NUMERICAL ALGEBRA,
CONTROL AND OPTIMIZATION
Volume **3**, Number **1**, March **2013**

doi:10.3934/naco.2013.3.77

pp. **77–94**

HAHN'S SYMMETRIC QUANTUM VARIATIONAL CALCULUS

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Abstract. We introduce and develop the Hahn symmetric quantum calculus with applications to the calculus of variations. Namely, we obtain a necessary optimality condition of Euler–Lagrange type and a sufficient optimality condition for variational problems within the context of Hahn's symmetric calculus. Moreover, we show the effectiveness of Leitmann's direct method when applied to Hahn's symmetric variational calculus. Illustrative examples are provided.

1. Introduction.

Due to its many applications, quantum operators are recently subject to an increase number of investigations [24–26]. The use of quantum differential operators, instead of classical derivatives, is useful because they allow to deal with sets of nondifferentiable functions [4, 10]. Applications include several fields of physics, such as cosmic strings and black holes [27], quantum mechanics [12, 29], nuclear and high energy physics [18], just to mention a few. In particular, the q-symmetric quantum calculus has applications in quantum mechanics [17].

In 1949, Hahn introduced his quantum difference operator [13], which is a generalization of the quantum q-difference operator defined by Jackson [14]. However, only in 2009, Aldwoah [1] defined the inverse of Hahn's difference operator, and short after, Malinowska and Torres [24] introduced and investigated the Hahn quantum variational calculus. For a deep understanding of quantum calculus, we refer the reader to [2, 5, 6, 11, 15, 16] and references therein.

For a fixed $q \in]0,1[$ and an $\omega \geq 0$, we introduce here the Hahn symmetric difference operator of function f at point $t \neq \dfrac{\omega}{1-q}$ by

$$\tilde{D}_{q,\omega}\left[y\right](t) = \frac{f\left(qt+\omega\right) - f\left(q^{-1}\left(t-\omega\right)\right)}{\left(q-q^{-1}\right)t + \left(1+q^{-1}\right)\omega}.$$

2010 *Mathematics Subject Classification.* 39A13, 49K05.

Key words and phrases. Hahn's symmetric calculus, Euler–Lagrange difference equations, calculus of variations, Leitmann's principle, quantum calculus.

Part of first author's Ph.D., which is carried out at the University of Aveiro under the *Doctoral Programme in Mathematics and Applications* of Universities of Aveiro and Minho.

Our main aim is to establish a necessary optimality condition and a sufficient optimality condition for the Hahn symmetric variational problem

$$\begin{gathered}\mathcal{L}(y) = \int_a^b L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right) \tilde{d}_{q,\omega}t \longrightarrow \text{extremize} \\ y \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right) \\ y(a) = \alpha, \quad y(b) = \beta,\end{gathered} \tag{P}$$

where α and β are fixed real numbers, and extremize means maximize or minimize. Problem (P) will be clear and precise after definitions of Section 2. We assume that the Lagrangian L satisfies the following hypotheses:

(H1) $(u,v) \to L(t,u,v)$ is a $C^1\left(\mathbb{R}^2, \mathbb{R}\right)$ function for any $t \in I$;

(H2) $t \to L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right)$ is continuous at ω_0 for any admissible function y;

(H3) functions $t \to \partial_{i+2}L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right)$ belong to $\mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ for all admissible y, $i = 0, 1$;

where I is an interval of $\mathbb{R}$ containing $\omega_0 := \dfrac{\omega}{1-q}$, $a, b \in I$, $a < b$, and $\partial_j L$ denotes the partial derivative of L with respect to its jth argument.

In Section 2 we introduce the necessary definitions and prove some basic results for the Hahn symmetric calculus. In Section 3 we formulate and prove our main results for the Hahn symmetric variational calculus. New results include a necessary optimality condition (Theorem 3.8) and a sufficient optimality condition (Theorem 3.10) to problem (P). In Section 3.3 we show that Leitmann's direct method can also be applied to variational problems within Hahn's symmetric variational calculus. Leitmann introduced his direct method in the sixties of the 20th century [19], and the approach has recently proven to be universal: see, e.g., [3, 8, 9, 20–23, 28].

2. **Hahn's symmetric calculus.**

Let $q \in]0,1[$ and $\omega \geq 0$ be real fixed numbers. Throughout the text, we make the assumption that I is an interval (bounded or unbounded) of $\mathbb{R}$ containing $\omega_0 := \dfrac{\omega}{1-q}$. We denote by $I^{q,\omega}$ the set $I^{q,\omega} := qI + \omega := \{qt + \omega : t \in I\}$. Note that $I^{q,\omega} \subseteq I$ and, for all $t \in I^{q,\omega}$, one has $q^{-1}(t-\omega) \in I$. For $k \in \mathbb{N}_0$,

$$[k]_q := \frac{1-q^k}{1-q}.$$

Definition 2.1. Let f be a real function defined on I. The Hahn symmetric difference operator of f at a point $t \in I^{q,\omega} \setminus \{\omega_0\}$ is defined by

$$\tilde{D}_{q,\omega}[f](t) = \frac{f(qt+\omega) - f\left(q^{-1}(t-\omega)\right)}{\left(q - q^{-1}\right)t + \left(1 + q^{-1}\right)\omega},$$

while $\tilde{D}_{q,\omega}[f](\omega_0) := f'(\omega_0)$, provided f is differentiable at ω_0 (in the classical sense). We call to $\tilde{D}_{q,\omega}[f]$ the Hahn symmetric derivative of f.

Remark 1. If $\omega = 0$, then the Hahn symmetric difference operator $\tilde{D}_{q,\omega}$ coincides with the q-symmetric difference operator $\tilde{D}_q$: if $t \neq 0$, then

$$\tilde{D}_{q,0}[f](t) = \frac{f(qt) - f\left(q^{-1}t\right)}{\left(q-q^{-1}\right)t} =: \tilde{D}_q[f](t);$$

for $t = 0$ and f differentiable at 0, $\tilde{D}_{q,0}[f](0) = f'(0) =: \tilde{D}_q[f](0)$.

Remark 2. If $\omega > 0$ and we let $q \to 1$ in Definition 2.1, then we obtain the well known symmetric difference operator $\tilde{D}_\omega$:

$$\tilde{D}_\omega[f](t) = \frac{f(t+\omega) - f(t-\omega)}{2\omega}.$$

Remark 3. If f is differentiable at $t \in I^{q,\omega}$ in the classical sense, then

$$\lim_{(q,\omega)\to(1,0)} \tilde{D}_{q,\omega}[f](t) = f'(t).$$

In what follows we make use of the operator σ defined by $\sigma(t) := qt + \omega$, $t \in I$. Note that the inverse operator of σ, σ^{-1}, is defined by $\sigma^{-1}(t) := q^{-1}(t-\omega)$. Moreover, Aldwoah [1, Lemma 6.1.1] proved the following useful result.

Lemma 2.2 ([1]). *Let $k \in \mathbb{N}$ and $t \in I$. Then,*

1. $\sigma^k(t) = \underbrace{\sigma \circ \sigma \circ \cdots \circ \sigma}_{k \text{ times}}(t) = q^k t + \omega[k]_q$;
2. $\left(\sigma^k(t)\right)^{-1} = \sigma^{-k}(t) = q^{-k}\left(t - \omega[k]_q\right)$.

