$$S_- = \overline{\operatorname{spt}\nu \cap \{x + y < \pi/2\}} \quad \text{and} \quad S_+ = \operatorname{spt}\nu \cap \{x + y \geq \pi/2\}.$$

We have $\operatorname{spt}\nu = S_- \cup S_+$.

Corollary 2. *Let ν be an optimal measure. Then*

(a) the corresponding set S_+ is either empty or the graph of a continuous monotone decreasing function defined on a segment.

(b) The set S_- is either empty or the graph of a continuous monotone increasing function defined on a segment (let it be $[x^i,\, x^f]$).

(c) Take a point $(x_0, y_0) \in S_-$ with $x_0 \neq x^f$ and consider the vertical line $x = x_0$ and the horizontal line $y = y_0$.

(i) If $2x_0 + y_0 < \pi/2$ then $\{x = x_0\} \cap S_+ = \emptyset$.

(ii) If $2x_0 + y_0 \geq \pi/2$ then $\{x = x_0\} \cap S_+ = \{(x_0,\, \pi - 2x_0 - y_0)\}$.

(iii) If $x_0 + 2y_0 < \pi - T$ then $\{y = y_0\} \cap S_+ = \emptyset$.

(iv) If $x_0 + 2y_0 \geq \pi - T$ then $\{y = y_0\} \cap S_+ = \{(\pi - x_0 - 2y_0,\, y_0)\}$.

Corollary 3. *The set S_1^- contains the point $(0, K)$. Further,*

(a) if $0 < T < \pi/2$, only the cases $\langle 1\rangle$, $\langle 12\rangle$, or $\langle 124\rangle$ can be realized;

(b) if $T = \pi/2$, only the cases $\langle 1\rangle$ or $\langle 14\rangle$ can be realized.

The points $(x, y) \in S_1^-$ satisfy the equation

$$\sin x - \sin y + 1 - \sin T = 0, \tag{42}$$

the points $(x, y) \in S_2^-$ satisfy the equation

$$\sin x - \sin y + \sin(2x + y) - \sin T = 0, \tag{43}$$

and the points $(x, y) \in S_4^-$ satisfy the equation

$$x = y = 0. \tag{44}$$

Lemmas 4 – 11 are all valid in our case, with corresponding modifications.

By Corollaries 2 and 3, S_1^- is the graph of a monotone increasing function on a segment $[0,\, \tau]$ defined implicitly by $\sin x = \sin y - \sin K$. Thus, S_1^- coincides with $\mathcal{S}_1^-(\tau)$. One easily checks that the condition $2x+y \le \pi/2$ imposes the restriction $\tau \le x_*$. In the cases $\langle 12\rangle$, $\langle 124\rangle$, and $\langle 14\rangle$ the final endpoint of S_1^- lies on the line $2x+y=\pi/2$, and therefore $\tau = x_*$. On the other hand, if $\tau < x_*$ then S_1^- coincides with S_-, that is, the case $\langle 1\rangle$ is realized.

If the set S_2^- is more than a singleton (that is, in the cases $\langle 12\rangle$ and $\langle 124\rangle$), it is a part of the graph of the function $y(x)$ defined implicitly by

$$\sin x - \sin y + \sin(2x+y) - \sin T = 0, \quad x_* \le x \le T, \quad 0 \le y \le \pi/2$$

The endpoints of the graph can be indicated explicitly: the original one, $(\pi/2 - x_*, x_*)$, coincides with the final endpoint of $\mathcal{S}_1^-(x_*)$, and the final one is $(T, \pi/2 - T)$.

