Isoperimetric problems of the calculus of variations with fractional derivatives

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Abstract

In this paper we study isoperimetric problems of the calculus of variations with left and right Riemann-Liouville fractional derivatives. Both situations when the lower bound of the variational integrals coincide and do not coincide with the lower bound of the fractional derivatives are considered.

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1 Introduction

Isoperimetric problems consist in maximizing or minimizing a cost functional subject to integral constraints [5]. They have found a broad class of important applications throughout the centuries. Areas of application include astronomy, physics, geometry, algebra, and analysis [6, 17]. Concrete isoperimetric problems in engineering have been also investigated by a number of authors [18].

The study of isoperimetric problems is nowadays done, in an elegant and rigorously way, by means of the theory of the calculus of variations. This is possible through a powerful tool known as the Euler-Lagrange equation [33]. Recently the theory of the calculus of variations has been considered in the fractional context [7, 8, 10, 11, 12, 13, 14, 23, 25, 27, 32]. The fractional calculus allows to generalize the ordinary differentiation and integration to an arbitrary (non-integer) order, and provides a powerful tool for modeling and solving various problems in science and engineering [28, 29, 31]. The problems considered are more general, and hold for a bigger class of admissible functions which are not necessarily differentiable in the classical sense [30]. Several results were proved for the new calculus of variations. They include: Euler-Lagrange equations for fractional variational problems with Riemann-Liouville

[1], Riesz [3], Caputo, and (α, β) derivatives [19]; transversality conditions [2]; and Noether's symmetry theorem [21, 22]. For a state of the art of the fractional variational theory see the recent papers [4, 9, 15, 16, 20, 24] and references therein. In this paper we develop further the theory of the fractional variational calculus by studying isoperimetric problems.

The paper is organized as follows. In Section 2 we shortly review the necessary background on fractional calculus. Our results are given in Section 3. In Section 3.1 we introduce the basic fractional isoperimetric problem and prove correspondent necessary optimality conditions, both for normal and abnormal extremizers (Theorems 6 and 7, respectively). In Section 3.2 we generalize our results for functionals where the lower bound of the integral is greater than the lower bound of the Riemann-Liouville derivatives. Finally, in Section 3.3 we present a necessary condition of optimality for the case where the order of the derivative is taken as a free variable.

2 Preliminaries of fractional calculus

A fractional derivative is a generalization of the ordinary differentiation, which allows real number powers of the differential operator. There exist numerous applications of fractional derivatives to several fields, like geometry, physics, engineering, etc. In the literature we may find a great number of definitions for fractional derivatives (see, e.g., [28, 29, 31]). In this paper we deal with the left and right Riemann-Liouville fractional derivatives, which are defined in the following way.

Definition 1. *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. The left and right Riemann-Liouville fractional derivatives of order $\alpha > 0$ are defined respectively by*

$$ {}_a\mathcal{D}_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-t)^{n-\alpha-1} f(t)dt \,, \quad x \in (a, b] \,, $$

and

$$ {}_x\mathcal{D}_b^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_x^b (t-x)^{n-\alpha-1} f(t)dt \,, \quad x \in [a, b) \,, $$

where Γ is the Euler gamma function, α is the order of the derivative, and $n = [\alpha] + 1$ with $[\alpha]$ being the integer part of α.

If $\alpha \geq 1$ is an integer, these fractional derivatives are understood in the sense of usual differentiation, that is,

$$ {}_a\mathcal{D}_x^\alpha f(x) = \left(\frac{d}{dx}\right)^\alpha f(x) \quad \text{and} \quad {}_x\mathcal{D}_b^\alpha f(x) = \left(-\frac{d}{dx}\right)^\alpha f(x). $$

¿From the physical point of view, if $f(x)$ describes a certain process through time x, then the left derivative is related to the past of this process, while the right derivative belongs to the future.

These operations are linear, in the sense that

$$ {}_a\mathcal{D}_x^\alpha (\mu f(x) + \nu g(x)) = \mu \, {}_a\mathcal{D}_x^\alpha f(x) + \nu \, {}_a\mathcal{D}_x^\alpha g(x) $$

and

$$ {}_x\mathcal{D}_b^\alpha (\mu f(x) + \nu g(x)) = \mu \, {}_x\mathcal{D}_b^\alpha f(x) + \nu \, {}_x\mathcal{D}_b^\alpha g(x). $$

We now present the integration by parts formula for fractional derivatives.

Lemma 2. ([31, p. 46]) *If f and g and the fractional derivatives ${}_a\mathcal{D}_x^\alpha g$ and ${}_x\mathcal{D}_b^\alpha f$ are continuous at every point $x \in [a,b]$, then for $0 < \alpha < 1$ we have*

$$\int_a^b f(x)_a\mathcal{D}_x^\alpha g(x)dx = \int_a^b g(x)_x\mathcal{D}_b^\alpha f(x)dx. \tag{1}$$

Moreover, formula (1) is still valid for $\alpha = 1$ provided f or g are zero at $x = a$ and $x = b$.