Furthermore, $\{\sigma^k(t)\}_{k=1}^{\infty}$ is a decreasing (resp. an increasing) sequence in k when $t > \omega_0$ (resp. $t < \omega_0$) with

$$\omega_0 = \inf_{k\in\mathbb{N}} \sigma^k(t) \quad \left(\text{resp. } \omega_0 = \sup_{k\in\mathbb{N}} \sigma^k(t)\right).$$

The sequence $\{\sigma^{-k}(t)\}_{k=1}^{\infty}$ is increasing (resp. decreasing) when $t > \omega_0$ (resp. $t < \omega_0$) with

$$\infty = \sup_{k\in\mathbb{N}} \sigma^{-k}(t) \quad \left(\text{resp. } -\infty = \inf_{k\in\mathbb{N}} \sigma^{-k}(t)\right).$$

For simplicity of notation, we write $f(\sigma(t)) := f^\sigma(t)$.

Remark 4. With above notations, if $t \in I^{q,\omega} \setminus \{\omega_0\}$, then the Hahn symmetric difference operator of f at point t can be written as

$$\tilde{D}_{q,\omega}[f](t) = \frac{f^\sigma(t) - f^{\sigma^{-1}}(t)}{\sigma(t) - \sigma^{-1}(t)}.$$

Lemma 2.3. *Let $n \in \mathbb{N}_0$ and $t \in I$. Then,*

$$\sigma^{n+1}(t) - \sigma^{n-1}(t) = q^n\left(\sigma(t) - \sigma^{-1}(t)\right),$$

where $\sigma^0 \equiv id$ is the identity function.

Proof. The equality follows by direct calculations:

$$\begin{aligned}\sigma^{n+1}(t) - \sigma^{n-1}(t) &= q^{n+1}t + \omega[n+1]_q - q^{n-1}t - \omega[n-1]_q \\ &= q^n\left(q - q^{-1}\right)t + \omega\left(q^n + q^{n-1}\right) \\ &= q^n\left(qt + \omega - q^{-1}t + q^{-1}\omega\right) \\ &= q^n\left(\sigma(t) - \sigma^{-1}(t)\right).\end{aligned}$$

□

The Hahn symmetric difference operator has the following properties.

Theorem 2.4. *Let $\alpha, \beta \in \mathbb{R}$ and $t \in I^{q,\omega}$. If f and g are Hahn symmetric differentiable on I, then*

1. $\tilde{D}_{q,\omega}\left[\alpha f+\beta g\right](t)=\alpha\tilde{D}_{q,\omega}\left[f\right](t)+\beta\tilde{D}_{q,\omega}\left[g\right](t)$;
2. $\tilde{D}_{q,\omega}\left[fg\right](t)=\tilde{D}_{q,\omega}\left[f\right](t)\,g^{\sigma}(t)+f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}\left[g\right](t)$;
3. $\tilde{D}_{q,\omega}\left[\frac{f}{g}\right](t)=\dfrac{\tilde{D}_{q,\omega}\left[f\right](t)\,g^{\sigma^{-1}}(t)-f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}\left[g\right](t)}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)}$ *if* $g^{\sigma}(t)\,g^{\sigma^{-1}}(t)\neq 0$;
4. $\tilde{D}_{q,\omega}\left[f\right]\equiv 0$ *if, and only if,* f *is constant on* I.

Proof. For $t=\omega_0$ the equalities are trivial (note that $\sigma(\omega_0)=\omega_0=\sigma^{-1}(\omega_0)$). We do the proof for $t\neq\omega_0$:

1.
$$\begin{aligned}\tilde{D}_{q,\omega}\left[\alpha f+\beta g\right](t)&=\frac{(\alpha f+\beta g)^{\sigma}(t)-(\alpha f+\beta g)^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\&=\alpha\frac{f^{\sigma}(t)-f^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}+\beta\frac{g^{\sigma}(t)-g^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\&=\alpha\tilde{D}_{q,\omega}\left[f\right](t)+\beta\tilde{D}_{q,\omega}\left[g\right](t).\end{aligned}$$

2.
$$\begin{aligned}\tilde{D}_{q,\omega}\left[fg\right](t)&=\frac{(fg)^{\sigma}(t)-(fg)^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\&=\frac{f^{\sigma}(t)-f^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}g^{\sigma}(t)+f^{\sigma^{-1}}(t)\,\frac{g^{\sigma}(t)-g^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\&=\tilde{D}_{q,\omega}\left[f\right](t)\,g^{\sigma}(t)+f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}\left[g\right](t).\end{aligned}$$

3. Because
$$\begin{aligned}\tilde{D}_{q,\omega}\left[\frac{1}{g}\right](t)&=\frac{\frac{1}{g^{\sigma}(t)}-\frac{1}{g^{\sigma^{-1}}(t)}}{\sigma(t)-\sigma^{-1}(t)}\\&=-\frac{1}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)}\frac{g^{\sigma}(t)-g^{\sigma^{-1}}(t)}{\sigma(t)-\sigma^{-1}(t)}\\&=-\frac{\tilde{D}_{q,\omega}\left[g\right](t)}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)},\end{aligned}$$
one has
$$\begin{aligned}\tilde{D}_{q,\omega}\left[\frac{f}{g}\right](t)&=\tilde{D}_{q,\omega}\left[f\frac{1}{g}\right](t)\\&=\tilde{D}_{q,\omega}\left[f\right](t)\,\frac{1}{g^{\sigma}(t)}+f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}\left[\frac{1}{g}\right](t)\\&=\frac{\tilde{D}_{q,\omega}\left[f\right](t)}{g^{\sigma}(t)}-f^{\sigma^{-1}}(t)\,\frac{\tilde{D}_{q,\omega}\left[g\right](t)}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)}\\&=\frac{\tilde{D}_{q,\omega}\left[f\right](t)\,g^{\sigma^{-1}}(t)-f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}\left[g\right](t)}{g^{\sigma}(t)\,g^{\sigma^{-1}}(t)}.\end{aligned}$$

4. If f is constant on I, then it is clear that $\tilde{D}_{q,\omega}\left[f\right]\equiv 0$. Suppose now that $\tilde{D}_{q,\omega}\left[f\right]\equiv 0$. Then, for each $t\in I$, $\left(\tilde{D}_{q,\omega}\left[f\right]\right)^{\sigma}(t)=0$ and, therefore, $f(t)=f^{\sigma^2}(t)$. Hence, $f(t)=f^{\sigma^2}(t)=\cdots=f^{\sigma^{2n}}(t)$ for each $n\in\mathbb{N}$ and $t\in I$. Because $\lim_{n\to+\infty}f(t)=\lim_{n\to+\infty}f^{\sigma^{2n}}(t)$, $\lim_{n\to+\infty}\sigma^{2n}(t)=\omega_0$ (by Lemma 2.2), and f is continuous at ω_0, then $f(t)=f(\omega_0)$ for all $t\in I$. □

Lemma 2.5. *For $t \in I$ one has $\tilde{D}_{q,\omega}[f^\sigma](t) = q\tilde{D}_{q,\omega}[f](\sigma(t))$.*

Proof. For each $t \in I \backslash \{\omega_0\}$ we have

$$\tilde{D}_{q,\omega}[f^\sigma](t) = \frac{f^{\sigma^2}(t) - f(t)}{\sigma(t) - \sigma^{-1}(t)}$$

and

$$\begin{aligned} \tilde{D}_{q,\omega}[f](\sigma(t)) &= \frac{f^{\sigma^2}(t) - f(t)}{\sigma^2(t) - t} \\ &= \frac{f^{\sigma^2}(t) - f(t)}{q(\sigma(t) - \sigma^{-1}(t))} \quad \text{(see Lemma 2.3)}. \end{aligned}$$

We conclude that $\tilde{D}_{q,\omega}[f^\sigma](t) = q\tilde{D}_{q,\omega}[f](\sigma(t))$. Finally, the intended result follows from the fact that $\tilde{D}_{q,\omega}[f^\sigma](\omega_0) = q\tilde{D}_{q,\omega}[f](\omega_0)$. □

Definition 2.6. Let $a, b \in I$ and $a < b$. For $f : I \to \mathbb{R}$ the Hahn symmetric integral of f from a to b is given by

$$\int_a^b f(t)\,\tilde{d}_{q,\omega}t = \int_{\omega_0}^b f(t)\,\tilde{d}_{q,\omega}t - \int_{\omega_0}^a f(t)\,\tilde{d}_{q,\omega}t,$$

where

$$\int_{\omega_0}^x f(t)\,\tilde{d}_{q,\omega}t = \left(\sigma^{-1}(x) - \sigma(x)\right)\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(x), \quad x \in I,$$

provided the series converges at $x = a$ and $x = b$. In that case, f is said to be Hahn symmetric integrable on $[a, b]$. We say that f is Hahn symmetric integrable on I if it is Hahn symmetric integrable over $[a, b]$ for all $a, b \in I$.