The derivative of the function $y(x)$ equals

$$\frac{dy}{dx} = \frac{\cos x + 2\cos(2x+y)}{\cos y - \cos(2x+y)};$$

the denominator in the quotient is always positive, and the derivative at the initial point is positive, $\frac{dy}{dx}(x_*) > 0$. The zero of the derivative, $\frac{dy}{dx} = 0$, is obtained from the system of equations

$$\sin x - \sin y + \sin(2x+y) - \sin T = 0, \quad \cos x + 2\cos(2x+y) = 0.$$

After some algebra one obtains

$$3\sin^4 x + 2\sin T \sin^3 x - \sin^2 T = 0.$$

Further calculation shows that this equation has no zeros on the interval $[x_*,\, T)$, if $0 < T \le T_1 = \arcsin(1/\sqrt{5})$, and has exactly one zero on this interval, if $T_1 < T \le \pi/2$. This implies that the function $y(x)$ monotone increases, if $T \le T_1$, and first increases and then decreases, if $T > T_1$.

The set $S_2^- = \{(x, y(x)) : x \in [x_*,\, \tau]\}$ is in general an initial part of the graph corresponding to monotone increase of $y(x)$. This means that $S_2^- = \mathcal{S}_2^-(\tau)$ for some $x_* < \tau \le T$, and $\frac{dy}{dx}(\tau) \ge 0$.

One derives from the equation in (35) that each point $(x,y) \in S_4^-$ belongs to the union of lines $x = y$ and $x + y = \pi/2$. If S_4^- is more than a singleton (that is, in the cases $\langle 124\rangle$ and $\langle 14\rangle$), it is the graph of a continuous monotone increasing function, and therefore is a segment of the line $x = y$. The initial point of this segment lies on the line $x+2y = \pi - T$, and thus, is equal to $((\pi - T)/3, (\pi - T)/3)$. Notice that $(\pi - T)/3 > x_*$.

Suppose that the case $\langle 124\rangle$ is realized; then S_2^- is the monotone increasing subgraph of $y(x)$ with the final endpoint $((\pi-T)/3,(\pi-T)/3)$. Thus, one has $\tau=(\pi-T)/3$ and $\frac{dy}{dx}((\pi-T)/3)\geq 0$. The last inequality is equivalent to

$$\cos x+2\cos(2x+y)\geq 0,\quad \text{with } x=y=(\pi-T)/3,$$

and after a simple algebra one comes to the solution: $T\geq T_3=\arcsin\sqrt{27/32}$. Thus, the case $\langle 124\rangle$ can only be realized if $T_3\leq T<\pi/2$.

Consider the case $\langle 1\rangle$. The set $S_-=S_1^-$ is the graph of a continuous monotone increasing function with the endpoints at $(0,K)$ and (τ,τ^+), where τ^+ is found from the equation

$$\sin\tau-\sin\tau^++1-\sin T=0. \tag{45}$$

This set is contained in the intersection of half-planes $2x+y\leq\pi/2,\ x+2y\leq\pi-T$. Each vertical line $\{x\}\times\mathbb{R},\ 0\leq x<\tau$ and each horizontal line $\mathbb{R}\times\{y\},\ K\leq y<\tau^+$ has nonempty intersection with S_-. By claims (i) and (iii) of Corollary 2, these lines do not intersect S_+.

On the other hand, each vertical line $\{x\}\times\mathbb{R},\ \tau<x\leq T$ and each horizontal line $\mathbb{R}\times\{y\},\ \tau^+<y\leq\pi/2$ does not intersect S_-. By claim (a) of Lemma 5, each of these lines has nonempty intersection with S_+. Using now claim (a) of Corollary 2, one concludes that S_+ is the graph of a continuous monotone decreasing function g with the endpoints at $(\tau,\pi/2)$ and (T,τ^+). Since the marginal measures of ν coincide with μ_1 and μ_2, for $\tau\leq x\leq T$ one has

$$\sin x-\sin\tau=\mu_1([\tau,\,x])=\nu([\tau,\,x]\times\mathbb{R})=\nu(\mathbb{R}\times[g(x),\,\pi/2])=\mu_2([g(x),\,\pi/2])=1-\sin g(x);$$

thus,

$$\sin x+\sin g(x)=\sin\tau+1.$$

We conclude that S_+ coincides with $\mathcal{S}_3^+(\tau,T,\pi/2)$, and so,

$$\operatorname{spt}\nu=\mathcal{S}_1^-(\tau)\cup\mathcal{S}_3^+(\tau,T,\pi/2).$$