3 Main results

¿From now on we fix $\alpha, \beta \in (0,1)$. We consider functionals $\mathcal{J}$ of the form

$$\mathcal{J}(y) = \int_a^b L(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)dx \tag{2}$$

defined on the set of admissible functions y that have continuous left fractional derivatives of order α and continuous right fractional derivatives of order β in $[a,b]$, and where $(x,y,u,v) \to L(x,y,u,v)$ is a function with continuous first and second partial derivatives with respect to all its arguments such that $\frac{\partial L}{\partial u}(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)$ has continuous right fractional derivative of order α for all $x \in [a,b]$ and $\frac{\partial L}{\partial v}(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)$ has continuous left fractional derivative of order β in $[a,b]$.

Remark 1. *The left Riemann-Liouville fractional derivative is infinite at $x = a$ if $y(a) \neq 0$. If $y(b) \neq 0$, then the right Riemann-Liouville fractional derivative is also not finite at $x = b$ [30]. For this reason, by considering that the admissible functions y have continuous left fractional derivatives, then necessarily $y(a) = 0$; by considering that the admissible functions y have continuous right fractional derivatives, then necessarily $y(b) = 0$. This fact seems to have been neglected in some previous work on the calculus of variations with Riemann-Liouville fractional derivatives. Alternatively, we can consider the general case of boundary conditions, say $y(a) = y_a$ and $y(b) = y_b$, and study functionals of type*

$$\mathcal{J}(y) = \int_a^b L(x, y(x), {}_a\mathcal{D}_x^\alpha(y(x) - y_a), {}_x\mathcal{D}_b^\beta(y(x) - y_b))dx.$$

This needs, however, a modified fractional calculus [26].

Definition 3. *The functional $\mathcal{J}$ is said to have a* local minimum *(resp.* local maximum*) at y if there exists a $\delta > 0$ such that $\mathcal{J}(y) \leq \mathcal{J}(y_1)$ (resp. $\mathcal{J}(y) \geq \mathcal{J}(y_1)$) for all y_1 such that $\|y - y_1\| < \delta$.*

In [1] the following problem is addressed: among all curves $y(x)$ satisfying the boundary conditions, find the ones that maximize or minimize a given functional $\mathcal{J}$. An answer to this question is given in the next theorem.

Theorem 4 ([1])**.** *Let $\mathcal{J}$ be a functional as in (2) and y an extremum of $\mathcal{J}$. Then, y satisfies the following Euler-Lagrange equation:*

$$\frac{\partial L}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial L}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial L}{\partial v} = 0. \tag{3}$$

3.1 The fractional isoperimetric problem

We introduce the *fractional isoperimetric problem* as follows: find the functions y that satisfy boundary conditions

$$y(a) = y_a, \quad y(b) = y_b \tag{4}$$

($y_a = 0$ if left Riemann-Liouville fractional derivatives are present in (2); $y_b = 0$ if right Riemann-Liouville fractional derivatives are present in (2) — cf. Remark 1), the integral constraint

$$\mathcal{I}(y) = \int_a^b g(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)dx = l\,, \tag{5}$$

and give a minimum or a maximum to (2). We assume that l is a specified real constant, functions y have continuous left and right fractional derivatives (if present in (2)), and $(x, y, u, v) \to g(x, y, u, v)$ is a function with continuous first and second partial derivatives with respect to all its arguments such that $\frac{\partial g}{\partial u}(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)$ has continuous right fractional derivative of order α for all $x \in [a, b]$ and $\frac{\partial g}{\partial v}(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y)$ has continuous left fractional derivative of order β in $[a, b]$. Theorem 4 motivates the following definition.

Definition 5. *An admissible function y is an* extremal *for $\mathcal{I}$ in (5) if it satisfies the equation*

$$\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v} = 0$$

for all $x \in [a, b]$.

The following theorem gives a necessary condition for y to be a solution of the fractional isoperimetric problem defined by (2)-(4)-(5) under the assumption that y is not an extremal for $\mathcal{I}$.

Theorem 6. *Suppose that $\mathcal{J}$ given by (2) has a local minimum or a local maximum at y subject to the boundary conditions (4) and the isoperimetric constraint (5). Further, suppose that y is not an extremal for the functional $\mathcal{I}$. Then there exists a constant λ such that y satisfies the fractional differential equation*

$$\frac{\partial F}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial F}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial F}{\partial v} = 0 \tag{6}$$

with $F = L - \lambda g$.

Proof. Consider neighboring functions of the form

$$\hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2, \tag{7}$$

where for each $i \in \{1, 2\}$ ϵ_i is a sufficiently small parameter, η_i have continuous left and right fractional derivatives, and $\eta_i(a) = \eta_i(b) = 0$.