We now present two technical results that will be useful to prove the fundamental theorem of Hahn's symmetric integral calculus (Theorem 2.8).

Lemma 2.7 (cf. [1])**.** *Let $a, b \in I$, $a < b$. If $f : I \to \mathbb{R}$ is continuous at ω_0, then, for $s \in [a, b]$, the sequence $\left(f^{\sigma^{2n+1}}(s)\right)_{n\in\mathbb{N}}$ converges uniformly to $f(\omega_0)$ on I.*

The next result tell us that if a function f is continuous at ω_0, then f is Hahn's symmetric integrable.

Corollary 1 (cf. [1])**.** *Let $a, b \in I$, $a < b$, and $f : I \to \mathbb{R}$ be continuous at ω_0. Then, for $s \in [a, b]$, the series $\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(s)$ is uniformly convergent on I.*

Theorem 2.8 (Fundamental theorem of the Hahn symmetric integral calculus)**.** *Assume that $f : I \to \mathbb{R}$ is continuous at ω_0 and, for each $x \in I$, define*

$$F(x) := \int_{\omega_0}^x f(t)\,\tilde{d}_{q,\omega}t.$$

Then F is continuous at ω_0. Furthermore, $\tilde{D}_{q,\omega}[F](x)$ exists for every $x \in I^{q,\omega}$ with $\tilde{D}_{q,\omega}[F](x) = f(x)$. Conversely,

$$\int_a^b \tilde{D}_{q,\omega}[f](t)\,\tilde{d}_{q,\omega}t = f(b) - f(a)$$

for all $a, b \in I$.

Proof. We note that function F is continuous at ω_0 by Corollary 1. Let us begin by considering $x \in I^{q,\omega}\backslash\{\omega_0\}$. Then,

$$\begin{aligned}
&\tilde{D}_{q,\omega}\left[\tau \mapsto \int_0^\tau f(t)\,\tilde{d}_{q,\omega}t\right](x)\\
&= \frac{\int_{\omega_0}^{\sigma(x)} f(t)\,\tilde{d}_{q,\omega}t - \int_{\omega_0}^{\sigma^{-1}(x)} f(t)\,\tilde{d}_{q,\omega}t}{\sigma(x)-\sigma^{-1}(x)}\\
&= \frac{1}{\sigma(x)-\sigma^{-1}(x)}\Bigg\{\left[\sigma^{-1}(\sigma(x)) - \sigma(\sigma(x))\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\sigma(x))\\
&\qquad - \left[\sigma^{-1}\left(\sigma^{-1}(x)\right) - \sigma\left(\sigma^{-1}(x)\right)\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}\left(\sigma^{-1}(x)\right)\Bigg\}\\
&= \sum_{n=0}^{+\infty} q^{2n} f^{\sigma^{2n}}(x) - \sum_{n=0}^{+\infty} q^{2n+2} f^{\sigma^{2n+2}}(x)\\
&= \sum_{n=0}^{+\infty}\left(q^{2n} f^{\sigma^{2n}}(x) - q^{2(n+1)} f^{\sigma^{2(n+1)}}(x)\right)\\
&= f(x).
\end{aligned}$$

If $x = \omega_0$, then

$$\begin{aligned}
&\tilde{D}_{q,\omega}[F](\omega_0)\\
&= \lim_{h\to 0}\frac{F(\omega_0+h)-F(\omega_0)}{h}\\
&= \lim_{h\to 0}\frac{1}{h}\left[\sigma^{-1}(\omega_0+h) - \sigma(\omega_0+h)\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0+h)\\
&= \lim_{h\to 0}\frac{1}{h}\left[q^{-1}(\omega_0+h-\omega) - q(\omega_0+h) - \omega\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0+h)\\
&= \lim_{h\to 0}\frac{1}{h}\left[\left(q^{-1}-q\right)\omega_0 + \left(-q^{-1}-1\right)\omega + \left(q^{-1}-q\right)h\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0+h)\\
&= \lim_{h\to 0}\frac{1}{h}\left[\frac{\left(q^{-1}-q\right)\omega}{1-q} + \left(-q^{-1}-1\right)\omega + \left(q^{-1}-q\right)h\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0+h)\\
&= \lim_{h\to 0}\frac{1}{h}\left[\left(\frac{1+q}{q} + \frac{-1-q}{q}\right)\omega + \left(q^{-1}-q\right)h\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0+h)\\
&= \lim_{h\to 0}\frac{1-q^2}{q}\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(\omega_0+h)\\
&= \left(1-q^2\right)\sum_{n=0}^{+\infty} q^{2n} f(\omega_0)\\
&= \left(1-q^2\right)\frac{1}{1-q^2} f(\omega_0)\\
&= f(\omega_0).
\end{aligned}$$

Finally, since for $x \in I \backslash \{\omega_0\}$ we have

$$\begin{aligned}\int_{\omega_0}^{x} \tilde{D}_{q,\omega}[f](t)\tilde{d}_{q,\omega}t &= \left[\sigma^{-1}(x) - \sigma(x)\right] \sum_{n=0}^{+\infty} q^{2n+1} \tilde{D}_{q,\omega}[f]^{\sigma^{2n+1}}(x) \\ &= \left[\sigma^{-1}(x) - \sigma(x)\right] \sum_{n=0}^{+\infty} q^{2n+1} \frac{f^{\sigma}\left(\sigma^{2n+1}(x)\right) - f^{\sigma^{-1}}\left(\sigma^{2n+1}(x)\right)}{\sigma\left(\sigma^{2n+1}(x)\right) - \sigma^{-1}\left(\sigma^{2n+1}(x)\right)} \\ &= \left[\sigma^{-1}(x) - \sigma(x)\right] \sum_{n=0}^{+\infty} q^{2n+1} \frac{f^{\sigma}\left(\sigma^{2n+1}(x)\right) - f^{\sigma^{-1}}\left(\sigma^{2n+1}(x)\right)}{q^{2n+1}\left(\sigma(x) - \sigma^{-1}(x)\right)} \\ &= \sum_{n=0}^{+\infty} \left[f^{\sigma^{2n}}(x) - f^{\sigma^{2(n+1)}}(x)\right] \\ &= f(x) - f(\omega_0),\end{aligned}$$

where in the third equality we use Lemma 2.3, then

$$\begin{aligned}\int_a^b \tilde{D}_{q,\omega}[f](t)\,\tilde{d}_{q,\omega}t &= \int_{\omega_0}^b \tilde{D}_{q,\omega}[f](t)\,\tilde{d}_{q,\omega}t - \int_{\omega_0}^a \tilde{D}_{q,\omega}[f](t)\,\tilde{d}_{q,\omega}t \\ &= f(b) - f(a).\end{aligned}$$

□

The Hahn symmetric integral has the following properties.