One has $2\tau+\tau^+\leq\pi/2$, and since $(T,\tau^+)\in S_+$, we get $T+\tau^+\geq\pi/2$. From these two inequalities one concludes that $\tau\leq T/2$. Taking into account (45), one has $\sin\tau+1-\sin T=\sin\tau^+\geq\cos T$. Thus,

$$\sin T+\cos T\leq 1+\sin\tau\leq 1+\sin(T/2). \tag{46}$$

From this inequality after a simple algebra one derives that $T>\pi/4$.

The minimum of $x+g(x)$ is attained at the point x_0 satisfying $x_0=g(x_0)$. Using (46), one obtains for this point $2\sin x_0=\sin\tau+1\leq\sin x_*+1$. If $T<T_2=\arcsin(5-3\sqrt{2})$, after a simple algebra one finds that $x_0<\pi/4$. That is, if $\pi/4<T<T_2$, one has, on

one side, $\tau \le x_0 < T$ and hence $(x_0, x_0) \in \mathcal{S}_3^+(\tau)$, and on the other side, $2x_0 < \pi/2$, in contradiction to the condition that the set $\mathcal{S}_3^+(\tau)$ lies in the half-plane $x + y \ge \pi/2$.

We conclude that the case $\langle 1 \rangle$ can only be realized if $T_2 \le T \le \pi/2$. The condition $\mathcal{S}_3^+(\tau) \subset \{x + y \ge \pi/2\}$ is equivalent to the inequality $2x_0 \ge \pi/2$. Thus we obtain the restriction on the parameter τ,

$$\sin \tau = 2 \sin x_0 - 1 \ge \sqrt{2} - 1.$$

Note that in the limiting case $T = \pi/2$ the set $\mathcal{S}_1^-(\tau)$ becomes the diagonal $\{(x, y) : x = y,\ 0 \le x \le \tau\}$.

Consider the case $\langle 12 \rangle$. Then S_- coincides with $\mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-(\tau),\ x_* < \tau \le T$. Notice that the set $\mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-(T)$ is the graph of a function joining the points $(0, K)$ and $(T, \pi/2 - T)$ and contains S_-. When $0 < T \le T_1$, the function is monotone increasing, and if $T_1 < T \le \pi/2$, the function first increases and then decreases.

Assume that $0 < T \le T_1$ and let $(x, y) \in (\mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-(T)) \setminus \{(T, \pi/2 - T)\}$. One has $K \le y < \pi/2 - T$, hence the horizontal segment $y = y,\ 0 \le x \le T$ lies in the half-plane $x + y < \pi/2$ and thus does not intersect S_+. Therefore by Lemma 5 (a) it has nonempty intersection with S_-. Since $S_- \subset \mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-(T)$ and (x, y) is the unique point of intersection of the segment with $\mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-(T)$, we conclude that S_- contains (x, y). This argument shows that S_- coincides with $\mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-(T)$.

The set S_+ can be determined using claims (i) and (ii) of Corollary 2. If $0 \le x_0 < x_*$, the intersection of the vertical line $x = x_0$ with S_+ is empty, and if $x_* \le x_0 \le T$ then $\{x = x_0\} \cap S_+ = \{(x_0, \pi - 2x_0 - y(x_0))\}$. Using the definition of the function $y(x)$, one easily verifies that S_+ coincides with $\mathcal{S}_1^+(T)$. It is the graph of a continuous monotone decreasing function with the endpoints at $(x_*, \pi/2)$ and $(T, \pi/2 - T)$. Thus, $\operatorname{spt} \nu = \mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-(T) \cup \mathcal{S}_1^+(T)$.