First we will show that (7) has a subset of admissible functions for the fractional isoperimetric problem. Consider the quantity

$$\mathcal{I}(\hat{y}) = \int_a^b g(x, y + \epsilon_1\eta_1 + \epsilon_2\eta_2, {}_a\mathcal{D}_x^\alpha y + \epsilon_1 {}_a\mathcal{D}_x^\alpha \eta_1 + \epsilon_2 {}_a\mathcal{D}_x^\alpha \eta_2, {}_x\mathcal{D}_b^\beta y + \epsilon_1 {}_x\mathcal{D}_b^\beta \eta_1 + \epsilon_2 {}_x\mathcal{D}_b^\beta \eta_2)dx.$$

Then we can regard $\mathcal{I}(\hat{y})$ as a function of ϵ_1 and ϵ_2. Define $\hat{I}(\epsilon_1, \epsilon_2) = \mathcal{I}(\hat{y}) - l$. Thus,

$$\hat{I}(0,0) = 0. \tag{8}$$

On the other hand, we have

$$\begin{aligned}
\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} &= \int_a^b \left[\frac{\partial g}{\partial y}\eta_2 + \frac{\partial g}{\partial u}{}_a\mathcal{D}_x^\alpha \eta_2 + \frac{\partial g}{\partial v}{}_x\mathcal{D}_b^\beta \eta_2\right] dx \\
&= \int_a^b \left[\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right] \eta_2 dx,
\end{aligned} \tag{9}$$

where (9) follows from (1). Since y is not an extremal for $\mathcal{I}$, by the fundamental lemma of the calculus of variations (see, e.g., [33, p. 32]), there exists a function η_2 such that

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} \neq 0. \tag{10}$$

Using (8) and (10), the implicit function theorem asserts that there exists a function $\epsilon_2(\cdot)$, defined in a neighborhood of zero, such that $\hat{I}(\epsilon_1, \epsilon_2(\epsilon_1)) = 0$. We are now in a position to derive the necessary condition (6). Consider the real function $\hat{J}(\epsilon_1, \epsilon_2) = \mathcal{J}(\hat{y})$. By hypothesis, $\hat{J}$ has minimum (or maximum) at $(0,0)$ subject to the constraint $\hat{I}(0,0) = 0$, and we have proved that $\nabla \hat{I}(0,0) \neq \mathbf{0}$. We can appeal to the Lagrange multiplier rule (see, e.g., [33, p. 77]) to assert the existence of a number λ such that $\nabla(\hat{J}(0,0) - \lambda \hat{I}(0,0)) = \mathbf{0}$. Repeating the calculations as before,

$$\left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \left[\frac{\partial L}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial L}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right] \eta_1(x) dx$$

and

$$\left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} = \int_a^b \left[\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right] \eta_1(x) dx\,.$$

Therefore, one has

$$\int_a^b \left[\frac{\partial L}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial L}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - \lambda\left(\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right)\right] \eta_1(x) dx = 0. \tag{11}$$

Since (11) holds for any function η_1, we obtain (6):

$$\frac{\partial L}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial L}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - \lambda\left(\frac{\partial g}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial g}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right) = 0.$$

□

Remark 2. *Theorem 6 holds true in the case when α or β are equal to* 1*. Indeed, in the proof we imposed the condition $\eta_2(a) = \eta_2(b) = 0$, and formula (1) is valid.*

Example 1. *Let α be a given number in the interval $(0,1)$. We consider the following fractional isoperimetric problem:*

$$\begin{gathered}\int_0^1 (x^4 + ({}_0\mathcal{D}_x^\alpha y)^2)dx \longrightarrow \min \\ \int_0^1 x^2 {}_0\mathcal{D}_x^\alpha y \, dx = \frac{1}{5} \\ y(0)=0\,, \quad y(1) = \frac{2}{2\alpha+3\alpha^2+\alpha^3}\,.\end{gathered} \tag{12}$$

The augmented Lagrangian is

$$F(x, y, {}_0\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_1^\beta y) = x^4 + ({}_0\mathcal{D}_x^\alpha y)^2 - \lambda\, x^2 {}_0\mathcal{D}_x^\alpha y$$

and it is a simple exercise to see that

$$y(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{t^2}{(x-t)^{1-\alpha}} dt = \frac{1}{\Gamma(\alpha)} \frac{2x^{\alpha+2}}{2\alpha+3\alpha^2+\alpha^3} \tag{13}$$

(i) is not an extremal for the isoperimetric functional, (ii) satisfy ${}_0\mathcal{D}_x^\alpha y = x^2$, (iii) (6) holds for $\lambda = 2$, i.e., ${}_x\mathcal{D}_1^\alpha(2\,{}_0\mathcal{D}_x^\alpha y - 2x^2) = 0$. We remark that for $\alpha = 1$ (13) gives $y(x) = x^3/3$, which coincides with the solution of the associated classical variational problem (Fig. 1). Indeed, for