Theorem 2.9. *Let $f, g : I \to \mathbb{R}$ be Hahn's symmetric integrable on I, $a, b, c \in I$, and $\alpha, \beta \in \mathbb{R}$. Then,*

1. $\int_a^a f(t)\,\tilde{d}_{q,\omega}t = 0$;
2. $\int_a^b f(t)\,\tilde{d}_{q,\omega}t = -\int_b^a f(t)\,\tilde{d}_{q,\omega}t$;
3. $\int_a^b f(t)\,\tilde{d}_{q,\omega}t = \int_a^c f(t)\,\tilde{d}_{q,\omega}t + \int_c^b f(t)\,\tilde{d}_{q,\omega}t$;
4. $\int_a^b (\alpha f + \beta g)(t)\,\tilde{d}_{q,\omega}t = \alpha \int_a^b f(t)\,\tilde{d}_{q,\omega}t + \beta \int_a^b g(t)\,\tilde{d}_{q,\omega}t$;
5. *if $\tilde{D}_{q,\omega}[f]$ and $\tilde{D}_{q,\omega}[g]$ are continuous at ω_0, then*

$$\int_a^b f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t)\,\tilde{d}_{q,\omega}t = f(t)\,g(t)\Big|_a^b - \int_a^b \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t)\,\tilde{d}_{q,\omega}t. \tag{1}$$

Proof. Properties 1 to 4 are trivial. Property 5 follows from Theorem 2.4 and Theorem 2.8: since

$$\tilde{D}_{q,\omega}[fg](t) = \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t) + f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t),$$

then

$$f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t) = \tilde{D}_{q,\omega}[fg](t) - \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t)$$

and hence,

$$\int_a^b f^{\sigma^{-1}}(t)\,\tilde{D}_{q,\omega}[g](t)\,\tilde{d}_{q,\omega}t = f(t)\,g(t)\Big|_a^b - \int_a^b \tilde{D}_{q,\omega}[f](t)\,g^{\sigma}(t)\,\tilde{d}_{q,\omega}t.$$

□

Remark 5. Relation (1) gives a Hahn's symmetric integration by parts formula.

Remark 6. Using Lemma 2.5 and the Hahn symmetric integration by parts formula (1), we conclude that

$$\int_a^b f(t)\,\tilde{D}_{q,\omega}[g](t)\,\tilde{d}_{q,\omega}t = f^{\sigma}(t)\,g(t)\Big|_a^b - q\int_a^b \left(\tilde{D}_{q,\omega}[f]\right)^{\sigma}(t)\,g^{\sigma}(t)\,\tilde{d}_{q,\omega}t. \qquad (2)$$

Proposition 1. *Let $c \in I$, f and g be Hahn's symmetric integrable on I. Suppose that $|f(t)| \le g(t)$ for all $t \in \left\{\sigma^{2n+1}(c) : n \in \mathbb{N}_0\right\} \cup \{\omega_0\}$.*

1. *If $c \ge \omega_0$, then*
$$\left|\int_{\omega_0}^{c} f(t)\,\tilde{d}_{q,\omega}t\right| \le \int_{\omega_0}^{c} g(t)\,\tilde{d}_{q,\omega}t.$$
2. *If $c < \omega_0$, then*
$$\left|\int_{c}^{\omega_0} f(t)\,\tilde{d}_{q,\omega}t\right| \le \int_{c}^{\omega_0} g(t)\,\tilde{d}_{q,\omega}t.$$

Proof. If $c \ge \omega_0$, then

$$\begin{aligned}
\left|\int_{\omega_0}^{c} f(t)\,\tilde{d}_{q,\omega}t\right| &= \left|\left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(c)\right| \\
&\le \left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1}\left|f^{\sigma^{2n+1}}(c)\right| \\
&\le \left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1} g^{\sigma^{2n+1}}(c) \\
&= \int_{\omega_0}^{c} g(t)\,\tilde{d}_{q,\omega}t.
\end{aligned}$$

If $c < \omega_0$, then

$$\begin{aligned}
\left|\int_{c}^{\omega_0} f(t)\,\tilde{d}_{q,\omega}t\right| &= \left|-\left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(c)\right| \\
&\le \left|\sigma^{-1}(c) - \sigma(c)\right|\sum_{n=0}^{+\infty} q^{2n+1}\left|f^{\sigma^{2n+1}}(c)\right| \\
&= -\left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1}\left|f^{\sigma^{2n+1}}(c)\right| \\
&\le -\left[\sigma^{-1}(c) - \sigma(c)\right]\sum_{n=0}^{+\infty} q^{2n+1} g^{\sigma^{2n+1}}(c) \\
&= -\int_{\omega_0}^{c} g(t)\,\tilde{d}_{q,\omega}t, \\
&= \int_{c}^{\omega_0} g(t)\,\tilde{d}_{q,\omega}t
\end{aligned}$$

providing the desired equality. □

As an immediate consequence, we have the following result.

Corollary 2. *Let $c \in I$ and f be Hahn's symmetric integrable on I. Suppose that $f(t) \ge 0$ for all $t \in \left\{\sigma^{2n+1}(c) : n \in \mathbb{N}_0\right\} \cup \{\omega_0\}$.*

1. *If* $c \geq \omega_0$, *then*
$$\int_{\omega_0}^{c} f(t)\, \tilde{d}_{q,\omega} t \geq 0.$$
2. *If* $c < \omega_0$, *then*
$$\int_{c}^{\omega_0} f(t)\, \tilde{d}_{q,\omega} t \geq 0.$$

Remark 7. In general it is not true that if f is a nonnegative function on $[a, b]$, then
$$\int_{a}^{b} f(t)\, \tilde{d}_{q,\omega} t \geq 0.$$
As an example, consider the function f defined in $[-5, 5]$ by
$$f(x) = \begin{cases} 6 & \text{if} \quad t = 3 \\ 1 & \text{if} \quad t = 4 \\ 0 & \text{if} \quad t \in [-5, 5] \setminus \{3, 4\}. \end{cases}$$
For $q = \frac{1}{2}$ and $\omega = 1$, this function is Hahn's symmetric integrable because is continuous at $\omega_0 = 2$. However,
$$\begin{aligned} \int_{4}^{6} f(t)\, \tilde{d}_{q,\omega} t &= \int_{2}^{6} f(t)\, \tilde{d}_{q,\omega} t - \int_{2}^{4} f(t)\, \tilde{d}_{q,\omega} t \\ &= (10-4) \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^{2n+1} f^{\sigma^{2n+1}}(6) - (6-3) \sum_{n=0}^{+\infty} \left(\frac{1}{2}\right)^{2n+1} f^{\sigma^{2n+1}}(4) \\ &= 6\left(\frac{1}{2}\right) \times 1 - 3\left(\frac{1}{2}\right) \times 6 \\ &= -6. \end{aligned}$$
This example also proves that, in general, it is not true that
$$\left| \int_{a}^{b} f(t)\, \tilde{d}_{q,\omega} t \right| \leq \int_{a}^{b} |f(t)|\, \tilde{d}_{q,\omega} t$$
for any $a, b \in I$.