Let now $T_1 < T \le \pi/2$. The function $y(x)$ first increases and then decreases on the segment $[x_*, T]$. The set S_2^- coincides with $\mathcal{S}_2^-(\tau)$, where $x_* < \tau < \min\{T, (\pi - T)/3\}$. Actually the interval of variation of τ can be further diminished by imposing the restrictions $\frac{dy}{dx}(\tau) \ge 0$ and $y(\tau) \ge \pi/2 - T$.

The set S_+ can now be easily defined using claims (i) and (ii) of Corollary 2. For each $0 \le x < x_*$ there exists y_0 such that $(x_0, y_0) \in S_1^-$ and $2x_0 + y_0 < \pi/2$. Thus, by claim (i) of Corollary 2, $\{x = x_0\} \cap S_+ = \emptyset$.

Further, for each $x_* \le x \le \tau$ there exists y_0 such that $(x_0, y_0) \in S_2^-$ and hence $2x_0 + y_0 \ge \pi/2$. By claim (ii) of Corollary 2, $\{x = x_0\} \cap S_+ = \{(x_0, \pi - 2x_0 - y_0)\}$. One concludes from this that the set $S_+ \cap \{0 \le x < x_*\} = \emptyset$ and $S_+ \cap \{x_* \le x \le \tau\}$ is the graph of a monotone decreasing function satisfying the equation $\sin x + \sin y - \sin(2x + y) - \sin T = 0$, and therefore coincides with $\mathcal{S}_1^+(\tau)$.

The intersection of each vertical line $x = x_0,\ \tau < x_0 \le \pi/2$ is empty, therefore by Lemma 5 (a), $\{x = x_0\} \cap S_+$ is nonempty. Similarly, the intersection of each horizontal line $y = y_0,\ y(\tau) < x_0 < \pi - 2\tau - y(\tau)$ is empty, thus $\{y = y_0\} \cap S_+$ is nonempty.

Using claim (a) of Corollary 2, we conclude that the set $S_+ \cap \{x = x_0\}$ is the graph of a continuous monotone decreasing function (say g) with the endpoints $(\tau,\, \pi - 2\tau - y(\tau))$ and $(\pi/2,\, y(\tau))$. Using again that the marginal measures of ν coincide with μ_1 and μ_2, for $\tau \le x \le T$ one has

$$\sin x - \sin\tau = \mu_1([\tau,\, x]) = \nu([\tau,\, x] \times \mathbb{R}) = \nu(\mathbb{R} \times [g(x),\, \pi - 2\tau - y(\tau)])$$

$$= \mu_2([g(x),\, \pi - 2\tau - y(\tau)]) = \sin(2\tau + y(\tau)) - \sin g(x);$$

thus,

$$\sin x + \sin g(x) = \sin\tau + \sin(2\tau + y(\tau)).$$

We conclude that S_+ coincides with $\mathcal{S}_3^+(\tau, T, \pi/2)$, and so,

$$\operatorname{spt}\nu = \mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-(\tau) \cup \mathcal{S}_1^+(\tau) \cup \mathcal{S}_3^+(\tau, T, 2\tau + y(\tau)).$$

One has also take into account that the set $\mathcal{S}_3^+(\tau, T, 2\tau + y(\tau))$ is contained in the half-plane $x + y \ge \pi/2$. This condition on the parameter τ can further reduce the interval of variation of τ. However, it is much more time consuming just to check all the interval $\tau \in (x_*,\ \min\{T,\, (\pi - T)/3\})$, rather than do a laborious work of reducing it.