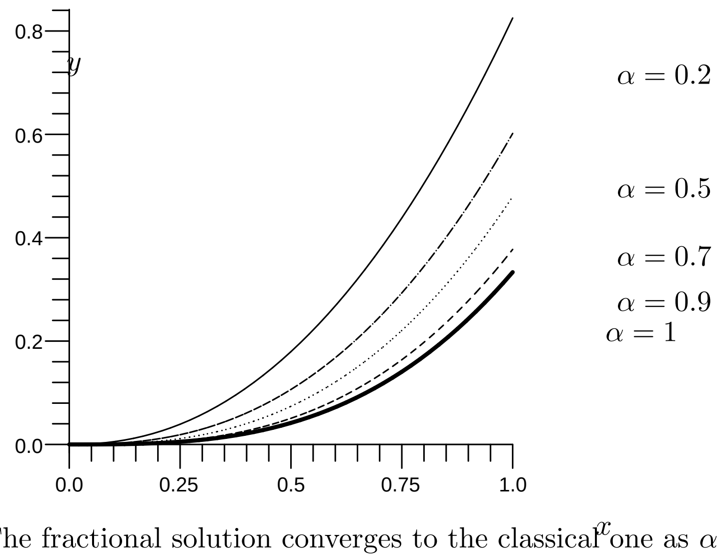

Figure 1: The fractional solution converges to the classical one as $\alpha \to 1$.

$\alpha \to 1$ our fractional problem (12) tends to the classical isoperimetric problem of minimizing the functional $\int_0^1 (x^4 + (y')^2)dx$ subject to the isoperimetric constraint $\int_0^1 x^2 y'\, dx = \frac{1}{5}$ and the boundary conditions $y(0) = 0$ and $y(1) = 1/3$. Then, $F = x^4 + (y')^2 - \lambda x^2 y'$ and the classical Euler-Lagrange equation is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0 \Leftrightarrow -2y'' + 2\lambda x = 0. \tag{14}$$

The solution of (14) subject to $y(0) = 0$, $y(1) = 1/3$, and $\int_0^1 x^2 y'\, dx = \frac{1}{5}$ is $\lambda = 2$ and $y = x^3/3$.

Introducing a multiplier λ_0 associated with the cost functional (2), we can easily include in Theorem 6 the situation when the solution of the fractional isoperimetric problem defined by (2)-(4)-(5) is an extremal for the fractional isoperimetric functional. This is done in Theorem 7.

Theorem 7. *If y is a local minimizer or a local maximizer of (2) subject to the boundary conditions (4) and the isoperimetric constraint (5), then there exist two constants λ_0 and λ, not both zero, such that*

$$\frac{\partial K}{\partial y} + {}_x\mathcal{D}_b^\alpha \frac{\partial K}{\partial u} + {}_a\mathcal{D}_x^\beta \frac{\partial K}{\partial v} = 0 \tag{15}$$

with $K = \lambda_0 L - \lambda g$.

Proof. Using the same notation as in the proof of Theorem 6, we have that $(0,0)$ is an extremal of $\hat{J}$ subject to the constraint $\hat{I} = 0$. Then, by the abnormal Lagrange multiplier rule (see, e.g., [33, p. 82]) there exist two reals λ_0 and λ, not both zero, such that $\nabla(\lambda_0 \hat{J}(0,0) - \lambda \hat{I}(0,0)) = \mathbf{0}$. Therefore,

$$\lambda_0 \left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} - \lambda \left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} = 0.$$

Applying the same reasoning as in the proof of Theorem 6, we end up with (15). □

3.2 An extension

In [9] a fractional functional

$$\mathcal{L}(y) = \int_A^B L(x, y, {}_a\mathcal{D}_x^\alpha y) dx \tag{16}$$

is considered with $[A, B] \subset [a, b]$, i.e., with the lower bound of the integral not coinciding with the lower bound of the fractional derivative. The main result of [9] is a new Euler-Lagrange equation for the functional (16). We now extend the techniques of [9] to prove an Euler-Lagrange equation for functionals containing both left and right Riemann-Liouville fractional derivatives, i.e., for fractional functionals of the form

$$\mathcal{J}(y) = \int_A^B L(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y) dx, \tag{17}$$

where the integrand L satisfies the same conditions as before. Let y be a local extremum of $\mathcal{J}$ such that $y(a) = y_a$ and $y(b) = y_b$, and let $\hat{y} = y + \epsilon\eta$ with $\eta(a) = \eta(b) = 0$. Consider the function $\hat{J}(\epsilon) = \mathcal{J}(y + \epsilon\eta)$. Since $\hat{J}(\epsilon)$ has a local extremum at $\epsilon = 0$, then