3. **Hahn's symmetric variational calculus.**

We begin this section with some useful definitions and notations. For $s \in I$ we set
$$[s]_{q,\omega} := \left\{\sigma^{2n+1}(s) : n \in \mathbb{N}_0\right\} \cup \{\omega_0\}.$$
Let $a, b \in I$ with $a < b$. We define the Hahn symmetric interval from a to b by
$$[a, b]_{q,\omega} := \left\{\sigma^{2n+1}(a) : n \in \mathbb{N}_0\right\} \cup \left\{\sigma^{2n+1}(b) : n \in \mathbb{N}_0\right\} \cup \{\omega_0\},$$
that is,
$$[a, b]_{q,\omega} = [a]_{q,\omega} \cup [b]_{q,\omega}.$$
Let $r \in \{0, 1\}$. We denote the linear space
$$\left\{ y : I \to \mathbb{R} \mid \tilde{D}^{i}_{q,\omega}[y],\, i = 0, r, \text{ are bounded on } [a, b]_{q,\omega} \text{ and continuous at } \omega_0 \right\}$$
endowed with the norm
$$\|y\|_r = \sum_{i=0}^{r} \sup_{t \in [a,b]_{q,\omega}} \left| \tilde{D}^{i}_{q,\omega}[y](t) \right|,$$
where $\tilde{D}^{0}_{q,\omega}[y] = y$, by $\mathcal{Y}^r\left([a, b]_{q,\omega}, \mathbb{R}\right)$.

Definition 3.1. We say that y is an admissible function to problem (P) if $y \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ and y satisfies the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$.

Definition 3.2. We say that y_* is a local minimizer (resp. local maximizer) to problem (P) if y_* is an admissible function and there exists $\delta > 0$ such that

$$\mathcal{L}(y_*) \leq \mathcal{L}(y) \qquad (\text{resp. } \mathcal{L}(y_*) \geq \mathcal{L}(y)\,)$$

for all admissible y with $\|y_* - y\|_1 < \delta$.

Definition 3.3. We say that $\eta \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ is an admissible variation to problem (P) if $\eta(a) = 0 = \eta(b)$.

Before proving our main results, we begin with three basic lemmas.

3.1. **Basic Lemmas.** The following results are useful to prove Theorem 3.8.

Lemma 3.4 (Fundamental lemma of the Hahn symmetric variational calculus)**.** *Let $f \in \mathcal{Y}^0\left([a,b]_{q,\omega}, \mathbb{R}\right)$. One has*

$$\int_a^b f(t)\, h^{\sigma}(t)\, \tilde{d}_{q,\omega} t = 0$$

for all $h \in \mathcal{Y}^0\left([a,b]_{q,\omega}, \mathbb{R}\right)$ with $h(a) = h(b) = 0$ if, and only if, $f(t) = 0$ for all $t \in [a,b]_{q,\omega}$.

Proof. The implication "$\Leftarrow$" is obvious. Let us prove the implication "$\Rightarrow$". Suppose, by contradiction, that exists $p \in [a,b]_{q,\omega}$ such that $f(p) \neq 0$.

1. If $p \neq \omega_0$, then $p = \sigma^{2k+1}(a)$ or $p = \sigma^{2k+1}(b)$ for some $k \in \mathbb{N}_0$.
   (a) Suppose that $a \neq \omega_0$ and $b \neq \omega_0$. In this case we can assume, without loss of generality, that $p = \sigma^{2k+1}(a)$. Define

   $$h(t) = \begin{cases} f^{\sigma^{2k+1}}(a) & \text{if } t = \sigma^{2k+2}(a) \\ 0 & \text{otherwise.} \end{cases}$$

   Then,

   $$\begin{aligned} &\int_a^b f(t)\, h^{\sigma}(t)\, \tilde{d}_{q,\omega} t \\ &\quad = \left[\sigma^{-1}(b) - \sigma(b)\right] \sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(b)\, h^{\sigma^{2n+2}}(b) \\ &\qquad - \left[\sigma^{-1}(a) - \sigma(a)\right] \sum_{n=0}^{+\infty} q^{2n+1} f^{\sigma^{2n+1}}(a)\, h^{\sigma^{2n+2}}(a) \\ &\quad = -\left[\sigma^{-1}(a) - \sigma(a)\right] q^{2k+1} \left[f^{\sigma^{2k+1}}(a)\right]^2 \neq 0, \end{aligned}$$

   which is a contradiction.
   (b) Suppose that $a \neq \omega_0$ and $b = \omega_0$. Therefore, $p = \sigma^{2k+1}(a)$ for some $k \in \mathbb{N}_0$. Define

   $$h(t) = \begin{cases} f^{\sigma^{2k+1}}(a) & \text{if } t = \sigma^{2k+2}(a) \\ 0 & \text{otherwise.} \end{cases}$$

   We obtain a contradiction with a similar proof as in case (a).

(c) The case $a = \omega_0$ and $b \neq \omega_0$ is similar to (b).

2. If $p = \omega_0$, we assume, without loss of generality, that $f(p) > 0$. Since

$$\lim_{n\to+\infty} \sigma^{2n+2}(a) = \lim_{n\to+\infty} \sigma^{2n+2}(b) = \omega_0$$

and f is continuous at ω_0,

$$\lim_{n\to+\infty} f^{\sigma^{2n+1}}(a) = \lim_{n\to+\infty} f^{\sigma^{2n+1}}(b) = f(\omega_0).$$

Therefore, there exists an order $n_0 \in \mathbb{N}$ such for all $n > n_0$ the inequalities

$$f^{\sigma^{2n+1}}(a) > 0 \quad\text{and}\quad f^{\sigma^{2n+1}}(b) > 0$$

hold.

(a) If $a, b \neq \omega_0$, then for some $k > n_0$ we define

$$h(t) = \begin{cases} -\frac{f^{\sigma^{2k+1}}(b)}{\sigma^{-1}(a)-\sigma(a)} & \text{if } t = \sigma^{2k+2}(a) \\ \frac{f^{\sigma^{2k+1}}(a)}{\sigma^{-1}(b)-\sigma(b)} & \text{if } t = \sigma^{2k+2}(b) \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\int_a^b f(t)\, h^{\sigma}(t)\, \tilde{d}_{q,\omega}t = 2q^{2k+1} f^{\sigma^{2k+1}}(a)\, f^{\sigma^{2k+1}}(b) > 0.$$

(b) If $a = \omega_0$, then we define

$$h(t) = \begin{cases} f^{\sigma^{2k+1}}(b) & \text{if } t = \sigma^{2k+2}(b) \\ 0 & \text{otherwise.} \end{cases}$$

Therefore,

$$\int_{\omega_0}^b f(t)\, h^{\sigma}(t)\, \tilde{d}_{q,\omega}t = \left[\sigma^{-1}(b) - \sigma(b)\right] q^{2k+1} \left[f^{\sigma^{2k+1}}(b)\right]^2 \neq 0.$$

(c) If $b = \omega_0$, the proof is similar to the previous case.

□

Definition 3.5. Let $s \in I$ and $g : I \times \left]-\bar{\theta}, \bar{\theta}\right[\to \mathbb{R}$. We say that $g(t, \cdot)$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$0 < |\theta - \theta_0| < \delta \Rightarrow \left| \frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t, \theta_0) \right| < \varepsilon$$

for all $t \in [s]_{q,\omega}$, where $\partial_2 g = \frac{\partial g}{\partial \theta}$.