Consider the case $\langle 124\rangle$. We have $T_3 < T < \pi/2$ and $S_1^- = \mathcal{S}_1^-(x_*)$, $S_2^- = \mathcal{S}_2^-((\pi - T)/3)$, $S_3^- = \mathcal{S}_3^-(\tau)$, where $\tau < (\pi - T)/3$. The condition $\mathcal{S}_3^-(\tau) \subset \{x + y \le \pi/2\}$ imposes the restriction $\tau \le \pi/4$. The set S_+ is determined similarly to the cases considered above. The intersection $S_+ \cap \{0 \le x < x_*\}$ is empty, the intersection $S_+ \cap \{x_* \le x \le (\pi - T)/3\}$ coincides with $\mathcal{S}_1^+((\pi - T)/3)$, the set $S_+ \cap (\{(\pi - T)/3 \le x \le \pi/4\} \cup \{(\pi - T)/3 \le y \le \pi/4\})$ coincides with $\mathcal{S}_2^+(\tau)$, and the set $S_+ \cap \{\tau \le x < \pi - 3\tau\}$ coincides with $\mathcal{S}_3^+(\tau, \pi - 3\tau, 3\tau)$. Thus,

$$\operatorname{spt}\nu = \mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-((\pi - T)/3) \cup \mathcal{S}_3^-(\tau) \cup \mathcal{S}_1^+((\pi - T)/3) \cup \mathcal{S}_2^+(\tau) \cup \mathcal{S}_3^+(\tau, \pi - 3\tau, 3\tau).$$

In the limiting case $T = \pi/2$ the set $\mathcal{S}_1^-(x_*) \cup \mathcal{S}_2^-((\pi - T)/3) \cup \mathcal{S}_3^-(\tau)$ becomes the diagonal $\{(x, y) : x = y,\ 0 \le x \le \tau\}$, the set $\mathcal{S}_2^-((\pi - T)/3)$ shrinks to the point $(\pi/6, \pi/6)$, and the set $\mathcal{S}_1^+((\pi - T)/3)$ shrinks to the point $(\pi/6, \pi/2)$.

The measure ν can always be reconstructed by its support $\operatorname{spt}\nu$. All we need is to use that the projections of the measure $P_1^\#\nu$ and $P_2^\#\nu$ coincide with μ_1 and μ_2, respectively. Here $P_1(x, y) = x$ and $P_2(x, y) = y$ are the projections on the coordinate axes.

Take, for example, the case $\langle 124\rangle$, where we have the union of sets $\mathcal{S}_1^-(x_*)$, $\mathcal{S}_2^-((\pi - T)/3)$, $\mathcal{S}_3^-(\tau)$, $\mathcal{S}_1^+((\pi - T)/3)$, $\mathcal{S}_2^+(\tau)$, and $\mathcal{S}_3^+(\tau, \pi - 3\tau, 3\tau)$. (The other cases are considered analogously.) It suffices to determine $\nu(A)$, where A belongs to one of these sets.

If $A \subset \mathcal{S}_1^-(x_*)$ or $A \subset \mathcal{S}_3^+(\tau, \pi - 3\tau, 3\tau)$, we take $\nu(A) = \mu_1(P_1(A))$ or, equivalently, $\nu(A) = \mu_2(P_2(A))$. The set $\mathcal{S}_2^+(\tau) = L_1 \cup L_2$ is the union of two line segments

$$L_1 = \{(x, \pi - 3x) :\ (\pi - T)/3 \le x \le \tau\} \quad \text{and} \quad L_2 = \{(\pi - 3x, x) :\ (\pi - T)/3 \le x \le \tau\}$$

If $A \subset L_1 \cup \mathcal{S}_1^+((\pi - T)/3)$, we have $\nu(A) = \mu_1(P_1(A))$, and if $A \subset L_2$, $\nu(A) = \mu_2(P_2(A))$. Finally, if $A \subset \mathcal{S}_2^-((\pi - T)/3) \cup \mathcal{S}_3^-(\tau)$, the ν-measure of the set $\hat{A} = P_1^{-1}(P_1(A)) \setminus A \subset L_1 \cup \mathcal{S}_1^+((\pi - T)/3)$ is already defined, and we put $\nu(A) = \mu_1(P_1(A)) - \nu(\hat{A})$.

The proof of Theorem 5 is complete.

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