$$\begin{aligned}
0 &= \int_A^B \left[\frac{\partial L}{\partial y} \cdot \eta + \frac{\partial L}{\partial u} \cdot {}_a\mathcal{D}_x^\alpha \eta + \frac{\partial L}{\partial v} \cdot {}_x\mathcal{D}_b^\beta \eta\right] dx \\
&= \int_A^B \frac{\partial L}{\partial y} \cdot \eta dx + \left[\int_a^B \frac{\partial L}{\partial u} \cdot {}_a\mathcal{D}_x^\alpha \eta dx - \int_a^A \frac{\partial L}{\partial u} \cdot {}_a\mathcal{D}_x^\alpha \eta dx\right] \\
&\quad + \left[\int_A^b \frac{\partial L}{\partial v} \cdot {}_x\mathcal{D}_b^\beta \eta dx - \int_B^b \frac{\partial L}{\partial v} \cdot {}_x\mathcal{D}_b^\beta \eta dx\right] \\
&= \int_A^B \frac{\partial L}{\partial y} \cdot \eta dx + \left[\int_a^B \eta \cdot {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} dx - \int_a^A \eta \cdot {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u} dx\right] \\
&\quad + \left[\int_A^b \eta \cdot {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} dx - \int_B^b \eta \cdot {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v} dx\right].
\end{aligned}$$

Continuing in a similar way,

$$
\begin{aligned}
0 &= \int_A^B \frac{\partial L}{\partial y}\cdot \eta dx + \left[\int_a^A \eta \cdot {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} dx + \int_A^B \eta \cdot {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} dx - \int_a^A \eta \cdot {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u} dx\right] \\
&+ \left[\int_A^B \eta \cdot {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} dx + \int_B^b \eta \cdot {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} dx - \int_B^b \eta \cdot {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v} dx\right] \\
&= \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u}\right] \eta dx + \int_A^B \left[\frac{\partial L}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right] \eta dx \\
&+ \int_B^b \left[{}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right] \eta dx \,.
\end{aligned}
$$

Let $\eta_1 : [a, A] \to \mathbb{R}$ be any function satisfying $\eta_1(a) = 0$, and η be given by

$$
\eta(x) = \begin{cases} \eta_1(x) & \text{if } x \in [a, A]\,, \\ 0 & \text{elsewhere.} \end{cases}
$$

Therefore,

$$
0 = \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u}\right] \eta_1 dx.
$$

By the arbitrariness of η_1 and the fundamental lemma of calculus of variations,

$$
{}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u} = 0 \text{ for all } x \in [a, A].
$$

Analogously, we have

$$
\frac{\partial L}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} = 0 \text{ for all } x \in [A, B]\,,
$$

and

$$
{}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v} = 0 \text{ for all } x \in [B, b].
$$

We have just proved the following.

Theorem 8. *Let y be a local extremizer of (17). Then, y satisfies the following equations:*

$$
\begin{cases}
\frac{\partial L}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} = 0 & \text{for all } x \in [A, B]\,, \\
{}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u} = 0 & \text{for all } x \in [a, A]\,, \\
{}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v} = 0 & \text{for all } x \in [B, b].
\end{cases}
$$

Remark 3. *Theorem 8 simplifies to the result proved in [9] in the case the Lagrangian L in (17) does not depend on the right Riemann-Liouville fractional derivative ${}_x\mathcal{D}_b^\beta y$.*

We will study now the fractional isoperimetric problem for functionals of type (17) subject to an integral constraint

$$
\mathcal{I}(y) = \int_A^B g(x, y, {}_a\mathcal{D}_x^\alpha y, {}_x\mathcal{D}_b^\beta y) dx = l\,. \tag{18}
$$

Definition 9. *We say that y is an* extremal *for functional $\mathcal{I}$ given in (18) if*

$$\frac{\partial g}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v} = 0 \qquad \text{for all } x \in [A, B].$$

Theorem 10. *Let y give a local minimum or a local maximum to the fractional functional (17) subject to the constraint (18). If y is not an extremal for $\mathcal{I}$, then there exists a constant λ such that*

$$\begin{cases} \frac{\partial F}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial F}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial F}{\partial v} = 0 & \text{for all } x \in [A, B] \\ {}_x\mathcal{D}_B^\alpha \frac{\partial F}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial F}{\partial u} = 0 & \text{for all } x \in [a, A] \\ {}_A\mathcal{D}_x^\beta \frac{\partial F}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial F}{\partial v} = 0 & \text{for all } x \in [B, b] \end{cases} \tag{19}$$

with $F = L - \lambda g$.