Lemma 3.6 (cf. [24]). *Let $s \in I$ and assume that $g : I \times \left]-\bar{\theta}, \bar{\theta}\right[\to \mathbb{R}$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$. If $\int_{\omega_0}^s g(t, \theta_0)\, \tilde{d}_{q,\omega}t$ exists, then $G(\theta) := \int_{\omega_0}^s g(t, \theta)\, \tilde{d}_{q,\omega}t$ for θ near θ_0, is differentiable at θ_0 with*

$$G'(\theta_0) = \int_{\omega_0}^s \partial_2 g(t, \theta_0)\, \tilde{d}_{q,\omega}t.$$

Proof. For $s > \omega_0$ the proof is similar to the proof given in Lemma 3.2 of [24]. The result is trivial for $s = \omega_0$. Suppose that $s < \omega_0$ and let $\varepsilon > 0$ be arbitrary. Since $g(t, \cdot)$ is differentiable at θ_0 uniformly in $[s]_{q,\omega}$, then there exists $\delta > 0$ such that for all $t \in [s]_{q,\omega}$ and for $0 < |\theta - \theta_0| < \delta$ the following inequality holds:

$$\left|\frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0)\right| < \frac{\varepsilon}{2(\omega_0 - s)}. \tag{3}$$

Since, for $0 < |\theta - \theta_0| < \delta$, we have

$$\begin{aligned}
&\left|\frac{G(\theta) - G(\theta_0)}{\theta - \theta_0} - \int_{\omega_0}^{s} \partial_2 g(t,\theta_0)\, \tilde{d}_{q,\omega} t\right| \\
&= \left|\frac{\int_{\omega_0}^{s} g(t,\theta)\, \tilde{d}_{q,\omega} t - \int_{\omega_0}^{s} g(t,\theta_0)\, \tilde{d}_{q,\omega} t}{\theta - \theta_0} - \int_{\omega_0}^{s} \partial_2 g(t,\theta_0)\, \tilde{d}_{q,\omega} t\right| \\
&= \left|\int_{\omega_0}^{s} \left[\frac{g(t,\theta) - g(t,\theta_0)}{\theta - \theta_0} - \partial_2 g(t,\theta_0)\right] \tilde{d}_{q,\omega} t\right| \\
&< \int_{s}^{\omega_0} \frac{\varepsilon}{2(\omega_0 - s)} \tilde{d}_{q,\omega} t \quad \text{(using Proposition 1 and inequality (3))} \\
&= \frac{\varepsilon}{2(\omega_0 - s)} \int_{s}^{\omega_0} 1 \tilde{d}_{q,\omega} t = \frac{\varepsilon}{2} < \varepsilon
\end{aligned}$$

then we can conclude that $G'(\theta) = \displaystyle\int_{\omega_0}^{s} \partial_2 g(t,\theta_0)\, \tilde{d}_{q,\omega} t.$ □

For an admissible variation η and an admissible function y, we define $\phi :]-\bar{\epsilon}, \bar{\epsilon}[\to \mathbb{R}$ by $\phi(\epsilon) := \mathcal{L}(y + \epsilon\eta)$. The first variation of functional $\mathcal{L}$ of problem (P) is defined by $\delta\mathcal{L}(y,\eta) := \phi'(0)$. Note that

$$\begin{aligned}
\mathcal{L}(y + \epsilon\eta) &= \int_a^b L\left(t, y^\sigma(t) + \epsilon\eta^\sigma(t), \tilde{D}_{q,\omega}[y](t) + \epsilon\tilde{D}_{q,\omega}[\eta](t)\right) \tilde{d}_{q,\omega} t \\
&= \mathcal{L}_b(y + \epsilon\eta) - \mathcal{L}_a(y + \epsilon\eta),
\end{aligned}$$

where

$$\mathcal{L}_\xi(y + \epsilon\eta) = \int_{\omega_0}^{\xi} L\left(t, y^\sigma(t) + \epsilon\eta^\sigma(t), \tilde{D}_{q,\omega}[y](t) + \epsilon\tilde{D}_{q,\omega}[\eta](t)\right) \tilde{d}_{q,\omega} t$$

with $\xi \in \{a, b\}$. Therefore, $\delta\mathcal{L}(y,\eta) = \delta\mathcal{L}_b(y,\eta) - \delta\mathcal{L}_a(y,\eta)$.

The following lemma is a direct consequence of Lemma 3.6.

Lemma 3.7. *For an admissible variation η and an admissible function y, let*

$$g(t,\epsilon) := L\left(t, y^\sigma(t) + \epsilon\eta^\sigma(t), \tilde{D}_{q,\omega}[y](t) + \epsilon\tilde{D}_{q,\omega}[\eta](t)\right).$$

Assume that

1. *$g(t, \cdot)$ is differentiable at ω_0 uniformly in $[a,b]_{q,\omega}$;*
2. *$\mathcal{L}_\xi(y + \epsilon\eta) = \int_{\omega_0}^{\xi} g(t,\epsilon)\, \tilde{d}_{q,\omega} t$, $\xi \in \{a,b\}$, exist for $\epsilon \approx 0$;*
3. *$\int_{\omega_0}^{a} \partial_2 g(t,0)\, \tilde{d}_{q,\omega} t$ and $\int_{\omega_0}^{b} \partial_2 g(t,0)\, \tilde{d}_{q,\omega} t$ exist.*

Then,

$$\phi'(0) := \delta\mathcal{L}(y,\eta) = \int_a^b \Big[\partial_2 L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right)\eta^\sigma(t) + \partial_3 L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right)\tilde{D}_{q,\omega}[\eta](t)\Big]\tilde{d}_{q,\omega}t.$$

3.2. **Optimality Conditions.** In this section we present a necessary optimality condition (the Hanh symmetric Euler–Lagrange equation) and a sufficient optimality condition to problem (P).

Theorem 3.8 (The Hahn symmetric Euler–Lagrange equation)**.** *Under hypotheses (H1)–(H3) and conditions 1 to 3 of Lemma 3.7 on the Lagrangian L, if $y_* \in \mathcal{Y}^1\left([a,b]_{q,\omega}, \mathbb{R}\right)$ is a local extremizer to problem (P), then y_* satisfies the Hahn symmetric Euler–Lagrange equation*

$$\partial_2 L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right) = \tilde{D}_{q,\omega}\left[\tau \mapsto \partial_3 L\left(\sigma(\tau), y^{\sigma^2}(\tau), \left(\tilde{D}_{q,\omega}[y]\right)^\sigma(\tau)\right)\right](t) \tag{4}$$

for all $t \in [a,b]_{q,\omega}$.

Proof. Let y_* be a local minimizer (resp. maximizer) to problem (P) and η an admissible variation. Define $\phi : \mathbb{R} \to \mathbb{R}$ by $\phi(\epsilon) := \mathcal{L}(y_* + \epsilon\eta)$. A necessary condition for y_* to be an extremizer is given by $\phi'(0) = 0$. By Lemma 3.7,

$$\int_a^b \Big[\partial_2 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right)\eta^\sigma(t) + \partial_3 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right)\tilde{D}_{q,\omega}[\eta](t)\Big]\tilde{d}_{q,\omega}t = 0.$$

Using the integration by parts formula (2), we get

$$\begin{aligned}
&\int_a^b \partial_3 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right)\tilde{D}_{q,\omega}[\eta](t)\,\tilde{d}_{q,\omega}t \\
&\quad = \partial_3 L\left(\sigma(t), y_*^{\sigma^2}(t), \left(\tilde{D}_{q,\omega}[y_*]\right)^\sigma(t)\right)\eta(t)\Big|_a^b \\
&\qquad - q\int_a^b \left(\tilde{D}_{q,\omega}\left[\tau \mapsto \partial_3 L\left(\tau, y_*^\sigma(\tau), \left(\tilde{D}_{q,\omega}[y_*]\right)(\tau)\right)\right]\right)^\sigma(t)\,\eta^\sigma(t)\,\tilde{d}_{q,\omega}t.
\end{aligned}$$

Since $\eta(a) = \eta(b) = 0$, then

$$\int_a^b \Big[\partial_2 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right) - q\left(\tilde{D}_{q,\omega}\left[\tau \mapsto \partial_3 L\left(\tau, y_*^\sigma(\tau), \left(\tilde{D}_{q,\omega}[y_*]\right)(\tau)\right)\right]\right)^\sigma(t)\Big]\eta^\sigma(t)\,\tilde{d}_{q,\omega}t = 0$$

and by Lemma 3.4 we get

$$\partial_2 L\left(t, y_*^\sigma(t), \tilde{D}_{q,\omega}[y_*](t)\right) = q\left(\tilde{D}_{q,\omega}\left[\tau \mapsto \partial_3 L\left(\tau, y_*^\sigma(\tau), \tilde{D}_{q,\omega}[y_*](\tau)\right)\right]\right)^\sigma(t)$$

for all $t \in [a,b]_{q,\omega}$. Finally, using Lemma 2.5, we conclude that

$$\begin{aligned}\partial_2 L\left(t, y_*^{\sigma}(t), \tilde{D}_{q,\omega}[y_*](t)\right)& \\ &= \tilde{D}_{q,\omega}\left[\tau \mapsto \partial_3 L\left(\sigma(\tau), y_*^{\sigma^2}(\tau), \left(\tilde{D}_{q,\omega}[y_*]\right)^{\sigma}(\tau)\right)\right](t).\end{aligned}$$

□

The particular case $\omega = 0$ gives the q-symmetric Euler–Lagrange equation.