Proof. Consider a variation $(\epsilon_1, \epsilon_2) \mapsto \hat{y} = y + \epsilon_1\eta_1 + \epsilon_2\eta_2$ where $\eta_1(a) = \eta_1(b) = \eta_2(a) = \eta_2(b) = 0$. Let

$$\hat{I}(\epsilon_1, \epsilon_2) = \int_A^B g(x, \hat{y}, {}_a\mathcal{D}_x^\alpha \hat{y}, {}_x\mathcal{D}_b^\beta \hat{y}) dx - l.$$

Then, $\hat{I}(0, 0) = 0$ and

$$\begin{aligned} \left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} &= \int_A^B \left[\frac{\partial g}{\partial y}\eta_2 + \frac{\partial g}{\partial u}{}_a\mathcal{D}_x^\alpha \eta_2 + \frac{\partial g}{\partial v}{}_x\mathcal{D}_b^\beta \eta_2\right] dx \\ &= \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial g}{\partial u}\right] \eta_2 dx + \int_A^B \left[\frac{\partial g}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right] \eta_2 dx \\ &+ \int_B^b \left[{}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right] \eta_2 dx. \end{aligned}$$

Since y is not an extremal for $\mathcal{I}$, there exists a function η_2 such that $\left.\frac{\partial \hat{I}}{\partial \epsilon_2}\right|_{(0,0)} \neq 0$. By the implicit function theorem, there exists a subset of curves $\{y + \epsilon_1\eta_1 + \epsilon_2\eta_2 \,|\, (\epsilon_1, \epsilon_2) \in \mathbb{R}^2\}$ admissible for the fractional isoperimetric problem. Let $\hat{J}(\epsilon_1, \epsilon_2) = \mathcal{J}(\hat{y})$. Then, there exists a real λ such that $\nabla(\hat{J}(0, 0) - \lambda\hat{I}(0, 0)) = \mathbf{0}$. Because

$$\begin{aligned} \left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} &= \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial L}{\partial u}\right] \eta_1 dx + \int_A^B \left[\frac{\partial L}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial L}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right] \eta_1 dx \\ &+ \int_B^b \left[{}_A\mathcal{D}_x^\beta \frac{\partial L}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial L}{\partial v}\right] \eta_1 dx, \\ \left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} &= \int_a^A \left[{}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial g}{\partial u}\right] \eta_1 dx + \int_A^B \left[\frac{\partial g}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial g}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right] \eta_1 dx \\ &+ \int_B^b \left[{}_A\mathcal{D}_x^\beta \frac{\partial g}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial g}{\partial v}\right] \eta_1 dx, \end{aligned}$$

$$\left.\frac{\partial \hat{J}}{\partial \epsilon_1}\right|_{(0,0)} - \lambda \left.\frac{\partial \hat{I}}{\partial \epsilon_1}\right|_{(0,0)} = 0,$$

and η_1 is an arbitrary function, it follows (19). □

Similarly as before, we can include in Theorem 10 the situation when the solution y is an extremal for $\mathcal{I}$ (abnormal extremizer). For that we introduce a new multiplier λ_0 that will be zero when the solution y is an extremal for $\mathcal{I}$ and one otherwise.

Theorem 11. *If y is a local minimizer or a local maximizer of (17) subject to the isoperimetric constraint (18), then there exist two constants λ_0 and λ, not both zero, such that*

$$\begin{cases} \frac{\partial K}{\partial y} + {}_x\mathcal{D}_B^\alpha \frac{\partial K}{\partial u} + {}_A\mathcal{D}_x^\beta \frac{\partial K}{\partial v} = 0 & \text{for all } x \in [A,B] \\ {}_x\mathcal{D}_B^\alpha \frac{\partial K}{\partial u} - {}_x\mathcal{D}_A^\alpha \frac{\partial K}{\partial u} = 0 & \text{for all } x \in [a,A] \\ {}_A\mathcal{D}_x^\beta \frac{\partial K}{\partial v} - {}_B\mathcal{D}_x^\beta \frac{\partial K}{\partial v} = 0 & \text{for all } x \in [B,b] \end{cases}$$

with $K = \lambda_0 L - \lambda g$.

3.3 Dependence on a parameter

Consider the following fractional problem of the calculus of variations: to extremize the functional

$$\Psi(y) = \int_0^1 \left[\frac{x^\alpha}{\Gamma(\alpha+1)}({}_0D_x^\alpha y)^2 - 2\overline{y}\,{}_0D_x^\alpha y\right]^2 dx$$

when subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 1.$$

Here, $\overline{y} := x^\alpha, x \in [0,1]$. The fractional Euler-Lagrange associated to this problem is

$$ {}_xD_1^\alpha \left(2\left[\frac{x^\alpha}{\Gamma(\alpha+1)}({}_0D_x^\alpha y)^2 - 2\overline{y}\,{}_0D_x^\alpha y\right] \cdot \left[\frac{2x^\alpha}{\Gamma(\alpha+1)}{}_0D_x^\alpha y - 2\overline{y}\right]\right) = 0. \tag{20}$$

Replacing y by $\overline{y}$, and since ${}_0D_x^\alpha \overline{y} = \Gamma(\alpha+1)$, we conclude that $\overline{y}$ is a solution of (20).

Consider now the following problem: what is the order of the derivative α, such that $\Psi(\overline{y})$ attains a maximum or a minimum? In other words, find the extremizers for $\psi(\alpha) = \Psi(\overline{y})$. Direct computations show that

$$\psi(\alpha) = \int_0^1 [x^\alpha \Gamma(\alpha+1)]^2\, dx.$$

Evaluating its derivative,

$$\begin{aligned} \psi'(\alpha) &= \int_0^1 \frac{d}{d\alpha}[x^\alpha\Gamma(\alpha+1)]^2\, dx \\ &= \int_0^1 2x^\alpha\Gamma(\alpha+1)\left[x^\alpha \ln x\,\Gamma(\alpha+1) + x^\alpha \int_0^\infty t^\alpha \ln t\, e^{-t} dt\right] dx. \end{aligned}$$

We have that $\alpha \approx 0.901$ is a solution of the equation $\psi'(\alpha) = 0$, and such value is precisely where $\Psi(\overline{y})$ attains a minimum.

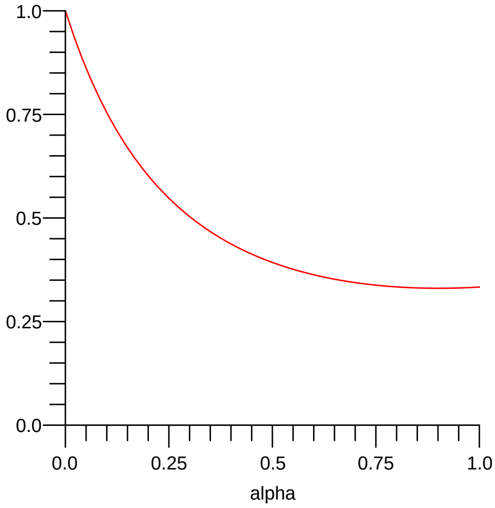

Figure 2: Graph of $\Psi(\overline{y})$ for $\alpha \in [0,1]$

More generally, consider the functional

$$\Phi(y,\alpha) = \int_a^b L(x, y(x), {}_aD_x^\alpha y(x))dx. \tag{21}$$

Functional (21) contains the left Riemann-Liouville derivative only, but we can consider functionals containing right Riemann-Liouville derivatives or both in a similar way. Let h be a curve such that $h(a) = h(b) = 0$, δ be a real number, and (y, α) be an extremal for Φ. Then,

$$\Phi(y+h, \alpha+\delta) - \Phi(y,\alpha)$$
$$= \int_a^b \frac{\partial L}{\partial y} \cdot h + \frac{\partial L}{\partial u} \cdot {}_aD_x^{\alpha+\delta} h + \frac{\partial L}{\partial u} \cdot ({}_aD_x^{\alpha+\delta} y - {}_aD_x^\alpha y)dx + O|(h,\delta)|^2 .$$

For $\delta = 0$, using the fractional integration by parts formula and the fundamental lemma of the calculus of variations, we obtain the known fractional Euler-Lagrange equation:

$$\frac{\partial L}{\partial y}(x, y(x), {}_aD_x^\alpha y(x)) + {}_xD_b^\alpha \frac{\partial L}{\partial u}(x, y(x), {}_aD_x^\alpha y(x)) = 0.$$

For $h = 0$, we obtain the relation

$$\int_a^b \frac{\partial L}{\partial u}(x, y(x), {}_aD_x^\alpha y(x))\phi'(\alpha)dx = 0,$$

where $\phi(\alpha) = {}_aD_x^\alpha y(x)$. In summary, we have:

Theorem 12. *If (y,α) is an extremal of Φ given by (21), satisfying the boundary conditions $y(a) = 0$ and $y(b) = y_b$, then y satisfies the system*

$$\begin{cases} \dfrac{\partial L}{\partial y}(x, y(x), {}_aD_x^\alpha y(x)) + {}_xD_b^\alpha \dfrac{\partial L}{\partial u}(x, y(x), {}_aD_x^\alpha y(x)) = 0 \\ \displaystyle\int_a^b \frac{\partial L}{\partial u}(x, y(x), {}_aD_x^\alpha y(x))\phi'(\alpha)dx = 0 \end{cases} \tag{22}$$

where $\phi(\alpha) = {}_aD_x^\alpha y(x)$.

In the previous example the solution obtained satisfies system (22) since

$$\frac{\partial L}{\partial u}(x, \overline{y}(x), {}_aD_x^\alpha \overline{y}(x))) = 0.$$

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References

[1] O. P. Agrawal, Formulation of Euler-Lagrange equations for fractional variational problems, J. Math. Anal. Appl. **272** (2002), no. 1, 368–379.

[2] O. P. Agrawal, Fractional variational calculus and the transversality conditions, J. Phys. A **39** (2006), no. 33, 10375–10384.

[3] O. P. Agrawal, Fractional variational calculus in terms of Riesz fractional derivatives, J. Phys. A **40** (2007), no. 24, 6287–6303.

[4] R. Almeida, A. B. Malinowska and D. F. M. Torres, A fractional calculus of variations for multiple integrals with application to vibrating string, J. Math. Phys. **51** (2010), no. 3, 033503, 12 pp.