Corollary 3 (The q-symmetric Euler–Lagrange equation [7])**.** *Let $\omega = 0$. Under hypotheses (H1)–(H3) and conditions 1 to 3 of Lemma 3.7 on the Lagrangian L, if $y_* \in \mathcal{Y}^1\left([a,b]_{q,0}, \mathbb{R}\right)$ is a local extremizer to problem (P) (with $\omega = 0$), then y_* satisfies the q-symmetric Euler–Lagrange equation*

$$\partial_2 L\left(t, y(qt), \tilde{D}_q[y](t)\right) = \tilde{D}_q\left[\tau \mapsto \partial_3 L\left(q\tau, y\left(q^2\tau\right), \tilde{D}_q[y](q\tau)\right)\right](t)$$

for all $t \in [a,b]_{q,0}$.

To conclude this section, we prove a sufficient optimality condition to (P).

Definition 3.9. Given a Lagrangian L, we say that $L(t,u,v)$ is jointly convex (resp. concave) in (u,v) if, and only if, $\partial_i L$, $i = 2,3$, exist and are continuous and verify the following condition:

$$L(t, u+u_1, v+v_1) - L(t,u,v) \geq \text{ (resp. } \leq) \ \partial_2 L(t,u,v)\, u_1 + \partial_3 L(t,u,v)\, v_1$$

for all $(t,u,v), (t, u+u_1, v+v_1) \in I \times \mathbb{R}^2$.

Theorem 3.10. *Suppose that $a < b$ and $a, b \in [c]_{q,\omega}$ for some $c \in I$. Also, assume that L is a jointly convex (resp. concave) function in (u,v). If y_* satisfies the Hahn symmetric Euler–Lagrange equation (4), then y_* is a global minimizer (resp. maximizer) to problem (P).*

Proof. Let L be a jointly convex function in (u,v) (the concave case is similar). Then, for any admissible variation η, we have

$$\begin{aligned}&\mathcal{L}(y_* + \eta) - \mathcal{L}(y_*) \\ &\quad= \int_a^b \Bigg(L\left(t, y^{\sigma}(t) + \eta^{\sigma}(t), \tilde{D}_{q,\omega}[y](t) + \tilde{D}_{q,\omega}[\eta](t)\right) \\ &\qquad\qquad - L\left(t, y^{\sigma}(t), \tilde{D}_{q,\omega}[y](t)\right)\Bigg)\tilde{d}_{q,\omega}t \\ &\quad\geq \int_a^b \Bigg(\partial_2 L\left(t, y^{\sigma}(t), \tilde{D}_{q,\omega}[y](t)\right)\eta^{\sigma}(t) \\ &\qquad\qquad + \partial_3 L\left(t, y^{\sigma}(t), \tilde{D}_{q,\omega}[y](t)\right)\tilde{D}_{q,\omega}[\eta](t)\Bigg)\tilde{d}_{q,\omega}t.\end{aligned}$$

Using the integration by parts formula (2) and Lemma 2.5, we get

$$\begin{aligned}\mathcal{L}(y_*+\eta)-\mathcal{L}(y_*) &\geq \partial_3 L\left(\sigma(t), y_*^{\sigma^2}(t), \left(\tilde{D}_{q,\omega}[y]\right)^{\sigma}(t)\right)\eta(t)\Big|_a^b \\ &+\int_a^b\Big[\partial_2 L\left(t, y_*^{\sigma}(t), \tilde{D}_{q,\omega}[y_*](t)\right) \\ &-\tilde{D}_{q,\omega}\left[\tau\mapsto\partial_3 L\left(\sigma(\tau), y^{\sigma^2}(\tau), \left(\tilde{D}_{q,\omega}[y]\right)^{\sigma}(\tau)\right)(t)\right]\Big]\eta^{\sigma}(t)\,\tilde{d}_{q,\omega}t.\end{aligned}$$

Since y_* satisfies (4) and η is an admissible variation, we obtain

$$\mathcal{L}(y_*+\eta)-\mathcal{L}(y_*)\geq 0,$$

proving that y_* is a minimizer to problem (P). □

Example 1. Let $q\in\,]0,1[$ and $\omega\geq 0$ be fixed real numbers. Also, let $I\subseteq\mathbb{R}$ be an interval such that $a:=\omega_0, b\in I$ and $a<b$. Consider the problem

$$\begin{gathered}\mathcal{L}(y)=\int_a^b\sqrt{1+\left(\tilde{D}_{q,\omega}[y](t)\right)^2}\,\tilde{d}_{q,\omega}t\longrightarrow\min \\ y\in\mathcal{Y}^1\left([a,b]_{q,\omega},\mathbb{R}\right) \\ y(a)=a,\quad y(b)=b.\end{gathered}\tag{5}$$

If y_* is a local minimizer to the problem, then y_* satisfies the Hahn symmetric Euler–Lagrange equation

$$\tilde{D}_{q,\omega}\left[\tau\mapsto\frac{\left(\tilde{D}_{q,\omega}[y]\right)^{\sigma}(\tau)}{\sqrt{1+\left(\left(\tilde{D}_{q,\omega}[y]\right)^{\sigma}(\tau)\right)^2}}\right](t)=0\quad\text{for all}\quad t\in[a,b]_{q,\omega}.\tag{6}$$

It is simple to check that function $y_*(t)=t$ is a solution to (6) satisfying the given boundary conditions. Since the Lagrangian is jointly convex in (u,v), then we conclude from Theorem 3.10 that function $y_*(t)=t$ is indeed a minimizer to problem (5).

3.3. **Leitmann's Direct Method.** Similarly to Malinowska and Torres [24], we show that Leitmann's direct method [19] has also applications in the Hahn symmetric variational calculus. Consider the variational functional integral

$$\bar{\mathcal{L}}(\bar{y})=\int_a^b\bar{L}\left(t,\bar{y}^{\sigma}(t),\tilde{D}_{q,\omega}[\bar{y}](t)\right)\tilde{d}_{q,\omega}t.$$

As before, we assume that function $\bar{L}: I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies the following hypotheses:

($\overline{\mathrm{H1}}$) $(u,v)\to\bar{L}(t,u,v)$ is a $C^1\left(\mathbb{R}^2,\mathbb{R}\right)$ function for any $t\in I$;

($\overline{\mathrm{H2}}$) $t\to\bar{L}\left(t,\bar{y}^{\sigma}(t),\tilde{D}_{q,\omega}[\bar{y}](t)\right)$ is continuous at ω_0 for any admissible function $\bar{y}$;

($\overline{\mathrm{H3}}$) functions $t\to\partial_{i+2}\bar{L}\left(t,\bar{y}^{\sigma}(t),\tilde{D}_{q,\omega}[\bar{y}](t)\right)$ belong to $\mathcal{Y}^1\left([a,b]_{q,\omega},\mathbb{R}\right)$ for all admissible $\bar{y}$, $i=0,1$.

Lemma 3.11 (Leitmann's fundamental lemma via Hahn's symmetric quantum operator)**.** *Let $y = z(t, \bar{y})$ be a transformation having a unique inverse $\bar{y} = \bar{z}(t, y)$ for all $t \in [a, b]_{q,\omega}$, such that there is a one-to-one correspondence*

$$y(t) \leftrightarrow \bar{y}(t)$$

for all functions $y \in \mathcal{Y}^1\left([a, b]_{q,\omega}, \mathbb{R}\right)$ satisfying the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$ and all functions $\bar{y} \in \mathcal{Y}^1\left([a, b]_{q,\omega}, \mathbb{R}\right)$ satisfying

$$\bar{y}(a) = \bar{z}(a, \alpha) \quad \text{and} \quad \bar{y}(b) = \bar{z}(b, \beta). \tag{7}$$

If the transformation $y = z(t, \bar{y})$ is such that there exists a function $G : I \times \mathbb{R} \to \mathbb{R}$ satisfying the identity

$$L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right) - \bar{L}\left(t, \bar{y}^\sigma(t), \tilde{D}_{q,\omega}[\bar{y}](t)\right) = \tilde{D}_{q,\omega}[\tau \mapsto G(\tau, \bar{y}(\tau))](t),$$

$\forall t \in [a, b]_{q,\omega}$, then if $\bar{y}_*$ is a maximizer (resp. minimizer) of $\bar{\mathcal{L}}$ with $\bar{y}_*$ satisfying (7), $y_* = z(t, \bar{y}_*)$ is a maximizer (resp. minimizer) of $\mathcal{L}$ for y_* satisfying $y_*(a) = \alpha$ and $y_*(b) = \beta$.

Proof. Suppose $y \in \mathcal{Y}^1\left([a, b]_{q,\omega}, \mathbb{R}\right)$ satisfies the boundary conditions $y(a) = \alpha$ and $y(b) = \beta$. Define function $\bar{y} \in \mathcal{Y}^1\left([a, b]_{q,\omega}, \mathbb{R}\right)$ through the formula $\bar{y} = \bar{z}(t, y)$, $t \in [a, b]_{q,\omega}$. Then, $\bar{y}$ satisfies (7) and

$$\begin{aligned}
&\mathcal{L}(y) - \bar{\mathcal{L}}(\bar{y})\\
&\quad= \int_a^b L\left(t, y^\sigma(t), \tilde{D}_{q,\omega}[y](t)\right) \tilde{d}_{q,\omega}t - \int_a^b \bar{L}\left(t, \bar{y}^\sigma(t), \tilde{D}_{q,\omega}[\bar{y}](t)\right) \tilde{d}_{q,\omega}t\\
&\quad= \int_a^b \tilde{D}_{q,\omega}[\tau \mapsto G(\tau, \bar{y}(\tau))](t)\, \tilde{d}_{q,\omega}t\\
&\quad= G(b, \bar{y}(b)) - G(a, \bar{y}(a))\\
&\quad= G(b, \bar{z}(b, \beta)) - G(a, \bar{z}(a, \alpha)).
\end{aligned}$$

The desired result follows immediately because the right-hand side of the above equality is a constant, depending only on the fixed-endpoint conditions $y(a) = \alpha$ and $y(b) = \beta$. $\square$

Example 2. Let $q \in]0, 1[$, $\omega \geq 0$, and $a := \omega_0, b$ with $\omega_0 < b$ be fixed real numbers. Also, let I be an interval of $\mathbb{R}$ such that $\omega_0, b \in I$. We consider the problem

$$\begin{gathered}
\mathcal{L}(y) = \int_a^b \left(\left(\tilde{D}_{q,\omega}[y](t)\right)^2 + q y^\sigma(t) + t\tilde{D}_{q,\omega}[y](t)\right) \tilde{d}_{q,\omega}t \longrightarrow \min\\
y \in \mathcal{Y}^1\left([a, b]_{q,\omega}, \mathbb{R}\right)\\
y(a) = \alpha, \quad y(b) = \beta,
\end{gathered} \tag{8}$$

where $\alpha, \beta \in \mathbb{R}$ and $\alpha \neq \beta$. We transform problem (8) into the trivial problem

$$\begin{gathered}
\bar{\mathcal{L}}(\bar{y}) = \int_a^b \left(\tilde{D}_{q,\omega}[\bar{y}](t)\right)^2 \tilde{d}_{q,\omega}t \longrightarrow \min\\
\bar{y} \in \mathcal{Y}^1\left([a, b]_{q,\omega}, \mathbb{R}\right)\\
\bar{y}(a) = 0, \quad \bar{y}(b) = 0,
\end{gathered}$$

which has solution $\bar{y} \equiv 0$. For that we consider the transformation

$$y(t) = \bar{y}(t) + ct + d,$$

where c, d are real constants that will be chosen later. Since $y^{\sigma}(t) = \bar{y}^{\sigma}(t) + c\sigma(t) + d$ and $\tilde{D}_{q,\omega}[y](t) = \tilde{D}_{q,\omega}[\bar{y}](t) + c$, we have

$$\begin{aligned}
&\left(\tilde{D}_{q,\omega}[y](t)\right)^2 + qy^{\sigma}(t) + t\tilde{D}_{q,\omega}[y](t) \\
&= \left(\tilde{D}_{q,\omega}[\bar{y}](t)\right)^2 + 2c\tilde{D}_{q,\omega}[\bar{y}](t) + c^2 + qd + q\bar{y}^{\sigma}(t) + t\tilde{D}_{q,\omega}[\bar{y}](t) + c(q\sigma(t) + t).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left[\left(\tilde{D}_{q,\omega}[y](t)\right)^2 + qy^{\sigma}(t) + t\tilde{D}_{q,\omega}[y](t)\right] - \left(\tilde{D}_{q,\omega}[\bar{y}](t)\right)^2 \\
&= \tilde{D}_{q,\omega}[2c\bar{y}](t) + \tilde{D}_{q,\omega}\left[\left(c^2 + qd\right) id\right](t) + \tilde{D}_{q,\omega}[\sigma \cdot \bar{y}](t) + c\tilde{D}_{q,\omega}[\sigma \cdot id](t) \\
&= \tilde{D}_{q,\omega}\left[2c\bar{y} + \left(c^2 + qd\right) id + \sigma \cdot \bar{y} + c(\sigma \cdot id)\right](t),
\end{aligned}$$

where id represents the identity function. In order to obtain the solution to the original problem, it suffices to choose c and d such that

$$\begin{cases} ca + d = \alpha \\ cb + d = \beta. \end{cases} \tag{9}$$

Solving the system of equations (9) we obtain $c = \dfrac{\alpha - \beta}{a - b}$ and $d = \dfrac{a\beta - b\alpha}{a - b}$. Hence, the global minimizer to problem (8) is

$$y(t) = \frac{\alpha - \beta}{a - b}t + \frac{a\beta - b\alpha}{a - b}.$$

Acknowledgments. Work supported by *FEDER* funds through *COMPETE* — Operational Programme Factors of Competitiveness ("Programa Operacional Factores de Competitividade") and by Portuguese funds through the *Center for Research and Development in Mathematics and Applications* (University of Aveiro) and the Portuguese Foundation for Science and Technology ("FCT — Fundação para a Ciência e a Tecnologia"), within project PEst-C/MAT/UI4106/2011 with COMPETE number FCOMP-01-0124-FEDER-022690. The first author was also supported by FCT through the Ph.D. fellowship SFRH/BD/33634/2009.

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Received December 2011; 1^{st} revision September 2012; final revision November 2012.

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