[5] R. Almeida and D. F. M. Torres, Hölderian variational problems subject to integral constraints, J. Math. Anal. Appl. **359** (2009), no. 2, 674–681.

[6] R. Almeida and D. F. M. Torres, Isoperimetric problems on time scales with nabla derivatives, J. Vib. Control **15** (2009), no. 6, 951–958.

[7] R. Almeida and D. F. M. Torres, Calculus of variations with fractional derivatives and fractional integrals, Appl. Math. Lett. **22** (2009), no. 12, 1816–1820.

[8] R. Almeida and D. F. M. Torres, Leitmann’s direct method for fractional optimization problems, Appl. Math. Comput. **217** (2010), no. 3, 956–962.

[9] T. M. Atanacković, S. Konjik and S. Pilipović, Variational problems with fractional derivatives: Euler-Lagrange equations, J. Phys. A **41** (2008), no. 9, 095201, 12 pp.

[10] D. Baleanu, New applications of fractional variational principles, Rep. Math. Phys. **61** (2008), no. 2, 199–206.

[11] D. Baleanu, Fractional variational principles in action, Phys. Scr. **T136** (2009), 014006.

[12] D. Baleanu, T. Maaraba and F. Jarad, Fractional variational principles with delay, J. Phys. A **41** (2008), no. 31, 315403, 8 pp.

[13] D. Baleanu, S. I. Muslih and E. M. Rabei, On fractional Euler-Lagrange and Hamilton equations and the fractional generalization of total time derivative, Nonlinear Dynam. **53** (2008), no. 1-2, 67–74.

[14] D. Baleanu and J. J. Trujillo, On exact solutions of a class of fractional Euler-Lagrange equations, Nonlinear Dynam. **52** (2008), no. 4, 331–335.

[15] D. Baleanu and J. I. Trujillo, A new method of finding the fractional Euler-Lagrange and Hamilton equations within Caputo fractional derivatives, Commun. Nonlinear Sci. Numer. Simul. **15** (2010), no. 5, 1111–1115.

[16] N. R. O. Bastos, R. A. C. Ferreira and D. F. M. Torres, Discrete-time fractional variational problems, Signal Process. **91** (2011), no. 3, 513–524.

[17] V. Blåsjö, The isoperimetric problem, Amer. Math. Monthly **112** (2005), no. 6, 526–566.

[18] J. P. Curtis, Complementary extremum principles for isoperimetric optimization problems, Optim. Eng. **5** (2004), no. 4, 417–430.

[19] R. A. El-Nabulsi and D. F. M. Torres, Necessary optimality conditions for fractional action-like integrals of variational calculus with Riemann-Liouville derivatives of order (α, β), Math. Methods Appl. Sci. **30** (2007), no. 15, 1931–1939.

[20] R. A. El-Nabulsi and D. F. M. Torres, Fractional actionlike variational problems, J. Math. Phys. **49** (2008), no. 5, 053521, 7 pp.

[21] G. S. F. Frederico and D. F. M. Torres, A formulation of Noether's theorem for fractional problems of the calculus of variations, J. Math. Anal. Appl. **334** (2007), no. 2, 834–846.

[22] G. S. F. Frederico and D. F. M. Torres, Fractional conservation laws in optimal control theory, Nonlinear Dynam. **53** (2008), no. 3, 215–222.

[23] G. S. F. Frederico and D. F. M. Torres, Fractional Noether's theorem in the Riesz-Caputo sense, Appl. Math. Comput. **217** (2010), no. 3, 1023–1033.

[24] M. A. E. Herzallah and D. Baleanu, Fractional-order Euler-Lagrange equations and formulation of Hamiltonian equations, Nonlinear Dynam. **58** (2009), no. 1-2, 385–391.

[25] F. Jarad, T. Maaraba and D. Baleanu, Fractional variational principles with delay within caputo derivatives, Rep. Math. Phys. **65** (2010), no. 1, 17–28.

[26] G. Jumarie, Table of some basic fractional calculus formulae derived from a modified Riemann-Liouville derivative for non-differentiable functions, Appl. Math. Lett. **22** (2009), no. 3, 378–385.

[27] A. B. Malinowska and D. F. M. Torres, Generalized natural boundary conditions for fractional variational problems in terms of the Caputo derivative, Comput. Math. Appl. **59** (2010), no. 9, 3110–3116.

[28] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

[29] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, CA, 1999.

[30] B. Ross, S. G. Samko and E. R. Love, Functions that have no first order derivative might have fractional derivatives of all orders less than one, Real Anal. Exchange **20** (1994/95), no. 1, 140–157.

[31] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional integrals and derivatives*, Translated from the 1987 Russian original, Gordon and Breach, Yverdon, 1993.

[32] V. E. Tarasov and G. M. Zaslavsky, Nonholonomic constraints with fractional derivatives, J. Phys. A **39** (2006), no. 31, 9797–9815.

[33] B. van Brunt, *The calculus of variations*, Springer, New York, 2004.